Kragujevac Journal of Mathematics
Volume 50(3) (2026), Pages 425–438.

ON THE GENERALIZED LEONARDO QUATERNIONS AND ASSOCIATED SPINORS

MUNESH KUMARI[1,2], KALIKA PRASAD[1,2*], HRISHIKESH MAHATO[1], AND PAULA MARIA MACHADO CRUZ CATARINO[3]

Abstract. In this paper, we introduce and study a new family of sequences called the generalized Leonardo spinors by defining a linear correspondence between the generalized Leonardo quaternions and spinors. We start with defining the generalized Leonardo quaternions and then present their some important properties such as Binet type formula, Catalan's identity, d'Ocagne's identity, series sums, etc. We give some interrelations of these quaternions with the Fibonacci and Lucas quaternions. Then, we present the generating functions, sum formulae, various well-known identities, etc. for the Leonardo spinors and show their connection with the Fibonacci and Lucas spinors.

1. Introduction

At the beginning of the 13th century, Leonardo of Pisa solved the famous rabbit growth problem based on idealized assumptions and that solution became a fascinating recursive integer sequence famed as the Fibonacci sequence [11]. For $n \geq 0$, the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given as $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$, $F_1 = 1$.

Recently, Catarino and Borges [2] studied the recurrence relations and various properties for the Leonardo numbers, in continuation Alp and Koçer [22] investigated their interesting properties. Kuhapatanakul and Juthamas [12] extended this study to the generalized Leonardo numbers along with their matrix representation and also in [20] the authors studied the matrix representation of Leonardo numbers. Karatas [10] defined the complex Leonardo numbers and studied their combinatorial properties.

Key words and phrases. Leonardo number, generalized Leonardo quaternions, generalized Leonardo spinors, Binet's formula, generating function, series sum.

2020 *Mathematics Subject Classification.* Primary: 11B39. Secondary: 15A66, 11R52.

DOI 10.46793/KgJMat2603.425K

Received: February 06, 2023.

Accepted: January 15, 2024.

İşbilir et al. [23] investigated the Pauli-Leonardo quaternions. Some recent developments on Leonardo numbers, their generalizations and interesting properties can be seen in [3, 5, 14–19, 22]. Here we restate some of them.

For $k \in \mathbb{Z}^+$, the generalized Leonardo numbers $\{\mathcal{L}_{k,n}\}$ are defined [12] by the recurrence relation

$$\mathcal{L}_{k,n+2} = \mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} + k, \quad n \geq 0, \quad \text{with } \mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1.$$

In negative subscript, these numbers are given as $\mathcal{L}_{k,-n} = (-1)^n(\mathcal{L}_{k,n-2} + k) - k$.

The Binet type formula for the generalized Leonardo numbers is

$$\mathcal{L}_{k,n} = (k+1)\Big(\frac{\lambda^{n+1} - \xi^{n+1}}{\lambda - \xi}\Big) - k, \tag{1.1}$$

where $\lambda = (1+\sqrt{5})/2$ and $\xi = (1-\sqrt{5})/2$.

In non-homogeneous form, the generalized Leonardo numbers satisfy the third order recurrence relation:

$$\mathcal{L}_{k,n+1} = 2\mathcal{L}_{k,n} - \mathcal{L}_{k,n-2}.$$

These numbers are associated with the Fibonacci numbers by the relation

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k.$$

In 1963, Horadam [8] defined quaternion sequences with components as Fibonacci and Lucas numbers. The Fibonacci quaternion Q_n is defined as

$$Q_n = F_n e_0 + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3 = (F_n, F_{n+1}, F_{n+2}, F_{n+3}), \quad n \geq 0, \tag{1.2}$$

and Lucas quaternion T_n as

$$T_n = L_n e_0 + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3 = (L_n, L_{n+1}, L_{n+2}, L_{n+3}), \quad n \geq 0,$$

where $\{e_0 = 1, e_1, e_2, e_3\}$ is the quaternion basis satisfying

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2.$$

Iyer [9] studied the relations between Fibonacci and Lucas quaternions. Further, Halici [7] obtained Binet's formula, generating functions and finite sum of these quaternions. The Binet's formulae for the Fibonacci and Lucas quaternions are given, respectively, by

$$Q_n = \frac{\lambda^*\lambda^n - \xi^*\xi^n}{\sqrt{5}} \quad \text{and} \quad T_n = \lambda^*\lambda^n + \xi^*\xi^n,$$

where $\lambda^* = 1 + \lambda e_1 + \lambda^2 e_2 + \lambda^3 e_3$ and $\xi^* = 1 + \xi e_1 + \xi^2 e_2 + \xi^3 e_3$.

For Fibonacci quaternions, Cassini's identity is given by

$$Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n(2Q_1 - 3e_3), \tag{1.3}$$

and Catalan's identity is

$$Q_{n-r}Q_{n+r} - Q_n^2 = (-1)^{n-r+1}(2F_rQ_r - 3F_{2r}e_3). \tag{1.4}$$

Theorem 1.1 (Sum formulae)**.** *For $n \geq 0$, we have*

(a) $\sum_{j=0}^{n} Q_j = Q_{n+2} - Q_1$;
(b) $\sum_{j=0}^{n} Q_{2j} = Q_{2n+1} - (1,0,1,1)$;
(c) $\sum_{j=0}^{n} Q_{2j+1} = Q_{2n} - Q_0$.

