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## UNI- AND BI-PARAMETRIC TWO-STEP ITERATIVE METHOD WITH MEMORY FOR SOLVING NONLINEAR EQUATIONS

NISHANT KUMAR<sup>1</sup> AND JAI PRAKASH JAISWAL<sup>2</sup>

**ABSTRACT.** In this paper, we have suggested a two-step with memory method for solving nonlinear equations by transforming an extant optimal fourth-order without memory method. The acceleration of the order of convergence is attained by employing a single and two self-accelerating parameters. These parameters are estimated by a Hermite interpolating polynomial to enhance the convergence order of iterative method without memory. This order of convergence acceleration is achieved without the use of any additional functional evaluations, precisely the convergence order of the suggested two-step with memory method is reached from 4 to 5.70156. The rate of convergence is also verified by Herzberger’s matrix method. Finally, various examples are taken into consideration to support the theoretical outcomes.

### 1. INTRODUCTION

In today’s real world, solving the nonlinear equation  $g(y) = 0$ , is a very momentous problem. Numerous iterative methods have been presented to find the nonlinear equation’s solution (see [1–4]). These iterative methods show a very important role in the area of numerical analysis because they are utilized in a wide range of pure and applied science fields. The most popular one-point without memory iterative technique among them is the Newton-Raphson method, which is described by

$$w_{n+1} = w_n - \frac{g(w_n)}{g'(w_n)},$$

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*Key words and phrases.* Iterative method with memory, Hermite interpolating polynomial,  $R$ -order of convergence, nonlinear equation, root finding, computational efficiency.

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for the solution of  $g(y) = 0$ ,  $w_0$  is the initial approximation and  $n = 0, 1, 2, \dots$ , whose convergence order is 2. One issue with this method is the presumption  $g'(w_n) \neq 0$ , which restricts its application. One-point iterative scheme established by Kumar et al. [5] is described as follows:

$$w_{n+1} = z_n - \frac{g(w_n)}{g'(w_n) - \lambda g(w_n)}.$$

Taking  $\lambda = 0$  in the above equation, we achieve the Newton-Raphson method. The error expression of the aforesaid scheme is

$$e_{n+1} = (\lambda - c_2)e_n^2 + O(e_n^3),$$

where  $e_n = w_n - \gamma$ ,  $c_i = \frac{g^{(i)}(\gamma)}{i!g'(\gamma)}$ ,  $i = 2, 3, \dots$ , and  $\gamma$  is a zero of nonlinear equation  $g(w) = 0$ . The convergence order of the aforesaid method can be increased by taking  $\lambda = c_2$  in the above error expression. For the classification of iterative methods one can go through the references [6, 7].

Several researchers are currently concentrating on creating with memory iterative techniques that uses one or more self-accelerating parameters. There are some excellent contributions dedicated to derivative free with memory iterative techniques, such as [8–12]. Unfortunately, there are very few memory-based derivative iterative techniques for solving nonlinear equations are available in the literature. The development of the multipoint iterative technique with memory is the main goal of this paper because it may raise the order of convergence of the optimal without memory methods without requiring any additional computations and has a high computational efficiency. In this paper, we present a uni- and bi-parametric two-step iterative method with memory for solving nonlinear equations, followed by a convergence analysis. The Hermite interpolating polynomial is used to calculate the parameters, and the order of convergence of the optimal two-point method is increased from 4 to 5 and 5.70156, respectively. The convergence rate is also verified by an alternate approach called Herzberger's matrix method [13]. At the last, the derived theoretical results are validated by numerical testing.

## 2. WITH MEMORY METHOD AND ITS CONVERGENCE ANALYSIS

In the following part, we will add the parameter  $\alpha$  to the iterative method presented by Khattri [14] to improve its convergence rate. First, we take into account the fourth-order without memory method, which is given in the article [14]:

$$(2.1) \quad \begin{aligned} z_n &= w_n - \frac{g(w_n)}{g'(w_n)}, \\ w_{n+1} &= z_n - \frac{g(z_n)}{2 \left( \frac{g(z_n) - g(w_n)}{z_n - w_n} \right) - g'(w_n)}. \end{aligned}$$

The error expressions for each sub-step of (2.1) are:

$$\begin{aligned}e_{n,z} &= z_n - \gamma = c_2 e_n^2 + O(e_n^3), \\ e_{n+1} &= (c_2^3 - c_2 c_3) e_n^4 + O(e_n^5),\end{aligned}$$

where  $e_{n,z} = z_n - \gamma$ ,  $e_n = w_n - \gamma$  and  $c_i = \frac{g^{(i)}(\gamma)}{i!g'(\gamma)}$ , for  $i = 2, 3, 4, \dots$ , and  $\gamma \in \mathbb{R}$ . After adding the parameter  $\alpha_n$  to the first sub-step of the above scheme, we can write the following with memory iterative scheme:

$$(2.2) \quad \begin{aligned}z_n &= w_n - \frac{g(w_n)}{g'(w_n) - \alpha_n g(w_n)}, \\ w_{n+1} &= z_n - \frac{g(z_n)}{2 \left( \frac{g(z_n) - g(w_n)}{z_n - w_n} \right) - g'(w_n)}.\end{aligned}$$

The error expressions for each sub- step of (2.2) are:

$$(2.3) \quad e_{n,z} = z_n - \gamma = (-\alpha_n + c_2) e_n^2 + O(e_n^3),$$

$$(2.4) \quad e_{n+1} = (\alpha_n - c_2)((\alpha_n - c_2)c_2 + c_3) e_n^4 + O(e_n^5),$$

where  $e_{n,z} = z_n - \gamma$ ,  $e_n = w_n - \gamma$  and  $c_i = \frac{g^{(i)}(\gamma)}{i!g'(\gamma)}$ , for  $i = 2, 3, 4, \dots$ , and  $\gamma \in \mathbb{R}$ . It is symbolized by **OWM4**. It is clear from (2.4) that the order of convergence of (2.2) is four for  $\alpha_n \neq c_2$  and when  $\alpha_n = c_2 = \frac{g''(\gamma)}{2!g'(\gamma)}$ , the convergence order of (2.2) is five. Now the issue is that the exact values of  $g'(\gamma)$  and  $g''(\gamma)$  are not available for this form of acceleration of convergence but we can use the data available from the most recent iteration and the one before it, and it satisfies the condition  $\lim_{n \rightarrow +\infty} \alpha_n = c_2 = \frac{g''(\gamma)}{2!g'(\gamma)}$  for the asymptotic error constant to be zero in the equation (2.4). For calculating  $\alpha_n$ , consider the best possible approximation:

$$(2.5) \quad \alpha_n = \frac{H_4''(w_n)}{2g'(w_n)},$$

where

$$\begin{aligned}H_4(w) &= g(w_n) + (w - w_n)g[w_n, w_n] + (w - w_n)^2 g[w_n, w_n, z_{n-1}] + (w - w_n)^2 \\ &\quad \times (w - z_{n-1})g[w_n, w_n, z_{n-1}, w_{n-1}] + (w - w_n)^2 (w - z_{n-1})(w - w_{n-1}) \\ &\quad \times g[w_n, w_n, z_{n-1}, w_{n-1}, w_{n-1}],\end{aligned}$$

and so,

$$\begin{aligned}H_4''(w_n) &= 2g[w_n, w_n, z_{n-1}] + (w_n - z_{n-1})(4g[w_n, w_n, z_{n-1}, w_{n-1}] \\ &\quad - 2g[w_n, z_{n-1}, w_{n-1}, w_{n-1}]).\end{aligned}$$

**Theorem 2.1.** *Let a Hermite interpolating polynomial  $H_m$  of degree  $m$  which interpolates a function  $g$  at nodes  $w_n, w_n, t_0, \dots, t_{m-2}$  located within an interval  $I$ , and the derivative  $g^{(m+1)}$  is continuous in  $I$ , as well as the Hermite interpolating polynomial satisfying the conditions  $H_m(w_n) = g(w_n)$ ,  $H_m'(w_n) = g'(w_n)$ ,  $H_m(t_i) = g(t_i)$ ,*

$i = 0, 1, \dots, m-2$ . Indicate the errors  $e_{t,i} = t_i - \gamma$ ,  $i = 0, 1, 2, \dots, m-2$ , and presume that

- (1) all nodes  $w_n, t_0, \dots, t_{m-2}$  are adequately near to the zero  $\gamma$ ;
- (2) the condition  $e_n = O(e_{t,0}, e_{t,1}, \dots, e_{t,m-2})$  holds.

Then

$$H_m''(w_n) = 2g'(\gamma) \left( c_2 - (-1)^{m-1} c_{m+1} \prod_{i=0}^{m-2} e_{t,i} + 3c_3 e_n \right),$$

$$\alpha_n = \frac{H_m''(w_n)}{2g'(w_n)} \sim \left( c_2 - (-1)^{m-1} c_{m+1} \prod_{i=0}^{m-2} e_{t,i} + (3c_3 - 2c_2^2) e_n \right)$$

and

$$\alpha_n - c_2 \sim \left( -(-1)^{m-1} c_{m+1} \prod_{i=0}^{m-2} e_{t,i} + (3c_3 - 2c_2^2) e_n \right).$$

*Proof.* The Hermite interpolation error expression can be written as follows:

$$g(w) - H_m(w) = \frac{g^{(m+1)}(\xi)}{(m+1)!} (w - w_n)^2 \prod_{i=0}^{m-2} (w - t_i), \quad \xi \in I.$$

After differentiating the aforementioned expression twice at the point  $w = w_n$ , we succeed

$$g''(w_n) - H_m''(w_n) = 2 \frac{g^{(m+1)}(\xi)}{(m+1)!} \prod_{i=0}^{m-2} (w_n - t_i), \quad \xi \in I,$$

or

$$(2.6) \quad H_m''(w_n) = g''(w_n) - 2 \frac{g^{(m+1)}(\xi)}{(m+1)!} \prod_{i=0}^{m-2} (w_n - t_i), \quad \xi \in I.$$

Using Taylor's expansion of derivative of  $g$  at the point  $w_n \in I$  and  $\xi \in I$  around the root  $\gamma$  of  $g$  gives

$$(2.7) \quad g'(w_n) = g'(\gamma)(1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)),$$

$$(2.8) \quad g''(w_n) = g'(\gamma)(2!c_2 + 3!c_3 e_n + O(e_n^2))$$

and

$$(2.9) \quad g^{(m+1)}(\xi) = g'(\gamma)((m+1)!c_{m+1} + (m+2)!c_{m+2}e_\xi + O(e_\xi^2)).$$

Putting the expressions (2.8), (2.9) in the equation (2.6), we obtain

$$(2.10) \quad H_m''(w_n) = 2g'(\gamma) \left( c_2 - (-1)^{m-1} c_{m+1} \prod_{i=0}^{m-2} e_{t,i} + 3c_3 e_n \right).$$

Now, dividing (2.10) by (2.7) and the simplifying we get

$$\frac{H_m''(w_n)}{2g'(w_n)} \sim \left( c_2 - (-1)^{m-1} c_{m+1} \prod_{i=0}^{m-2} e_{t,i} + (3c_3 - 2c_2^2) e_n \right).$$

Therefore,

$$\alpha_n \sim \left( c_2 - (-1)^{m-1} c_{m+1} \prod_{i=0}^{m-2} e_{t,i} + (3c_3 - 2c_2^2)e_n \right),$$

and so,

$$\alpha_n - c_2 \sim \left( -(-1)^{m-1} c_{m+1} \prod_{i=0}^{m-2} e_{t,i} + (3c_3 - 2c_2^2)e_n \right). \quad \square$$

**Theorem 2.2.** *If the errors of approximations  $e_i = w_i - \gamma$  generated by an iterative technique satisfy:*

$$e_{k+1} \sim \prod_{i=0}^{m-2} (e_{k-i})^{m_i}, \quad k \geq k(e_k),$$

then the  $R$ -order of convergence of iterative technique, denoted with  $O_R(\gamma)$ , satisfies the inequality  $O_R(\gamma) \geq q^*$ , where  $q^*$  is the unique positive solution of the equation  $q^{n+1} - \sum_{i=0}^n m_i q^{n-i} = 0$ .

As a result, we arrive at the following conclusion on the convergence theorem for the iterative technique with memory (2.2).

**Theorem 2.3.** *Let  $\alpha_n$  represent the variable in the iterative technique (2.2), which is calculated by (2.5). If an initial approximation  $w_0$  is close enough to a simple root of  $g(w)$ , the iterative method (2.2)–(2.5) with memory has an  $R$ -order of convergence of at least 5.*

*Proof.* Initially, we will suppose that the  $R$ -order convergence of the sequences  $\{w_n\}$  and  $\{z_n\}$  is at least  $r$  and  $p$ . Hence,  $e_{n+1} \sim E_{n,r} e_n^r$ , where  $E_{n,r}$  is an asymptotic error constant. The above relation may be also re-written as

$$(2.11) \quad e_{n+1} \sim E_{n,r} (E_{n-1,r} e_{n-1}^r)^r \sim E_{n,r} E_{n-1,r}^r e_{n-1}^{r^2}$$

and

$$e_{n,z} \sim E_{n,p} e_n^p,$$

or

$$(2.12) \quad e_{n,z} \sim E_{n,p} (E_{n-1,r} e_{n-1}^r)^p \sim E_{n,p} E_{n-1,r}^p e_{n-1}^{rp}.$$

By error expressions (2.3) and (2.4), it may be written as

$$(2.13) \quad e_{n,z} \sim z_n - \alpha \sim (-\alpha_n + c_2)e_n^2 + O(e_n^3),$$

$$(2.14) \quad e_{n+1} \sim D_{n,4}(\alpha_n - c_2)e_n^4 + O(e_n^5),$$

where  $D_{n,4}$  is a varying quantity. Now, applying Theorem 2.1 for the case of  $m = 4$ , where  $t_0 = z_{n-1}$ ,  $t_1 = w_{n-1}$  and  $t_2 = w_{n-1}$ , we get

$$(2.15) \quad \alpha_n - c_2 \sim c_5 e_{t,0} e_{t,1} e_{t,2} = c_5 e_{n-1,z} e_{n-1}^2.$$

Substituting the relation (2.15) into the expressions (2.13) and (2.14), we obtain

$$\begin{aligned}
 e_{n,z} &\sim c_5 e_{n-1,z} e_{n-1}^2 e_n^2 \sim c_5 e_{n-1,z} e_{n-1}^2 (E_{n-1,r} e_{n-1}^r)^2 \sim c_5 e_{n-1,z} e_{n-1}^2 E_{n-1,r}^2 e_{n-1}^{2r} \\
 &\sim c_5 (E_{n-1,p} e_{n-1}^p) e_{n-1}^2 E_{n-1,r}^2 e_{n-1}^{2r} \\
 (2.16) \quad &\sim c_5 E_{n-1,r}^2 E_{n-1,p} e_{n-1}^{2r+p+2}
 \end{aligned}$$

and

$$\begin{aligned}
 e_{n+1} &\sim D_{n,4} c_5 e_{n-1,z} e_{n-1}^2 e_n^4 \sim D_{n,4} c_5 (E_{n-1,p} e_{n-1}^p) e_{n-1}^2 (E_{n-1,r} e_{n-1}^r)^4 \\
 (2.17) \quad &\sim D_{n,4} c_5 E_{n-1,p} E_{n-1,r}^4 e_{n-1}^{4r+p+2}.
 \end{aligned}$$

By comparing the components of  $e_{n-1}$  in the two sets of relations (2.12)–(2.16) and (2.11)–(2.17), we arrive at the following system of equations:

$$\begin{aligned}
 (2.18) \quad &2r + p + 2 = rp, \\
 &4r + p + 2 = r^2.
 \end{aligned}$$

The positive solution to the system (2.18) is provided by the values  $p = 3$  and  $r = 5$ . As a result, when  $\alpha_n$  is determined by (2.5), the  $R$ -order of the method with memory (2.2) is reached to at least 5. □

*An alternative proof.* The method discussed in reference [15], known as the Herzberger’s matrix approach, is now being utilized on the order of single step  $s$ -point method  $x_k = \Psi(x_{k-1}, x_{k-2}, \dots, x_{k-s})$ . A matrix  $A^{(s)} = (a_{ij})$ , associated with this method, has the elements

$$\begin{aligned}
 &a_{1,j} = \text{amount of information required at point } x_{k-j}, \quad j = 1, 2, 3, \dots, s, \\
 &a_{i,i-1} = 1, \quad i = 2, 3, \dots, s, \\
 (2.19) \quad &a_{i,j} = 0, \quad \text{otherwise.}
 \end{aligned}$$

The order of an  $s$ -step method  $\Psi = \Psi_1 \circ \Psi_2 \circ \dots \circ \Psi_s$  is the spectral radius of the product of matrices  $A^{(s)} = A_1 \cdot A_2 \cdot \dots \cdot A_s$ . We may express each estimate  $w_{n+1}, z_n$  as a function of available information  $g(z_n)$  and  $g(w_n)$  from the  $n$ -th iteration and  $g(z_{n-1})$  and  $g(w_{n-1})$  from the previous iteration, depending on the accelerating technique. We construct the relevant matrices from the relations (2.2), (2.5) and (2.19) as follows:

$$\begin{aligned}
 w_{n+1} = \Psi_1(z_n, w_n, z_{n-1}) &\Rightarrow A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
 z_n = \Psi_2(w_n, z_{n-1}, w_{n-1}) &\Rightarrow A_2 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
 \end{aligned}$$

Thus, we acquire

$$A^{(2)} = A_1 \cdot A_2 = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix},$$

whose eigenvalues are  $(5, 0, 0)$  and spectral radius of the matrix  $A^{(2)}$  is 5. Therefore, the order of convergence of with memory method (2.2) is five.  $\square$

Now, by making some more modification in the scheme (2.2) at this time, we are attempting to enhance its convergence order. Consider the following new updated version of the scheme (2.2), where an additional parameter  $\beta_n$  is added in the second sub-step, we get a new bi-parametric two-step iterative method with memory given by:

$$(2.20) \quad \begin{aligned} \alpha_n &= \frac{H_4''(w_n)}{2g'(w_n)}, \\ z_n &= w_n - \frac{g(w_n)}{g'(w_n) - \alpha_n g(w_n)}, \\ w_{n+1} &= z_n - \frac{g(z_n)}{2 \left( \frac{g(z_n) - g(w_n)}{z_n - w_n} \right) - g'(w_n) - \beta_n g(w_n)^2}. \end{aligned}$$

It's error equation is:

$$(2.21) \quad e_{n+1} = (\alpha_n - c_2)(g'(\gamma)\beta_n + (\alpha_n - c_2)c_2 + c_3)e_n^4 + O(e_n^5),$$

where  $e_n = w_n - \gamma$  and  $c_i = \frac{g^{(i)}(\gamma)}{i!g'(\gamma)}$ , for  $i = 2, 3, 4, \dots$ , and  $\gamma \in \mathbb{R}$ . It is symbolised by **OWM6**. It is clear from (2.21) that the order of convergence of scheme (2.20) is five for  $\beta_n \neq \frac{-c_3}{g'(\gamma)}$  and when  $\beta_n = -\frac{c_3}{g'(\gamma)} = -\frac{g'''(\gamma)}{3!g'(\gamma)^2}$ , the convergence order of method (2.20) would be higher. However, exact values of  $g'(\gamma)$  and  $g'''(\gamma)$  are not available for this type of convergence acceleration and so we can use the data available from the most recent iteration and the one before it, and it satisfies the condition  $\lim_{n \rightarrow +\infty} \beta_n = -\frac{c_3}{g'(\gamma)} = -\frac{g'''(\gamma)}{3!g'(\gamma)^2}$  for the asymptotic constant to be zero in the relation (2.21). For the calculation of  $\beta_n$ , we consider the following best possible expression:

$$(2.22) \quad \beta_n = -\frac{H_5'''(z_n)}{3!g'(w_n)^2},$$

where

$$\begin{aligned} H_5(w) &= g(z_n) + (w - z_n)g[z_n, w_n] + (w - z_n)(w - w_n)g[z_n, w_n, w_n] \\ &\quad + (w - z_n)(w - w_n)^2g[z_n, w_n, w_n, z_{n-1}] + (w - z_n)(w - w_n)^2(w - z_{n-1}) \\ &\quad \times g[z_n, w_n, w_n, z_{n-1}, w_{n-1}] + (w - z_n)(w - w_n)^2(w - z_{n-1})(w - w_{n-1}) \\ &\quad \times g[z_n, w_n, w_n, z_{n-1}, w_{n-1}, w_{n-1}], \end{aligned}$$

and so,

$$\begin{aligned}
 H_5'''(z_n) = & 6g[z_n, w_n, w_n, z_{n-1}] + (12(z_n - w_n) + 6(z_n - z_{n-1}))g[z_n, w_n, w_n, z_{n-1}, w_{n-1}] \\
 & + (6(z_n - w_n)^2 + 12(z_n - w_n)(z_n - z_{n-1}) + 12(z_n - w_n)(z_n - w_{n-1}) \\
 & + 6(z_n - z_{n-1})(z_n - w_{n-1}))g[z_n, w_n, w_n, z_{n-1}, w_{n-1}, w_{n-1}].
 \end{aligned}$$

