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A STUDY ON SOME CONFORMABLE FRACTIONAL IMPLICIT HYBRID DIFFERENTIAL EQUATIONS WITH DELAY

ABDELKRIM SALIM[1,2], SALIM KRIM[3,2], JAMAL EDDINE LAZREG[1],
AND MOUFFAK BENCHOHRA[1]

Abstract. This paper deals with some existence results for a class of conformable implicit fractional differential Hybrid equations with delay. The results are based on some suitable fixed point theorems. In the last section, we provide different examples to illustrate our obtained results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). Its versatility has made it a crucial tool in the field. In the previous decades more and more researchers have paid their attentions to fractional calculus, since they found that the fractional order integrals and derivatives were more suitable for the description of the phenomena in the real world, such as viscoelastic systems, dielectric polarization, electromagnetic waves, heat conduction, robotics, biological systems, finance and so on. For some details and recent publication on the subject, see the monographs [1, 5–8, 21, 32, 34–36] and the papers [2–4]. The study of implicit differential equations has received great attention in the last years; see [1, 13, 26–28].

As models of equations, functional differential equations with delay are commonly used. Several authors studied differential equations with delay [14, 15, 17, 19]. For more details, see the papers which are concerned with finite delay [29, 30], infinite delay [1, 10, 14, 18], and state-dependent delay [1, 17].

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The fractional derivative of an unknown function hybrid with nonlinearity is used in hybrid differential equations. This class of equations derives from several fields of practical mathematics and physics, such as the deflection of a curved beam with a constant or variable cross-section, a three-layer beam, electromagnetic waves, or gravity-driven flows, and so forth. For more details on the subject, we recommend readers to the publications [13, 25, 27, 31].

The authors of [33] studied the nonlinear fractional differential hybrid system with periodic boundary conditions, given by:

$$\begin{cases} {}^{C}\mathfrak{D}_{a^+}^{\varrho,\Psi}\left(v(\vartheta)\mathrm{g}_1(\vartheta, v(\vartheta))\right) = \mathrm{g}_2(\vartheta, v(\vartheta)), \quad \varrho \in [a,b], \\ v(a) = v(b), \end{cases}$$

where $\vartheta \in [a,b]$, ${}^{C}\mathfrak{D}_{a^+}^{\varrho,\Psi}$ is the Ψ-Caputo fractional derivative, $g_1 : [a,b] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $\mathrm{g}_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Their arguments are based on Dhage's fixed point theorem.

In [20], Khalil et al. provided a unique concept of fractional derivative, which is a natural extension of the traditional first derivative. The conformable fractional derivative is natural, and it contains most of the features of the classical integral derivative, such as product rule, quotient rule, linearity, chain rule, and power rule, and it is very useful for modelling different physical problems. Indeed, several publications have been produced since that time, and various equations have been solved using that notion [9, 12, 24].

In [22], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}_{\zeta_\jmath}^{\vartheta} x(\zeta) = f\left(\zeta, x_\zeta, \mathcal{T}_\jmath^{\vartheta} x(\zeta)\right), \quad \zeta \in \Omega_\jmath, \quad \jmath = 0, 1, \ldots, \beta, \\ \Delta x|_{\zeta=\zeta_\jmath} = \Upsilon_\jmath(x_{\zeta_\jmath^-}), \quad \jmath = 1, 2, \ldots, \beta, \\ x(\zeta) = \mu(\zeta), \quad \zeta \in (-\infty, \varkappa], \end{cases}$$

where $0 \le \varkappa = \zeta_0 < \zeta_1 < \cdots < \zeta_\beta < \zeta_{\beta+1} = \bar{\varkappa} < \infty$, $\mathcal{T}_{\zeta_\jmath}^{\vartheta} x(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $f : \Omega \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Omega := [\varkappa, \bar{\varkappa}]$, $\Omega_0 := [\varkappa, \zeta_1]$, $\Omega_\jmath := (\zeta_\jmath, \zeta_{\jmath+1}]$, $\jmath = 1, 2, \ldots, \beta$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_\jmath : \mathcal{Q} \to \mathbb{R}$ are given continuous functions, and $\mathcal{Q}$ is called a phase space.

In this paper, first we investigate the following class of conformable fractional differential Hybrid equation with finite delay:

$$\mathcal{T}_\varepsilon^\varsigma\left(\Phi(t)x(t)\right) = f\left(t, x_t, \mathcal{T}_\varepsilon^\varsigma\left(\Phi(t)x(t)\right)\right), \quad t \in \Theta := (\varepsilon, \beta], \tag{1.1}$$

$$x(t) = \zeta(t), \quad t \in (\varepsilon - \kappa, \varepsilon], \tag{1.2}$$

where $\mathcal{T}_\varepsilon^\varsigma x(t)$ is the conformable fractional derivative starting from the initial time ε of the function f of order $\varsigma \in (0,1)$, $f : \Theta \times C([\varepsilon - \kappa, \varepsilon], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\zeta \in C((\varepsilon - \kappa, \beta], \mathbb{R})$, $\Phi \in C\left(\Theta, \mathbb{R}\backslash\{0\}\right)$, $\varepsilon < \beta < +\infty$ and $\kappa > 0$ is the time delay. For any $t \in \Theta$, we give x_t by

$$x_t(\vartheta) = x(t + \vartheta), \quad \text{for } \vartheta \in [\varepsilon - \kappa, \varepsilon].$$

Next, we consider the following infinite delay problem:

$$\mathcal{T}^{\varsigma}_{\varepsilon}\left(\Phi(t)x(t)\right) = f\left(t, x_t, \mathcal{T}^{\varsigma}_{\varepsilon}\left(\Phi(t)x(t)\right)\right), \quad t \in \Theta, \tag{1.3}$$

$$x(t) = \zeta(t), \quad t \in (-\infty, \varepsilon], \tag{1.4}$$

where $f : \Theta \times \mathcal{G} \times \mathbb{R} \to \mathbb{R}$, $\Phi \in C\left(\Theta, \mathbb{R}\backslash\{0\}\right)$, $\zeta : (-\infty, \varepsilon] \to \mathbb{R}$, $\varepsilon < \beta < +\infty$ are given continuous functions, and $\mathcal{G}$ is called a phase space that will be determined later.