This paper will relate a new sequence of the generalized Leonardo quaternions and spinors that was motivated by a recent study of spinors with the Fibonacci numbers by Erişir and Güngör [6] and with the k-Fibonacci numbers by Kumari et al. [13].

This paper is structured as follows. In Section 2 we present a new quaternions sequence - the generalized Leonardo quaternions and study some properties of them. Section 3 is dedicated to the introduction of the generalized Leonardo spinors by defining a correspondence between the generalized Leonardo quaternions and spinors. We start the section recalling the important results involving spinors and we finish showing the relationship among the generalized Leonardo spinors and Fibonacci and Lucas spinors.

2. The Generalized Leonardo Quaternions

In this section, we first define the generalized Leonardo quaternions and then investigate their algebraic properties which we need later to prove some identities for spinors.

Definition 2.1. For $n \geq 0$, nth generalized Leonardo quaternion $\mathcal{QL}_{k,n}$ is defined as

$$\mathcal{QL}_{k,n} = \mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+1}e_1 + \mathcal{L}_{k,n+2}e_2 + \mathcal{L}_{k,n+3}e_3.$$

And the conjugate $\overline{\mathcal{QL}}_{k,n}$ is defined as

$$\overline{\mathcal{QL}}_{k,n} = \mathcal{L}_{k,n}e_0 - \mathcal{L}_{k,n+1}e_1 - \mathcal{L}_{k,n+2}e_2 - \mathcal{L}_{k,n+3}e_3. \tag{2.1}$$

The above defined generalized Leonardo quaternions can be written in recurrence form as

$$\mathcal{QL}_{k,n+2} = \mathcal{QL}_{k,n+1} + \mathcal{QL}_{k,n} + k\gamma, \quad \text{where } \gamma = e_0 + e_1 + e_2 + e_3. \tag{2.2}$$

From Definition 2.1 and (2.1), we get

$$\mathcal{QL}_{k,n}\overline{\mathcal{QL}}_{k,n} = \overline{\mathcal{QL}}_{k,n}\mathcal{QL}_{k,n} = (k+1)[3(k+1)F_{2n+5} - 2kL_{n+4}] + 4k^2.$$

Similar to the generalized Leonardo numbers, the generalized Leonardo quaternions can also be extended in negative indices given in the following definition.

Definition 2.2. For $n > 0$, the generalized Leonardo quaternions with negative subscript $\mathcal{QL}_{k,-n}$ are defined as

$$\mathcal{QL}_{k,-n} = (-1)^n \sum_{r=0}^{3} (-1)^r (\mathcal{L}_{k,n-2-r} + k)e_r - k\gamma.$$

The conjugate of $\mathcal{QL}_{k,-n}$ is given as

$$\overline{\mathcal{QL}}_{k,-n} = (-1)^n(\mathcal{L}_{k,n-2} + k) + (-1)^{n+1} \sum_{r=1}^{3} (-1)^r (\mathcal{L}_{k,n-2-r} + k)e_r - k\overline{\gamma},$$

where $\overline{\gamma} = e_0 - e_1 - e_2 - e_3$.

Now we give some relations among generalized Leonardo quaternion, Fibonacci quaternion and Lucas quaternion in the next theorem.

Theorem 2.1. *For $n \geq 0$, the following identities are verified:*

(a) $\mathcal{QL}_{k,n+3} = 2\mathcal{QL}_{k,n+2} - \mathcal{QL}_{k,n}$;
(b) $\mathcal{QL}_{k,n} = (k+1)Q_{n+1} - k\gamma$;
(c) $\mathcal{QL}_{k,-n} = 2\mathcal{QL}_{k,-n+2} - \mathcal{QL}_{k,-n+3}$;
(d) $\mathcal{QL}_{k,-n} = (k+1)Q_{-n+1} - k\gamma$;
(e) $\mathcal{QL}_{k,n} + \mathcal{QL}_{k,-n} = \begin{cases} (k+1)F_nT_1 - 2k\gamma, & n \text{ is odd}, \\ (k+1)L_nQ_1 - 2k\gamma, & n \text{ is even}; \end{cases}$
(f) $\mathcal{QL}_{k,-n} + \overline{\mathcal{QL}}_{k,-n} = 2\mathcal{L}_{k,-n}$.

Proof. The first identity follows from expression (2.2) and the second identity uses $\mathcal{L}_{k,n} = (k+1)F_{n+1} - k$ and (1.2). To prove third and fourth statements, we use Definition 2.2 and definition of the Fibonacci quaternions, respectively. Fifth and sixth identities follow from the definitions of $\mathcal{QL}_{k,-n}$ and its conjugate. □

In the next theorem, we present the Binet type formula for the generalized Leonardo quaternions and with the help of that we investigate some well known identities and properties of these quaternions.

Theorem 2.2 (Binet type formula)**.** *For $n \geq 0$, we have*

$$\mathcal{QL}_{k,n} = \frac{k+1}{\sqrt{5}}\Big(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*\Big) - k\gamma,$$

where $\lambda^* = 1 + \lambda e_1 + \lambda^2 e_2 + \lambda^3 e_3$, $\xi^* = 1 + \xi e_1 + \xi^2 e_2 + \xi^3 e_3$ and $\gamma = e_0 + e_1 + e_2 + e_3$.