**Theorem 2.4.** *Let a Hermite interpolating polynomial  $H_m$  of degree  $m$  which interpolates a function  $g$  at nodes  $z_n, w_n, w_n, t_0, \dots, t_{m-3}$  located within an interval  $I$ , and the derivative  $g^{(m+1)}$  is continuous in  $I$ , as well as the Hermite interpolating polynomial satisfying the conditions  $H_m(z_n) = g(z_n)$ ,  $H_m'(z_n) = g'(z_n)$ ,  $H_m(w_n) = g(w_n)$ ,  $H_m'(w_n) = g'(w_n)$ ,  $H_m(t_i) = f(t_i)$ ,  $i = 0, 1, \dots, m-3$ . Indicate the errors  $e_{t,i} = t_i - \gamma$ ,  $i = 0, 1, 2, \dots, m-3$ , and presume that*

- (1) *all nodes  $z_n, w_n, t_0, \dots, t_{m-3}$  are adequately near to the zero  $\gamma$ ;*
- (2) *the condition  $e_n = O(e_{t,0}, e_{t,1}, \dots, e_{t,m-3})$  and  $e_{n,z} = z_n - \gamma = O(e_n^2, e_{t,0}, \dots, e_{t,m-3})$  hold.*

Then

$$H_m'''(z_n) = 3!g'(\gamma) \left( c_3 - (-1)^{m-2}c_{m+1} \prod_{i=0}^{m-3} e_{t,i} + 4c_4e_{n,z} \right)$$

and

$$g'(\gamma)\beta_n + c_3 \sim \left( -(-1)^{m-2}c_{m+1} \prod_{i=0}^{m-3} e_{t,i} \right).$$

*Proof.* The error expression for Hermite interpolating polynomial can be written as

$$g(w) - H_m(w) = \frac{g^{(m+1)}(\xi)}{(m+1)!} (w - z_n)(w - w_n)^2 \prod_{i=0}^{m-3} (w - t_i), \quad \xi \in I.$$

After differentiating the aforementioned expression thrice at the point  $w = z_n$  will give

$$g'''(z_n) - H_m'''(z_n) = 3! \frac{g^{(m+1)}(\xi)}{(m+1)!} \prod_{i=0}^{m-3} (z_n - t_i), \quad \xi \in I,$$

or

$$(2.23) \quad H_m'''(z_n) = g'''(z_n) - 3! \frac{g^{(m+1)}(\xi)}{(m+1)!} \prod_{i=0}^{m-3} (z_n - t_i), \quad \xi \in I.$$

Using Taylor's series expansion for derivatives of  $g$  at the point  $z_n \in I$  and  $\xi \in I$  about the root  $\gamma$  of  $g$  gives

$$\begin{aligned}
 g'(z_n) &= g'(\gamma) \left( 1 + 2c_2e_{n,z} + 3c_3e_{n,z}^2 + O(e_{n,z}^3) \right), \\
 g''(z_n) &= g'(\gamma) \left( 2c_2 + 3!c_3e_{n,z} + O(e_{n,z}^2) \right), \\
 (2.24) \quad g'''(z_n) &= g'(\gamma) \left( 3!c_3 + 4!c_4e_{n,z} + O(e_{n,z}^2) \right)
 \end{aligned}$$

and

$$(2.25) \quad g^{(m+1)}(\xi) = g'(\gamma) \left( (m+1)!c_{m+1} + (m+2)!c_{m+2}e_\xi + O(e_\xi^2) \right).$$

Putting the expansions (2.24) and (2.25) in the relation (2.23), we get

$$(2.26) \quad H_m'''(z_n) = 3!g'(\gamma) \left( c_3 - (-1)^{m-2}c_{m+1} \prod_{i=0}^{m-3} e_{t,i} + 4c_4e_{n,z} \right).$$

Now, dividing the relation (2.26) by  $g'(w_n)^2$ , we obtain

$$-\frac{H_m'''(z_n)}{3!g'(w_n)^2} \sim -\frac{1}{g'(\gamma)} \left( c_3 - (-1)^{m-2}c_{m+1} \prod_{i=0}^{m-3} e_{t,i} \right)$$

or

$$\beta_n \sim -\frac{1}{g'(\gamma)} \left( c_3 - (-1)^{m-2}c_{m+1} \prod_{i=0}^{m-3} e_{t,i} \right),$$

and hence

$$g'(\gamma)\beta_n + c_3 \sim (-1)^{m-2}c_{m+1} \prod_{i=0}^{m-3} e_{t,i}.$$

Thus the proof is finished.  $\square$

**Theorem 2.5.** *Let  $\beta_n$  be the changing parameter in the iterative method (2.20), and be computed by (2.22). If an initial approximation  $w_0$  is close enough to a simple root of  $g(w)$ , the iterative method (2.20)–(2.22) with memory has  $R$ -order of convergence of at least 5.70156.*

*Proof.* Initially, let us suppose that the  $R$ -order convergence of the sequences  $\{w_n\}$  and  $\{z_n\}$  is at least  $r$  and  $p$ . So, that

$$(2.27) \quad e_{n+1} \sim E_{n,r}e_n^r,$$

where  $E_{n,r}$  is an asymptotic error constant. Now, the relation (2.27) may be also re-written as

$$(2.28) \quad e_{n+1} \sim E_{n,r}(E_{n-1,r}e_{n-1}^r)^r \sim E_{n,r}E_{n-1,r}^r e_{n-1}^{r^2}$$

and

$$(2.29) \quad \begin{aligned} e_{n,z} &\sim E_{n,p}e_n^p, \\ e_{n,z} &\sim E_{n,p}(E_{n-1,p}e_{n-1}^p)^p \sim E_{n,p}E_{n-1,p}^p e_{n-1}^{p^2}. \end{aligned}$$

By error expressions (2.3) and (2.21), it may be written as

$$(2.30) \quad e_{n,z} \sim z_n - \gamma \sim (-\alpha_n + c_2)e_n^2 + O(e_n^3),$$

$$(2.31) \quad e_{n+1} \sim (\alpha_n - c_2) \left( g'(\gamma)\beta_n + (\alpha_n - c_2)c_2 + c_3 \right) e_n^4 + O(e_n^5).$$

Using Theorem 2.4 for  $m = 5$ ,  $t_0 = z_{n-1}$ ,  $t_1 = w_{n-1}$  and  $t_2 = w_{n-1}$ , we obtain

$$(2.32) \quad g'(\gamma)\beta_n + c_3 \sim c_6 e_{t,0} e_{t,1} e_{t,2} = c_6 e_{n-1,z} e_{n-1}^2.$$

Using the relations (2.15) into (2.30) and (2.32) in the expression (2.31), it can be written as

$$\begin{aligned}
 e_{n,z} &\sim c_5 e_{n-1,z} e_{n-1}^2 e_n^2 \sim c_5 e_{n-1,z} e_{n-1}^2 (E_{n-1,r} e_{n-1}^r)^2, \\
 &\sim c_5 e_{n-1,z} e_{n-1}^2 E_{n-1,r}^2 e_{n-1}^{2r} \sim c_5 (E_{n-1,p} e_{n-1}^p) e_{n-1}^2 E_{n-1,r}^2 e_{n-1}^{2r} \\
 (2.33) \quad &\sim c_5 E_{n-1,r}^2 E_{n-1,p} e_{n-1}^{2r+p+2}
 \end{aligned}$$

and

$$\begin{aligned}
 e_{n+1} &\sim (c_5 e_{n-1,z} e_{n-1}^2) (c_6 e_{n-1,z} e_{n-1}^2 + c_2 c_5 e_{n-1,z} e_{n-1}^2) e_n^4 \\
 &\sim c_5 (c_6 + c_2 c_5) e_{n-1,z}^2 e_{n-1}^4 e_n^4 \sim c_5 (c_6 + c_2 c_5) (E_{n-1,p} e_{n-1}^p)^2 e_{n-1}^4 (E_{n-1,r} e_{n-1}^r)^4 \\
 (2.34) \quad &\sim c_5 (c_6 + c_2 c_5) E_{n-1,p}^2 E_{n-1,r}^4 e_{n-1}^{4r+2p+4}.
 \end{aligned}$$

By comparing the components of  $e_{n+1}$  in (2.34)–(2.28) and (2.33)–(2.29), we arrive at the following system of equations:

$$\begin{aligned}
 (2.35) \quad &4r + 2p + 4 = r^2, \\
 &2r + p + 2 = rp.
 \end{aligned}$$

The positive solution to the system (2.35) is provided by  $p = 2.85078$ ,  $r = 5.70156$ . As a result, when  $\beta_n$  is determined by formula (2.22), the  $R$ -order of the method with memory scheme (2.20) is at least 5.70156.  $\square$

*An alternative proof.* From the relations (2.20), (2.22) and similar to that used in the alternative proof of the previous Theorem 2.3, we derive the corresponding matrices:

$$\begin{aligned}
 w_{n+1} = \Psi_1(z_n, w_n, z_{n-1}, w_{n-1}) &\Rightarrow A_1 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
 z_n = \Psi_2(w_n, z_{n-1}, w_{n-1}, z_{n-2}) &\Rightarrow A_2 = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
 \end{aligned}$$

Thus, we acquire

$$A^{(2)} = A_1 \cdot A_2 = \begin{bmatrix} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

whose eigenvalues are  $(5.70156, -0.701562, 0, 0)$  and spectral radius of the matrix  $A^{(2)}$  is 5.70156. Therefore, the order of convergence of with memory method (2.20) is 5.70156.  $\square$

## 3. NUMERICAL RESULTS AND COMPARISONS

In this part, we will numerically compare the considered uni-parametric two-step with memory scheme *OWM4* along with the similar nature schemes *XW(16–18)*, *XW(16–19)*, *XW(16–20)*, *XW(17–18)*, *XW(17–19)* and *XW(17–20)* considered in [16] and *NC4(2.4–2.5)*, *NC4(2.4–2.6)* and *NC4(2.4–2.7)* presented in the article [17] and presented bi-parametric with memory method *OWM6* along with the proposed in [18]. Wang [18] presented two bi-parametric iterative methods with memory as mentioned below:

$$(3.1) \quad \begin{aligned} z_n &= w_n - \frac{g(w_n)}{g'(w_n) - \alpha_n g(w_n)}, \\ w_{n+1} &= z_n - \frac{g(z_n)}{2g[w_n, z_n] - g'(w_n)} \cdot \left( \frac{g'(w_n)^2}{g'(w_n)^2 - \beta_n g(w_n)^2} \right), \end{aligned}$$

which is represented by the symbol **XW1** and

$$(3.2) \quad \begin{aligned} z_n &= w_n - \frac{g(w_n)}{g'(w_n) - \alpha_n g(w_n)}, \\ w_{n+1} &= z_n - \frac{g(z_n)}{2g[w_n, z_n] - g'(w_n)} \left( 1 + \beta_n \left( \frac{g(w_n)}{g'(w_n)} \right)^2 \right), \end{aligned}$$

which is denoted by **XW2**. In the following form, they have captured the values of the two parameters  $\alpha_n$  and  $\beta_n$  for both methods.

**Method 1:**

$$(3.3) \quad \alpha_n = \frac{H_4''(w_n)}{2g'(w_n)} \quad \text{and} \quad \beta_n = -\frac{H_4'''(w_n)}{6g'(w_n)},$$

where

$$H_4'''(w_n) = 6g[w_n, w_n, z_{n-1}, w_{n-1}] + 6(2w_n - w_{n-1} - z_{n-1})g[w_n, z_n, z_{n-1}, w_{n-1}, w_{n-1}].$$

**Method 2:**

$$(3.4) \quad \alpha_n = \frac{H_4''(w_n)}{2g'(w_n)} \quad \text{and} \quad \beta_n = -\frac{H_3'''(z_n)}{6g'(w_n)},$$

where  $H_3'''(z_n) = 6g[z_n, w_n, w_n, z_{n-1}]$ .

**Method 3:**

$$(3.5) \quad \alpha_n = \frac{H_4''(w_n)}{2g'(w_n)} \quad \text{and} \quad \beta_n = -\frac{H_4'''(z_n)}{6g'(w_n)},$$

where  $H_4'''(z_n) = H_3'''(z_n) + 6(3z_n - z_{n-1} - 2w_n)g[z_n, w_n, w_n, z_{n-1}, w_{n-1}]$ .

**Method 4:**

$$(3.6) \quad \alpha_n = \frac{H_4''(w_n)}{2g'(w_n)} \quad \text{and} \quad \beta_n = -\frac{H_5'''(z_n)}{6g'(w_n)},$$

where  $H_5'''(z_n) = H_4'''(z_n) + 6[(z_n - z_{n-1})(z_n - w_{n-1}) + (z_n - w_n)^2 + 2(z_n - w_n)(2z_n - z_{n-1} - w_{n-1})]g[z_n, w_n, w_n, z_{n-1}, w_{n-1}]$ .

TABLE 1. Test functions and their roots.

Nonlinear function	Root
$g_1 = we^{w^2} - \sin^2 w + 3 \cos w + 5$	-1.2676...
$g_2 = w^5 + w^4 + 4w^2 - 15$	1.3474...
$g_3 = w^3 - w^2 - 1$	1.4655...

Table 1 includes the roots of three nonlinear test functions (taken from [18, 19]). The numerical results shown in Table 2 and 3 are consistent with the theory presented in this discussion. The absolute errors  $|w_n - \gamma|$  upto three iterate have been calculated. For numerical computation MATHEMATICA 8 is used. Now, according to Weerakoon [20], the formula below can be used to estimate the computational order of convergence,

$$COC \approx \frac{\log |g(w_{n+1})/g(w_n)|}{\log |g(w_n)/g(w_{n-1})|},$$

to verify the established theoretical rate of convergence. Table 2 and 3 confirms the significance of the presented with memory scheme over some well published similar nature algorithms.

#### 4. CONCLUSION

In this article, we have presented a two-step with memory iterative method for finding the solution of nonlinear equations. Because our goal is to develop the method of higher-order convergence without any extra functional computations. To obtain higher-order convergence without any extra computations, we have employed one and two self-accelerating parameters that are constructed by Hermite interpolating polynomials in the well-established optimal fourth-order without memory scheme. The order of convergence for the new suggested two-step iterative with memory has risen from 4 to 5.70156, which is also verified by an alternate approach called Herzberger's matrix method. The numerical results have been provided to validate the theoretical outcomes.

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TABLE 2. Numerical comparison of single parametric two-point with memory method

Method	$ w_1 - \gamma $	$ w_2 - \gamma $	$ w_3 - \gamma $	COC
Example $g_1$ , initial guess $w_0 = -1.6$				
XW (16) – (18), $\alpha_0 = -0.01$	$3.5037e - 2$	$2.8246e - 6$	$1.2080e - 25$	4.7042
XW (16) – (19), $\alpha_0 = -0.01$	$3.5037e - 2$	$4.7605e - 7$	$1.7515e - 27$	4.1782
XW (16) – (20), $\alpha_0 = -0.01$	$3.5037e - 2$	$3.4949e - 7$	$4.1255e - 28$	4.1649
XW (17) – (18), $\alpha_0 = -0.01$	$1.8398e - 2$	$2.1773e - 7$	$1.3052e - 30$	4.7018
XW (17) – (19), $\alpha_0 = -0.01$	$1.8398e - 2$	$3.0276e - 8$	$1.7117e - 32$	4.1837
XW (17) – (20), $\alpha_0 = -0.01$	$1.8398e - 2$	$3.5032e - 8$	$2.7935e - 32$	4.2025
NC (2.4) – (2.5), $\alpha_0 = -0.01$	$1.8880e - 2$	$2.3820e - 7$	$1.9513e - 30$	4.7005
NC (2.4) – (2.6), $\alpha_0 = -0.01$	$1.8880e - 2$	$3.3604e - 8$	$2.6359e - 32$	4.1835
NC (2.4) – (2.7), $\alpha_0 = -0.01$	$1.8880e - 2$	$3.8273e - 8$	$4.0253e - 32$	4.2025
OWM4 (2.4) – (2.7), $\alpha_0 = -0.01$	$1.8309e - 2$	$2.9275e - 8$	$5.2296e - 38$	5.1217
Example $g_2$ , initial guess $w_0 = 1.4$				
XW (16) – (18), $\alpha_0 = -0.01$	$1.8371e - 5$	$1.3038e - 22$	$7.0315e - 101$	4.5640
XW (16) – (19), $\alpha_0 = -0.01$	$1.8371e - 5$	$1.5797e - 22$	$5.6795e - 107$	4.3243
XW (16) – (20), $\alpha_0 = -0.01$	$1.8371e - 5$	$6.7085e - 25$	$1.5685e - 108$	4.3026
XW (17) – (18), $\alpha_0 = -0.01$	$3.8040e - 6$	$2.3630e - 25$	$3.2074e - 113$	4.5748
XW (17) – (19), $\alpha_0 = -0.01$	$3.8040e - 6$	$4.3246e - 27$	$9.8591e - 118$	4.3278
XW (17) – (20), $\alpha_0 = -0.01$	$3.8040e - 6$	$2.2214e - 28$	$7.0328e - 124$	4.2953
NC (2.4) – (2.5), $\alpha_0 = -0.01$	$3.7144e - 6$	$2.1871e - 25$	$2.2845e - 113$	4.5752
NC (2.4) – (2.6), $\alpha_0 = -0.01$	$3.7144e - 6$	$3.9924e - 27$	$7.0907e - 118$	4.3279
NC (2.4) – (2.7), $\alpha_0 = -0.01$	$3.7144e - 6$	$1.9614e - 28$	$4.0581e - 124$	4.2951
OWM4 (2.4) – (2.7), $\alpha_0 = -0.01$	$3.8734e - 6$	$2.4314e - 28$	$2.8924e - 139$	4.9961
Example $g_3$ , initial guess $w_0 = 1.3$				
XW (16) – (18), $\alpha_0 = -0.01$	$1.5319e - 2$	$4.8667e - 10$	$2.6217e - 44$	4.5743
XW (16) – (19), $\alpha_0 = -0.01$	$1.5319e - 2$	$1.7723e - 10$	$1.6516e - 45$	4.4174
XW (16) – (20), $\alpha_0 = -0.01$	$1.5319e - 2$	$1.7723e - 10$	$3.3034e - 45$	4.3794
XW (17) – (18), $\alpha_0 = -0.01$	$7.1877e - 4$	$7.3695e - 16$	$1.0978e - 70$	4.5729
XW (17) – (19), $\alpha_0 = -0.01$	$7.1877e - 4$	$3.5313e - 17$	$5.5819e - 75$	4.3430
XW (17) – (20), $\alpha_0 = -0.01$	$7.1877e - 4$	$3.5313e - 17$	$1.1164e - 74$	4.3204
NC (2.4) – (2.5), $\alpha_0 = -0.01$	$7.1305e - 4$	$7.3404e - 16$	$1.0912e - 70$	4.5737
NC (2.4) – (2.6), $\alpha_0 = -0.01$	$7.1305e - 4$	$3.3934e - 17$	$4.6559e - 75$	4.3431
NC (2.4) – (2.7), $\alpha_0 = -0.01$	$7.1305e - 4$	$3.3934e - 17$	$9.3119e - 75$	4.3205
OWM4 (2.4) – (2.7), $\alpha_0 = -0.01$	$0.7357e - 4$	$3.9816e - 17$	$1.8529e - 83$	4.9998

TABLE 3. Numerical comparison of bi-parametric two-point with memory scheme

Method	$ w_1 - \gamma $	$ w_2 - \gamma $	$ w_3 - \gamma $	COC
Example $g_1$ , initial guess $w_0 = -1.5$				
XW1 (3.1) – (3.3), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$1.5696e - 11$	$2.0984e - 58$	5.4952
XW1 (3.1) – (3.4), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$4.1038e - 12$	$3.1195e - 62$	5.5000
XW1 (3.1) – (3.5), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$6.2388e - 12$	$8.6664e - 64$	5.8067
XW1 (3.1) – (3.6), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$2.6379e - 12$	$1.5543e - 68$	6.0433
XW2 (3.2) – (3.3), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$1.5694e - 11$	$2.0977e - 58$	5.4952
XW2 (3.2) – (3.4), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$4.1074e - 12$	$3.1303e - 62$	5.5002
XW2 (3.2) – (3.5), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$6.2364e - 12$	$8.6536e - 64$	5.8067
XW2 (3.2) – (3.6), $\alpha_0 = \beta_0 = 0.01$	$0.527e - 2$	$2.6351e - 12$	$1.5360e - 68$	6.0435
OWM6 (2.30) – (2.32), $\alpha_0 = \beta_0 = 0.01$	$0.957e - 2$	$1.086e - 12$	$3.0103e - 87$	5.9901
Example $g_2$ , initial guess $w_0 = 1.5$				
XW1 (3.1) – (3.3), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$3.1146e - 22$	$9.2151e - 116$	5.2365
XW1 (3.1) – (3.4), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$1.0210e - 21$	$7.3405e - 117$	5.4852
XW1 (3.1) – (3.5), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$1.7755e - 22$	$2.1424e - 124$	5.6292
XW1 (3.1) – (3.6), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$1.5437e - 22$	$2.7763e - 128$	5.8211
XW2 (3.2) – (3.3), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$3.1146e - 22$	$9.2152e - 116$	5.2365
XW2 (3.2) – (3.4), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$1.0211e - 21$	$7.3412e - 117$	5.4852
XW2 (3.2) – (3.5), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$1.7755e - 22$	$2.1423e - 124$	5.6292
XW2 (3.2) – (3.6), $\alpha_0 = \beta_0 = 0.01$	$0.2261e - 3$	$1.5436e - 22$	$2.7762e - 128$	5.8210
OWM6 (2.30) – (2.32), $\alpha_0 = \beta_0 = 0.01$	$0.1794e - 4$	$1.8535e - 29$	$4.8802e - 166$	5.6942
Example $g_3$ , initial guess $w_0 = 1.3$				
XW1 (3.1) – (3.3), $\alpha_0 = \beta_0 = 0.2$	$0.1813e - 3$	$1.0922e - 23$	$5.2152e - 139$	6.0000
XW1 (3.1) – (3.4), $\alpha_0 = \beta_0 = 0.2$	$0.1813e - 3$	$1.0922e - 23$	$5.2152e - 139$	6.0000
XW1 (3.1) – (3.5), $\alpha_0 = \beta_0 = 0.2$	$0.1813e - 3$	$1.0922e - 23$	$5.2152e - 139$	6.0000
XW1 (3.1) – (3.6), $\alpha_0 = \beta_0 = 0.2$	$0.1813e - 3$	$1.0922e - 23$	$5.2152e - 139$	6.0000
XW2 (3.2) – (3.3), $\alpha_0 = \beta_0 = 0.2$	$0.1830e - 3$	$1.1536e - 23$	$7.2406e - 139$	5.9999
XW2 (3.2) – (3.4), $\alpha_0 = \beta_0 = 0.2$	$0.1830e - 3$	$1.1536e - 23$	$7.2406e - 139$	5.9999
XW2 (3.2) – (3.5), $\alpha_0 = \beta_0 = 0.2$	$0.1830e - 3$	$1.1536e - 23$	$7.2406e - 139$	5.9999
XW2 (3.2) – (3.6), $\alpha_0 = \beta_0 = 0.2$	$0.1830e - 3$	$1.1536e - 23$	$7.2406e - 139$	5.9999
OWM6 (2.30) – (2.32), $\alpha_0 = \beta_0 = 0.2$	$0.4008e - 4$	$2.114e - 28$	$4.5555e - 168$	6.0000

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## SOME $q$ -ANALOGUES OF GRANVILLE AND SUN'S CONGRUENCES

WEI-WEI QI<sup>1</sup>

ABSTRACT. In this paper, we use  $q$ -binomial theorem to establish some new  $q$ -analogues of Granville and Sun's congruence:

$$\sum_{k=1}^{p-1} \frac{x^k}{k} \equiv \frac{1 - x^p - (x-1)^p}{p} \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p} \left( \frac{1 + (x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k} \right) \pmod{p},$$

where  $x$  is a variable and  $p$  is an odd prime.