For any $t \in \Theta$, we define $x_t \in \mathcal{G}$ by $x_t(\vartheta) = x(t + \vartheta)$ for $\vartheta \in (-\infty, \varepsilon]$. In the next segment, we look into the following state-dependent finite delay problem:

$$\mathcal{T}^{\varsigma}_{\varepsilon}\left(\Phi(t)x(t)\right) = f\left(t, x_{\rho(t,x_t)}, \mathcal{T}^{\varsigma}_{\varepsilon}\left(\Phi(t)x(t)\right)\right), \quad t \in \Theta, \tag{1.5}$$

$$x(t) = \zeta(t), \quad t \in (\varepsilon - \kappa, \varepsilon], \tag{1.6}$$

where $f \in C((\varepsilon, \beta], \mathbb{R})$, $\Phi \in C\left(\Theta, \mathbb{R}\backslash\{0\}\right)$, $\zeta \in C((\varepsilon - \kappa, \beta], \mathbb{R})$, $\varepsilon < \beta < +\infty$ are given continuous functions.

Finally, we study the following problem with state dependent infinite delay:

$$\mathcal{T}^{\varsigma}_{\varepsilon}\left(\Phi(t)x(t)\right) = f\left(t, x_{\rho(t,x_t)}, \mathcal{T}^{\varsigma}_{\varepsilon}\left(\Phi(t)x(t)\right)\right), \quad t \in \Theta, \tag{1.7}$$

$$x(t) = \zeta(t), \quad t \in (-\infty, \varepsilon], \tag{1.8}$$

where $f : \Theta \times \mathcal{G} \times \mathbb{R} \to \mathbb{R}$, $\Phi \in C\left(\Theta, \mathbb{R}\backslash\{0\}\right)$, $\zeta : (-\infty, \varepsilon] \to \mathbb{R}$, $\varepsilon < \beta < +\infty$ are given continuous functions.

2. Preliminaries

Let $C(\Theta)$ be the Banach space of all real continuous functions on Θ with the norm

$$\|x\|_{\infty} = \sup_{t \in \Theta} |x(t)|.$$

Let $\mathcal{C} := C([\varepsilon - \kappa, \beta])$ be a Banach space with the norm

$$\|x\|_{C} := \sup_{t \in [\varepsilon - \kappa, \beta]} |x(t)|.$$

By $L^1(\Theta)$ we denote the Banach space of measurable functions $x : \Theta \to \mathbb{R}$ which are Lebesgue integrable, equipped with the norm

$$\|x\|_{L^1} = \int_{\varepsilon}^{\beta} |x(t)| dt.$$

Definition 2.1 (The conformable fractional derivative [9])**.** Let $f : [0, +\infty) \to \mathbb{R}$ be a given function, the conformable fractional derivative of f of order ς is defined by

$$\mathcal{T}^{\varsigma}(f)(t) = \lim_{\alpha \to 0} \frac{f\left(t + \alpha t^{1-\varsigma}\right) - f(t)}{\alpha},$$

for $t > 0$ and $\varsigma \in (0, 1]$. If f is ς-differentiable in some $(0, \varepsilon)$, $\varepsilon > 0$, and $\lim_{t \to 0+} \mathcal{T}^{\varsigma}(f)(t)$ exists, then define $\mathcal{T}^{\varsigma}(f)(0) = \lim_{t \to 0+} \mathcal{T}^{\varsigma}(f)(t)$. If the conformable fractional derivative of f of order ς exists, then we simply say that f is ς-differentiable. It is easy to see that if f is differentiable, then $\mathcal{T}^{\varsigma}(f)(t) = t^{1-\varsigma} f'(t)$.

Definition 2.2 (The conformable fractional integral [9])**.** The conformable fractional integral starting from ε of the function $f : [\varepsilon, +\infty) \to \mathbb{R}$ of order $\varsigma \in (0, 1]$ is defined as

$$I_{\varepsilon}^{r} f(t) = \int_{\varepsilon}^{t} (\vartheta - \varepsilon)^{1-\varsigma} f(t) d\vartheta.$$

Lemma 2.1 ([9])**.** *Let $\varsigma \in (0, 1]$ and $f : [\varepsilon, +\infty) \to \mathbb{R}$ be a continuous function. Then, for all $t > \varepsilon$,*

$$\mathfrak{T}_{\varepsilon}^{r} I_{\varepsilon}^{r} f(t) = f(t).$$

Further, if f is differentiable on $(\varepsilon, +\infty)$, then, for all $t > \varepsilon$,

$$I_{\varepsilon}^{r} \mathfrak{T}_{\varepsilon}^{r} f(t) = f(t) - f(\varepsilon).$$

By following the same approach as in the paper [9], we may obtain the following result.

Lemma 2.2 ([9])**.** *A function x is a solution of problem* (1.1)–(1.2)*, if and only if x satisfies the following integral equation*

$$x(t) = \begin{cases} \dfrac{1}{\Phi(t)} \left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t} (\vartheta - \varepsilon)^{\varsigma - 1} \widehat{f}(\vartheta) d\vartheta \right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in [\varepsilon - \kappa, \varepsilon], \end{cases} \tag{2.1}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$.

For our purpose we will need the following fixed point theorems.

3. Existence of Solutions with Finite Delay

In this section, we are concerned with the existence results of the problem (1.1)–(1.2).

Let us introduce the following hypothesis.

(H_1) There exist constants $\omega_1, \mathcal{M} > 0,\ 0 < \omega_2 < 1$ such that:

$$|f(t, x_1, \Im_1) - f(t, x_2, \Im_2)| \leq \omega_1 \|x_1 - x_2\|_{[\varepsilon-\kappa,\varepsilon]} + \omega_2 |\Im_1 - \Im_2|,$$

and

$$|\Phi(t)| \geq \mathcal{M},$$

for any $x_1, x_2 \in \mathcal{C},\ \Im_1, \Im_2 \in \mathbb{R}$, and each $t \in \Theta$.

Remark 3.1. We note that by taking: $\varpi_1 = \omega_1,\ \varpi_2 = \omega_2$ and $\varpi_3 = f^*$, where $f^* = \sup\limits_{t \in [\varepsilon, \beta]} f(t, 0, 0)$, hypothesis (H_1) implies that

$$|f(t, x, \Im)| \leq \varpi_1 \|x\|_{[\varepsilon-\kappa,\varepsilon]} + \varpi_2 |\Im| + \varpi_3.$$

Now, we will give our first existence and uniqueness result that is based on Banach's fixed point theorem.