Proof. Using (2.1) and Binet's formula (1.1), we write

$$\begin{aligned}
\mathcal{QL}_{k,n} &= \sum_{r=0}^{3} \mathcal{L}_{k,n+r} e_r, \\
&= \sum_{r=0}^{3} \frac{(k+1)(\lambda^{n+r+1} - \xi^{n+r+1}) - \sqrt{5}k}{\sqrt{5}} e_r, \\
&= (k+1)\sum_{r=0}^{3} \frac{(\lambda^{n+r+1} - \xi^{n+r+1})}{\sqrt{5}} e_r - k\sum_{r=0}^{3} e_r \\
&= \frac{k+1}{\sqrt{5}}\Big(\lambda^{n+1}\sum_{r=0}^{3}\lambda^r e_r - \xi^{n+1}\sum_{r=0}^{3}\xi^r e_r\Big) - k\sum_{r=0}^{3} e_r, \\
&= \frac{k+1}{\sqrt{5}}\Big(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*\Big) - k\gamma.
\end{aligned}$$
□

Theorem 2.3. *For $n \geq 0$, the following identities are verified:*

(a) $\mathcal{QL}_{k,n} = \frac{k+1}{2}\Big(T_{n+2} - Q_{n+2}\Big) - k\gamma$;

(b) $\mathcal{QL}_{k,n} = \frac{k+1}{5}\Big(T_n + T_{n+2}\Big) - k\gamma.$

Proof. (a) Using the Binet's formula of Fibonacci and Lucas quaternions, we have

$$\begin{aligned}
T_{n+2} - Q_{n+2} &= \Big(\lambda^*\lambda^{n+2} + \xi^*\xi^{n+2}\Big) - \Big(\frac{\lambda^*\lambda^{n+2} - \xi^*\xi^{n+2}}{\sqrt{5}}\Big) \\
&= \frac{1}{\sqrt{5}}(\lambda^*\lambda^{n+2}(\sqrt{5}-1) + \xi^*\xi^{n+2}(\sqrt{5}+1)) \\
&= \frac{1}{\sqrt{5}}(-2\xi\lambda^*\lambda^{n+2} + 2\lambda\xi^*\xi^{n+2}) \\
&= \frac{2}{\sqrt{5}}(-\lambda\xi(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*)) \\
&= \frac{2}{\sqrt{5}}(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*) \\
&= 2\Big(\frac{\mathcal{QL}_{k,n} + k\gamma}{k+1}\Big).
\end{aligned}$$

Thus, on simplification

$$\mathcal{QL}_{k,n} = \frac{k+1}{2}\Big(T_{n+2} - Q_{n+2}\Big) - k\gamma.$$

(b) Similar to (a), we have

$$\begin{aligned}
T_n + T_{n+2} &= (\lambda^*\lambda^n + \xi^*\xi^n) + (\lambda^*\lambda^{n+2} + \xi^*\xi^{n+2}) \\
&= \lambda\xi\Big(\frac{\lambda^*\lambda^n + \xi^*\xi^n}{\lambda\xi}\Big) + (\lambda^*\lambda^{n+2} + \xi^*\xi^{n+2}) \\
&= -\lambda^{n+1}\lambda^*(\xi - \lambda) + \xi^{n+1}\xi^*(\xi - \lambda) \\
&= \sqrt{5}(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*) \\
&= \sqrt{5}\Big(\frac{\sqrt{5}}{k+1}(\mathcal{QL}_{k,n} + k\gamma)\Big),
\end{aligned}$$

as required. ∎

Theorem 2.4 (Catalan's identity)**.** *For $n, r \in \mathbb{N}$ such that $n \geq r$, we have*

$$\begin{aligned}
\mathcal{QL}_{k,n-r}\mathcal{QL}_{k,n+r} - \mathcal{QL}_{k,n}^2 =&(k+1)^2[(-1)^{n-r+2}(2F_rQ_r - 3F_{2r}e_3)] \\
&+ k(k+1)(Q_{n+1} - Q_{n+1-r})\gamma \\
&+ k(k+1)\gamma(Q_{n+1} - Q_{n+1+r}).
\end{aligned}$$

Proof. Using (b) of Theorem 2.1 and identity of (1.4), we have

$$\begin{aligned}
&\mathcal{QL}_{k,n-r}\mathcal{QL}_{k,n+r} - \mathcal{QL}_{k,n}^2 \\
=&\Big((k+1)Q_{n-r+1} - k\gamma\Big)\Big((k+1)Q_{n+r+1} - k\gamma\Big) - \Big((k+1)Q_{n+1} - k\gamma\Big)^2
\end{aligned}$$

$$
\begin{aligned}
=&\Big((k+1)^2Q_{n-r+1}Q_{n+r+1}-(k+1)kQ_{n-r+1}\gamma-k(k+1)\gamma Q_{n+r+1}\\
&+k^2(\gamma)^2\Big)-\Big((k+1)^2Q_{n+1}^2+k^2(\gamma)^2-(k+1)kQ_{n+1}\gamma-(k+1)k\gamma Q_{n+1}\Big)\\
=&(k+1)^2(Q_{n+1-r}Q_{n+1+r}-Q_{n+1}^2)+k(k+1)(Q_{n+1}-Q_{n+1-r})\gamma\\
&+k(k+1)\gamma(Q_{n+1}-Q_{n+1+r})\\
=&(k+1)^2[(-1)^{n-r+2}(2F_rQ_r-3F_{2r}e_3)]\\
&+k[(\mathcal{QL}_{k,n}-\mathcal{QL}_{k,n-r})\gamma-\gamma(\mathcal{QL}_{k,n}-\mathcal{QL}_{k,n+r})]. \qquad \square
\end{aligned}
$$

If we substitute $r=1$ in the above identity, the Cassini's identity for Leonardo quaternions $\mathcal{QL}_{k,n}$ is obtained which is stated in the next theorem.