### 1. INTRODUCTION

In recent years,  $q$ -analogues of congruences involving harmonic numbers have been widely studied by many authors. In 2004, Granville [1] showed that for any prime  $p \geq 5$ ,

$$(1.1) \quad \sum_{k=1}^{p-1} \frac{x^k}{k} \equiv \frac{1 - x^p - (x-1)^p}{p} \pmod{p}.$$

For any positive integer  $n$ , the  $q$ -integer is defined as

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

It is easy to see that  $\lim_{q \rightarrow 1} [n]_q = n$ .

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Pan [3, (5.1)] showed that Granville [1] confirmed a conjecture of Skula: For any prime  $p \geq 5$ ,

$$(1.2) \quad \left(\frac{2^{p-1} - 1}{p}\right)^2 \equiv -\sum_{j=1}^{p-1} \frac{2^j}{j^2} \pmod{p}.$$

Pan [3, Theorem 5.1] established a  $q$ -analogue of (1.2) as follows:

$$\begin{aligned} & \sum_{j=1}^{p-1} \frac{q^j (-q; q)_j}{[j]_q^2} + Q_p(2, q)^2 \\ & \equiv -(p-1) Q_p(2, q)(1-q) - \frac{(7p-5)(p-1)(1-q)^2}{24} \pmod{[p]_q}, \end{aligned}$$

where  $Q_p(2, q) = \frac{(-q; q)_{p-1-1}}{[p]_q}$ .

The harmonic numbers are given by

$$H_n = \sum_{k=1}^n \frac{1}{k} \quad \text{and} \quad H_0 = 0.$$

A  $q$ -analogues of harmonic numbers  $H_n$  is given by

$$H_n(q) = \sum_{k=1}^n \frac{q^k}{[k]_q}.$$

The  $q$ -Pochhammer symbol is given by

$$(x; q)_n = (1-x)(1-xq) \cdots (1-xq^{n-1}), \quad (x; q)_0 = 1.$$

The  $q$ -binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear to see that  $\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ .

In [4, Theorem 1], Shi and Pan showed that for prime  $p \geq 5$ ,

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{(p-1)(1-q)}{2} + \frac{(p^2-1)(1-q)^2 [p]_q}{24} \pmod{[p]_q^2}.$$

In [5, Lemma 4.1], Sun established an interesting congruence:

$$\sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p} \left( \frac{1 + (x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k} \right) + p \sum_{k=2}^{p-1} \frac{x^k}{k^2} H_{k-1} \pmod{p^2},$$

where  $x$  is a real number and  $p$  is an odd prime. Thus, it is easy to get the following congruence:

$$(1.3) \quad \sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p} \left( \frac{1 + (x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k} \right) \pmod{p}.$$

In [6, Theorem 2.1], Tauraso proved that if  $p$  is an odd prime, and  $a, b, d$  are integers such that  $a, d > 0, b \geq 0$  and  $\gcd(a, p)=1$ , then

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^d} \equiv \frac{(1-q)^d}{p^d} \left( (-1)^d p \sum_{s=0}^{d-1} c_s \binom{r_0 + sp}{d} - \sum_{s=0}^d (-1)^s \binom{d}{s} \binom{sp}{2d} \right) \pmod{[p]_q},$$

where

$$r_0 \equiv \frac{-b}{a} \pmod{p}, \quad r_0 \in \{0, 1, \dots, p-1\},$$

$$c_s = \sum_{k=0}^s (-1)^{s-k} \binom{r_0 + kp + d - 1}{d-1} \binom{d}{s-k}.$$

The  $n$ th cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ (n,k)=1}} (q - \zeta^k),$$

where  $\zeta$  denotes a primitive  $n$ th root of unity.

The first aim of the paper is to give two new  $q$ -analogues of (1.1).

**Theorem 1.1.** *For any positive integer  $n$  and variable  $x$ , we have*

$$(1.4) \quad \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} \equiv \frac{1 - (qx)^n - (qx; q)_n}{[n]_q} + \frac{(n-1)(1-q)(x^n + 1)}{2} \pmod{\Phi_n(q)},$$

and

$$(1.5) \quad \sum_{k=1}^{n-1} \frac{q^{-k}(x; q^{-1})_k}{[k]_q} \equiv \frac{1 - (qx)^n - (qx; q)_n}{[n]_q} + \frac{(q-1)(x-1)}{x} + \frac{x(n-1)(1-q)(x^n + 1) - 2(1-q)(1-xq^{1-n})(x; q^{-1})_{n-1}}{2x} \pmod{\Phi_n(q)}.$$

Letting  $q \rightarrow 1$  and  $n = p$  in Theorem 1.1, (1.4) and (1.5) will reduce to (1.1). So (1.4) and (1.5) are  $q$ -analogues of (1.1).

When  $x = -q$  and  $n = p$ , we will see that Theorem 1.1 is a  $q$ -analogue of [3, (5.5)]:

$$\frac{2^{p-1} - 1}{p} \equiv - \sum_{j=1}^{p-1} \frac{2^{j-1}}{j} \pmod{p}.$$

The second aim of the paper is to examine two  $q$ -analogues of (1.3).

**Theorem 1.2.** *For any odd positive integer  $n$  and variable  $x$ , we have*

$$(1.6) \quad \sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2} \equiv \left( \frac{(n-1)(1-q)}{2} - \frac{1}{[n]_q} \right) \sum_{k=1}^{n-1} \frac{(q^{-1}x; q^{-1})_k - 1}{[k]_q} + \frac{t_0(1 - (q^{-1}x; q^{-1})_n) - (q^{-1}x)^n}{[n]_q^2} \pmod{\Phi_n(q)},$$

and

$$(1.7) \quad \sum_{k=1}^{n-1} \frac{q^k(x; q)_k}{[k]_q^2} \equiv \left( \frac{(n-1)(1-q)}{2} - \frac{1}{[n]_q} \right) \sum_{k=1}^{n-1} \frac{x^k - 1}{[k]_q} + \frac{t_0(1 - x^n) - (x; q)_n}{[n]_q^2} \pmod{\Phi_n(q)},$$

where  $t_0 = 1 - \frac{(n-1)(1-q^n)}{2} + \frac{(n-1)(n-3)(1-q^n)^2}{8}$ .

In particular, it is easy to see that (1.6) is a  $q$ -analogue of (1.2) when  $x = 2$ , and (1.7) is a  $q$ -analogue of (1.2) when  $x = -q$ .

We will prove Theorems 1.1 and 1.2 in Sections 2 and 4. In addition, we establish some generalizations of Theorems 1.1 and 1.2 in Sections 3 and 5.

## 2. PROOF OF THEOREM 1.1

Firstly, we need to build some lemmas.

**Lemma 2.1.** *For any positive integer  $n$  and variable  $x$ , we have*

$$(2.1) \quad \sum_{k=1}^{n-1} \frac{x^k - 1}{[k]_q} = \sum_{k=1}^{n-1} \frac{(-q)^k (x; q)_k q^{\binom{k+1}{2} - nk}}{[k]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

*Proof.* By [7, (1.1)], for any positive integer  $j$  and  $n$

$$(2.2) \quad \sum_{k=j}^n q^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_q.$$

Letting  $j \rightarrow j - 1$ ,  $n \rightarrow n - 2$  and  $q \rightarrow q^{-1}$  in (2.2) gives

$$q^{(j-1)^2+j-1} \sum_{k=j-1}^{n-2} q^{-jk} \begin{bmatrix} k \\ j-1 \end{bmatrix}_q = q^{j^2-(n-1)j} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q.$$

By  $q$ -binomial theorem: for variable  $x$ , positive integer  $n$

$$(2.3) \quad x^n = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \prod_{j=0}^{i-1} (x - q^j).$$

Letting  $n \rightarrow k$  and  $q \rightarrow q^{-1}$  in (2.3), we have

$$(2.4) \quad x^k - 1 = \sum_{i=1}^k \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^i (x; q)_i q^{\binom{i+1}{2} - ki}.$$

Therefore, we have

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{x^k - 1}{[k]_q} &= \sum_{k=1}^{n-1} \frac{1}{[k]_q} \sum_{i=1}^k (-1)^i (x; q)_i \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i+1}{2} - ik} \\ &= \sum_{i=1}^{n-1} (-1)^i (x; q)_i q^{\binom{i+1}{2}} \sum_{k=i}^{n-1} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i \end{bmatrix}_q \\ &= \sum_{i=1}^{n-1} (-1)^i (x; q)_i q^{\binom{i+1}{2} - i} \sum_{k=i-1}^{n-2} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q \\ &= \sum_{i=1}^{n-1} \frac{(-q)^i (x; q)_i}{[i]_q} q^{\binom{i+1}{2} - ni} \begin{bmatrix} n-1 \\ i \end{bmatrix}_q. \end{aligned} \quad \square$$

As Shi and Pan [4, Theorem 1] mentioned, the following  $q$ -congruence lemma can be proved by the same method. We replace  $p$  with  $n$  and  $[p]$  with  $\Phi_n(q)$  in the proof of [4, Theorem 1].

**Lemma 2.2.** *For any positive integer  $n$ , we have*

$$(2.5) \quad \sum_{k=1}^{n-1} \frac{1}{[k]_q} \equiv \frac{(n-1)(1-q)}{2} + \frac{(n^2-1)(1-q)^2 [n]_q}{24} \pmod{\Phi_n(q)^2}.$$

**Lemma 2.3.** *For any positive integer  $n$  and variable  $x$ , we have*

$$(2.6) \quad \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \equiv \frac{1 - x^n - (x; q)_n}{[n]_q} + x^n \frac{(n-1)(1-q)}{2} \pmod{\Phi_n(q)}.$$

*Proof.* Note that

$$(2.7) \quad \begin{bmatrix} n-1 \\ i \end{bmatrix}_q = (-1)^i q^{-\binom{i+1}{2}} \prod_{j=1}^i \left( 1 - \frac{[n]_q}{[j]_q} \right) \equiv (-1)^i q^{-\binom{i+1}{2}} \pmod{\Phi_n(q)},$$

and

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{q^k (x; q)_k}{[k]_q} &\equiv \sum_{k=1}^{n-1} \frac{q^{n(n-1)/2 - nk + k} (x; q)_k}{[k]_q} \\ (2.8) \quad &\equiv -\frac{1}{[n]_q} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n}{2} - nk + \binom{k}{2}} (x; q)_k \\ &= -\frac{1}{[n]_q} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (x; q)_k \\ &= -\frac{1}{[n]_q} \left( q^{\binom{n}{2}} x^n - q^{\binom{n}{2}} - (x; q)_n \right) \pmod{\Phi_n(q)}. \end{aligned}$$

With the help of (2.1) and (2.7), we have

$$(2.9) \quad \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \equiv \sum_{k=1}^{n-1} \frac{q^k (x; q)_k}{[k]_q} + \sum_{k=1}^{n-1} \frac{1}{[k]_q} \pmod{\Phi_n(q)}.$$

Since

$$\begin{aligned} q^{tn} &= 1 - (1 - q^{tn}) \\ &= 1 - (1 - q^n) (1 + q^n + q^{2n} + \dots + q^{(t-1)n}) \\ &\equiv 1 - t(1 - q^n) \pmod{\Phi_n(q)^2}, \end{aligned}$$

we have

$$(2.10) \quad q^{\binom{n}{2}} \equiv 1 - \frac{(n-1)(1-q^n)}{2} \pmod{\Phi_n(q)^2}.$$

Combining (2.8), (2.9) and (2.10), we get Lemma 2.3. □

**Lemma 2.4.** *For any positive integer  $n$  and variable  $x$ , we have*

$$(2.11) \quad \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k - 1}{[k]_q} = \sum_{k=1}^{n-1} \frac{(-qx)^k}{[k]_q} q^{\binom{k+1}{2} - nk} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

*Proof.* The  $q$ -binomial theorem [2, (3.3.6)]: For variable  $x$ , positive integer  $n$ ,

$$(2.12) \quad (x; q)_n = \sum_{i=0}^n (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i,$$

letting  $q \rightarrow q^{-1}$  in (2.12), we have

$$(2.13) \quad (x; q^{-1})_n = \sum_{i=0}^n (-1)^i q^{\binom{i+1}{2} - ni} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i.$$

Using (2.2) and (2.13), we get

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k - 1}{[k]_q} &= \sum_{k=1}^{n-1} \frac{1}{[k]_q} \sum_{i=1}^k (-x)^i \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i+1}{2} - ik} \\ &= \sum_{i=1}^{n-1} (-x)^i q^{\binom{i+1}{2}} \sum_{k=i}^{n-1} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i \end{bmatrix}_q \\ &= \sum_{i=1}^{n-1} (-x)^i q^{\binom{i+1}{2} - i} \sum_{k=i-1}^{n-2} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q \\ &= \sum_{i=1}^{n-1} \frac{(-qx)^i}{[i]_q} q^{\binom{i+1}{2} - ni} \begin{bmatrix} n-1 \\ i \end{bmatrix}_q. \end{aligned}$$
□

Next we will proof (1.4).

By (2.11) we have

$$\sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1} \frac{(-qx)^k}{[k]_q} q^{\binom{k+1}{2} - nk} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \sum_{k=1}^{n-1} \frac{1}{[k]_q}.$$

Using (2.7), we get

$$\sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} \equiv \sum_{k=1}^{n-1} \frac{(xq)^k}{[k]_q} + \sum_{k=1}^{n-1} \frac{1}{[k]_q} \pmod{\Phi_n(q)},$$

where we have used the fact that  $q^{kn} \equiv 1 \pmod{\Phi_n(q)}$ . Finally, use (2.5) and (2.6), letting  $x \rightarrow qx$  in (2.6), we have

$$\sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} \equiv \frac{1 - (qx)^n - (qx; q)_n}{[n]_q} + \frac{(n-1)(1-q)(x^n + 1)}{2} \pmod{\Phi_n(q)}.$$

Next we will give the proof of (1.5).

Let

$$M(k, x, q) = \frac{(x; q^{-1})_k}{[k]_q}.$$

For  $k > 1$ , it is easy to check that

$$(2.14) \quad (1 - q^k)M(k, x, q) = (1 - q^{k-1})(1 - xq^{1-k})M(k - 1, x, q).$$

Summing both sides of (2.14) over  $k$  from 1 to  $n - 1$ , we have

$$\sum_{k=1}^{n-1} (1 - q^k)M(k, x, q) = \sum_{k=1}^{n-1} (1 - q^{k-1})(1 - xq^{1-k})M(k - 1, x, q).$$

Then

$$\sum_{k=0}^{n-1} (1 - q^k)M(k, x, q) = \sum_{k=0}^{n-2} (1 - q^k)(1 - xq^{-k})M(k, x, q).$$

After simplifying, we get

$$(2.15) \quad \sum_{k=1}^{n-1} \frac{q^{-k}(x; q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} - \frac{(1-q)(1-xq^{1-n})(x; q^{-1})_{n-1} - (x-1)(q-1)}{x}.$$

Finally, combining (1.4), we get (1.5).

So we complete the proof of Theorem 1.1.

### 3. COROLLARY 1

**Corollary 3.1.** *For any positive integer  $n$  and variable  $x$ , we have*

$$(3.1) \quad \sum_{1 \leq k \leq j \leq n-1} q^{-j} \frac{(x; q^{-1})_k}{[k]_q} \equiv q^{1-n} (1 - (xq; q)_n) - q(x^n - 1) + \frac{2q(1 - xq^{1-n})(x; q^{-1})_{n-1} - x(q^{1-n} - q)(x^n + 1)(n-1) - 2q}{2x} \pmod{\Phi_n(q)}.$$

When  $q \rightarrow 1$ ,  $x = 2$  and  $n = p$ , the corollary reduce to the following congruence:

$$\sum_{k=1}^{n-1} \sum_{j=1}^k \frac{(-1)^j}{j} \equiv -2q_p(2) \pmod{p},$$

where  $q_p(2) = \frac{2^{p-1}-1}{p}$ .

*Proof.* Note that

$$\begin{aligned} \sum_{1 \leq k \leq j \leq n-1} q^{-j} \frac{(x; q^{-1})_k}{[k]_q} &= \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} \sum_{j=k}^{n-1} q^{-j} \\ &= \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} \cdot \frac{q^{-k} - q^{-n}}{1 - q^{-1}} \\ &= \frac{q}{q-1} \sum_{k=1}^{n-1} \frac{q^{-k} (x; q^{-1})_k}{[k]_q} - \frac{q}{(q-1)q^n} \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q}. \end{aligned}$$

Then combining Theorem 1.1, we get (3.1). □

**Corollary 3.2.** *For any positive integer  $n$ , and variable  $x$ , we have*

$$\begin{aligned} \sum_{1 \leq k \leq j \leq n-1} q^{j-k} \frac{(x; q^{-1})_k}{[k]_q} &\equiv \frac{2q^n ((x; q^{-1})_{n-1} - 1) + x(x^n + 1)(n-1)(1 - q^n)}{2x} \\ &\quad - (xq; q)_n - q(x; q^{-1})_{n-1} - q^n(x^n - 1) + 1 \pmod{\Phi_n(q)^2}. \end{aligned}$$

When  $q \rightarrow 1$ ,  $x = 2$  and  $n = p$ , the corollary reduce to the following congruence:

$$\sum_{k=1}^{n-1} \sum_{j=1}^k \frac{(-1)^j}{j} \equiv -2q_p(2) + q_p(2)^2 p \pmod{p^2}.$$

*Proof.*

$$\begin{aligned} \sum_{1 \leq k \leq j \leq n-1} q^{j-k} \frac{(x; q^{-1})_k}{[k]_q} &= \sum_{k=1}^{n-1} \frac{q^{-k} (x; q^{-1})_k}{[k]_q} \sum_{j=k}^{n-1} q^j \\ &= \sum_{k=1}^{n-1} \frac{q^{-k} (x; q^{-1})_k}{[k]_q} \cdot \frac{q^k - q^n}{1 - q} \\ &= \frac{1}{1 - q} \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} - \frac{q^n}{1 - q} \sum_{k=1}^{n-1} \frac{q^{-k} (x; q^{-1})_k}{[k]_q}. \end{aligned}$$

Combining (1.4) and (2.15), we get Corollary 3.2, where we have used the fact that  $q^n \equiv 1 \pmod{\Phi_n(q)}$ , so the result can modulo  $\Phi_n(q)^2$ . □

#### 4. PROOF OF THEOREM 1.2

In order to prove Theorem 1.2, we need the following lemma.