Theorem 3.1. *Assume that hypothesis* (H_1) *holds. If*

$$\ell := \frac{(\beta-\varepsilon)^{\varsigma}\omega_1}{\mathcal{M}\varsigma(1-\omega_2)} < 1, \tag{3.1}$$

then the problem (1.1)–(1.2) *has a unique solution on* $[\varepsilon-\kappa, \beta]$.

Proof. Consider the operator $\mathcal{H} : C(\Theta) \to C(\Theta)$ such that,

$$(\mathcal{H}x)(t) = \begin{cases} \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in [\varepsilon-\kappa, \varepsilon], \end{cases} \tag{3.2}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$.

Let $x, \Im \in C(\Theta)$. Then, for each $t \in [\varepsilon-\kappa, \varepsilon]$, we get

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| = 0,$$

and for each $t \in \Theta$, we obtain

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| \leq \frac{1}{|\Phi(t)|}\left[\int_{\varepsilon}^{t}(\vartheta-\varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta) - \Upsilon(\vartheta)|d\vartheta\right],$$

where $\widehat{f}, \Upsilon \in C(\Theta)$ such that

$$\widehat{f}(t) = f(t, x_t, \widehat{f}(t)) \quad \text{and} \quad \Upsilon(t) = f(t, \Im_t, \Upsilon(t)).$$

From (H_1), we have

$$\begin{aligned} |\widehat{f}(t) - \Upsilon(t)| &= |f(t, x_t, \widehat{f}(t)) - f(t, \Im_t, \Upsilon(t))| \\ &\leq \omega_1\|x_t - \Im_t\|_{[\varepsilon-\kappa,\varepsilon]} + \omega_2|\widehat{f}(t) - \Upsilon(t)| \\ &\leq \omega_1\|x_t - \Im_t\|_{[\varepsilon-\kappa,\varepsilon]} + \omega_2|\widehat{f}(t) - \Upsilon(t)|. \end{aligned}$$

Thus,

$$\|\widehat{f} - \Upsilon\|_{\infty} \leq \frac{\omega_1}{1-\omega_2}\|(x - \Im)\|_C.$$

Then, for each $t \in \Theta$, we get

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| \leq \frac{(\beta-\varepsilon)^{\varsigma}\omega_1}{\mathcal{M}\varsigma(1-\omega_2)}\|x - \Im\|_C \leq \ell\|x - \Im\|_C.$$

Hence, we get

$$\|\mathcal{H}(x) - \mathcal{H}(\Im)\|_C \leq \ell\|x - \Im\|_C.$$

Consequently, by Banach's fixed point theorem, the operator $\mathcal{H}$ has a unique fixed point, which is the unique solution of our problem (1.1)–(1.2) on $[\varepsilon-\kappa, \beta]$. $\square$

Theorem 3.2. *If* (H_1) *holds, and*

$$\frac{(\beta-\varepsilon)^{\varsigma}\varpi_1}{\mathcal{M}\varsigma(1-\varpi_2)} < 1,$$

then problem (1.1)–(1.2) *has at least one solution on* $[\varepsilon-\kappa, \beta]$.

Proof. Consider $\mathcal{H} : C(\Theta) \to C(\Theta)$ defined in (3.2). Let $\delta > 0$ such that

$$\delta \geq \max\left\{ \|\zeta\|_{C([\varepsilon-\kappa,\beta])}, \frac{\frac{|\Phi(\varepsilon)\zeta(\varepsilon)|}{\mathcal{M}} + \frac{(\beta-\varepsilon)^{\varsigma}\varpi_3}{\mathcal{M}\varsigma(1-\varpi_2)}}{1 - \frac{(\beta-\varepsilon)^{\varsigma}\varpi_1}{\mathcal{M}\varsigma(1-\varpi_2)}} \right\}. \tag{3.3}$$

Define the ball $\Omega_\delta = \{\xi \in C(\Theta) : \|\xi\|_C \leq \delta\}$.

Step 1. $\mathcal{H}$ is continuous.

Let $\{x_n\}_n$ be a sequence such that $x_n \to x$ on Ω_δ. For each $t \in [\varepsilon - \kappa, \varepsilon]$, we have

$$|(\mathcal{H}x_n)(t) - (\mathcal{H}x)(t)| = 0,$$

and for each $t \in \Theta$, we have

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| \leq \frac{1}{|\Phi(t)|}\left[\int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}|\widehat{f}_n(\vartheta) - \widehat{f}(\vartheta)|d\vartheta\right], \tag{3.4}$$

where $\widehat{f}_n, \widehat{f} \in C(\Theta)$ such that

$$\widehat{f}_n(t) = f(t, x_{nt}, \widehat{f}_n(t)) \quad \text{and} \quad \widehat{f}(t) = f(t, x_t, \widehat{f}(t)).$$

Since

$$\|x_n - x\|_C \to 0, \qquad \text{as } n \to +\infty,$$

and $f, \widehat{f}$ and $\widehat{f}_n$ are continuous, then by Lebesgue dominated convergence theorem, we deduce that

$$\|\mathcal{H}(x_n) - \mathcal{H}(x)\|_C \to 0 \quad \text{as } n \to +\infty.$$

Hence, $\mathcal{H}$ is continuous.

Step 2. $\mathcal{H}(\Omega_\delta) \subset \Omega_\delta$.

Let $x \in \Omega_\delta$. If $t \in [\varepsilon - \kappa, \varepsilon]$, then $|(\mathcal{H}x)(t)| \leq \|\zeta\|_C \leq \delta$. From Remark 3.1, for each $t \in \Theta$, we have

$$\begin{aligned}
|\widehat{f}(t)| &\leq |f(t, x_t, \widehat{f}(t))| \leq \varpi_1\|x_t\|_{[\varepsilon-\kappa,\beta]} + \varpi_2|\widehat{f}(t)| + \varpi_3\\
&\leq \varpi_1\|x\|_C + \varpi_2\|\widehat{f}\|_\infty + \varpi_3 \leq \varpi_1\delta + \varpi_2\|\widehat{f}\|_\infty + \varpi_3.
\end{aligned}$$

Then,

$$\|\widehat{f}\|_\infty \leq \frac{\delta\varpi_1 + \varpi_3}{1 - \varpi_2}.$$

Thus,

$$\begin{aligned}
|(\mathcal{H}x)(t)| &\leq \frac{1}{|\Phi(t)|}\left|\Phi(\varepsilon)\zeta(\varepsilon) + \int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right|\\
&\leq \frac{1}{|\Phi(t)|}\left[|\Phi(\varepsilon)\zeta(\varepsilon)| + \int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta)|d\vartheta\right]\\
&\leq \frac{1}{\mathcal{M}}\left[|\Phi(\varepsilon)\zeta(\varepsilon)| + \frac{(\beta - \varepsilon)^{\varsigma}(\delta\varpi_1 + \varpi_3)}{\varsigma(1 - \varpi_2)}\right]\\
&\leq \delta.
\end{aligned}$$

Hence, $\|\mathcal{H}(x)\|_C \leq \delta$. Consequently, $\mathcal{H}(\Omega_\delta) \subset \Omega_\delta$.

