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A QUALITATIVE STUDY ON FRACTIONAL LOGISTIC INTEGRO-DIFFERENTIAL EQUATIONS IN AN ARBITRARY TIME SCALE

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Abstract. This manuscript deals with the investigation related to uniqueness and existence of solution of fractional order nonlinear pantograph integro-differential equation in arbitrary time scale. The fractional derivatives are defined in Riemann-Liouville sense, the primary tools are taken as Banach contraction principle and Schauder's fixed point theory to establish the theoretical outcomes. Finally, we give examples to show the efficiency of our results.

1. Introduction

The theory on time scale calculus is a new field of interest for the mathematicians. This particular branch was first encountered by Aulbach and Hilger [20] in the year 1990. To unify the calculus for both discrete and continuous problems, time scale theory was introduced. Due to the unification nature the topic "time scale", frequently appears in numerous physical modelling problems, where the discrete and continuous data are simultaneously involved. In current time, it is a topic of vigorous research in diverse areas, such as economics, control theory, robotics, biology, quantum calculus and many other fields. Different problems in engineering and natural phenomenons are extensively modelled into fractional equations. See, for example [13, 16, 17, 27, 28, 33–35, 38] and references cited therein. Over the past few years different researchers have qualitatively studied fractional integro-differential equations and time scales integro-differential equations separately, see [1, 4, 5, 11, 12, 21, 32] and references cited

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therein. But as per our knowledge, a less number of work has been reported in the combined treatment for time scale fractional integro-differential equations.

In this study, our main objective is to provide a theoretical platform to investigate existence and uniqueness of solution for the considered types of equations. With this motivation, we introduce the following nonlinear equation

$$ {}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta} w(\theta) = \mu w(\theta)\Big(u(\theta) - a(\theta)w(\theta) - b(\theta)\int_{\theta_0}^{\Theta} k_*(s)\mathcal{N}(w(s))\Delta s\Big), \quad \theta \in (\theta_0, \Theta), \tag{1.1} $$

we treat $\gamma \in (0,1)$ as the order of fractional derivative, μ is nonzero constant; ${}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta}$ is the left differentiation operator of order γ in Riemann-Liouville sense (**L**-RLFD); ${}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{\xi}_{\theta}$ is the left integral operator of order ξ in Riemann-Liouville fractional sense (**L**-RLFI). Moreover, $\mathbb{T}$ represents arbitrary time scale; $u(\theta), a(\theta), b(\theta)$ are continuous, nonnegative and bounded functions; $k_* : [\theta_0, \Theta]_{\mathbb{T}} \to [0, \infty)$ is a smooth function on $[\theta_0, \Theta]_{\mathbb{T}}$. This manuscript deals with Equation (1.1) subject to the initial condition defined as

$$ {}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{\xi}_{\theta} w(\theta_0) = 0, \quad \text{for all } \xi \in (0,1). \tag{1.2} $$

One can relate Equation (1.1) and Condition (1.2) as model problem for mix of stop-start phenomenon, such as insect populations that are smooth during the incubating season and die out in winter. If we include toxic effect in the insect population model, then the considered equation perfectly support the concerned physical event. Consequently, $u(\theta)$ denotes the birth rate, $a(\theta)$ is crowding coefficient, $b(\theta)$ is the toxicity coefficient. $\mu w(\theta)\int_{\theta_0}^{\Theta} k_*(s)\mathcal{N}(w(s))\Delta s$ represents the effect of toxin accumulation on the species. $w(\theta)$ denotes insect population at time θ. However, the initial condition is taken as homogeneous (die out season) to avoid the complexity. Although, this condition is often nonzero in real life situation.

By construction, Equation (1.1) together with condition (1.2) takes the form of a logistic integro-differential equation of fractional order in an arbitrary time scale $\mathbb{T}$. In literature, the particular cases have been discussed; if $\mathbb{T} = \mathbb{R}$ then Equation (1.1) and condition (1.2) leads to classical fractional order logistic integro-differential equation [15, 25, 26, 36]. If $\mathbb{T} = \mathbb{Z}$, that leads to logistic integro-difference equation of fractional order [9, 22, 24]. In either case, if one concentrate upon integer order operations only, then that leads to classical pantograph integro-differential [23,29] and integro-difference [6, 37] equations, respectively. For the above cases the theoretical investigation related to existence and uniqueness of solution have been carried out by J. M. Cushing [14], A. Aghajani et al. [2], K. Balachandran et al. [7], Sarkar et al. [31], Saha et al. [30], Gupta et al. [19], S. Etemad et al. [8], B. Ahmad et al. [3] and many other researchers. However, all the previously published works are primarily focused on separate cases only. In this paper, we establish a unified theory on an arbitrary time scale. Banach contraction principle and Schauder's fixed point theory are adopted along with some fundamental results from time scale theory.