**Lemma 4.1.** *For any positive integer  $n$ , variable  $x$ , we have*

$$(4.1) \quad \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} = q^{\binom{n}{2}} \sum_{k=1}^n \frac{(x; q^{-1})_k - 1}{[k]_q}$$

and

$$(4.2) \quad \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(x; q)_k}{[k]_q} = q^{\binom{n}{2}} \sum_{k=1}^n \frac{x^k - 1}{[k]_q}.$$

*Proof.* The  $q$ -binomial coefficients satisfy the following recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

It is easy to see that

$$(4.3) \quad q^{\binom{n-k}{2}} = q^{\binom{n}{2} + \binom{k+1}{2} - nk}.$$

So, we have

$$\begin{aligned} (4.4) \quad & \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} \\ &= \sum_{k=1}^n (-1)^k \left( q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \right) q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} \\ &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{1}{[n]_q} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} x^k. \end{aligned}$$

By induction and using (2.13), (4.3) and (4.4), we have

$$\begin{aligned} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{q^{\binom{n}{2}}}{[n]_q} ((x; q^{-1})_n - 1) \\ &= q^{\binom{n}{2}} \sum_{k=1}^n \frac{(x; q^{-1})_k - 1}{[k]_q}. \end{aligned}$$

Using (2.4), and (4.3), we have

$$(4.5) \quad q^{\binom{n}{2}} x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (-1)^k (x; q)_k.$$

So, we have

$$\begin{aligned} (4.6) \quad & \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{(x; q)_k}{[k]_q} \\ &= \sum_{k=1}^n (-1)^k \left( q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \right) q^{\binom{n-k}{2}} \frac{(x; q)_k}{[k]_q} \\ &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{\binom{n-1-k}{2}} \frac{(x; q)_k}{[k]_q} + \frac{1}{[n]_q} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (x; q)_k. \end{aligned}$$

By induction and using (4.5) and (4.6), we have

$$\begin{aligned} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{(x; q)_k}{[k]_q} &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{q^{\binom{n}{2}}}{[n]_q} (x^n - 1) \\ &= q^{\binom{n}{2}} \sum_{k=1}^n \frac{x^k - 1}{[k]_q}. \quad \square \end{aligned}$$

Next we will proof Theorem 1.2.

For any odd positive integer  $n$ , variable  $x$ , we have

$$\begin{aligned}
 (4.7) \quad \sum_{i=1}^{n-1} \frac{(xq)^i}{[i]_q^2} &\equiv \sum_{i=1}^{n-1} \frac{q^{n(n-1)/2-ni}(xq)^i}{[i]_q^2} \\
 &\equiv -\frac{1}{[n]_q} \sum_{i=1}^{n-1} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{n-i}{2}} \frac{x^i}{[i]_q} \\
 &= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^n \frac{(x; q^{-1})_i - 1}{[i]_q} - \frac{x^n}{[n]_q^2} \\
 &= -\frac{q^{\binom{n}{2}}}{[n]_q} \left( \sum_{i=1}^{n-1} \frac{(x; q^{-1})_i - 1}{[i]_q} + \frac{(x; q^{-1})_n - 1}{[n]_q} \right) - \frac{x^n}{[n]_q^2} \\
 &= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^{n-1} \frac{(x; q^{-1})_i - 1}{[i]_q} + \frac{q^{\binom{n}{2}}(1 - (x; q^{-1})_n) - x^n}{[n]_q^2} \pmod{\Phi_n(q)}.
 \end{aligned}$$

In the first step we need  $n$  to be an odd positive integer, otherwise it doesn't work, where in the third step we used (4.1).

Obviously, we have,

$$\begin{aligned}
 q^{tn} &= 1 - (1 - q^n) (1 + q^n + q^{2n} + \dots + q^{(t-1)n}) \\
 &\equiv 1 - (1 - q^n) \left( t - \frac{t(t-1)(1 - q^n)}{2} \right) \pmod{\Phi_n(q)^3}.
 \end{aligned}$$

Note that

$$(4.8) \quad q^{\binom{n}{2}} \equiv 1 - \frac{(n-1)(1 - q^n)}{2} + \frac{(n-1)(n-3)(1 - q^n)^2}{8} \pmod{\Phi_n(q)^3},$$

where we let  $t_0 = 1 - \frac{(n-1)(1 - q^n)}{2} + \frac{(n-1)(n-3)(1 - q^n)^2}{8}$ . Then, letting  $x \rightarrow q^{-1}x$  in (4.7) and combining (2.10), (4.7) and (4.8), we get(1.6).

Furthermore,

$$\begin{aligned}
 (4.9) \quad \sum_{i=1}^{n-1} \frac{q^i(x; q)_i}{[i]_q^2} &\equiv \sum_{i=1}^{n-1} \frac{q^{n(n-1)/2-ni}(x; q)_i}{[i]_q^2} \\
 &\equiv -\frac{1}{[n]_q} \sum_{i=1}^{n-1} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{n-i}{2}} \frac{(x; q)_i}{[i]_q} \\
 &= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^n \frac{x^i - 1}{[i]_q} - \frac{(x; q)_n}{[n]_q^2} \\
 &= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^{n-1} \frac{x^i - 1}{[i]_q} + \frac{q^{\binom{n}{2}}(1 - x^n) - (x; q)_n}{[n]_q^2} \pmod{\Phi_n(q)},
 \end{aligned}$$

where in the third step we have used (4.2). Combining (2.10), (4.8) and (4.9), we get (1.7).

5. COROLLARY 2

**Corollary 5.1.** *For any odd positive integer  $n$  and variable  $x$ , we have*

$$\begin{aligned}
 (5.1) \quad \sum_{j=1}^{n-1} \frac{x^j}{[j]_q} \sum_{k=1}^j \frac{x^k}{[k]_q} &\equiv \frac{1}{2} \left( \frac{1 - x^n - (x; q)_n}{[n]_q} + \frac{x^n(n-1)(1-q)}{2} \right)^2 \\
 &+ \frac{t_0(1 - (q^{-1}x^2; q^{-1})_n) - (q^{-1}x^2)^n}{2[n]_q^2} \\
 &- \left( \frac{1}{2[n]_q} - \frac{(n-1)(1-q)}{4} \right) \sum_{k=1}^{n-1} \frac{(q^{-1}x^2; q^{-1})_k - 1}{[k]_q} \pmod{\Phi_n(q)}.
 \end{aligned}$$

*Proof.* Note that

$$\begin{aligned}
 \sum_{j=1}^{n-1} \frac{x^j}{[j]_q} \sum_{k=1}^j \frac{x^k}{[k]_q} &= \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \sum_{j=k}^{n-1} \frac{x^j}{[j]_q} \\
 &= \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \left( \sum_{j=1}^{n-1} \frac{x^j}{[j]_q} - \sum_{j=1}^k \frac{x^j}{[j]_q} + \frac{x^k}{[k]_q} \right) \\
 &= \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \sum_{j=1}^{n-1} \frac{x^j}{[j]_q} - \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \sum_{j=1}^k \frac{x^j}{[j]_q} + \sum_{k=1}^{n-1} \frac{x^{2k}}{[k]_q^2}.
 \end{aligned}$$

Then,

$$(5.2) \quad \sum_{j=1}^{n-1} \frac{x^j}{[j]_q} \sum_{k=1}^j \frac{x^k}{[k]_q} = \frac{1}{2} \left( \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \right)^2 + \frac{1}{2} \sum_{k=1}^{n-1} \frac{x^{2k}}{[k]_q^2}.$$

By (1.6), letting  $x \rightarrow x^2$ , we get,

$$\begin{aligned}
 (5.3) \quad \sum_{k=1}^{n-1} \frac{x^{2k}}{[k]_q^2} &\equiv \left( \frac{(n-1)(1-q)}{2} - \frac{1}{[n]_q} \right) \sum_{k=1}^{n-1} \frac{(q^{-1}x^2; q^{-1})_k - 1}{[k]_q} \\
 &+ \frac{q^{\binom{n}{2}}(1 - (q^{-1}x^2; q^{-1})_n) - (q^{-1}x^2)^n}{[n]_q^2} \pmod{\Phi_n(q)}.
 \end{aligned}$$

Then combining (2.6), (2.10), (4.8), (5.2) and (5.3), we get Corollary 5.1. □

For example, let  $x = -1$ . We see that for any odd positive integer  $n$ ,

$$\begin{aligned}
 (5.4) \quad \sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \sum_{j=1}^k \frac{(-1)^j}{[j]_q} &\equiv 2Q_n^2(2, q) + (n-1)(1-q) \\
 &+ \frac{(n^2 - 1)(1-q)^2}{12} \pmod{\Phi_n(q)},
 \end{aligned}$$

where  $Q_n(2, q) = \frac{(-q; q)_{n-1-1}}{[n]_q}$ . The case  $n = p \geq 5$  in (5.4) is Theorem 1.4 in [8].

We replace  $x$  with  $-1$  in (5.2), and obtain

$$(5.5) \quad \sum_{j=1}^{n-1} \frac{(-1)^j}{[j]_q} \sum_{k=1}^j \frac{(-1)^k}{[k]_q} = \frac{1}{2} \left( \sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \right)^2 + \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{[k]_q^2}.$$

As Shi and Pan [4, (5)] mentioned, the following  $q$ -congruence can be proved by the same method. We replace  $p$  with  $n$  and  $[p]$  with  $\Phi_n(q)$  in the proof of [4, (5)]:

$$(5.6) \quad \sum_{k=1}^{n-1} \frac{1}{[k]_q^2} \equiv -\frac{(n-1)(n-5)(1-q)^2}{12} \pmod{\Phi_n(q)}.$$

Meanwhile, as B. He [8, (1.7)] mentioned, We replace  $p$  with  $n$  and  $[p]$  with  $\Phi_n(q)$  in the proof of [8, (1.7)]:

$$(5.7) \quad \sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \equiv -2Q_n(2, q) - \frac{(n-1)(1-q)}{2} \pmod{\Phi_n(q)}.$$

Finally, combining (5.5), (5.6) and (5.7), we get (5.4). Meanwhile, let  $x = -1$  in (5.1), we also can simplify to get (5.4), so we omit this part. We replace  $n$  with  $p$  and  $\Phi_n(q)$  with  $[p]_q$  in (5.4). This consequence will be [8, Theorem 1.4].

**Corollary 5.2.** *For any odd positive integer  $n$  and variable  $x$ ,*

$$(5.8) \quad \sum_{1 \leq k \leq j \leq n-1} \frac{q^j x^k}{[k]_q^2} \equiv \frac{t_0(1 - (q^{-1}x; q^{-1})_n) - (q^{-1}x)^n}{[n]_q} - \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} - \frac{2 - (n-1)(1-q^n)}{2} \sum_{k=1}^{n-1} \frac{(q^{-1}x; q^{-1})_k - 1}{[k]_q} \pmod{\Phi_n(q)^2}.$$

*Proof.* Note that

$$(5.9) \quad \begin{aligned} \sum_{1 \leq k \leq j \leq n-1} \frac{q^j x^k}{[k]_q^2} &= \sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2} \sum_{j=k}^{n-1} q^j \\ &= \sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2} \left( \frac{1-q^n}{1-q} - \frac{1-q^k}{1-q} \right) \\ &= \frac{1-q^n}{1-q} \sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2} - \sum_{k=1}^{n-1} \frac{x^k}{[k]_q}. \end{aligned}$$

Finally, combining (1.6) and (5.9), we get the desired result. □

For example, when  $x = 1$ ,  $x = q^k$ ,  $x = (-1)^k$  in (5.8), we will get

$$(5.10) \quad \sum_{j=1}^{n-1} q^j \sum_{k=1}^j \frac{1}{[k]_q^2} \equiv -\frac{(n-1)(1-q)}{2} - \frac{(n-1)(n-3)(1-q)^2[n]_q}{8} \pmod{\Phi_n(q)^2},$$

$$(5.11) \quad \sum_{j=1}^{n-1} q^j \sum_{k=1}^j \frac{q^k}{[k]_q^2} \equiv \frac{(n-1)(1-q)}{2} - \frac{(n^2-1)(1-q)^2[n]_q}{8} \pmod{\Phi_n(q)^2},$$

and

$$(5.12) \quad \sum_{j=1}^{n-1} q^j \sum_{k=1}^j \frac{(-1)^k}{[k]_q^2} \equiv 2Q_n(2, n) + \frac{(n-1)(1-q)}{2} - \left( Q_n(2, q)^2 + 3Q_n(2, q)(1-q) + \frac{(n+7)(n-1)(1-q)^2}{12} \right) [n]_q \pmod{\Phi_n(q)^2},$$

where  $n$  is an odd positive integer.

As B. He [8, Theorem 1.2] mentioned, we replace  $p$  with  $n$  and  $[p]$  with  $\Phi_n(q)$  in the proof of [8, Theorem 1.2]:

$$(5.13) \quad \sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \equiv -2Q_n(2, q) - \frac{(n-1)(1-q)}{2} + \left( Q_n(2, q)^2 + Q_n(2, q)(1-q) + \frac{(n^2-1)(1-q)}{12} \right) [n]_q \pmod{\Phi_n(q)^2}$$

and

$$(5.14) \quad \sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q^2} \equiv -2Q_n(2, q)(1-q) + \frac{(1-n)(1-q)^2}{2} \pmod{\Phi_n(q)},$$

Furthermore, note that

$$(5.15) \quad \sum_{k=1}^{n-1} \frac{1}{[k]_q} = \sum_{k=1}^{n-1} \frac{1 - q^k + q^k}{[k]_q} = (1-q)(n-1) + \sum_{k=1}^{n-1} \frac{q^k}{[k]_q}$$

and

$$(5.16) \quad \sum_{k=1}^{n-1} \frac{1}{[k]_q^2} = (1-q) \sum_{k=1}^{n-1} \frac{1}{[k]_q} + \sum_{k=1}^{n-1} \frac{q^k}{[k]_q^2}.$$

Then combining (2.6), (5.6), (5.9), (5.13), (5.14), (5.15) and (5.16), it is not difficult for us to get (5.10), (5.11) and (5.12). Meanwhile, letting  $x = 1$ ,  $x = q$  and  $x = -1$  in (5.8), we also can simplify to get (5.10), (5.11) and (5.12), so we omit this part. We

replace  $n$  with  $p$  and  $\Phi_n(q)$  with  $[p]_q$  in (5.10), (5.11) and (5.12). These consequences will be [8, (1.3)], [8, (1.4)] and [8, Theorem 1.3].

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## ON DOUBLE $q$ -LAPLACE TRANSFORM AND APPLICATIONS

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**ABSTRACT.** We introduce four  $q$ -analogues of the double Laplace transform and prove some of their main properties. Next we show how they can be used to solve some  $q$ -functional equations and partial  $q$ -differential equations.

### 1. INTRODUCTION

The classical Laplace transform of a function  $f$  is given by

$$(1.1) \quad \mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t) dt, \quad s = a + ib \in \mathbb{C},$$

and plays a fundamental role in pure and applied analysis. Laplace transform has been studied very extensively and has found to have a wide variety of applications in mathematical, physical, statistical, and engineering sciences and also in other sciences. There is a very extensive literature available of the Laplace transform of a function  $f(t)$  of one variable  $t$  and its applications (see for example Churchill [9], Schiff [21], Debnath and Bhatta [10] and the references therein).

The double Laplace transform of a function  $f(x, y)$  of two variables was first introduced in 1939 by Bernstein in his dissertation [5] (later published as an article [6]) as

$$(1.2) \quad \mathcal{L}_2(f(x, y))(r, s) = \int_0^{+\infty} \int_0^{+\infty} f(x, y) e^{-(rx+sy)} dx dy,$$

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where  $x$  and  $y$  are two positive numbers,  $r$  and  $s$  are complex numbers. Very recently, several interesting properties and applications of the double Laplace transform to functional, integral and partial differential equations have been studied in [11].

The development of  $q$ -analysis started in the 1740s, when Euler initiated the theory of partitions, also called additive analytic number theory. Euler always wrote in Latin and his collected works were published only at the beginning of the 1800s, under the legendary Jacobi. In 1829 Jacobi presented his triple product identity (sometimes called the Gauss-Jacobi triple product identity), and his  $\theta$  and elliptic functions, which in principle are equivalent to  $q$ -analysis. The progress of  $q$ -calculus continued under C. F. Gauss (1777–1855), who in 1812 invented the hypergeometric series and their contiguity relations. Gauss would later invent the  $q$ -binomial coefficients and prove an identity for them, which forms the basis for  $q$ -analysis.

The theory of  $q$ -analysis have been applied in recent past in many areas of mathematics and physics like ordinary fractional calculus, optimal control problems, quantum calculus,  $q$ -transform analysis and in finding solutions of the  $q$ -difference and  $q$ -integral equations. In 1910, Jackson [15] presented a precise definition of the so-called the  $q$ -Jackson integral and developed  $q$ -calculus in a systematic way.

In order to deal with  $q$ -difference equations,  $q$ -versions of the classical Laplace transform have been consecutively introduced in the literature. Studies of  $q$ -versions of Laplace transform go back to Hahn [14]. Abdi [1–3] published also many results in this domain. In a recent paper [8] two very interesting versions of  $q$ -Laplace transform are introduced as follows

$$(1.3) \quad L_q(f(t))(s) = \int_0^{+\infty} E_q(-qst) f(t) d_q t, \quad s > 0,$$

for the first kind and

$$(1.4) \quad \mathcal{L}_q(f(t))(s) = \int_0^{+\infty} e_q(-st) f(t) d_q t, \quad s > 0,$$

for the second kind. Note that both (1.3) and (1.4) generalize (1.1). We will frequently use some properties of (1.3) and (1.4) and will refer the reader to the paper [8] for more details.

In this paper, we introduce four kinds of double  $q$ -Laplace transforms and prove their main properties. Next, applications are done to solve some classical partial  $q$ -differential equations that appear in the litterature. The double  $q$ -Laplace transform introduced here are clearly generalization of the one given in [5].

## 2. BASIC DEFINITIONS AND MISCELLANEOUS RESULTS

**2.1.  $q$ -number,  $q$ -factorial,  $q$ -binomial,  $q$ -power,  $q$ -addition.** For any complex number  $a$ , the basic or  $q$ -number is defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad q \neq 1.$$

For any non negative integer  $n$ , the  $q$ -factorial is defined by

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q = \prod_{k=1}^n [k]_q, \quad n \in \mathbb{N}, \quad [0]_q! = 1,$$

and the  $q$ -pochhammer is defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N}.$$

The limit,  $\lim_{n \rightarrow +\infty} (a; q)_n$  is denoted by  $(a; q)_\infty$ , provided that  $|q| < 1$ . Then,

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad n \in \mathbb{N}_0, \quad |q| < 1,$$

and for any complex number  $\alpha$ , this definition can be extended by

$$(a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty}, \quad |q| < 1,$$

where the principal value of  $q^\alpha$  is taken.

The  $q$ -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad 0 \leq k \leq n.$$

It is worth noting that  $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$ .

The  $q$ -power basis is defined by

$$(x \ominus y)_q^n = \begin{cases} (x-y)(x-yq) \cdots (x-yq^{n-1}), & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

In the same line we introduce the following notation

$$(x \oplus y)_q^n = \begin{cases} (x+y)(x+yq) \cdots (x+yq^{n-1}), & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

It is not difficult to proved that (see [19])

$$(x \oplus y)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} x^k y^{n-k}.$$

In [22], Schork has studied Ward's "Calculus of Sequences" and introduced a  $q$ -addition  $x \oplus_q y$  by

$$(x \oplus_q y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k},$$

and although this  $q$ -addition was already known to Jackson, it was generalized later on by Ward and Al-Salam. For more information about different  $q$ -additions, see e.g.

[13]. Similarly the  $q$ -subtraction can be defined in the same way by [16]

$$(2.1) \quad (x \ominus_q y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (-y)^{n-k} = (x \oplus_q (-y))^n.$$

Al-Salam introduced in [4] the following  $q$ -coaddition

$$(x \boxplus_q y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k y^{n-k}.$$

We introduce the following  $q$ -cosubtraction [13, p. 233]

$$(x \boxminus_q y)^n = (x \boxplus_q (-y))^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k (-y)^{n-k}.$$

**2.2. The  $q$ -derivative and the  $q$ -integral.** The  $q$ -derivative operator is defined by [17, 18]

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0,$$

satisfying the important product rule

$$D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x).$$

In this sense, note that when we deal with functions  $f(x_1, x_2, \dots, x_n)$  of more than one variable, we denote  $D_q f$  by  $D_{q, x_i} f$  or  $\frac{\partial_q}{\partial x_i} f$  to make clear that the derivative is taken with respect to the variable  $x_i$ . For the case of two variables  $x$  and  $y$  for example, the  $q$ -partial derivative with respect to  $x$  is given by [20]

$$D_{q, x} f(x, y) = \frac{f(x, y) - f(qx, y)}{(1-q)x}, \quad x \neq 0,$$

and

$$D_{q, x} f(x, y) \Big|_{x=0} = \lim_{x \rightarrow 0} D_{q, x} f(x, y).$$

The  $q$ -integral operator is defined by [17, 18]

$$\int_0^z f(z) d_q t = z(1-q) \sum_{k=0}^{+\infty} q^k f(zq^k).$$

This definition can be established based on a simple geometric series. Note that for  $a < b$  two real numbers, one has

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

and the  $q$ -integration by part is

$$\int_a^b f(x) D_q g(x) d_q x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_q f(x) d_q x.$$

Note that in this  $q$ -integration by part,  $b = +\infty$  is allowed as well [17].