Step 3. $\mathcal{H}(\Omega_\delta)$ is equicontinuous.

For $\varepsilon \leq t_1 \leq t_2 \leq \beta$, and $x \in \Omega_\delta$, we get

$$\begin{aligned}|\mathcal{H}(x)(t_1) - \mathcal{H}(x)(t_2)| &\leq \frac{1}{|\Phi(t)|}\left|\int_\varepsilon^{t_1}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta - \int_\varepsilon^{t_2}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right| \\ &\leq \frac{\delta\varpi_1+\varpi_3}{\mathcal{M}\varsigma(1-\varpi_2)}\left|(t_2-\varepsilon)^\varsigma - (t_1-\varepsilon)^\varsigma\right|.\end{aligned}$$

As $t_2 \to t_1$ then $|\mathcal{H}(x)(t_1) - \mathcal{H}(x)(t_2)| \to 0$. We deduce that $\mathcal{H}(\Omega_\delta)$ is equicontinuous.

Consequently, Arzelá-Ascoli theorem implies that $\mathcal{H}$ is continuous and compact. Thus, by Schauder's fixed point theorem [37], we deduce that $\mathcal{H}$ has at least a fixed point which is a solution of (1.1)–(1.2). □

4. Existence of Solutions with Infinite Delay

In this section, we are concerned with the existence results of (1.3)–(1.4). Let the space $(\mathcal{G}, \|\cdot\|_\mathcal{G})$ is a seminormed linear space of functions mapping $(-\infty, \varepsilon]$ into $\mathbb{R}$, and verifying the following axioms which were derived from Hale and Kato's originals [14].

(Ax_1) If $x : (-\infty, \beta] \to \mathbb{R}$, and $x_0 \in \mathcal{G}$, then there exist constants $\xi_1, \xi_2, \xi_3 > 0$, such that for each $t \in \Theta$; we have:

(i) x_t is in $\mathcal{G}$;

(ii) $\|x_t\|_\mathcal{G} \leq \xi_1\|x_1\|_\mathcal{G} + \xi_2 \sup_{\vartheta\in[\varepsilon,t]}|x(\vartheta)|$;

(iii) $\|x(t)\| \leq \xi_3\|x_t\|_\mathcal{G}$.

(Ax_2) For the function $x(\cdot)$ in (Ax_1), y_t is a $\mathcal{G}$-valued continuous function on Θ.

(Ax_3) The space $\mathcal{G}$ is complete.

Consider the space $\Omega = \{x : (-\infty, \beta] \to \mathbb{R},\ x|_{(-\infty,\varepsilon]} \in \mathcal{G},\ x|_\Theta \in C([\varepsilon-\kappa, \beta], \mathbb{R})\}$.

Definition 4.1. By a solution of problem (1.3)–(1.4), we mean a function $x \in \Omega$ such that

$$x(t) = \begin{cases} \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_\varepsilon^t (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in (-\infty, \varepsilon], \end{cases}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$.

The following hypothesis will be used in the sequel.

(H_2) The functions f and Φ verify:

$$|f(t, x_1, \Im_1) - f(t, x_2, \Im_2)| \leq b_1\|x_1 - x_2\|_\mathcal{G} + b_2|\Im_1 - x_2|$$

and

$$|\Phi(t)| \geq \mathcal{M},$$

for any $x_1, \Im_1 \in \mathcal{G}$, $x_2, \Im_2 \in \mathbb{R}$, and each $t \in \Theta$, where $b_1, \mathcal{M} > 0$ and $0 < b_2 < 1$.

Remark 4.1. We note that by taking: $B_1 = b_1$, $B_2 = b_2$ and $B_3 = f^*$, where $f^* = \sup_{t\in[\varepsilon,\beta]} f(t, 0, 0)$, hypothesis (H_2) implies that

$$|f(t, x, \Im)| \leq B_1\|x\|_\mathcal{G} + B_2|\Im| + B_3,$$

for any $x \in \mathcal{G}$, $\Im \in \mathbb{R}$, and each $t \in \Theta$.

Theorem 4.1. *Assume that the hypothesis* (H_2) *holds. If*

$$\lambda := \frac{(\beta-\varepsilon)^{\varsigma} b_1}{\mathcal{M}\varsigma(1-b_2)} < 1, \tag{4.1}$$

then the problem (1.3)–(1.4) *has a unique solution on* $(-\infty, \beta]$.

Proof. Consider the operator $N_1 : \Omega \to \Omega$ such that,

$$(N_1 x)(t) = \begin{cases} \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in (-\infty, \varepsilon], \end{cases}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$.

Let $\varkappa_1 : (-\infty, \beta] \to \mathbb{R}$ be a function given by

$$\varkappa_1(t) = \begin{cases} \zeta(t), & t \in (-\infty, \varepsilon], \\ \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon)\right], & t \in \Theta. \end{cases}$$

For each $\varkappa_2 \in C(\Theta)$, with $\varkappa_2(0) = 0$, we denote by $\overline{\varkappa_2}$ the function defined by

$$\overline{\varkappa_2} = \begin{cases} 0, & t \in (-\infty, \varepsilon], \\ \varkappa_2(t), & t \in \Theta. \end{cases}$$