Theorem 2.5 (Cassini's identity)**.** *For any natural number n, we have*

$$
\begin{aligned}
&\mathcal{QL}_{k,n-1}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,n}^2\\
=&(k+1)^2[(-1)^{n+1}(2Q_1-3e_3)]+k[\mathcal{QL}_{k,n-2}\gamma+\gamma\mathcal{QL}_{k,n-1}+2k\overline{\gamma}].
\end{aligned}
$$

Theorem 2.6 (d'Ocagne's identity)**.** *For $n,r\in\mathbb{N}$ such that $n\geq r$, we have*

$$
\begin{aligned}
&\mathcal{QL}_{k,r}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,r+1}\mathcal{QL}_{k,n}\\
=&(k+1)^2[(-1)^{r+1}(F_{n-r}T_0+L_{n-r}(Q_0-3e_3))]+k(k+1)[Q_r\gamma-\gamma Q_n].
\end{aligned}
$$

Proof. Using identity (2) of Theorem 2.1, we have

$$
\begin{aligned}
\mathcal{QL}_{k,r}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,r+1}\mathcal{QL}_{k,n}=&\Big((k+1)Q_{r+1}-k\gamma\Big)\Big((k+1)Q_{n+2}-k\gamma\Big)\\
&-\Big((k+1)Q_{r+2}-k\gamma\Big)\Big((k+1)Q_{n+1}-k\gamma\Big)\\
=&(k+1)^2(Q_{r+1}Q_{n+2}-Q_{r+2}Q_{n+1})\\
&-k(k+1)[(Q_{r+1}-Q_{r+2})\gamma+\gamma(Q_{n+2}-Q_{n+1})]\\
=&(k+1)^2(Q_{r+1}Q_{n+2}-Q_{r+2}Q_{n+1})\\
&+k(k+1)[Q_r\gamma-\gamma Q_n].
\end{aligned}
$$

Now, using the d'Ocagne's identity for the Fibonacci quaternions, i.e., $Q_{r+1}Q_{n+2}-Q_{r+2}Q_{n+1}=(-1)^{r+1}[F_{n-r}T_0+L_{n-r}(Q_0-3e_3)]$, we get

$$
\begin{aligned}
\mathcal{QL}_{k,r}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,r+1}\mathcal{QL}_{k,n}=&(k+1)^2[(-1)^{r+1}(F_{n-r}T_0+L_{n-r}(Q_0-3e_3))]\\
&+k(k+1)[Q_r\gamma-\gamma Q_n]. \qquad \square
\end{aligned}
$$

Theorem 2.7. *The generating function for the generalized Leonardo quaternions is*

$$
\phi(t)=\frac{\mathcal{QL}_{k,0}-\mathcal{QL}_{k,-2}t-\mathcal{QL}_{k,-1}t^2}{1-2t+t^3}.
$$

Proof. Let the generating function for sequence $\{\mathcal{QL}_{k,n}\}_{n=0}^{+\infty}$ is $\phi(t)=\sum_{n=0}^{+\infty}\mathcal{QL}_{k,n}t^n$.

Now using relation (2) of Theorem 2.1, we write

$$\phi(t)=\sum_{n=0}^{+\infty}((k+1)Q_{n+1}-k\gamma)t^n=(k+1)\sum_{n=0}^{+\infty}Q_{n+1}t^n-k\gamma\sum_{n=0}^{+\infty}t^n.$$

Taking into account the generating function of the Fibonacci quaternions, i.e., $\sum_{n=0}^{+\infty}Q_{n+1}t^n=\frac{Q_1+Q_0t}{1-t-t^2}$, we have

$$\begin{aligned}\phi(t)&=(k+1)\frac{Q_0+Q_0t}{1-t-t^2}-k\gamma\frac{1}{1-t}\\&=\frac{(k+1)[Q_1-(Q_1-Q_0)t-Q_0t^2]-k\gamma(1-t-t^2)}{(1-t-t^2)(1-t)}\\&=\frac{\mathcal{QL}_{k,0}-\mathcal{QL}_{k,-2}t-\mathcal{QL}_{k,-1}t^2}{1-2t+t^3}.\end{aligned}$$ □

In the next theorem, we give the sum of finite terms of $\mathcal{QL}_{k,n}$ and also with even and odd subscripts.

Theorem 2.8 (Finite sum formulae)**.** *For any positive integer n, we have*

(a) $\sum_{r=0}^{n}\mathcal{QL}_{k,r}=\mathcal{QL}_{k,n+2}-\mathcal{QL}_{k,1}-k(n+1)\gamma$;
(b) $\sum_{r=0}^{n}\mathcal{QL}_{k,2r}=\mathcal{QL}_{k,2n+1}-\mathcal{QL}_{k,-1}-k(n+1)\gamma$;
(c) $\sum_{r=1}^{n}\mathcal{QL}_{k,2r-1}=\mathcal{QL}_{k,2n}-\mathcal{QL}_{k,0}-kn\gamma$.