**2.3. The  $q$ -hypergeometric, the  $q$ -exponential and  $q$ -trigonometric functions.**

The basic hypergeometric or  $q$ -hypergeometric function  ${}_r\phi_s$  is defined by the series

$${}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{k=0}^{+\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \frac{z^k}{(q; q)_k}.$$

where

$$(a_1, \dots, a_r)_k := (a_1; q)_k \cdots (a_r; q)_k,$$

The usual exponential function may have two different natural  $q$ -extensions, denoted by  $e_q(z)$  and  $E_q(z)$ , which are defined, respectively, by

$$e_q(z) := {}_1\phi_0 \left( \begin{matrix} 0 \\ - \end{matrix} \middle| q; (1-q)z \right) = \sum_{n=0}^{+\infty} \frac{z^n}{[n]_q!}, \quad 0 < |q| < 1, |z| < 1,$$

and

$$E_q(z) := {}_0\phi_0 \left( \begin{matrix} - \\ - \end{matrix} \middle| q, -(1-q)z \right) = \sum_{n=0}^{+\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} z^n, \quad 0 < |q| < 1.$$

It is worth noting that  $e_q(z)$  and  $E_q(z)$  are linked by the well known relation

$$e_q(z)E_q(-z) = 1.$$

They fulfil the  $q$ -derivative rules

$$\begin{aligned} D_q e_q(\lambda x) &= \lambda e_q(\lambda x), \\ D_q E_q(\lambda x) &= \lambda E_q(\lambda qx). \end{aligned}$$

It is not difficult to see that [4, 8, 13]

$$(2.2) \quad e_q(x)e_q(t) = e_q(x \oplus_q y), \quad \text{for all } x, y \in \mathbb{C},$$

and

$$E_q(x)E_q(t) = E_q(x \boxplus_q y), \quad \text{for all } x, y \in \mathbb{C}.$$

From these definitions of the  $q$ -exponential functions, we derive the following  $q$ -trigonometric functions [8, 17]

$$\begin{aligned} \cos_q(z) &= \frac{e_q(iz) + e_q(-iz)}{2} = \sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n}}{[2n]_q!}, \\ \sin_q(z) &= \frac{e_q(iz) - e_q(-iz)}{2i} = \sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n+1}}{[2n+1]_q!}, \\ \text{Cos}_q(z) &= \frac{E_q(iz) + E_q(-iz)}{2} = \sum_{n=0}^{+\infty} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_q!} z^{2n}, \\ \text{Sin}_q(z) &= \frac{E_q(iz) - E_q(-iz)}{2i} = \sum_{n=0}^{+\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_q!} z^{2n+1}, \end{aligned}$$

and the hyperbolic  $q$ -trigonometric functions

$$\begin{aligned} \cosh_q(z) &= \frac{e_q(z) + e_q(-z)}{2} = \sum_{n=0}^{+\infty} \frac{z^{2n}}{[2n]_q!}, \\ \sinh_q(z) &= \frac{e_q(z) - e_q(-z)}{2} = \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{[2n+1]_q!}, \\ \text{Cosh}_q(z) &= \frac{E_q(z) + E_q(-z)}{2} = \sum_{n=0}^{+\infty} \frac{q^{\binom{2n}{2}}}{[2n]_q!} z^{2n}, \\ \text{Sinh}_q(z) &= \frac{E_q(z) - E_q(-z)}{2} = \sum_{n=0}^{+\infty} \frac{q^{\binom{2n+1}{2}}}{[2n+1]_q!} z^{2n+1}. \end{aligned}$$

**2.4. The  $q$ -Gamma functions.** The  $q$ -Gamma function of the first kind [17] is defined for  $0 < q < 1$  as

$$\Gamma_q(t) = \int_0^{+\infty} x^{t-1} E_q(-qx) d_q x, \quad t > 0.$$

It satisfies the fundamental relation

$$\Gamma_q(t + 1) = [t]_q \Gamma_q(t), \quad t > 0.$$

Since for any non-negative integer  $n$

$$\Gamma_q(n + 1) = [n]_q!,$$

it is clear that the  $q$ -Gamma function is a generalization of the  $q$ -factorial.

The  $q$ -Gamma function of the second kind [8, 12] is defined by

$$\gamma_q(t) = \int_0^{+\infty} x^{t-1} e_q(-x) d_q x, \quad t > 0,$$

and satisfied

$$\gamma_q(1) = 1, \quad \gamma_q(t + 1) = q^{-t} [t]_q \gamma_q(t), \quad \gamma_q(n) = q^{-\binom{n}{2}} \Gamma_q(n), \quad n \in \mathbb{N}.$$

### 3. DOUBLE $q$ -LAPLACE TRANSFORM OF THE FIRST KIND

Based on definitions (1.2) and (1.3) we define the double  $q$ -Laplace transform of the first kind as

$$(3.1) \quad \mathcal{L}_{2,q}^{(1)}[f(x, y)](r, s) = \int_0^{+\infty} \int_0^{+\infty} f(x, y) E_q(-qrx) E_q(-qsy) d_q x d_q y, \quad r, s > 0.$$

Note that if  $f(x, y) = g(x)h(y)$ , then

$$(3.2) \quad \mathcal{L}_{2,q}^{(1)}[f(x, y)](r, s) = L_q\{g(x)\}(r) L_q\{h(y)\}(s).$$

In particular, if  $h(y) = 1$  or  $g(x) = 1$ , then (3.2) reads

$$(3.3) \quad \mathcal{L}_{2,q}^{(1)}[f(y)](r, s) = L_q\{1\}(r) L_q\{f(y)\}(s) = \frac{1}{r} L_q\{f(y)\}(s)$$

and

$$(3.4) \quad \mathcal{L}_{2,q}^{(1)}[f(x)](r, s) = L_q\{g(x)\}(r)L_q\{1\}(s) = \frac{1}{s}L_q\{g(x)\}(r).$$

**Proposition 3.1.** *For any two complex numbers  $\alpha$  and  $\beta$ , we have*

$$\mathcal{L}_{2,q}^{(1)}\{\alpha f(x, y) + \beta g(x, y)\} = \alpha\mathcal{L}_{2,q}^{(1)}\{f(x, y)\} + \beta\mathcal{L}_{2,q}^{(1)}\{g(x, y)\}.$$

*Proof.* The proof follows from (3.1). □

In what follows, we give some examples. From (3.1), we note that:

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)}\{1\}(r, s) &= \int_0^{+\infty} \int_0^{+\infty} E_q(-qrx)E_q(-qsy)d_qx d_qy \\ &= \left(\int_0^{+\infty} E_q(-qrx)d_qx\right) \left(\int_0^{+\infty} E_q(-qsy)d_qy\right) \\ &= \frac{1}{r} \cdot \frac{1}{s} = \frac{1}{rs}. \\ \mathcal{L}_{2,q}^{(1)}\{xy\}(r, s) &= \int_0^{+\infty} \int_0^{+\infty} xyE_q(-qrx)E_q(-qsy)d_qx d_qy \\ &= \left(\int_0^{+\infty} xE_q(-qrx)d_qx\right) \left(\int_0^{+\infty} yE_q(-qsy)d_qy\right) \\ &= \frac{1}{r^2} \cdot \frac{1}{s^2} = \frac{1}{(rs)^2} \end{aligned}$$

and

$$\mathcal{L}_{2,q}^{(1)}\{1 + 4xy\}(r, s) = \mathcal{L}_{2,q}^{(1)}\{1\}(r, s) + 4\mathcal{L}_{2,q}^{(1)}\{xy\}(r, s) = \frac{1}{rs} + \frac{4}{(rs)^2}.$$

We recall the following important relation [17],

$$(3.5) \quad \int_0^{+\infty} f(\alpha x)d_qx = \frac{1}{\alpha} \int_0^{+\infty} f(x)d_qx,$$

where  $\alpha$  is a non zero complex number and  $f$  is a one variable function.

Now we state the scaling theorem for  $\mathcal{L}_{2,q}^{(1)}$ .

**Theorem 3.1.** *Let  $a$  and  $b$  be two non zero complex numbers,  $f$  a two variable function, then the following formula applies*

$$(3.6) \quad \mathcal{L}_{2,q}^{(1)}\{f(ax, by)\}(r, s) = \frac{1}{ab}\mathcal{L}_{2,q}^{(1)}\{f(x, y)\}\left(\frac{r}{a}, \frac{s}{b}\right).$$

*Proof.* Using relation (3.5), we have

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)}\{f(ax, by)\}(r, s) &= \int_0^{+\infty} \int_0^{+\infty} f(ax, by)E_q(-qrx)E_q(-qsy)d_qx d_qy \\ &= \int_0^{+\infty} \left(\int_0^{+\infty} f(ax, by)E_q(-qrx)d_qx\right) E_q(-qsy)d_qy \\ &= \frac{1}{a} \int_0^{+\infty} \left(\int_0^{+\infty} f(x, by)E_q\left(-qx\frac{r}{a}\right)d_qx\right) E_q(-qsy)d_qy \end{aligned}$$

$$\begin{aligned} &= \frac{1}{a} \int_0^{+\infty} \left( \int_0^{+\infty} f(x, by) E_q(-qsy) d_q y \right) E_q \left( -qx \frac{r}{a} \right) d_q x \\ &= \frac{1}{ab} \int_0^{+\infty} \left( \int_0^{+\infty} f(x, y) E_q \left( -qy \frac{s}{b} \right) d_q y \right) E_q \left( -qx \frac{r}{a} \right) d_q x \\ &= \frac{1}{ab} \int_0^{+\infty} \int_0^{+\infty} f(x, y) E_q \left( -qx \frac{r}{a} \right) E_q \left( -qy \frac{s}{b} \right) d_q x d_q y, \end{aligned}$$

and the proof of the theorem is completed. □

**Theorem 3.2.** *For  $\alpha > -1, \beta > -1$ , we have the following*

$$\mathcal{L}_{2,q}^{(1)}\{x^\alpha y^\beta\}(r, s) = \frac{\Gamma_q(\alpha + 1)}{r^{\alpha+1}} \cdot \frac{\Gamma_q(\beta + 1)}{s^{\beta+1}}.$$

*In particular, for  $\alpha = n \in \mathbb{N}$  and  $\beta = m \in \mathbb{N}$ , we get*

$$\mathcal{L}_{2,q}^{(1)}\{x^n y^m\}(r, s) = \frac{[n]_q! [m]_q!}{r^{n+1} s^{m+1}}.$$

*Proof.* The proof follows from the relation  $L_q\{t^\alpha\}(s) = \frac{\Gamma_q(\alpha+1)}{s^{\alpha+1}}$  (see [8]) and the obvious equation

$$\mathcal{L}_{2,q}^{(1)}\{x^\alpha y^\beta\}(r, s) = L_q\{x^\alpha\}(r) \times L_q\{y^\beta\}(s). \quad \square$$

Let us take for example  $\alpha = -\frac{1}{2}$  and  $\beta = \frac{1}{2}$ . Then we see that

$$\mathcal{L}_{2,q}^{(1)}\left(\sqrt{\frac{y}{x}}\right)(r, s) = L_q\{x^{-\frac{1}{2}}\}(r) \times L_q\{y^{\frac{1}{2}}\}(s) = \Gamma_q\left(\frac{1}{2}\right) \Gamma_q\left(\frac{3}{2}\right) \frac{1}{s\sqrt{rs}},$$

and for  $\alpha = -\frac{1}{2}$  and  $\beta = -\frac{1}{2}$  we have

$$\mathcal{L}_{2,q}^{(1)}\left(\frac{1}{\sqrt{xy}}\right)(r, s) = L_q\{x^{-\frac{1}{2}}\}(r) \times L_q\{y^{-\frac{1}{2}}\}(s) = \left[\Gamma_q\left(\frac{1}{2}\right)\right]^2 \frac{1}{\sqrt{rs}}.$$

**Proposition 3.2.** *Let  $a$  and  $b$  be two real numbers. Then we have:*

$$(3.7) \quad \mathcal{L}_{2,q}^{(1)}\{(ax \oplus_q by)^n\}(r, s) = \frac{[n]_q!}{br - as} \left( \left(\frac{b}{s}\right)^{n+1} - \left(\frac{a}{r}\right)^{n+1} \right).$$

*Proof.* Combining the scaling property (see equation (3.6)) and (2.2) we have

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)}\{(ax \oplus_q by)^n\}(r, s) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{L}_{2,q}^{(1)}\{(ax)^k (by)^{n-k}\}(r, s) \\ &= \frac{1}{ab} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{L}_{2,q}^{(1)}\{x^k y^{n-k}\}\left(\frac{r}{a}, \frac{s}{b}\right) \\ &= \frac{1}{ab} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q! [n-k]_q!}{r^{k+1} s^{n-k+1}} a^{k+1} b^{n-k+1} \\ &= \frac{[n]_q!}{ab} \left(\frac{a}{r}\right) \left(\frac{b}{s}\right)^{n+1} \sum_{k=0}^n \left(\frac{as}{rb}\right)^k \end{aligned}$$

$$= \frac{[n]_q!}{br - as} \left( \left(\frac{b}{s}\right)^{n+1} - \left(\frac{a}{r}\right)^{n+1} \right).$$

This ends the proof of the proposition. □

**Theorem 3.3.** *Let  $a$  and  $b$  be two complex numbers, then*

$$\mathcal{L}_{2,q}^{(1)}\{e_q(ax \oplus_q by)\}(r, s) = \frac{1}{(r - a)(s - b)}, \quad r > \operatorname{Re}(a), s > \operatorname{Re}(b).$$

*Proof.* Using the definition of the  $q$ -addition (2.1), and Proposition 3.2 we have

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)}\{e_q(ax \oplus_q by)\}(r, s) &= \sum_{n=0}^{+\infty} \mathcal{L}_{2,q}^{(1)}\left\{\frac{(ax \oplus_q by)^n}{[n]_q!}\right\}(r, s) \\ &= \frac{1}{br - as} \sum_{n=0}^{+\infty} \left( \left(\frac{b}{s}\right)^{n+1} - \left(\frac{a}{r}\right)^{n+1} \right) \\ &= \frac{1}{br - as} \left( \frac{s}{s - b} - \frac{r}{r - a} \right) \\ &= \frac{1}{(r - a)(s - b)}. \end{aligned} \quad \square$$

Note also that this result can be obtained using equations (2.2), (3.2) and the fact that (see [8]):

$$(3.8) \quad L_q(e_q(ax))(s) = \frac{1}{s - a}.$$

**Proposition 3.3.** *The following formulas apply*

$$(3.9) \quad \mathcal{L}_{2,q}^{(1)}\{\cos_q(ax \oplus_q by)\}(r, s) = \frac{rs - ab}{(r^2 + a^2)(s^2 + b^2)},$$

$$(3.10) \quad \mathcal{L}_{2,q}^{(1)}\{\sin_q(ax \oplus_q by)\}(r, s) = \frac{as + br}{(r^2 + a^2)(s^2 + b^2)}.$$

*Proof.* We indicate two proofs of these equations. First we can use the relations (see [16])

$$\begin{aligned} \cos_q(x \oplus_q y) &= \cos_q(x) \cos_q(y) - \sin_q(x) \sin_q(y), \\ \sin_q(x \oplus_q y) &= \sin_q(x) \cos_q(y) + \cos_q(x) \sin_q(y), \end{aligned}$$

together with the equations (3.2) and (3.8).

For the second proof, we remark first that for any complex number  $\lambda$ , we have  $e_q(\lambda(x \oplus_q y)) = e_q(\lambda x \oplus_q \lambda y)$ , to write

$$\begin{aligned} \cos_q(ax \oplus_q by) &= \frac{1}{2} (e_q(i(ax \oplus_q by)) + e_q(-i(ax \oplus_q by))) \\ &= \frac{1}{2} (e_q((aix \oplus_q biy)) + e_q((-aix \oplus_q -biy))), \end{aligned}$$

$$\begin{aligned} \sin_q(ax \oplus_q by) &= \frac{1}{2i} (e_q(i(ax \oplus_q by)) - e_q(-i(ax \oplus_q by))) \\ &= \frac{1}{2i} (e_q((aix \oplus_q biy)) - e_q((-aix \oplus_q -biy))). \end{aligned}$$

Hence, using the linearity of  $\mathcal{L}_{2,q}^{(1)}$ , and equation (3.3), it follows that

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)}\{\cos_q(ax \oplus_q by)\}(r, s) &= \frac{1}{2} \left\{ \frac{1}{(r - ai)(s - bi)} + \frac{1}{(r + ai)(s + ib)} \right\} \\ &= \frac{rs - ab}{(r^2 + a^2)(s^2 + b^2)}. \end{aligned}$$

This proves again (3.9). (3.10) follows in the same way. □

**Proposition 3.4.** *The following equations apply*

$$\begin{aligned} \cosh_q(x \oplus_q y) &= \cosh_q(x) \cosh_q(y) + \sinh_q(x) \sinh_q(y), \\ \sinh_q(x \oplus_q y) &= \cosh_q(x) \sinh_q(y) + \sinh_q(x) \cosh_q(y). \end{aligned}$$

*Proof.* The proof uses the definitions of the involved functions. □

**Proposition 3.5.** *The following formulas apply*

$$(3.11) \quad \mathcal{L}_{2,q}^{(1)}\{\cosh_q(ax \oplus_q by)\}(r, s) = \frac{rs + ab}{(r^2 - a^2)(s^2 - b^2)},$$

$$(3.12) \quad \mathcal{L}_{2,q}^{(1)}\{\sinh_q(ax \oplus_q by)\}(r, s) = \frac{as + br}{(r^2 - a^2)(s^2 - b^2)}.$$

*Proof.* The proof follows from Proposition 3.4, equations (3.2) and (3.8). It can also be done using the fact that

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)}\{\cosh_q(ax \oplus_q by)\}(r, s) &= \frac{1}{2} \mathcal{L}_{2,q}^{(1)}\{(e_q(ax \oplus_q by) + e_q(-ax \oplus_q -by))\}(r, s) \\ &= \frac{1}{2} \left\{ \frac{1}{(r - a)(s - b)} + \frac{1}{(r + a)(s + b)} \right\} \\ &= \frac{rs + ab}{(r^2 - a^2)(s^2 - b^2)}, \end{aligned}$$

which proves (3.11). (3.12) can be obtained in a similar way. □

**Theorem 3.4.** *Let  $f$  be a one variable function that has a  $q$ -Laplace transform. Assume that  $f$  has the  $q$ -Taylor expansion*

$$f(x) = \sum_{n=0}^{+\infty} a_n \frac{x^n}{[n]_q!},$$

*then the following relation holds:*

$$(3.13) \quad \mathcal{L}_{2,q}^{(1)}[f(\alpha x \oplus_q \beta y)](r, s) = \frac{1}{\alpha s - \beta r} \left( L_q[f(x)]\left(\frac{r}{\alpha}\right) - L_q[f(x)]\left(\frac{s}{\beta}\right) \right),$$

*where  $\alpha, \beta \neq 0$  and  $\alpha s - \beta r \neq 0$ .*

*Proof.* We have the following

$$f(\alpha x \oplus_q \beta y) = \sum_{n=0}^{+\infty} a_n \frac{(\alpha x \oplus_q \beta y)^n}{[n]_q!} = \sum_{n=0}^{+\infty} \left( \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (\alpha x)^k (\beta y)^{n-k} \right) \frac{a_n}{[n]_q!}.$$

Hence, it follows that

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)} [f(x \oplus_q y)] (r, s) &= \sum_{n=0}^{+\infty} \left( \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\alpha^k [k]_q! \beta^{n-k} [n-k]_q!}{r^{k+1} s^{n+1-k}} \right) \frac{a_n}{[n]_q!} \\ &= \sum_{n=0}^{+\infty} \sum_{k=0}^n \frac{\alpha^k \beta^{n-k} a_n}{r^{k+1} s^{n+1-k}} \\ &= \frac{1}{\alpha s - \beta r} \left( \sum_{n=0}^{+\infty} a_n \left( \frac{\alpha}{r} \right)^{n+1} - \sum_{n=0}^{+\infty} a_n \left( \frac{\beta}{r} \right)^{n+1} \right) \\ &= \frac{1}{\alpha s - \beta r} \left( L_q[f(x)] \left( \frac{r}{\alpha} \right) - L_q[f(x)] \left( \frac{s}{\beta} \right) \right). \end{aligned}$$

This ends the proof of the theorem. □

The next two theorems provide formulas for the double  $q$ -Laplace transform of the partial  $q$ -derivative and the partial  $q$ -derivatives of the double  $q$ -Laplace transform. These results are of great importance in the resolution of partial  $q$ -differential equations as we will see in Section 5.

**Theorem 3.5.** *The following equations hold true*

(3.14)

$$\mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q f}{\partial_q x}(x, y) \right] (r, s) = r \mathcal{L}_{2,q}^{(1)} [f(x, y)] (r, s) - L_q [f(0, y)] (s),$$

(3.15)

$$\mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q f}{\partial_q y}(x, y) \right] (r, s) = s \mathcal{L}_{2,q}^{(1)} [f(x, y)] (r, s) - L_q [f(x, 0)] (r),$$

(3.16)

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q^2 f}{\partial_q x \partial_q y}(x, y) \right] (r, s) &= r s \mathcal{L}_{2,q}^{(1)} [f(x, y)] (r, s) - r \mathcal{L}_{2,q}^{(1)} [f(x, 0)] (r) \\ &\quad - s \mathcal{L}_{2,q}^{(1)} [f(0, y)] (s) + f(0, 0), \\ \mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q^2 f}{\partial_q x^2}(x, y) \right] (r, s) &= r^2 \mathcal{L}_{2,q}^{(1)} [f(x, y)] (r, s) - r \mathcal{L}_{2,q}^{(1)} [f(0, y)] (s) - L_q \left[ \frac{\partial_q f}{\partial_q x}(0, y) \right] (s), \\ \mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q^2 f}{\partial_q y^2}(x, y) \right] (r, s) &= s^2 \mathcal{L}_{2,q}^{(1)} [f(x, y)] (r, s) - s \mathcal{L}_{2,q}^{(1)} [f(x, 0)] (r) - L_q \left[ \frac{\partial_q f}{\partial_q y}(x, 0) \right] (r). \end{aligned}$$

*Proof.* From (3.1), and the formula of  $q$ -integration by parts, we have

$$\mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q f}{\partial_q x}(x, y) \right] (r, s) = \int_0^{+\infty} \int_0^{+\infty} \frac{\partial_q f}{\partial_q x}(x, y) E_q(-qrx) E_q(-qsy) d_q x d_q y$$

$$\begin{aligned} &= \int_0^{+\infty} \left( \int_0^{+\infty} \frac{\partial_q f}{\partial_q x}(x, y) E_q(-qrx) d_q x \right) E_q(-qsy) d_q y \\ &= \int_0^{+\infty} \left( -f(0, y) + r \int_0^{+\infty} f(x, y) E_q(-qrx) d_q x \right) E_q(-qry) d_q y \\ &= -L_q[f(0, y)](s) + r \mathcal{L}_{2,q}^{(1)}[f(x, y)](r, s). \end{aligned}$$

Hence (3.14) is proved. The proof of (3.16) uses (3.14), (3.15) and the fact that (see [16])

$$L_q \left[ \frac{\partial_q f}{\partial_q x}(x, 0) \right] (r) = r L_q[f(x, 0)](r) - f(0, 0).$$

The rest of the theorem is proved in the same way. □

The following theorem, which is obtained by induction from the previous one, is now stated without proof.