The paper is organized in the following manner. Section 2 deals with the fundamental concepts related to the present work, Section 3 covers the main results, in Section

4 we validate our findings through some examples and finally, in Section 5 we draw some concluding remarks.

2. Fundamental Concepts

This section is entirely dedicated to the recapitulation of basic concepts and definitions that are frequently used throughout the article.

As per the suggestion of Benkhettou et al. [10] we proceed with the notion of **L**-RLFI and **L**-RLFD operators on arbitrary measure chain. For local treatment to non integer order calculus on any measure chain we recommend the works of Benkhettou et al. [10]. In the present study, we only focus on delta derivative over $\mathbb{T}$, parallel investigations are gradually applicable for ∇-fractional case [35].

Definition 2.1 ([10])**.** Consider $[p, q]$ an interval from the arbitrary time scale $\mathbb{T}$. Then the **L**-RLFI on $\mathbb{T}$, for $f : \mathbb{T} \to \mathbb{R}$ is defined as

$$ {}_{p}^{\mathbb{T}}\mathcal{J}_{\theta}^{\xi} f(\theta) = \int_{p}^{\theta} \frac{(\theta - s)^{\xi - 1}}{\Gamma(\xi)} f(s) \Delta s, $$

with Γ as the stranded gamma function, $\xi \in (0, 1)$.

Definition 2.2 ([10])**.** Consider $[p, q]$ an interval from the arbitrary time scale $\mathbb{T}$. Then the **L**-RLFD on $\mathbb{T}$, for $f : \mathbb{T} \to \mathbb{R}$ is defined as

$$ {}_{p}^{\mathbb{T}}\mathcal{D}_{\theta}^{\gamma} f(\theta) = \Big(\int_{p}^{\theta} \frac{(\theta - s)^{-\gamma}}{\Gamma(1 - \gamma)} f(s) \Delta s \Big)^{\Delta}, $$

with $\gamma \in (0, 1)$.

Remark 2.1. If one choose real line as time scale, then from previous two definitions, we have the classical RL fractional operators.

Proposition 2.1 ([10])**.** *For $f : \mathbb{T} \to \mathbb{R}$ and $\gamma \in (0, 1)$ the following estimate holds*

$$ {}_{p}^{\mathbb{T}}\mathcal{D}_{\theta}^{\gamma} f = \Delta \circ_{p}^{\mathbb{T}} \mathcal{J}_{\theta}^{1-\gamma} f. $$

Proposition 2.2 ([10])**.** *For $\gamma > 0$ and $f \in C([p, q])$, the following estimate holds*

$$ {}_{p}^{\mathbb{T}}\mathcal{D}_{\theta}^{\gamma} \circ_{p}^{\mathbb{T}} \mathcal{J}_{\theta}^{\gamma} f = f. $$

Proposition 2.3 ([10])**.** *Under the initial condition ${}_{p}^{\mathbb{T}}\mathcal{J}_{\theta}^{1-\gamma} w(p) = 0$, $f \in C([p, q])$ and $\gamma \in (0, 1)$ leads to the following estimate*

$$ {}_{p}^{\mathbb{T}}\mathcal{J}_{\theta}^{\gamma} \circ_{p}^{\mathbb{T}} \mathcal{D}_{\theta}^{\gamma} f = f. $$

Theorem 2.1 ([10])**.** *Consider $f \in C([p, q])$, $\gamma > 0$, and ${}_{p}^{\mathbb{T}}\mathcal{J}_{\theta}^{\gamma}([p, q])$ be the space of functions that can be expressed as the Riemann-Liouville $\Delta-$integral with order γ. Then, $f \in_{p}^{\mathbb{T}} \mathcal{J}_{\theta}^{\gamma}([p, q])$ if and only if ${}_{p}^{\mathbb{T}}\mathcal{J}_{\theta}^{1-\gamma}([p, q]) f \in C^{1}([p, q])$ and ${}_{p}^{\mathbb{T}}\mathcal{J}_{\theta}^{1-\gamma} f(p) = 0$.*