If $x(\cdot)$ satisfies the integral equation

$$x(t) = \frac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right],$$

we can decompose $x(\cdot)$ as $x(t) = \overline{\varkappa_2}(t) + \varkappa_1(t)$, for $t \in \Theta$, which implies that $x_t = \overline{\varkappa_2}_t + \varkappa_{1t}$ for every $t \in \Theta$, and the function $\varkappa_2(\cdot)$ satisfies

$$\varkappa_2(t) = \frac{1}{\Phi(t)}\left[\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right],$$

where $\widehat{f}(t) = f(t, \overline{\varkappa_2}_t + \varkappa_{1t}, \widehat{f}(t))$, $t \in \Theta$. Set

$$C_0 = \{\varkappa_2 \in C(\Theta) : \varkappa_{2\varepsilon} = 0\},$$

and let $\|\cdot\|_T$ be the norm in C_0 defined by

$$\|\varkappa_2\|_T = \|\varkappa_{2\varepsilon}\|_{\mathcal{G}} + \sup_{t\in\Theta}|\varkappa_2(t)| = \sup_{t\in\Theta}|\varkappa_2(t)|, \quad \varkappa_2 \in C_0,$$

where C_0 is a Banach space with norm $\|\cdot\|_T$. Define the operator $\mathcal{K} : C_0 \to C_0$ by

$$(\mathcal{K}\varkappa_2)(t) = \frac{1}{\Phi(t)}\left[\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], \tag{4.2}$$

where $\widehat{f}(t) = f(t, \overline{\varkappa_2}_t + \varkappa_{1t}, \widehat{f}(t))$, $t \in \Theta$. We shall show that $\mathcal{K} : C_0 \to C_0$ is a contraction map. Let $\varkappa_2, \varkappa_2' \in C_0$, then we have for each $t \in \Theta$

$$|\mathcal{K}(\varkappa_2)(t) - \mathcal{K}(\varkappa_2')(t)| \le \frac{1}{|\Phi(t)|}\left[\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta) - \Upsilon(\vartheta)|d\vartheta\right], \tag{4.3}$$

where $\widehat{f}, \Upsilon \in C(\Theta)$ such that

$$\widehat{f}(t) = f(t, \overline{\varkappa_{2t}} + \varkappa_{1t}, \widehat{f}(t)) \quad \text{and} \quad \Upsilon(t) = f(t, \overline{\varkappa_2{}'}_t + \varkappa_{1t}, \Upsilon(t)).$$

Since, for each $t \in \Theta$, we have

$$|\widehat{f}(t) - \Upsilon(t)| \leq \frac{b_1}{1 - b_2} \|\overline{\varkappa_{2t}} - \overline{\varkappa_2{}'}_t\|_{\mathcal{G}}.$$

Then, for each $t \in \Theta$, we get

$$\begin{aligned} |\mathcal{K}(\varkappa_2)(t) - \mathcal{K}(\varkappa_2{}')(t)| &\leq \frac{(\beta - \varepsilon)^{\varsigma} b_1}{\mathcal{M}\varsigma(1 - b_2)} \|\overline{\varkappa_{2t}} - \overline{\varkappa_2{}'}_t\|_{\mathcal{G}} \leq \frac{(\beta - \varepsilon)^{\varsigma} b_1}{\mathcal{M}\varsigma(1 - b_2)} \|\overline{\varkappa_2} - \overline{\varkappa_2{}'}\|_{\beta} \\ &= \lambda \|\overline{\varkappa_2} - \overline{\varkappa_2{}'}\|_{\beta}. \end{aligned}$$

Thus, we get $\|\mathcal{K}(\varkappa_2)(t) - \mathcal{K}(\varkappa_2{}')(t)\|_T \leq \lambda \|\overline{\varkappa_2} - \overline{\varkappa_2{}'}\|_{\beta}$. Hence, from the Banach contraction principle, $\mathcal{K}$ admit a unique fixed point which is the unique solution of (1.3)–(1.4). □

Now, we demonstrate an existence result for problem (1.3)–(1.4) by using Scheafer's fixed point theorem [16].

Theorem 4.2. *Suppose that* (H_2) *holds. Then,* (1.3)–(1.4) *admit at least one solution on* $(-\infty, \beta]$.

Proof. Let $\mathcal{K} : C_0 \to C_0$ defined as in (4.2), For each given $\delta > 0$, we define the ball

$$\Omega_\delta = \{\varkappa_1 \in C_0 : \ \|\varkappa_1\|_\beta \leq \delta\}.$$

Step 1. $\mathcal{K}$ is continuous.

Let $\varkappa_{2n}$ be a sequence where $\varkappa_{2n} \to \varkappa_2$ in C_0. For each $t \in \Theta$, we have

$$|(\mathcal{K}\varkappa_{2n})(t) - (\mathcal{K}\varkappa_2)(t)| \leq \frac{1}{|\Phi(t)|} \left[\int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma - 1} |\widehat{f}_n(\vartheta) - \widehat{f}(\vartheta)| d\vartheta \right], \tag{4.4}$$

where $\widehat{f}_n, \widehat{f} \in C(\Theta)$ such that

$$\widehat{f}_n(t) = f(t, \overline{\varkappa_{2nt}} + \varkappa_{1t}, \widehat{f}_n(t)) \quad \text{and} \quad \widehat{f}(t) = f(t, \overline{\varkappa_{2t}} + \varkappa_{1t}, \widehat{f}(t)).$$

Since $\|\varkappa_{2n} - \varkappa_2\|_\beta \to 0,$ as $n \to \infty$ and $f, \widehat{f}$ and $\widehat{f}_n$ are continuous, then

$$\|\mathcal{K}(x_n) - \mathcal{K}(x)\|_\beta \to 0, \quad \text{as } n \to +\infty.$$

Hence, $\mathcal{K}$ is continuous.

Step 2. $\mathcal{K}(\Omega_\delta)$ is bounded.

Let $\varkappa_2 \in \Omega_\delta$, for $t \in \Theta$, we have

$$\begin{aligned} |\widehat{f}(t)| &\leq |f(t, \overline{\varkappa_{2t}} + \varkappa_{1t}, \widehat{f}(t))| \\ &\leq B_1 \|\overline{\varkappa_{2t}} + \varkappa_{1t}\|_{\mathcal{G}} + B_2 |\widehat{f}(t)| + B_3 \\ &\leq B_1 \Big[\|\overline{\varkappa_{2t}}\|_{\mathcal{G}} + \|\varkappa_{1t}\|_{\mathcal{G}} \Big] + B_2 \|\widehat{f}\|_\infty + B_3 \\ &\leq B_1 \xi_2 \delta + B_1 \xi_1 \|\varphi\|_{\mathcal{G}} + B_2 \|\widehat{f}\|_\infty + B_3. \end{aligned}$$

Then,

$$\|\widehat{f}\|_\infty \leq \frac{B_1\xi_2\delta + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3}{1-B_2}.$$

Thus,

$$\begin{aligned}|(\mathcal{K}\varkappa_2)(t)| &\leq \frac{1}{|\Phi(t)|}\left[\int_\varepsilon^t (\vartheta-\varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta)|d\vartheta\right]\\ &\leq \frac{(\beta-\varepsilon)^\varsigma(B_1\xi_2\delta + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3)}{\mathcal{M}\varsigma(1-B_2)} := \tilde{\ell}.\end{aligned}$$

Hence, $\|\mathcal{K}(\varkappa_2)\|_\beta \leq \tilde{\ell}$. Consequently, $\mathcal{K}$ maps bounded sets into bounded sets in C_0.