**Theorem 3.6** (Double Laplace transform of the Partial  $q$ -derivative). *The following equations are valid, where  $n$  is a non-negative integer,*

$$\begin{aligned} \mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q^n f}{\partial_q x^n}(x, y) \right] (r, s) &= r^n \mathcal{L}_{2,q}^{(1)}[f(x, y)](r, s) - \sum_{k=0}^{n-1} r^{n-1-k} L_q \left[ \frac{\partial_q^k f}{\partial_q x^k}(0, y) \right] (s), \\ \mathcal{L}_{2,q}^{(1)} \left[ \frac{\partial_q^n f}{\partial_q y^n}(x, y) \right] (r, s) &= s^n \mathcal{L}_{2,q}^{(1)}[f(x, y)](r, s) - \sum_{k=0}^{n-1} s^{n-1-k} L_q \left[ \frac{\partial_q^k f}{\partial_q y^k}(x, 0) \right] (r). \end{aligned}$$

*Remark 3.1.* Note that the expression

$$L_q \left[ \frac{\partial_q^n f}{\partial_q x^n}(0, y) \right] (s) = s^n L_q[f(0, y)](s) - \sum_{k=0}^{n-1} s^{n-1-k} \frac{\partial_q^k f}{\partial_q x^k}(0, 0)$$

is given in [16].

**Theorem 3.7** (Partial  $q$ -derivative of the double Laplace transform). *The following relation is valid*

(3.17)

$$\mathcal{L}_{2,q}^{(1)}[x^m y^n f(x, y)](r, s) = (-1)^{m+n} q^{\binom{m}{2} + \binom{n}{2}} \frac{\partial_q^{m+n}}{\partial_q s^n \partial_q r^m} \mathcal{L}_{2,q}^{(1)}[f(x, y)](q^{-m} r, q^{-n} s).$$

*Proof.* We recall the relation (see [16, Theorem 2.4])

$$L_q[x^n f(x)](s) = (-1)^n q^{\binom{n}{2}} \frac{\partial_q^n}{\partial_q s^n} L_q[f(x)](q^{-n} s),$$

from which we have:

$$\begin{aligned} &\mathcal{L}_{2,q}^{(1)}[x^m y^n f(x, y)](r, s) \\ &= \int_0^{+\infty} \int_0^{+\infty} x^m y^n f(x, y) E_q(-rqx) E_q(-sqy) d_q x d_q y \\ &= \int_0^{+\infty} y^n \left( \int_0^{+\infty} x^m f(x, y) E_q(-rqx) d_q x \right) E_q(-sqy) d_q y \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{+\infty} y^n \left( (-1)^m q^{\binom{m}{2}} \frac{\partial_q^m}{\partial_q r^m} \int_0^{+\infty} f(x, y) E_q(-q^{-m} r q x) d_q x \right) E_q(-s q y) d_q y \\
 &= (-1)^m q^{\binom{m}{2}} \frac{\partial_q^m}{\partial_q r^m} \int_0^{+\infty} \left( (-1)^n q^{\binom{n}{2}} \frac{\partial_q^n}{\partial_q s^n} \int_0^{+\infty} f(x, y) E_q(-q^{-n} s q y) d_q y \right) E_q(-q^{-m} r q x) d_q x \\
 &= (-1)^{m+n} q^{\binom{m}{2} + \binom{n}{2}} \frac{\partial_q^{m+n}}{\partial_q s^n \partial_q r^m} \int_0^{+\infty} \int_0^{+\infty} f(x, y) E_q(-q^{-n} r q x) E_q(-q^{-n} s q y) d_q x d_q y \\
 &= (-1)^{m+n} q^{\binom{m}{2} + \binom{n}{2}} \frac{\partial_q^{m+n}}{\partial_q s^n \partial_q r^m} \mathcal{L}_{2,q}^{(1)}[f(x, y)] (q^{-m} r, q^{-n} s).
 \end{aligned}$$

This proves the theorem. □

We summarize the previous results in Table 1.

TABLE 1. Some Laplace of the First Kind.

Originals	Transforms
$x^\alpha y^\beta \ (\alpha, \beta > -1)$	$\frac{\Gamma_q(\alpha + 1)}{r^{\alpha+1}} \cdot \frac{\Gamma_q(\beta + 1)}{s^{\beta+1}}$
$(ax \oplus_q by)^n$	$\frac{[n]_q!}{br - as} \left( \left(\frac{b}{s}\right)^{n+1} - \left(\frac{a}{r}\right)^{n+1} \right)$
$e_q(ax \oplus_q by)$	$\frac{1}{(r - a)(s - b)}, \ r > \text{Re}(a), \ s > \text{Re}(b)$
$\cos_q(ax \oplus_q by)$	$\frac{rs - ab}{(r^2 + a^2)(s^2 + b^2)}$
$\sin_q(ax \oplus_q by)$	$\frac{as + br}{(r^2 + a^2)(s^2 + b^2)}$
$\cosh_q(ax \oplus_q by)$	$\frac{rs + ab}{(r^2 - a^2)(s^2 - b^2)}$
$\sinh_q(ax \oplus_q by)$	$\frac{as - br}{(r^2 - a^2)(s^2 - b^2)}$

#### 4. DOUBLE $q$ -LAPLACE TRANSFORM OF THE SECOND KIND

The double  $q$ -Laplace transform of the second kind is defined as

$$(4.1) \quad \mathcal{L}_{2,q}^{(2)}[f(x, y)](r, s) = \int_0^{+\infty} \int_0^{+\infty} f(x, y) e_q(-rx) e_q(-sy) d_q x d_q y, \quad r, s > 0.$$

Note that if  $f(x, y) = g(x)h(y)$ , then

$$(4.2) \quad \mathcal{L}_{2,q}^{(2)}[f(x, y)](r, s) = \mathcal{L}_q\{g(x)\}(r) \mathcal{L}_q\{h(y)\}(s).$$

In particular, if  $h(y) = 1$ , or  $g(x) = 1$ , then (3.2) reads

$$(4.3) \quad \mathcal{L}_{2,q}^{(2)}[f(y)](r, s) = \mathcal{L}_q\{1\}(r)\mathcal{L}_q\{f(y)\}(s) = \frac{1}{r}\mathcal{L}_q\{f(y)\}(s)$$

and

$$(4.4) \quad \mathcal{L}_{2,q}^{(2)}[f(x)](r, s) = \mathcal{L}_q\{g(x)\}(r)\mathcal{L}_q\{1\}(s) = \frac{1}{s}\mathcal{L}_q\{g(x)\}(r).$$

**Proposition 4.1.** *For any two complex numbers  $\alpha$  and  $\beta$ , we have*

$$\mathcal{L}_{2,q}^{(2)}\{\alpha f(x, y) + \beta g(x, y)\} = \alpha\mathcal{L}_{2,q}^{(2)}\{f(x, y)\} + \beta\mathcal{L}_{2,q}^{(2)}\{g(x, y)\}.$$

*Proof.* The proof follows from (4.1). □

**Theorem 4.1.** *Let  $a$  and  $b$  be two non zero complex numbers,  $f$  a two variable function, then the following formula applies*

$$\mathcal{L}_{2,q}^{(2)}\{f(ax, by)\}(r, s) = \frac{1}{ab}\mathcal{L}_{2,q}^{(2)}\{f(x, y)\}\left(\frac{r}{a}, \frac{s}{b}\right).$$

*Proof.* Using relation (3.5), we have

$$\begin{aligned} \mathcal{L}_{2,q}^{(2)}\{f(ax, by)\}(r, s) &= \int_0^{+\infty} \int_0^{+\infty} f(ax, by)e_q(-rx)e_q(-sy)d_qx d_qy \\ &= \int_0^{+\infty} \left( \int_0^{+\infty} f(ax, by)e_q(-rx)d_qx \right) e_q(-sy)d_qy \\ &= \frac{1}{a} \int_0^{+\infty} \left( \int_0^{+\infty} f(x, by)e_q\left(-x\frac{r}{a}\right) d_qx \right) e_q(-sy)d_qy \\ &= \frac{1}{a} \int_0^{+\infty} \left( \int_0^{+\infty} f(x, by)e_q(-sy)d_qy \right) e_q\left(-x\frac{r}{a}\right) d_qx \\ &= \frac{1}{ab} \int_0^{+\infty} \left( \int_0^{+\infty} f(x, y)e_q\left(-y\frac{s}{b}\right) d_qy \right) e_q\left(-x\frac{r}{a}\right) d_qx \\ &= \frac{1}{ab} \int_0^{+\infty} \int_0^{+\infty} f(x, y)e_q\left(-x\frac{r}{a}\right) e_q\left(-y\frac{s}{b}\right) d_qx d_qy, \end{aligned}$$

and the proof of the theorem is completed. □

**Theorem 4.2.** *For  $\alpha > -1$ ,  $\beta > -1$ , we have the following*

$$\mathcal{L}_{2,q}^{(2)}\{x^\alpha y^\beta\}(r, s) = \frac{\gamma_q(\alpha + 1)}{r^{\alpha+1}} \cdot \frac{\gamma_q(\beta + 1)}{s^{\beta+1}}.$$

*In particular, for  $\alpha = n \in \mathbb{N}$  and  $\beta = m \in \mathbb{N}$ , we get*

$$\mathcal{L}_{2,q}^{(2)}\{x^n y^m\}(r, s) = \frac{[n]_q!}{q^{\binom{n+1}{2}} r^{n+1}} \cdot \frac{[m]_q!}{q^{\binom{m+1}{2}} s^{m+1}}.$$

*Proof.* The proof follows from the relation  $\mathcal{L}_q\{t^\alpha\}(s) = \frac{\gamma_q(\alpha+1)}{s^{\alpha+1}}$  (see [8]) and the obvious equation

$$\mathcal{L}_{2,q}^{(2)}\{x^\alpha y^\beta\}(r, s) = \mathcal{L}_q\{x^\alpha\}(r) \times \mathcal{L}_q\{y^\beta\}(s). \quad \square$$

**Theorem 4.3.** *Let  $a$  and  $b$  be two complex numbers, then the following relation holds*

$$\mathcal{L}_{2,q}^{(2)} \{(ax \boxplus_q by)^n\} (r, s) = \frac{q^{-\binom{n+1}{2}} [n]_q!}{br - as} \left( \left(\frac{b}{s}\right)^{n+1} - \left(\frac{a}{r}\right)^{n+1} \right).$$

*Proof.* From the definitions of the  $q$ -coaddition and the double  $q$ -Laplace transform of second kind, we have

$$\begin{aligned} \mathcal{L}_{2,q}^{(2)} \{(ax \boxplus_q by)^n\} (r, s) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} \mathcal{L}_{2,q}^{(2)} \left( (ax)^k (by)^{n-k} \right) (r, s) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} \frac{a^k [k]_q!}{q^{\binom{k+1}{2}} r^{k+1}} \cdot \frac{b^{n-k} [n-k]_q!}{q^{\binom{n-k+1}{2}} s^{n-k+1}} \\ &= \frac{q^{-\binom{n+1}{2}} [n]_q! b^n}{rs^{n+1}} \sum_{k=0}^n \left(\frac{as}{br}\right)^k \\ &= \frac{q^{-\binom{n+1}{2}} [n]_q! b^n}{rs^{n+1} (br)^n} \cdot \frac{(br)^{n+1} - (as)^{n+1}}{br - as} \\ &= \frac{q^{-\binom{n+1}{2}} [n]_q!}{br - as} \left( \left(\frac{b}{s}\right)^{n+1} - \left(\frac{a}{r}\right)^{n+1} \right). \end{aligned}$$

The theorem is then proved. □

**Theorem 4.4.** *Let  $a$  and  $b$  be two complex numbers, then the following relation holds*

$$\mathcal{L}_{2,q}^{(2)} \{E_q(ax \boxplus_q by)\} (r, s) = \frac{q^2}{(qr - a)(qs - b)}, \quad |r| > \left| \frac{a}{q} \right|, |s| > \left| \frac{b}{q} \right|.$$

*Proof.* From Theorem 4.3 and the definition of the big  $q$ -exponential function, we have

$$\begin{aligned} \mathcal{L}_{2,q}^{(2)} \{E_q(ax \boxplus_q by)\} (r, s) &= \sum_{n=0}^{+\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} \mathcal{L}_{2,q}^{(2)} \{(ax \boxplus_q by)^n\} (r, s) \\ &= \frac{1}{br - as} \left[ \frac{b}{s} \sum_{n=0}^{+\infty} \left(\frac{b}{qs}\right)^n - \frac{a}{r} \sum_{n=0}^{+\infty} \left(\frac{a}{qr}\right)^n \right] \\ &= \frac{1}{br - as} \left[ \frac{b}{s} \cdot \frac{qs}{qs - b} - \frac{a}{r} \cdot \frac{qr}{qr - a} \right] \\ &= \frac{q^2}{(qr - a)(qs - b)}. \end{aligned}$$

Note that this result can be also proved using the fact that

$$E_q(ax \boxplus_q by) = E_q(ax)E_q(by)$$

and the relation (see [8])  $\mathcal{L}_q(E_q(ax))(r) = \frac{q}{qr - a}$ . □

**Proposition 4.2.** *The following transforms hold*

$$(4.5) \quad \mathcal{L}_{2,q}^{(2)} \{ \text{Cos}_q(ax \boxplus_q by) \} (r, s) = \frac{q^2(q^2rs - ab)}{((qr)^2 + a^2)((qs)^2 + b^2)},$$

$$(4.6) \quad \mathcal{L}_{2,q}^{(2)} \{ \text{Sin}_q(ax \boxplus_q by) \} (r, s) = \frac{q^3(as + br)}{((qr)^2 + a^2)((qs)^2 + b^2)},$$

$$(4.7) \quad \mathcal{L}_{2,q}^{(2)} \{ \text{Cosh}_q(ax \boxplus_q by) \} (r, s) = \frac{q^2(q^2rs + ab)}{((qr)^2 - a^2)((qs)^2 - b^2)},$$

$$(4.8) \quad \mathcal{L}_{2,q}^{(2)} \{ \text{Sinh}_q(ax \boxplus_q by) \} (r, s) = \frac{q^3(as + br)}{((qr)^2 - a^2)((qs)^2 - b^2)}.$$

*Proof.* We have

$$\begin{aligned} \mathcal{L}_{2,q}^{(2)} \{ \text{Cos}_q(ax \boxplus_q by) \} (r, s) &= \frac{1}{2} \mathcal{L}_{2,q}^{(2)} [E_q(iax \boxplus_q iby) + E_q(-iax \boxplus_q -iby)] (r, s) \\ &= \frac{q^2}{(qr - ia)(qs - ib)} + \frac{q^2}{(qr + ia)(qs + ib)} \\ &= \frac{q^2(q^2rs - ab)}{((qr)^2 + a^2)((qs)^2 + b^2)}. \end{aligned}$$

So, (4.5) is proved. (4.6), (4.7) and (4.8) are proved in the same way.  $\square$

**Theorem 4.5.** *Let  $f$  be a one variable function that has a  $q$ -Laplace transform. Assume that  $f$  has the  $q$ -Taylor expansion*

$$f(x) = \sum_{n=0}^{+\infty} a_n q^{\binom{n}{2}} \frac{x^n}{[n]_q!},$$

*then the following relation holds*

$$(4.9) \quad \mathcal{L}_{2,q}^{(2)} [f(\alpha x \boxplus_q \beta y)] (r, s) = \frac{1}{\alpha s - \beta r} \left( \mathcal{L}_q [f(x)] \left( \frac{r}{\alpha} \right) - \mathcal{L}_q [f(x)] \left( \frac{s}{\beta} \right) \right).$$

*Proof.* Assume that  $f$  has the expansion as  $f(x) = \sum_{n=0}^{+\infty} a_n q^{\binom{n}{2}} \frac{x^n}{[n]_q!}$ . Then,

$$\begin{aligned} \mathcal{L}_{2,q}^{(2)} [f(\alpha x \boxplus_q \beta y)] (r, s) &= \sum_{n=0}^{+\infty} a_n \frac{q^{\binom{n}{2}}}{[n]_q!} \mathcal{L}_{2,q}^{(2)} \{ (\alpha x \boxplus_q \beta y)^n \} (r, s) \\ &= \sum_{n=0}^{+\infty} a_n \frac{q^{\binom{n}{2}}}{[n]_q!} \cdot \frac{q^{-\binom{n+1}{2}} [n]_q!}{\beta r - \alpha s} \left( \left( \frac{\beta}{s} \right)^{n+1} - \left( \frac{\alpha}{r} \right)^{n+1} \right) \\ &= \frac{1}{\beta r - \alpha s} \left( \frac{\beta}{s} \sum_{n=0}^{+\infty} a_n \left( \frac{\beta}{qs} \right)^n - \frac{\alpha}{r} \sum_{n=0}^{+\infty} a_n \left( \frac{\alpha}{qr} \right)^n \right) \\ &= \frac{1}{\alpha s - \beta r} \left( \mathcal{L}_q [f(x)] \left( \frac{r}{\alpha} \right) - \mathcal{L}_q [f(x)] \left( \frac{s}{\beta} \right) \right). \end{aligned}$$

So, the theorem is proved.  $\square$

**Theorem 4.6.** *The following equations hold true*

$$(4.10) \quad \mathcal{L}_{2,q}^{(2)} \left[ \frac{\partial_q f}{\partial_q x}(x, y) \right] (r, s) = r q^{-1} \mathcal{L}_{2,q}^{(2)} [f(x, y)] (r q^{-1}, s) - \mathcal{L}_q [f(0, y)] (s),$$

$$(4.11) \quad \mathcal{L}_{2,q}^{(2)} \left[ \frac{\partial_q f}{\partial_q y}(x, y) \right] (r, s) = s q^{-1} \mathcal{L}_{2,q}^{(2)} [f(x, y)] (r, s q^{-1}) - \mathcal{L}_q [f(x, 0)] (r),$$

$$(4.12) \quad \mathcal{L}_{2,q}^{(2)} \left[ \frac{\partial_q^2 f}{\partial_q x \partial_q y}(x, y) \right] (r, s) = r s q^{-2} \mathcal{L}_{2,q}^{(2)} [f(x, y)] (r q^{-1}, s q^{-1}) - r q^{-1} \mathcal{L}_q [f(x, 0)] (r q^{-1}) \\ - s q^{-1} \mathcal{L}_q [f(0, y q^{-1})] (s q^{-1}) + f(0, 0),$$

$$\mathcal{L}_{2,q}^{(2)} \left[ \frac{\partial_q^2 f}{\partial_q x^2}(x, y) \right] (r, s) = r^2 q^{-3} \mathcal{L}_{2,q}^{(2)} [f(x, y)] (r q^{-2}, s) - r q^{-1} \mathcal{L}_{2,q}^{(1)} [f(0, y)] (s) \\ - s q^{-1} \mathcal{L}_q [f(0, y)] (s q^{-1}) + f(0, 0),$$

$$\mathcal{L}_{2,q}^{(2)} \left[ \frac{\partial_q^2 f}{\partial_q y^2}(x, y) \right] (r, s) = s^2 q^{-3} \mathcal{L}_{2,q}^{(2)} [f(x, y)] (r, s q^{-2}) - r q^{-1} \mathcal{L}_{2,q}^{(1)} [f(x, 0)] (r q^{-1}) \\ - r q^{-1} \mathcal{L}_q [f(x, 0)] (r) + f(0, 0),$$

*Proof.* From (3.1), and the formula of  $q$ -integration by parts, we have

$$\mathcal{L}_{2,q}^{(2)} \left[ \frac{\partial_q f}{\partial_q x}(x, y) \right] (r, s) = \int_0^{+\infty} \int_0^{+\infty} \frac{\partial_q f}{\partial_q x}(x, y) e_q(-rx) e_q(-sy) d_q x d_q y \\ = \int_0^{+\infty} \left( \int_0^{+\infty} \frac{\partial_q f}{\partial_q x}(x, y) e_q(-rx) d_q x \right) e_q(-sy) d_q y \\ = \int_0^{+\infty} \left( -f(0, y) + r \int_0^{+\infty} f(qx, y) e_q(-rx) d_q x \right) e_q(-sy) d_q y \\ = -\mathcal{L}_q [f(0, y)] (s) + r q^{-1} \mathcal{L}_{2,q}^{(2)} [f(x, y)] (r q^{-1}, s).$$

Hence, (4.10) is proved. The proof of (4.12) uses (4.10), (4.11) and the fact that (see [16])

$$\mathcal{L}_q \left[ \frac{\partial_q f}{\partial_q x}(x, 0) \right] (r) = r q^{-1} \mathcal{L}_q [f(x, 0)] (r q^{-1}) - f(0, 0).$$

The rest of the theorem is proved in the same way. □

**Theorem 4.7** (Partial  $q$ -derivative of the double  $q$ -Laplace transform). *The following relation is valid*

$$(4.13) \quad \mathcal{L}_{2,q}^{(2)} [x^m y^n f(x, y)] (r, s) = (-1)^{m+n} \frac{\partial_q^{m+n}}{\partial_q s^n \partial_q r^m} \mathcal{L}_{2,q}^{(2)} [f(x, y)] (r, s).$$

*Proof.* We recall the relation (see [16, Theorem 3.5.])

$$\mathcal{L}_q [x^n f(x)] (s) = (-1)^n \frac{\partial_q^n}{\partial_q s^n} \mathcal{L}_q [f(x)] (s),$$

from which we have:

$$\begin{aligned}
& \mathcal{L}_{2,q}^{(2)}[x^m y^n f(x, y)](r, s) \\
&= \int_0^{+\infty} \int_0^{+\infty} x^m y^n f(x, y) e_q(-rx) e_q(-sy) d_q x d_q y \\
&= \int_0^{+\infty} y^n \left( \int_0^{+\infty} x^m f(x, y) e_q(-rx) d_q x \right) e_q(-sy) d_q y \\
&= \int_0^{+\infty} y^n \left( (-1)^m \frac{\partial_q^m}{\partial_q r^m} \int_0^{+\infty} f(x, y) e_q(-rx) d_q x \right) E_q(-s q y) d_q y \\
&= (-1)^m \frac{\partial_q^m}{\partial_q r^m} \int_0^{+\infty} \left( (-1)^n \frac{\partial_q^n}{\partial_q s^n} \int_0^{+\infty} f(x, y) e_q(-sy) d_q y \right) e_q(-rx) d_q x \\
&= (-1)^{m+n} \frac{\partial_q^{m+n}}{\partial_q s^n \partial_q r^m} \int_0^{+\infty} \int_0^{+\infty} f(x, y) e_q(-rx) e_q(-sy) d_q x d_q y \\
&= (-1)^{m+n} \frac{\partial_q^{m+n}}{\partial_q s^n \partial_q r^m} \mathcal{L}_{2,q}^{(2)}[f(x, y)](r, s).
\end{aligned}$$

This proves the theorem. □

We summarize the previous results in the following Table 2.