Proof. Using (b) of Theorem 2.1 and sum identity (a) of Theorem 1.1, we have

$$\begin{aligned}\sum_{r=0}^{n}\mathcal{QL}_{k,r}&=\sum_{r=0}^{n}((k+1)Q_{r+1}-k\gamma)\\&=(k+1)\sum_{r=0}^{n}Q_{r+1}-k\sum_{r=0}^{n}\gamma\\&=(k+1)(Q_{n+3}-Q_1-Q_0)-k\gamma(n+1)\\&=\mathcal{QL}_{k,n+2}-\mathcal{QL}_{k,1}-k(n+1)\gamma.\end{aligned}$$

Proof of identities (b) and (c) are similar using the finite sum formulae for even (b) and odd (c) of Theorem 1.1, respectively. □

Theorem 2.9. *For $n\in\mathbb{N}$, the following identity is verified.*

$$\sum_{r=0}^{n}\binom{n}{r}\mathcal{QL}_{k,r}=\mathcal{QL}_{k,2n}-k(2^n-1)\gamma. \tag{2.3}$$

Proof. The identity can be easily proved by making use of relation (2) of Theorem 2.1 and identity $\sum_{r=0}^{n}\binom{n}{r}Q_r=Q_{2n}$ for the Fibonacci quaternions. □

3. The Generalized Leonardo Spinors

Spinors are vectorial objects without their multilinear features for the mathematicians. A French Mathematician, Elie Cartan discovered the spinors first time. In 3-dimensional Euclidean space, there are many possible approaches to spinor theory like spinor ring algebra, Cartan's isotropic vectors, Clifford algebra, stereographic projection etc.

Here, we restate the definition given by the E. Cartan [1]. Consider a 3-dimensional space $\mathbb{C}^3$. Let $(x, y, z) \in \mathbb{C}^3$ be an isotropic vector. Then this vector can be associated with two numbers Ψ_1 and Ψ_2 given by

$$x = \Psi_1^2 - \Psi_2^2, \quad y = i(\Psi_1^2 + \Psi_2^2), \quad z = -2\Psi_1\Psi_2.$$

And solutions of these equations are

$$\Psi_1 = \pm\sqrt{\frac{x - iy}{2}} \quad \text{and} \quad \Psi_2 = \pm\sqrt{\frac{-x - iy}{2}}.$$

Thus spinor is the two-dimensional complex vectors described as

$$\Psi = (\Psi_1, \Psi_2) \equiv \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}.$$

A different approach to spinors derived from Euler's theorem was presented by Vivarelli [21] in 1984. He studied quaternions and one-index spinors by defining a linear and injective correspondence between them.

The correspondence $\Phi : \mathbb{H} \to \mathbb{S}$ between the set of quaternions $\mathbb{H}$ and spinors $\mathbb{S}$ is defined as

$$\Phi(a + be_1 + ce_2 + de_3) = \begin{bmatrix} d + ia \\ b + ic \end{bmatrix} \equiv Q, \quad a + be_1 + ce_2 + de_3 = p \in \mathbb{H}.$$

And, the product of two quaternions (qp) associated to a spinor-matrix product is given by $qp \mapsto -i\hat{Q}P$, where P is the spinor corresponding to the quaternion q and $\hat{Q}$ is the square(complex) unitary matrix given as

$$\begin{bmatrix} d + ia & b - ic \\ b + ic & -d + ia \end{bmatrix}. \tag{3.1}$$

E. Cartan [1] introduced the spinor conjugate to Ψ given as

$$\tilde{\Psi} = iA\overline{\Psi},$$

where $\overline{\Psi}$ is complex conjugate of Ψ and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Castillo [4] defined the mate of spinor Ψ as

$$\check{\Psi} = -A\overline{\Psi},$$

Recently, Erişir and Güngör [6] studied the spinors with Fibonacci and Lucas numbers components and obtained their properties. They defined the Fibonacci spinors sequence $\{S_n\}_{n\geq 0}$ as

$$S_0 = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}, \quad S_1 = \begin{bmatrix} 3+i \\ 1+2i \end{bmatrix}, \quad S_{n+2} = S_{n+1} + S_n,$$

and Lucas spinors sequence $\{S_n^{'}\}_{n\geq 0}$ as

$$S_0^{'} = \begin{bmatrix} 2+2i \\ 1+i \end{bmatrix}, \quad S_1^{'} = \begin{bmatrix} 4+i \\ 1+3i \end{bmatrix}, \quad S_{n+2}^{'} = S_{n+1}^{'} + S_n^{'}.$$

The Binet type formulae for the Fibonacci and Lucas spinors are, respectively,

$$S_n = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^n - \frac{1}{\sqrt{5}} \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^n$$

and

$$S_n^{'} = \begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^n + \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^n.$$

Motivated by the work of Erişir and Güngör [6], we are extending this study to the generalized Leonardo numbers and introducing a new sequence of the generalized Leonardo spinors.