Step 3. $\mathcal{K}(\Omega_\delta)$ is equicontinuous.

For $\varepsilon \leq t_1 \leq t_2 \leq \beta$, and $\varkappa_2 \in \Omega_\delta$, we have

$$\begin{aligned}|\mathcal{K}(x)(t_1) - \mathcal{K}(x)(t_2)| &\leq \frac{1}{|\Phi(t)|}\left|\int_\varepsilon^{t_1}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta - \int_\varepsilon^{t_2}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right|\\ &\leq \frac{B_1\xi_2\delta + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3}{\mathcal{M}\varsigma(1-B_2)}\left|(t_2-\varepsilon)^\varsigma - (t_1-\varepsilon)^\varsigma\right|.\end{aligned}$$

As $t_2 \to t_1$ we have that $|\mathcal{H}(x)(t_1) - \mathcal{H}(x)(t_2)| \to 0$. We deduce that $\mathcal{K}$ maps bounded sets into equicontinuous sets in C_0. Thus, $\mathcal{K} : C_0 \to C_0$ is completely continuous.

Step 4. The priori bounds.

We prove that the set

$$\mathcal{E} = \{x \in C_0 : \Im = \lambda\mathcal{K}(x), \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $\varkappa_2, u \in C_0$ such that $\varkappa_2 = \lambda\mathcal{K}(\varkappa_2)$, for some $\lambda \in (0,1)$. Then, for each $t \in \Theta$, we have

$$\varkappa_2(t) = \lambda(\mathcal{K}\varkappa_2)(t) = \lambda\frac{1}{\Phi(t)}\left[\int_\varepsilon^t(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right].$$

By Remark 4.1, we have

$$\begin{aligned}|\widehat{f}(t)| &\leq |f(t, \overline{\varkappa_{2t}} + \varkappa_{1t}, \widehat{f}(t))|\\ &\leq B_1\|\overline{\varkappa_{2t}} + \varkappa_{1t}\|_{\mathcal{G}} + B_2|\widehat{f}(t)| + B_3\\ &\leq B_1\left[\|\overline{\varkappa_{2t}}\|_{\mathcal{G}} + \|\varkappa_{1t}\|_{\mathcal{G}}\right] + B_2\|\widehat{f}\|_\infty + B_3\\ &\leq B_1\xi_2\|\varkappa_2\|_T + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_2\|\widehat{f}\|_\infty + B_3.\end{aligned}$$

This gives

$$\|\widehat{f}\|_\infty \leq \frac{B_1\xi_2\|\varkappa_2\|_T + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3}{1-B_2} := \eta.$$

Thus, for each $t \in \Theta$, we obtain

$$|\varkappa_2(t)| \leq \frac{1}{|\Phi(t)|}\left[\int_\varepsilon^t(\vartheta-\varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta)|d\vartheta\right] \leq \frac{\eta(\beta-\varepsilon)^\varsigma}{\mathcal{M}\varsigma} := \eta'.$$

Hence, $\|\varkappa_2\|_\beta \leq \eta'$. This shows that the set $\mathcal{E}$ is bounded. Thus, by Scheafer's fixed point theorem [16], $\mathcal{K}$ has a fixed point which is a solution of problem (1.3)–(1.4). □

5. Existence Results with State-Dependent Delay

5.1. **The Finite Delay Case.** We now consider the problem (1.5)-(1.6).

Definition 5.1. By a solution of problem (1.5)–(1.6), we mean a function $x \in C([\varepsilon-\kappa,\beta],\mathbb{R})$ such that

$$x(t)=\begin{cases} \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon)+\displaystyle\int_{\varepsilon}^{t}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t\in[\varepsilon,\beta],\\ \zeta(t), & t\in[\varepsilon-\kappa,\varepsilon],\end{cases}$$

where $\widehat{f}\in C(\Theta)$, with $\widehat{f}(t)=f(t,x_{\rho(t,x_t)},\widehat{f}(t))$.

For the next result we will make use of the following hypothesis.
(H_3) The functions f and Φ verify:

$$|f(t,x_1,\Im_1)-f(t,x_2,\Im_2)|\le\omega_3\|x_1-x_2\|_{[\varepsilon-\kappa,\varepsilon]}+\omega_4|\Im_1-\Im_2|$$

and

$$|\Phi(t)|\ge\mathcal{M},$$

for any $x_1,\Im_1\in C([\varepsilon-\kappa,\varepsilon],\mathbb{R}),\ x_2,\Im_2\in\mathbb{R}$, and each $t\in\Theta$, where $\omega_3,\mathcal{M}>0$ and $0<\omega_4<1$.

Remark 5.1. We note that by taking:

$$A_1=\omega_3,\quad A_2=\omega_4\quad\text{and}\quad A_3=f^*,$$

where $f^*=\sup\limits_{t\in[\varepsilon,\beta]}f(t,0,0)$. Then, hypothesis ($H_3$) implies that

$$|f(t,x,\Im)|\le A_1\|x\|_{[\varepsilon-\kappa,\varepsilon]}+A_2|\Im|+A_3,$$

for any $x\in C([\varepsilon-\kappa,\varepsilon],\mathbb{R}),\ v\in\Im$, and each $t\in\Theta$.

As in Theorems 3.1 and 3.2, we have the following results.

Theorem 5.1. *Assume that the hypothesis* (H_3) *holds. If*

$$\frac{(\beta-\varepsilon)^{\varsigma}\omega_3}{\mathcal{M}\varsigma(1-\omega_4)}<1,$$

then problem (1.5)–(1.6) *has a unique solution on* $[\varepsilon-\kappa,\beta]$.

Theorem 5.2. *Suppose that* (H_3) *holds. If*

$$\frac{(\beta-\varepsilon)^{\varsigma}A_1}{\mathcal{M}\varsigma(1-A_2)}<1,$$

then problem (1.5)–(1.6) *has at least one solution on* $[\varepsilon-\kappa,\beta]$.