Proposition 2.4 ([4]). *Let us consider an arbitrary measure chain* $\mathbb{T}$, *and an non-decreasing smooth function* f *on* $[p,q]$. *Suppose,* ϕ *is the extension of* f *in* $[p,q]$ *defined by*

$$\phi(s)=\begin{cases} f(s), & \text{if } s\in\mathbb{T},\\ f(\theta), & \text{if } s\in(\theta,\sigma(\theta))\not\subset\mathbb{T}.\end{cases}$$

Then,

$$\int_p^q f(\theta)\Delta\theta\le\int_p^q \phi(\theta)d\theta,$$

where $\sigma(\theta)=\inf\{s\in\mathbb{T}:s>\theta\}$ *is the forward jump operator on the consider measure chain.*

To avoid complexity, throughout the article $\theta=0$ is considered, the space of all continuous function on $[0,\Theta]$ is denoted by $C([0,\Theta])$, and $\|x\|_\infty=\sup_{\theta\in[0,\Theta]}\{|x(\theta)|\}$. Then, $\mathcal{X}=(C([0,\Theta]),\|\cdot\|_\infty)$ is a Banach space.

Theorem 2.2 ([4]). *A function* Ψ *is said to be primitive of* $\psi:\mathbb{T}\to\mathbb{R}$ *provided* $\Psi^\Delta(x)=\psi(x)$ *for each* $x\in\mathbb{T}$, *then the* Δ*-integral is given by*

$$\textstyle\int_{x_0}^{x}\psi(\tilde{x})\Delta\tilde{x}=\Psi(x)-\Psi(x_0).$$

A function $\psi:\mathbb{T}\to\mathbb{R}$ is called a rd-continuous on $\mathbb{T}$, if ψ is continuous at $x\in\mathbb{T}$ with $\sigma(x)=x$ and has finite left-sided limits at points $x\in\mathbb{T}$ with

$$\sup\{\tau\in\mathbb{T}:\tau<x\}=x,$$

and the set of all rd-continuous functions $\psi:\mathbb{T}\to\mathbb{R}$ is represented by $C_{rd}(\mathbb{T},\mathbb{R})$.

Theorem 2.3 (Schauder fixed point theorem). *Let* Ω *be a nonempty closed, bounded, convex subset of a Banach space* S *and* $\Phi:\Omega\to\Omega$ *be a continuous compact operator. Then* Φ *has a fixed point in* Ω.

Theorem 2.4 (Arzela-Ascoli theorem). *Let* S *be a compact Hausdroff metric space. Then* $Y\subset M(S)$ *is relatively compact if and only if* Y *is uniformly bounded and uniformly equi-continuous.*

3. Main Results

In this section, we present and illustrate the theoretical findings elaborately. We start with the corresponding integral representation of equation (1.1) with condition (1.2) and subsequently derive the main results.

Lemma 3.1. *Let* $0<\gamma<1$. *Then, Equation* (1.1) *is equivalent to*

(3.1)
$$w(\theta)=\frac{\mu}{\Gamma(\gamma)}\int_{\theta_0}^{\theta}(\theta-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_{\theta_0}^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s.$$

Proof. By definition we have

$$\begin{aligned}
&{}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta}w(\theta)\\
=&\frac{\mu}{\Gamma(\gamma)}\Big\{\int_{\theta_0}^{\theta}(\theta-s)^{\gamma-1}w(\theta)\Big(u(s)-a(s)w(s)-b(s)\int_{\theta_0}^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big\}^{\Delta}\\
=&\Big({}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{1-\gamma}_{\theta}w(\theta)\Big)^{\Delta}=\Big(\Delta\circ{}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{1-\gamma}_{\theta}\Big)w(\theta).
\end{aligned}$$

The conclusion follows from Proposition 2.3 that ${}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{\gamma}_{\theta}\circ({}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta}w(\theta))=w(\theta)$. □

To avoid complexity, we set $\theta_0=0$. It is convenient to observe that Equation (1.1) posses a solution w if and only if w turns out to be a fixed point of the operator $\Lambda:\mathcal{X}\to\mathcal{X}$, defined as

(3.2)
$$\Lambda w(\theta)=\frac{\mu}{\Gamma(\gamma)}\int_{\theta_0}^{\theta}(\theta-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_{0}^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s.$$

We consider the following two assumptions.