Let $\mathbb{L}$ and $\mathbb{S}$ denote the set of generalized Leonardo quaternions and set of spinors, respectively. Then the generalized Leonardo spinor $\mathfrak{L}_n$ given by a linear and injective correspondence $\Phi : \mathbb{L} \to \mathbb{S}$ is defined as

$$\Phi(\mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+1}e_1 + \mathcal{L}_{k,n+2}e_2 + \mathcal{L}_{k,n+3}e_3) = \begin{bmatrix} \mathcal{L}_{k,n+3} + i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} + i\mathcal{L}_{k,n+2} \end{bmatrix} \equiv \mathfrak{L}_n.$$

Spinor $\mathfrak{L}_n^*$ corresponding to the conjugate quaternion $\overline{\mathbb{Q}\mathcal{L}_{k,n}}$ is given as

$$\mathfrak{L}_n^* = \begin{bmatrix} -\mathcal{L}_{k,n+3} + i\mathcal{L}_{k,n} \\ -\mathcal{L}_{k,n+1} - i\mathcal{L}_{k,n+2} \end{bmatrix}.$$

Definition 3.1. For $n \geq 0$, the sequence of generalized Leonardo spinors $\{\mathfrak{L}_n\}_{n\geq 0}$ is defined recursively as $\mathfrak{L}_{n+2} = \mathfrak{L}_{n+1} + \mathfrak{L}_n + k\mathfrak{J}$, where $\mathfrak{L}_0 = \begin{bmatrix} (2k+3)+i \\ 1+i(k+2) \end{bmatrix}$, $\mathfrak{L}_1 = \begin{bmatrix} (4k+5)+i \\ (k+2)+i(2k+3) \end{bmatrix}$ and $\mathfrak{J} = \begin{bmatrix} 1+i \\ 1+i \end{bmatrix}$.

Lemma 3.1. *For generalized Leonardo spinors, the following identities are verified.*

$$\mathfrak{L}_{n+3} = 2\mathfrak{L}_{n+2} - \mathfrak{L}_n \quad \textit{and} \quad \mathfrak{L}_n = (k+1)S_{n+1} - k\mathfrak{J}. \tag{3.2}$$

In the next lemma, we present conjugates and mate of the generalized Leonardo spinors.

Lemma 3.2. *For generalized Leonardo spinors $\mathfrak{L}_n$, we have*

(a) *Complex Conjugate:* $\overline{\mathfrak{L}}_n = \begin{bmatrix} \mathcal{L}_{k,n+3} - i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} - i\mathcal{L}_{k,n+2} \end{bmatrix}$;

(b) *Spinor Conjugate:* $\widetilde{\mathfrak{L}}_n = \begin{bmatrix} \mathcal{L}_{k,n+2} + i\mathcal{L}_{k,n+1} \\ -\mathcal{L}_{k,n} - i\mathcal{L}_{k,n+3} \end{bmatrix}$;

(c) *Mate of Spinor:* $\check{\mathfrak{L}}_{k,n} = \begin{bmatrix} -\mathcal{L}_{k,n+1} + i\mathcal{L}_{k,n+2} \\ \mathcal{L}_{k,n+3} - i\mathcal{L}_{k,n} \end{bmatrix}$.

Proof. Using the definition of spinor conjugate and mate of spinor, above results can be easily established. □

Theorem 3.1 (Binet type formula)**.** *For $n \geq 0$, we have*

$$\mathfrak{L}_n = \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^{n+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^{n+1} \right) - k\mathfrak{J}. \tag{3.3}$$

Proof. We prove this theorem by induction on n. For $n = 1$, the R.H.S of (3.3) is

$$\mathfrak{L}_1 = (k+1)S_2 - k\mathfrak{J} = \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} (\lambda^5 - \xi^5) + i(\lambda^2 - \xi^2) \\ (\lambda^3 - \xi^3) + i(\lambda^4 - \xi^4) \end{bmatrix} \right) - k\mathfrak{J}.$$

Assume expression (3.3) is true for $n = m$, i.e.,

$$\mathfrak{L}_m = \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^{m+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^{m+1} \right) - k\mathfrak{J}.$$

By Definition 3.1 and taking into account the fact $\lambda^2 = \lambda + 1$ and $\xi^2 = \xi + 1$, we get

$$\begin{aligned}
\mathfrak{L}_{m+1} &= \mathfrak{L}_m + \mathfrak{L}_{m-1} + k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^{m+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^{m+1} \right) - k\mathfrak{J} \\
&\quad + \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^{m} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^{m} \right) - k\mathfrak{J} + k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} (\lambda^{m+1} + \lambda^m) - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} (\xi^{m+1} + \xi^m) \right) - k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^m(\lambda + 1) - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^m(\xi + 1) \right) - k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^{m+2} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^{m+2} \right) - k\mathfrak{J}.
\end{aligned}$$

This completes the proof. □

By replacing n with $-n$ in Binet type formula (3.3), we extend the generalized Leonardo spinors in negative direction. Thus,

$$\mathfrak{L}_{-n} = \frac{k+1}{\sqrt{5}} \left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^{-n+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^{-n+1} \right) - k\mathfrak{J} = (k+1)S_{-n+1} - k\mathfrak{J}.$$

In next theorem, we state some relations among $\mathfrak{L}_n$, $\mathfrak{L}_n^*, \mathfrak{L}_{-n}$ and $\mathfrak{L}_{-n}^*$ by omitting their proofs, as proofs of these identities can be seen easily using their definitions.

Theorem 3.2. *For the generalized Leonardo spinors, we have*

(a) $\mathfrak{L}_{-n} = 2\mathfrak{L}_{-n+2} - \mathfrak{L}_{-n+3}$;

(b) $\mathfrak{L}_{-n} = (k+1)S_{-n+1} - k\mathfrak{J}$;

(c) $\mathfrak{L}_n + \mathfrak{L}_{-n} = \begin{cases} (k+1)L_n S_1 - 2k\mathfrak{J}, & n \text{ is even}, \\ (k+1)F_n S_1' - 2\mathfrak{J}, & n \text{ is odd}; \end{cases}$

(d) $\mathfrak{L}_n + \mathfrak{L}_n^* = \begin{bmatrix} i2\mathcal{L}_{k,n} \\ 0 \end{bmatrix}$;

(e) $\mathfrak{L}_{-n} + \mathfrak{L}_{-n}^* = \begin{bmatrix} i2\mathcal{L}_{k,-n} \\ 0 \end{bmatrix}$.