TABLE 2. Some Laplace of the Second Kind.

Originals	Transforms
$x^\alpha y^\beta$ ( $\alpha, \beta > -1$ )	$\frac{\gamma_q(\alpha + 1)}{r^{\alpha+1}} \cdot \frac{\gamma_q(\beta + 1)}{s^{\beta+1}}$
$(ax \boxplus_q by)^n$	$\frac{q^{-\binom{n+1}{2}} [n]_q!}{br - as} \left( \left( \frac{b}{s} \right)^{n+1} - \left( \frac{a}{r} \right)^{n+1} \right)$
$E_q(ax \boxplus_q by)$	$\frac{q^2}{(qr - a)(qs - b)},  r  > \frac{a}{q},  s  > \frac{b}{q}$
$\text{Cos}_q(ax \boxplus_q by)$	$\frac{q^2(q^2rs - ab)}{((qr)^2 + a^2)((qs)^2 + b^2)}$
$\text{Sin}_q(ax \boxplus_q by)$	$\frac{q^3(as + br)}{((qr)^2 + a^2)((qs)^2 + b^2)}$
$\text{cosh}_q(ax \boxplus_q by)$	$\frac{q^2(q^2rs + ab)}{((qr)^2 - a^2)((qs)^2 - b^2)}$
$\text{Sinh}_q(ax \boxplus_q by)$	$\frac{q^3(as + br)}{((qr)^2 - a^2)((qs)^2 - b^2)}$

## 5. SOME APPLICATIONS

### 5.1. Application to some $q$ -Functional Equations.

5.1.1. *The first  $q$ -Cauchy's functional equation.* We consider the following  $q$ -Cauchy's functional equation

$$(5.1) \quad f(x \oplus_q y) = f(x) + f(y),$$

where  $f$  is an unknown function.

We apply the double  $q$ -Laplace transform  $\mathcal{L}_{2,q}^{(1)}$  to (5.1) combined with (3.13), (3.3) and (3.4), to get

$$\frac{1}{s-r} [L_q[f(x)](r) - L_q[f(y)](s)] = \frac{1}{s} L_q[f(x)](r) + \frac{1}{r} L_q[f(y)](s),$$

that is

$$L_q[f(x)](r) \left[ \frac{1}{s-r} - \frac{1}{s} \right] = L_q[f(y)](s) \left[ \frac{1}{s-r} + \frac{1}{r} \right].$$

Simplifying this equation, we obtain

$$r^2 L_q[f(x)](r) = q^2 L_q[f(y)](s),$$

where the left hand side is a function of  $r$  alone and the right hand side is a function of  $s$  alone. This equation is true provided each side is equal to an arbitrary constant  $k$  so that

$$r^2 L_q[f(x)](r) = k \quad \text{or} \quad L_q[f(x)](r) = \frac{k}{r^2}.$$

The inverse transform gives the solution of the  $q$ -Cauchy functional equation (5.1) as  $f(x) = kx$ , where  $k$  is an arbitrary constant.

5.1.2. *The second  $q$ -Cauchy's functional equation.* We consider the following  $q$ -Cauchy's functional equation

$$(5.2) \quad f(x \boxplus_q y) = f(x) + f(y),$$

where  $f$  is an unknown function.

We apply the double  $q$ -Laplace transform  $\mathcal{L}_{2,q}^{(2)}$  to (5.2) combined with (4.9), (4.3) and (4.4), to get

$$\frac{1}{s-r} [\mathcal{L}_q[f(x)](r) - \mathcal{L}_q[f(y)](s)] = \frac{1}{s} \mathcal{L}_q[f(x)](r) + \frac{1}{r} \mathcal{L}_q[f(y)](s),$$

that is

$$\mathcal{L}_q[f(x)](r) \left[ \frac{1}{s-r} - \frac{1}{s} \right] = \mathcal{L}_q[f(y)](s) \left[ \frac{1}{s-r} + \frac{1}{r} \right].$$

Simplifying this equation, we obtain

$$r^2 \mathcal{L}_q[f(x)](r) = q^2 \mathcal{L}_q[f(y)](s),$$

where the left hand side is a function of  $r$  alone and the right hand side is a function of  $s$  alone. This equation is true provided each side is equal to an arbitrary constant  $k$  so that

$$r^2 \mathcal{L}_q[f(x)](r) = k \quad \text{or} \quad \mathcal{L}_q[f(x)](r) = \frac{k}{r^2}.$$

The inverse transform gives the solution of the  $q$ -Cauchy functional equation (5.2) as  $f(x) = kqx$ , where  $k$  is an arbitrary constant.

5.1.3. *The first  $q$ -Cauchy-Abel's functional equation.* We consider the following  $q$ -Cauchy-Abel's functional equation

$$(5.3) \quad f(x \oplus_q y) = f(x)f(y),$$

where  $f$  is an unknown function.

We apply the double  $q$ -Laplace transform  $\mathcal{L}_{2,q}^{(1)}$  to (5.3) combined with (3.13) and (3.2) to get

$$\frac{1}{s-r} [L_q[f(x)](r) - L_q[f(y)](s)] = L_q[f(x)](r)L_q[f(y)](s),$$

that is

$$\frac{1 - rL_q[f(x)](r)}{L_q[f(x)](r)} = \frac{1 - sL_q[f(y)](s)}{L_q[f(y)](s)},$$

where the left hand side is a function of  $r$  alone and the right hand side is a function of  $s$  alone. This equation is true provided each side is equal to an arbitrary constant  $k$  so that

$$\frac{1 - rL_q[f(x)](r)}{L_q[f(x)](r)} = k \quad \text{or} \quad L_q[f(x)](r) = \frac{1}{r+k}.$$

The inverse transform gives the solution of the  $q$ -Cauchy-Abel's functional equation (5.3) as  $f(x) = e_q(-kx)$ , where  $k$  is an arbitrary constant.

5.1.4. *The second  $q$ -Cauchy-Abel's functional equation.* We consider the following  $q$ -Cauchy-Abel's functional equation

$$(5.4) \quad f(x \boxplus_q y) = f(x)f(y),$$

where  $f$  is an unknown function.

We apply the double  $q$ -Laplace transform  $\mathcal{L}_{2,q}^{(2)}$  to (5.4) combined with (4.9) and (4.2) to get

$$\frac{1}{s-r} [\mathcal{L}_q[f(x)](r) - \mathcal{L}_q[f(y)](s)] = \mathcal{L}_q[f(x)](r)\mathcal{L}_q[f(y)](s),$$

that is

$$\frac{1 - r\mathcal{L}_q[f(x)](r)}{\mathcal{L}_q[f(x)](r)} = \frac{1 - s\mathcal{L}_q[f(y)](s)}{\mathcal{L}_q[f(y)](s)},$$

where the left hand side is a function of  $r$  alone and the right hand side is a function of  $s$  alone. This equation is true provided each side is equal to an arbitrary constant  $k$  so that

$$\frac{1 - r\mathcal{L}_q[f(x)](r)}{\mathcal{L}_q[f(x)](r)} = k \quad \text{or} \quad \mathcal{L}_q[f(x)](r) = \frac{1}{r+k} = \frac{q}{qr + qk}.$$

The inverse transform gives the solution of the  $q$ -Cauchy-Abel's functional equation (5.4) as  $f(x) = E_q(-qkx)$ , where  $k$  is an arbitrary constant.

5.2. Application to some partial  $q$ -differential equations.

5.2.1. *The  $q$ -transport equation.* We introduce the following  $q$ -transport equation

$$(5.5) \quad \frac{\partial_q u}{\partial_q t}(x, t) + c \frac{\partial_q u}{\partial_q x}(x, t) = 0,$$

with

$$(5.6) \quad u(x, 0) = f(x), \quad x > 0, \quad \text{and} \quad u(0, t) = g(t), \quad t > 0.$$

Applying the double  $q$ -Laplace transform  $\mathcal{L}_{2,q}^{(1)}$  to (5.5) combined with (3.14), (3.15) and (5.6), we get

$$s\mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) - L_q[f(x)](r) + c \left[ r\mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) - L_q[g(t)](s) \right] = 0,$$

that is

$$\mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) = \frac{cL_q[g(t)](s) + L_q[f(x)](r)}{s + cr}.$$

Hence,

$$u(x, t) = \left( \mathcal{L}_{2,q}^{(1)} \right)^{-1} \left[ \frac{cL_q[g(t)](s) + L_q[f(x)](r)}{s + cr} \right].$$

In particular,

- if  $u(x, 0) = f(x) = 1$  and  $u(0, t) = g(t) = 1$ , then

$$\begin{aligned} u(x, t) &= \left( \mathcal{L}_{2,q}^{(1)} \right)^{-1} \left[ \frac{cL_q[g(t)](s) + L_q[f(x)](r)}{s + cr} \right] (x, t) \\ &= \left( \mathcal{L}_{2,q}^{(1)} \right)^{-1} \left[ \frac{c/s + 1/r}{s + cr} \right] (x, t) = \left( \mathcal{L}_{2,q}^{(1)} \right)^{-1} \left[ \frac{1}{sr} \right] (x, t) = 1; \end{aligned}$$

- if  $c = -1$ ,  $u(x, 0) = f(x) = x^n$  and  $u(0, t) = g(t) = t^n$  with  $n \in \mathbb{N}$ , then

$$\begin{aligned} u(x, t) &= \left( \mathcal{L}_{2,q}^{(1)} \right)^{-1} \left[ \frac{-L_q[t^n](s) + L_q[x^n](r)}{s - r} \right] (x, t) \\ &= \left( \mathcal{L}_{2,q}^{(1)} \right)^{-1} \left[ \frac{-[n]_q! / s^{n+1} + [n]_q! / r^{n+1}}{s - r} \right] (x, t) = (x \oplus_q t)^n, \end{aligned}$$

where (3.7) has been used.

5.2.2. *The non-homogenous space-time  $q$ -telegraph equation.* We consider the non-homogenous space-time  $q$ -telegraph equation

$$(5.7) \quad c^2 \frac{\partial_q^2 u}{\partial_q x^2}(x, t) - \frac{\partial_q^2 u}{\partial_q t^2}(x, t) - (\alpha + \beta) \frac{\partial_q u}{\partial_q t}(x, t) - \alpha\beta u(x, t) = [c^2 - (\alpha + 1)(\beta + 1)]e_q(x \oplus_q t),$$

with the conditions

$$\begin{aligned} u(0, t) &= e_q(t, ) \\ u(x, 0) &= e_q(x), \end{aligned}$$

$$\begin{aligned}\frac{\partial_q u}{\partial_q x}(0, t) &= e_q(t), \\ \frac{\partial_q u}{\partial_q t}(x, 0) &= e_q(x).\end{aligned}$$

Applying  $\mathcal{L}_{2,q}^{(1)}$  to (5.7), we obtain

$$\begin{aligned}& c^2 \left\{ r^2 \mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) - r L_q[u(0, t)](s) - L_q \left[ \frac{\partial_q u}{\partial_q x}(0, t) \right](s) \right\} \\ & - \left\{ s^2 \mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) - s L_q[u(x, 0)](r) - L_q \left[ \frac{\partial_q u}{\partial_q t}(x, 0) \right](r) \right\} \\ & - (\alpha + \beta) \left\{ s \mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) - L_q[u(x, 0)](r) \right\} - \alpha \beta \mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) \\ & = [c^2 - (\alpha + 1)(\beta + 1)] \mathcal{L}_{2,q}^{(1)}[e_q(x \oplus_q t)](r, s).\end{aligned}$$

Using the conditions and simplifying the result we obtain

$$\mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) = \frac{1}{(r-1)(s-1)},$$

and hence we have  $u(x, t) = e_q(x \oplus_q t)$ .

5.2.3. *The  $q$ -wave equation.* We consider the following  $q$ -wave equation in a quarter plane

$$\frac{\partial_q^2 u}{\partial_q t^2}(x, t) - c^2 \frac{\partial_q^2 u}{\partial_q x^2}(x, t) = 0,$$

with the initial condition

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial_q u}{\partial_q t}(x, 0) = g(x), \quad x > 0,$$

$$u(0, t) = 0 \quad \text{and} \quad \frac{\partial_q u}{\partial_q x}(0, t) = 0.$$

We apply the double  $q$ -Laplace transform  $\mathcal{L}_{2,q}^{(1)}$  to have

$$\begin{aligned}& s^2 \mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) - s L_q[u(x, 0)](r) - L_q \left[ \frac{\partial_q u}{\partial_q t}(x, 0) \right](r) \\ & \times c^2 \left\{ r^2 \mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) - r L_q[u(0, t)](s) - L_q \left[ \frac{\partial_q u}{\partial_q x}(0, t) \right](s) \right\} = 0.\end{aligned}$$

That is

$$\mathcal{L}_{2,q}^{(1)}[u(x, t)](r, s) = \frac{s L_q[f(x)](r) + L_q[g(x)](r)}{s^2 - c^2 r^2}.$$

Hence,

$$u(x, t) = \left( \mathcal{L}_{2,q}^{(1)} \right)^{-1} \left[ \frac{s L_q[f(x)](r) + L_q[g(x)](r)}{s^2 - c^2 r^2} \right](x, t).$$

*Remark 5.1.* Note that in [7], another  $q$ -wave equation is given combining the  $q$ -derivative with respect to  $t$  and the classical derivative with respect to  $x$  as

$$\frac{\partial_q^2 u}{\partial_q y^2}(x, y) - \frac{\partial^2 u}{\partial x^2}(x, t) = 0.$$

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## A STUDY ON LACUNARY STATISTICAL CONVERGENCE OF MULTISET SEQUENCES

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**ABSTRACT.** Statistical convergence developed rapidly after being defined independently by Fast and Steinhaus in 1951 and was studied in many fields. One of them is lacunary statistical convergence and it was defined by Fridy and Orhan in 1993. On the other hand, although there are various studies on multisets, which are sets that can repeat elements, the convergence of multiset sequences was defined by Pachilangode in 2021. In this study, lacunary statistical convergence of multiset sequences is examined and related examples and theorems are given.

### 1. INTRODUCTION

We know that in classical set theory an element is written only once in the set. Besides, in our daily life, we see a lot of situations where it is necessary to write more than one element. Some of these situations are computer coding, element formulas, and phone numbers. In each example, there are same numbers and same molecules that play different roles. If these numbers are used once rather than multiple times, it is clear that there will be problems. Weyl explained this situation as there can be more than one white ball, more than one red ball, and more than one green ball in the same sack and he tried to apply his notion of multiset (a set with an equivalence relation) to a variety of problems in physics, chemistry, and genetics [29]. For this reason, multisets have been found interesting and studied in many disciplines such as mathematics, physics, philosophy, logic, linguistics, computer science, etc. for many years. Looking at the literature, it is seen that studies on multisets date back to the 1970s. Many researchers have studied these sets under various names such as bags,

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occurrence set, weighted set, etc. Manna and Waldinger developed an elementary theory of bags using a primitive binary insertion symbol  $\odot$  [19]. If an atom  $u$  has multiplicity  $n \geq 0$  in bag  $x$ , then  $u$  has multiplicity  $n + 1$  in bag  $u \odot x$ . Their theory BAG admits only finite collections of atoms (no hierarchy of bags) and is developed to the point of a simple algebra of bags. On the other hand, Bender [1], Lake [17], Hickman [13], Meyer [20], and Monro [22] investigated some important properties of multisets. In 1981, Knuth [16] studied computer programming and multisets. Blizard studied on multisets in his doctoral thesis [4], [3], [2]. In the 2000s, important studies were carried out on multisets. Syropoulos published "Mathematics of multisets" in 2001 [28], Singh published "An overview of the application of multiset" in 2007 [26], Majumdar published "Soft multisets" in 2012 [18], Nazmul published "On multisets and multigroups" in 2013 [23] and İbrahim published "Multigroup actions on multisets" in 2017 [14].

Now we can give some basic information about multisets. As we said, the same element in a multiset can be written multiple times and plays a different role in each write. The order in which the element is written is not important, but it is very important how many times the elements are repeated in the set. For example,  $\{2, 3, 5, 3, 4, 2, 2, 4\}$  and  $\{5, 3, 4, 4, 3, 2, 2, 2\}$  sets are the same. We write  $\{2, 3, 5, 3, 4, 2, 2, 4\}$  multiset by  $\{2, 3, 4, 5\}_{3,2,2,1}$  or  $\{2|3, 3|2, 4|2, 5|1\}$  and it means 2 appearing 3 times, 3 appearing 2 times, 4 appearing 2 times and 5 appearing 1 times. The cardinality of a multiset is the sum of the multiplicities of its elements. Despite the long-term studies on multisets, studies on multiset sequences are quite new. In 2021, Pachilangode and John defined usual convergence of multiset sequences [24] and Debnath and Debnath defined statistical convergence of multiset sequences [5].

**Definition 1.1** ([24]). Let  $X$  be a set. A sequence in which all the terms are multisets is known as a multiset sequence. For any sequence  $x = (x_i) \in X$ , a multiset sequence is defined by

$$M = \{x_i | c_i : x_i \in X, c_i \in \mathbb{N}_0\}.$$

We can give the following two examples from Pachilangode's study to better understand multiset sequences.

*Example 1.1* ([24]). Let  $N_n = \{1|1, 2|2, \dots, n|n\}$ . Then  $\{N_n\}$  is an multiset sequence and  $n^{\text{th}}$  terms has  $\frac{n(n+1)}{2}$  elements.

*Example 1.2* ([24]). The prime factorises  $n$  completely, and let  $F_n$  be the multiset of these factors, including 1. Then,  $F_1 = \{1\}$ ,  $F_2 = \{1, 2\}$ ,  $F_3 = \{1, 3\}$ ,  $F_4 = \{1, 2, 2\}$  and  $F_{36} = \{1, 2, 2, 3, 3\}$ . In this case,  $\{F_n\}$  is an multiset sequence.

In 1935, Zygmund first mentioned the idea of statistical convergence in her monograph in Warsaw but it was formally introduced by Fast [7] and Steinhaus [27], independently. Later on, Schoenberg studied statistical convergence as a summability method [25]. After this date, it is seen that statistical convergence has been studied

in many mathematical fields [6, 8, 9, 21, 30]. This concept was also studied with ideals, weak convergence, modulus functions, complex uncertain sequences [11, 12, 15].

**Definition 1.2.** Let  $A \subseteq \mathbb{N}$ ,  $A_n = \{k \in A : k \leq n\}$  and  $|A_n|$  gives the cardinality of  $A_n$ . Then,  $d(A) = \lim_{n \rightarrow +\infty} \frac{|A_n|}{n}$  is natural density of the set  $A$ .