($\mathcal{A}1$) The nonlinear function $\mathcal{N}:\mathbb{R}^+\to\mathbb{R}^+$ is Lipschitz continuous function with Lipschitz constant $L_{\mathcal{N}}$.

($\mathcal{A}2$) The functions $u,a,b,k_*:\mathbb{R}^+\to\mathbb{R}^+$ are such that

$$\sup_{\theta\in\mathbb{R}^+}|u(\theta)|=U,\quad \sup_{\theta\in\mathbb{R}^+}|a(\theta)|=A,\quad \sup_{\theta\in\mathbb{R}^+}|b(\theta)|=B,\quad \sup_{\theta\in\mathbb{R}^+}|k_*(\theta)|=K.$$

With all these constructions, now we present our main result as follows.

Theorem 3.1. *If* ($\mathcal{A}1$) *and* ($\mathcal{A}2$) *are satisfied by Equation* (1.1) *together with condition* (1.2)*, then it admits at least one solution in* $\mathcal{X}$ *for all* $\mu>0$.

Proof. To prove the proposed claim, we proceed with the following three steps.

Step I. We take a convergent sequence w_n with limit $w\in\mathcal{X}$. Then,

$$\begin{aligned}
|\Lambda w_n(\theta)-\Lambda w(\theta)|\leq&\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|u(s)||w_n(s)-w(s)|\Delta s\\
&+\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|a(s)||(w_n(s))^2-(w(s))^2|\Delta s\\
&+\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|b(s)|\Big|w_n(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w_n(x))\Delta x\\
&-w(s)\int_0^{\Theta}k_*\mathcal{N}(w(x))\Delta x\Big|\Delta s.
\end{aligned}$$

This leads to the following estimate

$$|\Lambda w_n(\theta)-\Lambda w(\theta)|\leq J^1(\theta)+J^2(\theta)+J^3(\theta), \tag{3.3}$$

where

$$J^1(\theta)=\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|u(s)||w_n(s)-w(s)|\Delta s,$$

$$\begin{aligned}J^2(\theta) =&\frac{\mu}{\Gamma(\gamma)}\int_0^\theta(\theta-s)^{\gamma-1}|a(s)||(w_n(s))^2-(w(s))^2|\Delta s,\\ J^3(\theta) =&\frac{\mu}{\Gamma(\gamma)}\int_0^\theta(\theta-s)^{\gamma-1}|b(s)|\Big|w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w_n(x))\Delta x\\ &-w(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s.\end{aligned}$$

Now we separately consider all the right-hand terms of (3.3), and obtain the following inequalities:

$$\begin{aligned}J^1(\theta)&\le\frac{\mu U}{\Gamma(\gamma)}\int_0^\theta(\theta-s)^{\gamma-1}|w_n(s)-w(s)|\Delta s\\ &\le\frac{\mu U}{\Gamma(\gamma)}\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}\Delta s\\ &\le\frac{\mu U}{\Gamma(\gamma)}\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}ds,\end{aligned}$$

as $(\theta-s)^{\gamma-1}$ is nondecreasing. Therefore,

$$J^1(\theta)\le\frac{\mu U\Theta^\gamma}{\Gamma(\gamma+1)}\|w_n-w\|_\infty. \tag{3.4}$$

This implies

$$\begin{aligned}J^2(\theta)&\le\frac{\mu A}{\Gamma(\gamma)}\int_0^\theta(\theta-s)^{\gamma-1}|w_n(s)-w(s)||w_n(s)+w(s)|\Delta s\\ &\le\frac{\mu A}{\Gamma(\gamma)}\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}(|w_n(s)|+|w(s)|)\Delta s,\end{aligned}$$

since $w_n\to w$ in $\mathcal{X}$, it is bounded by some constant W. Thus we have

$$J^2(\theta)\le\frac{2\mu AW\Theta^\gamma}{\Gamma(\gamma+1)}\|w_n-w\|_\infty. \tag{3.5}$$