5.2. **The Infinite Delay Case.** In this part, we present the results concerning the last problem (1.7)–(1.8).

Definition 5.2. By a solution of (1.7)-(1.8), we mean a function $x \in \Omega$ such that

$$x(t) = \begin{cases} \dfrac{1}{\Phi(t)} \left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t} (\vartheta - \varepsilon)^{\varsigma - 1} \widehat{f}(\vartheta) d\vartheta \right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in (-\infty, \varepsilon], \end{cases}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_{\rho(t,x_t)}, \widehat{f}(t))$.

Set $\delta' := \delta'_{\rho^-} = \{\rho(t,x) : t \in \Theta,\ x \in \mathcal{G}\ \rho(t,x) < 0\}$. We suppose that $\rho : \Theta \times \mathcal{G} \to \mathbb{R}$ is continuous and $t \to x_t$ is continuous from δ' into $\mathcal{G}$.

(H_η) There exists a continuous bounded function $\varpi : \delta'_{\rho^-} \to (0, +\infty)$ where

$$\|\eta_t\|_{\mathcal{G}} \leq \varpi(t) \|\eta\|_{\mathcal{G}}, \quad \text{for any } t \in \delta'.$$

Lemma 5.1. *If $x \in \Omega$, then*

$$\|x_t\|_{\mathcal{G}} = (\xi_2 + \varpi')\|\eta\|_{\mathcal{G}} + \xi_1 \sup_{\tau \in [0, \max\{0,t\}]} \|x(\tau)\|,$$

where $\varpi' = \sup_{t \in \delta'} \varpi(t)$.

The following hypothesis will be used in the sequel.

(H_4) The functions f and Φ verify:

$$|f(t, x_1, \Im_1) - f(t, x_2, \Im_2)| \leq b_3 \|x_1 - x_2\|_{\mathcal{G}} + b_4 |\Im_1 - \Im_2|$$

and $|\Phi(t)| \geq \mathcal{M}$, for any $x_1, \Im_1 \in \mathcal{G}$, $x_2, \Im_2 \in \mathbb{R}$, and each $t \in \Theta$, where $b_3, \mathcal{M} > 0$ and $0 < b_4 < 1$.

Remark 5.2. We note that by taking: $B_4 = b_3$, $B_5 = b_4$ and $B_6 = f^*$, where $f^* = \sup_{t \in [\varepsilon, \beta]} f(t, 0, 0)$. Then, hypothesis (H_4) implies that

$$|f(t, x, \Im)| \leq B_4 \|x\|_{\mathcal{G}} + B_5 |\Im| + B_6,$$

for any $x \in \mathcal{G}$, $\Im \in \mathbb{R}$, and $t \in \Theta$.

As in Theorems 4.1 and 4.2, we have the following results.

Theorem 5.3. *Suppose that (H_4) holds. If*

$$\frac{(\beta - \varepsilon)^{\varsigma} b_3}{\mathcal{M}\varsigma(1 - b_4)} < 1,$$

then the problem (1.7)–(1.8) *has a unique solution on* $(-\infty, \beta]$.

Theorem 5.4. *Suppose that (H_ζ) and (H_4) hold. Then,* (1.7)–(1.8) *admit at least one solution on* $(-\infty, \beta]$.

6. Examples

We give now some examples that illustrate our obtained results throughout the paper.

Example 6.1. Consider the following problem

$$\begin{cases} (\mathfrak{T}_0^{1/2}\Phi x)(t) = \dfrac{1}{90\,(1+\|x_t\|)} + \dfrac{1}{30\left(1+|(\mathfrak{T}_0^{1/2}\Phi x)(t)|\right)}, & t\in[0,1],\\ x(t) = 1+t^2, & t\in[-1,0]. \end{cases} \tag{6.1}$$

Set

$$f(t,x,\Im) = \frac{1}{90\,(1+\|x\|)} + \frac{1}{30\,(1+|\Im|)}$$

and

$$\Phi(t) = \frac{\sqrt{2}}{3}\left(t^2+3|\sin(t)|+1\right),$$

where $t\in[0,1]$, $x\in\mathcal{C}$, $\Im\in\mathbb{R}$. Thus, f is continuous. For $x,\widetilde{x}\in\mathcal{C}$, $x,\widetilde{x}\in\mathbb{R}$, and $t\in[0,1]$, we have

$$|f(t,x,\Im)-f(t,\widetilde{x},\widetilde{\Im})| \le \frac{1}{90}\|x-\widetilde{x}\|_{[-1,0]} + \frac{1}{30}|\Im-\widetilde{\Im}|.$$

Hence, hypothesis (H_1) is satisfied with $\omega_1=\frac{1}{90}$, $\mathcal{M}=\frac{\sqrt{2}}{3}$ and $\omega_2=\frac{1}{30}$. Next, the condition (3.1) is verified with $\beta=1$ and $\varsigma=\frac{1}{2}$. Indeed,

$$\frac{(\beta-\varepsilon)^{\varsigma}\omega_1}{\mathcal{M}\varsigma(1-\omega_2)} = \frac{\frac{1}{90}}{\frac{\sqrt{2}}{3}(1-\frac{1}{30})\frac{1}{2}} \approx 0.0487659849094171 < 1.$$

Some calculations indicate that all of the requirements of Theorem 3.1 are verified. Thus, the problem (6.1) has a unique solution defined on $[-1,1]$.

Example 6.2. Consider now the following problem

$$\begin{cases} (\mathfrak{T}_0^{1/2}\Phi x)(t) = \dfrac{x_t e^{-\gamma t+t}}{180\,(e^t-e^{-t})\,(1+\|x_t\|)} \\ \qquad\qquad\qquad + \dfrac{x(t)e^{-\gamma t+t}}{60\,(e^t-e^{-t})\left(1+|(\mathfrak{T}_0^{1/2}\Phi x)(t)|\right)}, & t\in[0,1],\\ x(t) = t+1, & t\in(-\infty,0], \end{cases} \tag{6.2}$$

where Φ is the function defined in the first example. Let γ be a positive real constant and

$$B_\gamma = \left\{x\in C((-\infty,1],\mathbb{R},) : \lim_{\tau\to-\infty} e^{\gamma\tau}x(\tau) \text{ exists in } \mathbb{R}\right\}. \tag{6.3}$$