Theorem 3.3. *For $n \geq 0$, the following relations among the generalized Leonardo, Fibonacci and Lucas spinors hold*

$$\mathfrak{L}_n = \frac{k+1}{2}\left(S_{n+2}' - S_{n+2}\right) - k\mathfrak{J} \tag{3.4}$$

and

$$\mathfrak{L}_n = \frac{k+1}{5}\left(S_n' + S_{n+2}'\right) - k\mathfrak{J}. \tag{3.5}$$

Proof. By using relation $S_n' = S_{n-1} + S_{n+1}$ in the R.H.S. of expression (3.4), we write

$$\frac{k+1}{2}\left(S_{n+2}' - S_{n+2}\right) - k\mathfrak{J} = \frac{k+1}{2}\left(2S_{n+1}\right) - k\mathfrak{J} = \mathfrak{L}_n.$$

Similarly, (3.5) follows from the identity $5S_n = S_{n-1}' + S_{n+1}'$. □

Theorem 3.4. *The generating function for the generalized Leonardo spinors is*

$$f(t) = \frac{-1}{1-2t+t^2}\begin{bmatrix} [(k+2)t^2 + t - (2k+3)] - i[kt^2 - t + 1] \\ [t^2 - kt - 1] + i[t^2 + t - (k+2)] \end{bmatrix}.$$

Proof. Let $f(t) = \sum_{n=0}^{+\infty} \mathfrak{L}_n t^n$ be the ordinary generating function. Now consider the recurrence relation $\mathfrak{L}_{n+3} = 2\mathfrak{L}_{n+2} - \mathfrak{L}_n$. Then multiplying it by t^{n+3} and taking summation, we have

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \mathfrak{L}_{n+3}t^{n+3} - 2\sum_{n=0}^{+\infty} \mathfrak{L}_{n+2}t^{n+3} + \sum_{n=0}^{+\infty} \mathfrak{L}_n t^{n+3} = 0 \\
&\Longrightarrow (f(t) - \mathfrak{L}_0 - \mathfrak{L}_1 t - \mathfrak{L}_2 t^2) - 2t(f(t) - \mathfrak{L}_0 - \mathfrak{L}_1 t) + t^2 f(t) = 0 \\
&\Longrightarrow f(t)(1-2t+t^2) = \mathfrak{L}_0 + t(\mathfrak{L}_1 - 2\mathfrak{L}_0) + t^2(\mathfrak{L}_2 - 2\mathfrak{L}_1) \\
&\Longrightarrow f(t) = \frac{\mathfrak{L}_0 + t(\mathfrak{L}_1 - 2\mathfrak{L}_0) + t^2(\mathfrak{L}_2 - 2\mathfrak{L}_1)}{1-2t+t^2} \\
&\Longrightarrow f(t) = \frac{-1}{1-2t+t^2}\begin{bmatrix} [(k+2)t^2 + t - (2k+3)] - i[kt^2 - t + 1] \\ [t^2 - kt - 1] + i[t^2 + t - (k+2)] \end{bmatrix}.
\end{aligned}$$

□

Theorem 3.5 (Finite sum formulae)**.** *We have the following.*

Sum of first $n+1$ terms

$$\sum_{r=0}^{n} \mathfrak{L}_{k,r} = \mathfrak{L}_{k,n+2} - (k+1)S_2 - nk\mathfrak{J}. \tag{3.6}$$

Sum of first $n+1$ even indexed terms

$$\sum_{r=0}^{n} \mathfrak{L}_{k,2r} = \mathfrak{L}_{k,2n+1} - (k+1)S_0 - nk\mathfrak{J}. \tag{3.7}$$

Sum of first $n+1$ odd indexed terms

$$\sum_{r=0}^{n} \mathfrak{L}_{k,2r+1} = \mathfrak{L}_{k,2n+2} - (k+1)S_1 - nk\mathfrak{J}. \tag{3.8}$$

Proof. Using relation (3.2) and sum identity $\sum_{r=1}^{n} S_r = S_{n+2} - S_2$, we have

$$\begin{aligned}
\sum_{r=0}^{n} \mathfrak{L}_{k,r} &= \sum_{r=0}^{n} [(k+1)S_{r+1} - k\mathfrak{J}] = (k+1)\sum_{r=0}^{n} S_{r+1} - \sum_{r=0}^{n} k\mathfrak{J} \\
&= (k+1)[S_{n+3} - S_2] - (n+1)k\mathfrak{J} \\
&= \mathfrak{L}_{k,n+2} - (k+1)S_2 - nk\mathfrak{J}.
\end{aligned}$$

This proves expression (3.6).

The rest of the two expressions (3.7) and (3.8) follow directly from the identities $\sum_{r=1}^{n} S_{2r} = S_{2n+1} - S_1$ and $\sum_{r=1}^{n} S_{2r-1} = S_{2n} - S_0$, respectively. □

4. Conclusion

In summary, we defined and studied the generalized Leonardo quaternions and a new sequence of spinors by considering a linear and injective correspondence between the set of quaternions and the set of spinors. For generalized Leonardo quaternions, we obtained various identities, interrelations with the Fibonacci and Lucas quaternions, Catalan's identity, d'Ocagne's identity, generating functions, finite sum formulas, etc. For spinors, we presented the closed form formula, several identities, generating functions, finite sums with odd and even indexed terms, etc.