Therefore,

$$\begin{aligned}&J^3(\theta)\\ \le&\frac{\mu B}{\Gamma(\gamma)}\Big(\int_0^\theta(\theta-s)^{\gamma-1}\Big|w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w_n(x))\Delta x-w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s\\ &+\int_0^\theta(\theta-s)^{\gamma-1}\Big|w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x-w(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s\Big)\\ \le&\frac{\mu B}{\Gamma(\gamma)}\Big(\int_0^\theta(\theta-s)^{\gamma-1}\Big|w_n(s)\Big(\int_0^\Theta k_*(x)\Big(\mathcal{N}(w_n(x))-\mathcal{N}(w(x))\Big)\Delta x\Big)\Big|\Delta s\\ &+\int_0^\theta(\theta-s)^{\gamma-1}\Big|(w_n(s)-w(s))\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s\Big)\\ \le&\frac{\mu B}{\Gamma(\gamma)}\Big(WKL_{\mathcal{N}}\int_0^\theta(\theta-s)^{\gamma-1}\Big(\int_0^\Theta|w_n(x)-w(x)|\Delta x\Big)\Delta s\end{aligned}$$

$$+ KN\int_0^\theta(\theta-s)^{\gamma-1}|w_n(s)-w(s)|\Big(\int_0^\Theta\Delta x\Big)\Delta s\Big)$$
$$\le\frac{\mu B}{\Gamma(\gamma)}\Big(WKL_{\mathfrak{N}}\Theta\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}\Delta s+KN\Theta\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}\Delta s\Big)$$
$$\le\frac{\mu B\left(WL_{\mathfrak{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\|w_n-w\|_\infty.$$

It follows that

$$J^3(\theta)\le\frac{\mu B\left(WL_{\mathfrak{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\|w_n-w\|_\infty. \tag{3.6}$$

Bringing inequalities (3.4)–(3.6) in (3.3), we have

$$|\Lambda w_n(\theta)-\Lambda w(\theta)|\le\Big(\frac{\mu U\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{2\mu AW\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{\mu B\left(WL_{\mathfrak{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\Big)\|w_n-w\|_\infty.$$

Then,

$$\begin{aligned}&\|\Lambda w_n(\theta)-\Lambda w(\theta)\|_\infty\\ &\quad\le\Big(\frac{\mu U\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{2\mu AW\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{\mu B\left(WL_{\mathfrak{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\Big)\|w_n-w\|_\infty.\end{aligned} \tag{3.7}$$

Hence, the right side of (3.7) tends to 0 as $w_n\to w$. Therefore, $\Lambda w_n\to\Lambda w$. This concludes that the operator Λ is continuous.

Step II. In this step, our primary goal is to show that Λ preserves boundedness. For this purpose, we take $\Omega=[0,\Theta]$. We claim that for all $R>0$, $L>0$ and for all $w\in\mathbb{B}_R=\{w\in C(\Omega,\mathbb{R}):\|w\|_\infty\le R\}$ we have $\|\Lambda w\|_\infty\le L$. Consider $\tau\in\Omega$ and $w\in\mathbb{B}_R$. Then

$$\begin{aligned}|\Lambda w(\tau)|\le&\frac{\mu}{\Gamma(\gamma)}\Big(U\int_0^\tau(\tau-s)^{\gamma-1}|w(s)|\Delta s+A\int_0^\tau(\tau-s)^{\gamma-1}|w(s)|^2\Delta s\\ &+BKN\int_0^\tau(\tau-s)^{\gamma-1}|w(s)|\Big(\int_0^\Theta\Delta x\Big)\Delta s\Big)\\ \le&\frac{\mu}{\Gamma(\gamma)}\Big(UR+AR^2+BKNR\Theta\Big)\int_0^\tau(\tau-s)^{\gamma-1}\Delta s\\ \le&\frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\Theta^\gamma.\end{aligned}$$

Further, if we consider the supremum over τ, then the following result holds

$$\|\Lambda w\|_\infty\le\frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\Theta^\gamma. \tag{3.8}$$

This implies, Λw is bounded.