The norm of B_γ is given by

$$\|x\|_\gamma = \sup_{\tau\in(-\infty,1]} e^{\gamma\tau}|x(\tau)|.$$

Let $x : (-\infty, 0] \to \mathbb{R}$ be such that $x_0 \in B_\gamma$. Then

$$\begin{aligned}\lim_{\tau\to-\infty} e^{\gamma\tau} x_t(\tau) = \lim_{\tau\to-\infty} e^{\gamma\tau} x(t+\tau-1) &= \lim_{\tau\to-\infty} e^{\gamma(\tau-t+1)} x(\tau)\\ &= e^{\gamma(-t+1)} \lim_{\tau\to-\infty} e^{\gamma(\tau)} x_1(\tau) < +\infty.\end{aligned}$$

Hence, $x_t \in B_\gamma$. Finally, we prove that

$$\|x_t\|_\gamma \le \xi_1 \|x_1\|_\gamma + \xi_2 \sup_{\vartheta\in[0,t]} |x(\vartheta)|,$$

where $\xi_1 = \xi_2 = 1$ and $\xi_3 = 1$. We have $\|x_t(\tau)\| = |x(t+\tau)|$. If $t+\tau \le 1$, we get

$$\|x_t(\xi)\| \le \sup_{\vartheta\in(-\infty,0]} |x(\vartheta)|.$$

For $t + \tau \ge 0$, then we have

$$\|x_t(\xi)\| \le \sup_{\vartheta\in[0,t]} |x(\vartheta)|.$$

Thus, for all $t + \tau \in \Theta$, we get

$$\|x_t(\xi)\| \le \sup_{\vartheta\in(-\infty,0]} |x(\vartheta)| + \sup_{\vartheta\in[0,t]} |x(\vartheta)|.$$

Then,

$$\|x_t\|_\gamma \le \|x_0\|_\gamma + \sup_{\vartheta\in[0,t]} |x(\vartheta)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ is a phase space.

Set

$$f(t, x, \Im) = \frac{e^{-\gamma t+t}}{180\,(e^t - e^{-t})\left(1 + \|x\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\,(e^t - e^{-t})\,(1 + |\Im|)},$$

where $t \in [0, 1],\ x \in B_\gamma,\ \Im \in \mathbb{R}$.

For any $x, \tilde{x} \in B_\gamma,\ \Im, \tilde{\Im} \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$|f(t, x, \Im) - f(t, \tilde{x}, \tilde{\Im})| \le \frac{1}{180}\|x - \tilde{x}\|_{B_\gamma} + \frac{1}{60}|\Im - \tilde{\Im}|.$$

Hence, hypothesis (H_2) is satisfied with $b_1 = \frac{1}{180}$, $\mathcal{M} = \frac{\sqrt{2}}{3}$ and $b_2 = \frac{1}{60}$. All requirements of Theorem 4.2 are met. Then, the problem (6.2) has at least one solution defined on $(-\infty, 1]$.

Example 6.3. We consider the following problem

(6.4)
$$\begin{cases}(\mathfrak{T}_0^{1/2}\Phi x)(t) = \dfrac{1}{90(1 + |x(t - \sigma(x(t)))|)} + \dfrac{1}{30\left(1 + |(\mathfrak{T}_0^{1/2}\Phi x)(t)|\right)}, & t \in [0, 1],\\ x(t) = 1 + t^2, & t \in [-1, 0],\end{cases}$$

where Φ is the function defined in the first example and $\sigma \in C(\mathbb{R}, [0, 1])$. Set

$$\rho(t, \zeta) = t - \sigma(\zeta(0)), \quad (t, \zeta) \in [0, e] \times C([-1, 0], \mathbb{R}),$$

$$f(t,x,\Im) = \frac{1}{90(1+|x(t-\sigma(x(t)))|)} + \frac{1}{30\,(1+|\Im(t)|)}, \quad t \in [0,1],\ x \in \mathcal{C},\ \Im \in \mathbb{R}.$$

Obviously, f is jointly continuous. For any $x, \widetilde{x} \in \mathcal{C}$, $\Im, \widetilde{\Im} \in \mathbb{R}$ and $t \in [0,1]$, we have

$$|f(t,x,\Im) - f(t,\widetilde{x},\widetilde{\Im})| \leq \frac{1}{90}\|x-\widetilde{x}\|_{[-1,0]} + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence, hypothesis (H_3) is satisfied with $\omega_3 = \frac{1}{90}$, $\mathcal{M} = \frac{\sqrt{2}}{3}$ and $\omega_4 = \frac{1}{30}$. All requirements of Theorem 5.1 are verified. Thus, the problem (6.4) has a unique solution defined on $[-1,1]$.

Example 6.4. Consider now the problem

$$\begin{cases} (\mathcal{T}_0^{1/2}\Phi x)(t) = \dfrac{x(t-\lambda(x(t)))e^{-\gamma t+t}}{180\,(e^t-e^{-t})\,(1+|x(t-\sigma(x(t))|)} \\ \qquad\qquad + \dfrac{x(t)e^{-\gamma t+t}}{60\,(e^t-e^{-t})\left(1+|(\mathcal{T}_0^{1/2}\Phi x)(t)|\right)}, & t \in [0,2], \\ x(t) = t, & t \in (-\infty, 0], \end{cases} \tag{6.5}$$

where Φ is the function defined in the first example.

Let $\gamma > 0$ and B_γ be given in Example 2.

Let $\rho(t,\zeta) = t - \lambda(\zeta(0))$, $(t,\zeta) \in [0,2] \times B_\gamma$, and set

$$f(t,x,\Im) = \frac{e^{-\gamma t+t}}{180\,(e^t-e^{-t})\left(1+\|x\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\,(e^t-e^{-t})\,(1+|\Im|)},$$

where $t \in [0,2]$, $x \in B_\gamma$, $\Im \in \mathbb{R}$.

We can demonstrate that all conditions of Theorem 4.2 are verified. Then, the problem (6.5) has at least one solution defined on $(-\infty, 2]$.

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[1]Faculty of Technology,
Hassiba Benbouali University of Chlef,
P.O. Box 151 Chlef 02000, Algeria
Email address: salim.abdelkrim@yahoo.com

[2]Laboratory of Mathematics,
Djillali Liabes University of Sidi Bel-Abbes,
P.O. Box 89, Sidi Bel-Abbes 22000, Algeria
Email address: lazregjamal@yahoo.fr
Email address: benchohra@yahoo.com

[3]Ecole National Supérieure d'Oran,
BP 1063 SAIM MOHAMED, Oran 31003, Algeria
Email address: salimsalimkrim@gmail.com