Acknowledgements. The authors are grateful to the anonymous reviewers for their careful reading and insightful comments. The first and second authors would like to thank the University Grant Commission (UGC), India for the senior research fellowship.

References

[1] É. Cartan, *The Theory of Spinors*, Dover Publications, New York, 1981.

[2] P. Catarino and A. Borges, *On Leonardo numbers*, Acta Math. Univ. Comen. **89**(1) (2019), 75–86.

[3] P. Catarino and A. Borges, *A note on incomplete Leonardo numbers*, Integers **20**(#A43) (2020), 1–7.

[4] G. F. T. del Castillo, *3-D Spinors, Spin-weighted Functions and their Applications*, Springer Science & Business Media, 2003. https://doi.org/10.1007/978-0-8176-8146-3

[5] M. C. dos Santos Mangueira, R. P. M. Vieira, F. R. V. Alves and P. M. M. C. Catarino, *Leonardo's bivariate and complex polynomials*, Notes Number Theory Discrete Math. **28**(1) (2022), 115–123. https://doi.org/10.7546/nntdm.2022.28.1.115-123

[6] T. Erişir and M. A. Güngör, *On Fibonacci spinors*, Int. J. Geom. Methods Mod. Phys. **17**(4) (2020), Paper ID 2050065. https://doi.org/10.1142/S0219887820500656

[7] S. Halici, *On Fibonacci quaternions*, Adv. Appl. Clifford Algebras **22** (2012), 321–327. https://doi.org/10.1007/s00006-011-0317-1

[8] A. F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Am. Math. Mon. **70**(3) (1963), 289–291. https://doi.org/10.2307/2313129

[9] M. R. Iyer, *Some results on Fibonacci quaternions*, Fibonacci Quart. **7** (1969), 201–210.

[10] A. Karatas, *On complex Leonardo numbers*, Notes Number Theory Discrete Math. **28**(3) (2022), 458–465. https://doi.org/10.7546/nntdm.2022.28.3.458-465

[11] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley Online Library, 2017. https://doi.org/10.1002/9781118742327

[12] K. Kuhapatanakul and J. Chobsorn, *On the generalized Leonardo numbers*, Integers **22** (2022), A48.

[13] M. Kumari, K. Prasad and R. Frontczak, *On the k-Fibonacci and k-Lucas spinors*, Notes Number Theory Discrete Math. **29**(2) (2023), 322–335. https://doi.org/10.7546/nntdm.2023.29.2.322-335

[14] F. Kürüz, A. Dağdeviren and P. Catarino, *On Leonardo pisano hybrinomials*, Mathematics **9**(22) (2021), Paper ID 2923. https://doi.org/10.3390/math9222923

[15] K. Prasad, R. Mohanty, M. Kumari and H. Mahato, *Some new families of generalized k-Leonardo and Gaussian Leonardo Numbers*, Commun. Comb. Optim. **9**(3) (2024). https://doi.org/10.22049/CCO.2023.28236.1485

[16] K. Prasad, H. Mahato and M. Kumari, *On the generalized k-horadam-like sequences*, Algebra, Analysis and Associated Topics. Trends in Mathematics (2022), 11–26. https://doi.org/10.1007/978-3-031-19082-7_2

[17] K. Prasad, H. Mahato, M. Kumari and R. Mohanty, *On the generalized Leonardo Pisano octonions*, Natl Acad Sci Lett. (2023). https://doi.org/10.1007/s40009-023-01291-2

[18] Y. Soykan, *Generalized Leonardo numbers*, Journal of Progressive Research in Mathematics **18**(4) (2021), 58–84.

[19] R. P. M. Vieira, F. R. V. Alves and P. M. M. C. Catarino, *Bidimensional relationships and identities from Leonardo's sequence*, Sergipana Magazine of Mathematics and Mathematical Education **4** (2019), 156–173.

[20] R. P. M. Vieira, M. C. d. S. Mangueira, F. R. V. Alves and P. M. M. C. Catarino, *The matrix form of Leonardo's numbers*, Ciencia e Natura **42** (2020), 1–13.

[21] M. D. Vivarelli, *Development of spinor descriptions of rotational mechanics from Euler's rigid body displacement theorem*, Celestial Mechanics **32**(3) (1984), 193–207. https://doi.org/10.1007/BF01236599

[22] A. Yasemin and E. G. Koçer, *Some properties of Leonardo numbers*, Konuralp J. Math. **9**(1) (2021), 183–189.

[23] Z. İşbilir, M. Akyiğit and M. Tosun, *Pauli-Leonardo quaternions*, Notes Number Theory Discrete Math. **29**(1) (2023), 1–16. https://doi.org/10.7546/nntdm.2023.29.1.1-16

[1]Department of Mathematics,
Central University of Jharkhand,
Ranchi, 835205, India.
Email address: muneshnasir94@gmail.com
Email address: klkaprsd@gmail.com
Email address: hrishikesh.mahato@cuj.ac.in

[2]Department of Mathematics,
Government Engineering College Bhojpur,
Bihar, 802301, India.

[3]Department of Mathematics,
University of Trás-os-Montes e Alto Douro,
5001-801, Vila Real, Portugal.
Email address: pcatarin@utad.pt

*Corresponding Author