Step III. Our claim is the equi-continuity of Λ. We proceed by considering $\tau_1,\tau_2\in\Omega$ so that $0\le\tau_1<\tau_2\le\Theta$, $\mathbb{B}_R$ is a bounded set of $C(\Omega,\mathbb{R})$. For $w\in\mathbb{B}_R$, we get

$$|\Lambda w(\tau_2)-\Lambda w(\tau_1)|$$

$$
\begin{aligned}
\leq & \frac{\mu}{\Gamma(\gamma)}\Big|\int_0^{\tau_2}(\tau_2-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s \\
& -\int_0^{\tau_1}(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big| \\
\leq & \frac{\mu}{\Gamma(\gamma)}\Big|\int_0^{\tau_2}\Big((\tau_2-s)^{\gamma-1}-(\tau_1-s)^{\gamma-1}+(\tau_1-s)^{\gamma-1}\Big) \\
& \times w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s \\
& -\int_0^{\tau_1}(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big| \\
\leq & \frac{\mu}{\Gamma(\gamma)}\Big|\int_0^{\tau_2}\Big((\tau_2-s)^{\gamma-1}-(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s) \\
& -b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s \\
& +\int_{\tau_1}^{\tau_2}(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big| \\
\leq & \frac{\mu}{\Gamma(\gamma)}\Big(UR+AR^2+BKNR\Theta\Big)\Big|\int_0^{\tau_2}\Big((\tau_2-s)^{\gamma-1}-(\tau_1-s)^{\gamma-1}\Big)\Delta s \\
& +\int_{\tau_1}^{\tau_2}(\tau_1-s)^{\gamma-1}\Delta s\Big| \\
\leq & \frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\left|\tau_1^{\gamma}-\tau_2^{\gamma}+(\tau_1-\tau_2)^{\gamma}-(\tau_1-\tau_2)^{\gamma}\right| \\
\leq & \frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\left|\tau_2^{\gamma}-\tau_1^{\gamma}\right|.
\end{aligned}
$$

Thus, we get

$$
|\Lambda w(\tau_2)-\Lambda w(\tau_1)| \leq \frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\left|\tau_2^{\gamma}-\tau_1^{\gamma}\right|. \tag{3.9}
$$

Inequality (3.9) is independent of w and approaches to 0, when $\tau_2 \to \tau_1$. Therefore, $\Lambda(\tilde{\mathbb{B}}_R)$ is relatively compact. By Arzela-Ascoli theorem, it is compact. Subsequently, since Λ is continuous, the result follows from Schauder's fixed point theorem. This completes the proof. □

Theorem 3.2. *Under the considered assumptions, Equation* (1.1) *admits an unique solution if*

$$
0<\mu<\frac{\Gamma(\gamma+1)}{(U+2AW+B(WL_{\mathcal{N}}+N)K\Theta)\,\Theta^{\gamma}}.
$$

Proof. Consider ζ and ζ_1 as two solutions of considered problem. Then, from (3.7), we have

$$
\|\Lambda\zeta-\Lambda\zeta_1\|_{\infty} \leq \left(\frac{\mu U\Theta^{\gamma}}{\Gamma(\gamma+1)}+\frac{2\mu AW\Theta^{\gamma}}{\Gamma(\gamma+1)}+\frac{\mu B\left(WL_{\mathcal{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\right)\|\zeta-\zeta_1\|_{\infty}. \tag{3.10}
$$

If one choose μ as

$$0<\mu<\frac{\Gamma(\gamma+1)}{\Big(U+2AW+B(WL_{\mathcal{N}}+N)K\Theta\Big)\Theta^{\gamma}},$$

then (3.10) suggests that $\Lambda:\mathcal{X}\to\mathcal{X}$ is a contractive operator. Hence, by the Banach principle Λ has one and only one fixed point. That implies $\zeta=\zeta_1$, is the unique solution for (1.1)–(1.2). □

4. Numerical Examples

In this section two fractional order real time scale problems are studied to validate our theoretical outcomes.

Example 4.1. Consider the logistic equation

(4.1)

$$\begin{aligned}{}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{0.5}_{\theta}w(\theta)=&\frac{1}{100}w(\theta)\Big(2+\sin\left(\frac{\pi\theta}{2}\right)-w(\theta)-(2-\cos(\pi\theta))\int_{\theta_0}^{\Theta}2e^{-2s}w(s)\Delta s\Big),\\ {}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{0.5}_{\theta}w(\theta_0)=&0.\end{aligned}$$

Here $\gamma=0.5$, $u(\theta)=2+\sin(\frac{\pi\theta}{2})$, $a(\theta)=1$, $b(\theta)=2-\cos(\pi\theta)$, $k_*(s)=2$ and $\mathcal{N}(w(s))=e^{-2s}w(s)$. Since, in this problem $\mathbb{T}=\mathbb{R}$, therefore we take $[\theta_0,\Theta]\equiv[0,1]$.

Then we have the following estimates, $|u(\theta)|=|2+\sin(\frac{\pi\theta}{2})|\leq 2+1=3$, thus $U=3$. Following the same way, we have $B=3$, and $A=1,K=2$. We consider $W=4$, for this particular problem $N=4$. One can observe that

$$|\mathcal{N}(w(s))-\mathcal{N}(w_1(s))|=e^{-2s}|w(s)-w_1(s)|,$$

that implies for all $s\in[0,1]$, $\|\mathcal{N}(w(s))-\mathcal{N}(w_1(s))\|\leq\|w(s)-w_1(s)\|$. Consequently, $L_{\mathcal{N}}=1$. Thus, assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_2)$ are satisfied.

Finally, we calculate

$$\frac{\Gamma(\gamma+1)}{\Big(U+2AW+B(WL_{\mathcal{N}}+N)K\Theta\Big)\Theta^{\gamma}}=\frac{\Gamma(0.5+1)}{\Big(3+2.1.1+3.(4.1+4).2.1\Big)\cdot(1)^{0.5}}=0.0167,$$

thus, we have $\mu=0.01<0.0167$. Consequently, by Theorem 3.2, Equation (4.1) admits an unique solution.

Example 4.2. Consider the following equation on $[0,0.5]$, on the general time scale $\mathbb{T}$

$$\begin{aligned}{}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{0.25}_{\theta}w(\theta)=&w(\theta)\Big(\frac{1+\cos(\theta)}{25}-\frac{w(\theta)}{40e^{\theta^2+3}}-\frac{\theta}{20}\int_{\theta_0}^{\Theta}\frac{s}{s+e^s}\cdot\frac{\sin(w(s))}{e^{s^2}+5}\Delta s\Big),\\ {}^{\mathbb{T}}_{\theta}\mathcal{J}^{0.25}_{\theta}w(\theta_0)=&0.\end{aligned}\tag{4.2}$$

Here, $u(\theta)=\frac{1+\cos(\theta)}{25}$, $a(\theta)=\frac{1}{40e^{\theta^2+3}}$, $b(\theta)=\frac{\theta}{20}$, $k_*(s)=\frac{s}{s+e^s}$, $\mathcal{N}(w(s))=\frac{\sin(w(s))}{e^{s^2}+5}$, $\gamma=0.25$ and $\mu=1$.

Thus we have, $U=\frac{2}{25}$, $A=\frac{1}{40}$, $B=\frac{1}{20}$, $K=1$, $L_{\mathbb{N}}=\frac{1}{e^5}$. One can easily check that the concerned assumptions are fulfilled. For $W=2$ and $N=1$ we have the following estimates

$$\frac{\Gamma(\gamma+1)}{(U+2AW+B(WL_{\mathbb{N}}+N)K\Theta)\Theta^{\gamma}}=\frac{\Gamma(0.25+1)}{\left(\frac{2}{25}+2\frac{1}{40}\cdot 2+\frac{1}{20}(2\frac{1}{e^5}+1)\cdot 1\cdot\frac{1}{2}\right)(\frac{1}{2})^{0.25}}=5.594,$$

thus we have $\mu=1<5.594$. Consequently, by Theorem 3.2, the Equation (4.2) admits an unique solution.

5. Conclusion

This article deals with the investigation of existence and uniqueness of solution for the fractional order logistic integro differential equation in an arbitrary time scale. As per our knowledge, the considered class of equation has not yet been studied in theoretical point of view. The Banach contraction principle and Schauder's fixed point theory have been adopted to develop the theoretical findings. Two examples are considered to validate the main results.

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