**Definition 1.3** ([7]). A number sequence  $(x_i)$  is statistically convergent to  $L$  provided that for every  $\varepsilon > 0$ ,  $d(\{i \leq n : |x_i - L| \geq \varepsilon\}) = 0$ . In this case we write  $st\text{-}\lim x_i = L$  and usually the set of statistically convergent sequences is denoted by  $S$ .

Considering the definition of natural density, this definition can also be expressed as for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} |\{i \leq n : |x_i - L| \geq \varepsilon\}| = 0.$$

Lacunary statistical convergence was defined by Fridy and Orhan in 1993 [10]. Before giving this definition, let's remind the definition of a lacunary sequence.

**Definition 1.4.** A lacunary sequence is an increasing integer sequence  $\theta = (i_r)$  such that  $i_0 = 0$  and  $h_r = i_r - i_{r-1} \rightarrow +\infty$  as  $r \rightarrow +\infty$ . The intervals  $J_r = (i_{r-1}, i_r]$  are determined by  $\theta$  and the ratio is determined  $q_r = \frac{i_r}{i_{r-1}}$ .

*Example 1.3.*  $\theta = (r^2)$  is a lacunary sequence because  $i_0 = 0$  and  $h_r = i_r - i_{r-1} \rightarrow +\infty$  as  $r \rightarrow +\infty$ .

*Example 1.4.*  $\theta = (r)$  is not a lacunary sequence because  $i_0 = 0$  but  $h_r = i_r - i_{r-1} = 1$  for all  $r = 0, 1, \dots$

**Definition 1.5** ([10]). Let  $\theta = (i_r)$  be a lacunary sequence. The number sequence  $x = (x_i)$  is lacunary statistically convergent (or  $S_\theta$ -convergent) to  $L$  if for every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow +\infty} \frac{1}{h_r} |\{i \in J_r : |x_i - L| \geq \varepsilon\}| = 0.$$

In this case we write  $S_\theta - \lim x_i = L$  and usually the set of lacunary statistically convergent sequences is denoted by  $S_\theta$ .

Another concept closely related to statistical convergence is strong Cesàro summability:

$$|C_1| := \left\{ x : \text{for some } L, \lim_{n \rightarrow +\infty} \left( \frac{1}{n} \sum_{i=1}^n |x_i - L| \right) = 0 \right\}.$$

Similarly, there is a close relationship between strong Cesàro summability and  $N_\theta$  space:

$$N_\theta := \left\{ x : \text{for some } L, \lim_{r \rightarrow +\infty} \left( \frac{1}{h_r} \sum_{i \in J_r} |x_i - L| \right) = 0 \right\}.$$

After all this information, we can give definitions about the usual convergence and statistical convergence of multiset sequences. Throughout the paper, we study with multiset sequences of real numbers.

**Definition 1.6** ([5]). Let  $\mathbb{N}_0$  is the set of non-negative integers. The set

$$m\mathbb{R} = \{M = mx = x_i|c_i : x_i \in \mathbb{R} \text{ and } c_i \in \mathbb{N}_0\},$$

is called multiset of real numbers.

Due to repetitive elements of the multisets, it is necessary to define a new metric in order to work on multisets. Let  $(X, d)$  be a metric space and  $M$  be a multiset in this space. The  $d$  metric is not very functional on  $M$  because of the repetitive elements of  $M$ . Hence, if a new  $d_M$  metric is defined on  $M$ , then  $(M, d_M)$  is a metric space. In this study, it is defined as

$$d_M(mx, my) = d_M(x_i|c_i, y_i|t_i) = \sqrt{(x_i - y_i)^2 + (c_i - t_i)^2},$$

where  $d_M : M \times M \rightarrow \mathbb{R}$  for each  $i \in \mathbb{N}$ . It is easily seen that  $d_M$  satisfies the metric conditions with Minkowsky inequality.

**Definition 1.7.** A multiset sequence  $mx = (x_i|c_i)$  of  $m\mathbb{R}$  is convergent to  $l|c$  if given the set

$$\lim_{i \rightarrow +\infty} d_M(x_i|c_i, l|c) = \lim_{i \rightarrow +\infty} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0.$$

In this case,  $x_i \rightarrow l$  and  $c_i \rightarrow c$ , i.e., for each  $\varepsilon > 0$ ,  $|x_i - l| < \varepsilon$  and  $|c_i - c| < \varepsilon$ .

**Definition 1.8.** ([5]) Let  $x = (x_i)$  be a real sequence and  $c = (c_i)$  be a sequence of  $\mathbb{N}_0$ . A multiset sequence  $mx = (x_i|c_i)$  of  $m\mathbb{R}$  is statistically convergent to  $l|c$  of  $m\mathbb{R}$  if given for any  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left| \{i \leq n : d_M(x_i|c_i, l|c) \geq \varepsilon\} \right| = \lim_{n \rightarrow +\infty} \frac{1}{n} \left| \left\{ i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| = 0.$$

The set of all statistically convergent multiset sequences is denoted by  $S^{l/c}$  and is written by  $mx \rightarrow l|c(S)$ .

*Example 1.5* ([5]). Consider a multisequence  $mx = (x_i|c_i)$ , given by

$$x_i = \begin{cases} i, & i = n^2; n = 1, 2, 3, \dots, \\ 1, & \text{otherwise,} \end{cases} \quad \text{and} \quad c_i = \begin{cases} i, & i = n^3; n = 1, 2, 3, \dots, \\ 5, & \text{otherwise.} \end{cases}$$

Then for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{1}{n} \left| \left\{ i \leq n : \sqrt{(x_i - 1)^2 + (c_i - 5)^2} \geq \varepsilon \right\} \right| \\ & \leq \lim_{n \rightarrow +\infty} \frac{1}{n} \left( n^{\frac{1}{2}} + n^{\frac{1}{3}} - n^{\frac{1}{6}} \right) = 0. \end{aligned}$$

Therefore, the multisequence  $mx$  statistically converges to  $1|5$ .

2. MAIN RESULTS

We aim to define lacunary statistical convergence based on Debnath’s study on statistical convergence for multiset sequences. For this purpose, we need following definitions.

**Definition 2.1.** Let  $\theta = (i_r)$  be a lacunary sequence and  $mx = (x_i|c_i)$  be a multiset sequence of  $m\mathbb{R}$ .  $mx$  is said to be lacunary statistically convergent to  $l/c$ , if for each  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow +\infty} \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| = 0.$$

In this case, we write  $mx \rightarrow l/c(S_\theta)$ . The set of all lacunary statistically convergent multiset sequences is symbolized as  $S_\theta^{l/c}$ .

**Definition 2.2.** Let  $mx = (x_i|c_i)$  be a multiset sequence of  $m\mathbb{R}$ .  $mx$  is said to be statistically Cesàro summable to  $l/c$  if for each  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0.$$

In this case, we write  $mx \rightarrow l/c(\sigma)$ . The set of all statistically Cesàro summable multiset sequences is symbolized as  $\sigma^{l/c}$ .

**Definition 2.3.** Let  $\theta = (i_r)$  be a lacunary sequence and  $mx = (x_i|c_i)$  be a multiset sequence of  $m\mathbb{R}$ .  $mx$  is said to be lacunary strongly summable to  $l/c$  if for each  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow +\infty} \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0.$$

In this case, we write  $mx \rightarrow l/c(N_\theta)$ . The set of all lacunary strongly summable multiset sequences is symbolized as  $N_\theta^{l/c}$ .

Boundedness plays an important role in our results and proofs of theorems. So let’s give the definition of bounded multiset sequence.

**Definition 2.4** ([5]). A multiset sequence  $mx = (x_i|c_i)$  is said to be bounded provided that there exists a non-negative real number  $B$  such that  $\sqrt{x_i^2 + (c_i - 1)^2} \leq B$ .

**Theorem 2.1.** Let  $\theta = (i_r)$  be a lacunary sequence.

- i*) For any multiset sequence  $mx = (x_i|c_i)$ ,  $mx \in N_\theta^{l/c}$  implies  $mx \in S_\theta^{l/c}$ .
- ii*) If  $mx = (x_i|c_i)$  is a bounded multiset sequence then,  $mx \in S_\theta^{l/c}$  implies  $mx \in N_\theta^{l/c}$ .

*Proof.* *i*) Let  $mx \in N_\theta^{l/c}$  and  $\varepsilon > 0$  be given. Then,

$$\sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \sum_{\substack{i \in J_r \\ \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon}} \sqrt{(x_i - l)^2 + (c_i - c)^2}$$

$$\geq \varepsilon \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|,$$

and so,

$$\frac{1}{\varepsilon h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|.$$

If we take the limit of both sides

$$\lim_{r \rightarrow +\infty} \frac{1}{\varepsilon h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \lim_{r \rightarrow +\infty} \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|.$$

Then, we have the proof.

ii) This part shows in which case the inverse of *i*) is valid. Assume that  $mx \in S_\theta^{l/c}$  and  $mx$  be bounded. Then, there exists a non-negative real number  $B$  such that  $\sqrt{x_i^2 + (c_i - 1)^2} \leq B$  for all  $i \in \mathbb{N}$ . At the same time, from the fact that  $c, c_i \in \mathbb{N}_0$  and  $x_i \rightarrow l$ , we have,

$$\sqrt{(x_i - l)^2 + (c_i - c)^2} \leq \sqrt{x_i^2 + (c_i - 1)^2} \leq B.$$

Hence,

$$\begin{aligned} \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} &= \frac{1}{h_r} \sum_{\substack{i \in J_r \\ \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2}}} \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &\quad + \frac{1}{h_r} \sum_{\substack{i \in J_r \\ \sqrt{(x_i - l)^2 + (c_i - c)^2} < \frac{\varepsilon}{2}}} \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &\leq \frac{B}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2} \right\} \right| + \frac{\varepsilon}{2}. \end{aligned}$$

In that case

$$\lim_{r \rightarrow +\infty} \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \leq \lim_{r \rightarrow +\infty} \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2} \right\} \right|.$$

Since the limit of the right side is zero, the left side is also zero. So, the proof is completed. □

Now, let us investigate the relation between the spaces  $S^{l/c}$  and  $S_\theta^{l/c}$  with the following theorem.

**Theorem 2.2.** *Let  $\theta = (i_r)$  be a lacunary sequence.*

- i) If  $\liminf q_r > 1$ , then  $mx \in S^{l/c}$  implies  $x \in S_\theta^{l/c}$ .*
- ii) If  $\limsup_r q_r < +\infty$ , then  $mx \in S_\theta^{l/c}$  implies  $mx \in S^{l/c}$ .*

*Proof.* *i)* Assume that  $\liminf q_r > 1$ . In this case we know that for sufficiently large  $r$  there exists  $\lambda > 0$  such that  $q_r \geq 1 + \lambda$ . This implies that,

$$\frac{h_r}{i_r} \geq \frac{\lambda}{1 + \lambda}.$$

Since  $mx \in S^{l/c}$ , for each  $\varepsilon > 0$  and sufficiently large  $r$  from the definition of  $J_r$  we have,

$$\begin{aligned} & \frac{1}{i_r} \left| \left\{ i \leq i_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| \\ & \geq \frac{1}{i_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| \\ & \geq \frac{\lambda}{1 + \lambda} \cdot \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|. \end{aligned}$$

Hence, from the limit of both sides,

$$\begin{aligned} & \lim_r \frac{1}{h_r} \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| \\ & \leq \left( \frac{1 + \lambda}{\lambda} \right) \lim_{r \rightarrow +\infty} \frac{1}{i_r} \left| \left\{ i \leq i_r : \left| \sqrt{(x_i - l)^2 + (c_i - c)^2} \right| \geq \varepsilon \right\} \right| = 0. \end{aligned}$$

Then, we have the proof.

*ii)* Now, suppose that  $\limsup_r q_r < +\infty$  then, there is a  $K > 0$  such that  $q_r < K$  for all  $r$ . Let  $mx \in S_\theta^{l/c}$ . In order to facilitate transactions define the set for  $\varepsilon > 0$ ,

$$N_r := \left| \left\{ i \in J_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|.$$

From the definition of limit in  $S_\theta^{l/c}$ , for  $\varepsilon > 0$  there is an  $r_0 \in \mathbb{N}$  such that

$$\frac{N_r}{h_r} < \varepsilon, \quad \text{for each } r > r_0.$$

Now, let  $C := \max \{N_r : 1 \leq r \leq r_0\}$  and let  $i_{r-1} < n \leq i_r$ . Then,

$$\begin{aligned} & \frac{1}{n} \left| \left\{ i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{i_{r-1}} \left| \left\{ i \leq i_r : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right| \\ & = \frac{1}{i_{r-1}} \{N_1 + N_2 + \dots + N_{r_0}\} + \frac{i_{r_0+1} - i_{r_0}}{i_{r-1}} \cdot \frac{1}{h_{r_0+1}} N_{r_0+1} + \dots + \frac{i_r - i_{r-1}}{i_{r-1}} \cdot \frac{1}{h_r} N_r \\ & < \frac{C}{i_{r-1}} r_0 + \varepsilon \left\{ \frac{i_{r_0+1} - i_{r_0}}{i_{r-1}} + \dots + \frac{i_r - i_{r-1}}{i_{r-1}} \right\} \\ & = \frac{C}{i_{r-1}} r_0 + \varepsilon \frac{i_r - i_{r_0}}{i_{r-1}} = \frac{C}{i_{r-1}} r_0 + \varepsilon \left\{ \frac{i_r}{i_{r-1}} - \frac{i_{r_0}}{i_{r-1}} \right\} \end{aligned}$$

$$\left\langle \frac{C}{i_{r-1}} r_0 + \varepsilon q_r < \frac{C}{i_{r-1}} r_0 + \varepsilon K.$$

From this result we have the proof. □

The following theorem gives the relations between  $\sigma^{l/c}$  and  $N_\theta^{l/c}$ .

**Theorem 2.3.** *Let  $\theta = (i_r)$  be a lacunary sequence.*

- i) If  $\theta$  satisfies  $\liminf q_r > 1$ , then  $m x \in \sigma^{l/c}$  implies  $m x \in N_\theta^{l/c}$ .*
- ii) If  $\theta$  satisfies  $\limsup q_r < +\infty$ , then  $m x \in N_\theta^{l/c}$  implies  $m x \in \sigma^{l/c}$ .*

*Proof.* *i)* Assume that  $\liminf_r q_r > 1$ . Then, there exists  $\lambda > 0$  such that  $q_r \geq 1 + \lambda$  for sufficiently large  $r$ . Since  $h_r = i_r - i_{r-1}$ , we have  $\frac{i_r}{i_{r-1}} \geq 1 + \lambda$  for sufficiently large  $r$  which implies that  $\frac{h_r}{i_r} \geq \frac{\lambda}{1+\lambda}$

$$\begin{aligned} \frac{1}{i_r} \sum_{i=1}^{i_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} &\geq \frac{1}{i_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &= \left(\frac{h_r}{i_r}\right) \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &\geq \left(\frac{\lambda}{1 + \lambda}\right) \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2}. \end{aligned}$$

We know that from assumption  $\lim_{i_r \rightarrow +\infty} \frac{1}{i_r} \sum_{i=1}^{i_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0$ . Therefore,  $\lim_{r \rightarrow +\infty} \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0$  which implies  $m x \in N_\theta^{l/c}$ .

*ii)* If  $\limsup q_r < +\infty$ , then there exists  $K > 0$  such that  $q_r < K$  for all  $r \geq 1$ . Let  $m x \in N_\theta^{l/c}$  and denote  $L_r = \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2}$ . By the definition of convergence when  $r > r_0$  for every  $\varepsilon > 0$ ,

$$\frac{L_r}{h_r} = \frac{1}{h_r} \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon.$$

Choose  $D := \max \{L_r : 1 \leq r \leq r_0\}$  and  $i_{r-1} < n < i_r$ .

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &\leq \frac{1}{i_{r-1}} \sum_{i=1}^{i_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &= \frac{1}{i_{r-1}} \left( \sum_{i \in J_1} \sqrt{(x_i - l)^2 + (c_i - c)^2} + \sum_{i \in J_2} \sqrt{(x_i - l)^2 + (c_i - c)^2} \right. \\ &\quad \left. + \dots + \sum_{i \in J_r} \sqrt{(x_i - l)^2 + (c_i - c)^2} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{i_{r-1}} \{L_1 + L_2 + \dots + L_{r_0}\} + \frac{i_{r_0+1} - i_{r_0}}{i_{r-1}} \cdot \frac{1}{h_{r_0+1}} L_{r_0+1} + \dots + \frac{i_r - i_{r-1}}{i_{r-1}} \cdot \frac{1}{h_r} L_r \\
 &< \frac{D}{i_{r-1}} r_0 + \varepsilon \left\{ \frac{i_{r_0+1} - i_{r_0}}{i_{r-1}} + \dots + \frac{i_r - i_{r-1}}{i_{r-1}} \right\} \\
 &= \frac{D}{i_{r-1}} r_0 + \varepsilon \frac{i_r - i_{r_0}}{i_{r-1}} = \frac{D}{i_{r-1}} r_0 + \left\{ \frac{i_r}{i_{r-1}} - \frac{i_{r_0}}{i_{r-1}} \right\} \\
 &< \frac{D}{i_{r-1}} r_0 + \varepsilon q_r < \frac{D}{i_{r-1}} r_0 + \varepsilon K.
 \end{aligned}$$

This completes the proof of the theorem. □

**Theorem 2.4.** *i) For any multiset sequence  $mx = (x_i/c_i)$ ,  $mx \in \sigma^{l/c}$  implies  $mx \in S^{l/c}$ .*

*ii) If  $mx$  is a bounded multiset sequence, then  $mx \in S^{l/c}$  implies  $mx \in \sigma^{l/c}$ .*

*Proof.* *i)* We can prove this theorem in a similar way to the proof of Theorem 2.1. Let  $mx \in \sigma^{l/c}$  and  $\varepsilon > 0$  be given. Then,

$$\begin{aligned}
 \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} &\geq \sum_{\substack{i=1 \\ \sqrt{(x_i-l)^2+(c_i-c)^2} \geq \varepsilon}}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} \\
 &\geq \varepsilon \left| \left\{ i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|,
 \end{aligned}$$

and so,

$$\frac{1}{\varepsilon n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{1}{n} \left| \left\{ i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|.$$

If we take the limit of both sides,

$$\begin{aligned}
 &\lim_{n \rightarrow +\infty} \frac{1}{\varepsilon n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} \\
 &\geq \lim_{n \rightarrow +\infty} \frac{1}{n} \left| \left\{ i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \right|.
 \end{aligned}$$

Then, we have the proof.

*ii)* Now suppose that  $mx \in S^{l/c}$  and  $mx$  are bounded. Then, there exists a non-negative real number  $B$  such that  $\sqrt{x_i^2 + (c_i - 1)^2} \leq B$  for all  $i \in \mathbb{N}$ . We also know that,

$$\sqrt{(x_i - l)^2 + (c_i - c)^2} \leq \sqrt{x_i^2 + (c_i - 1)^2} \leq B.$$

Hence,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} &= \frac{1}{n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &\quad \left| \sqrt{(x_i - l)^2 + (c_i - c)^2} \right| \geq \frac{\varepsilon}{2} \\ &\quad + \frac{1}{n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &\quad \left| \sqrt{(x_i - l)^2 + (c_i - c)^2} \right| < \frac{\varepsilon}{2} \\ &\leq \frac{B}{n} \left| \left\{ i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2} \right\} \right| + \frac{\varepsilon}{2}, \end{aligned}$$

is obtained. In that case

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n \sqrt{(x_i - l)^2 + (c_i - c)^2} \\ &\leq \lim_{n \rightarrow +\infty} \frac{1}{n} \left| \left\{ i \leq n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \frac{\varepsilon}{2} \right\} \right| = 0. \end{aligned}$$

So, the proof is completed.  $\square$

### 3. CONCLUSIONS

In our daily life, we often encounter situations where an element of a set must be written more than once in the set. Some of these situations are computer coding, element formulas, and phone numbers. These sets are multisets and for this reason, multisets have been found interesting and studied in many disciplines such as mathematics, physics, philosophy, logic, linguistics, computer science, etc. for many years. In this situation, multiset sequences are also interesting and the studies on this subject are quite new. For this purpose, in this paper we introduce the lacunary statistical convergence of multiset sequences and we investigate some important relations.

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**FIXED POINT THEOREMS AND CONTINUITY  
CHARACTERIZATION FOR LINEAR MAPS IN COLOMBEAU  
ALGEBRAS**

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**ABSTRACT.** In this article, we present a novel characterization of the continuity of linear maps within Colombeau algebras. Additionally, we introduce an alternative representation for the contraction of these maps. Moreover, we put forth a new concept of fixed-point theorems in Colombeau algebra, extending classical fixed-point theorems, including those of Banach, Chatterjea, and Kannan. To underscore the practical relevance of our findings, we offer various examples and applications.

1. INTRODUCTION

The fixed point theorems are regarded as an effective tool for solving differential equations, for example, see [21–26]. In the literature, the embedding of fixed point theory in the framework of Colombeau algebra is based on the famous theorem of J. A. Marti see [16]. The idea of J. A. Marti is based on the Banach fixed point theorem in classical metric spaces, with several assumptions to make sense of the contraction of mappings. Our new idea is based on this result of J. A. Marti, but we will lighten the assumptions. Indeed: We have defined a new contraction in which we don't need all these assumptions. Our contraction is intended for mappings defined between the Colombeau algebras  $(\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$ , where  $(\mathcal{P}_i)_{i \in I}$ ,  $\mathcal{P}_i(u) = e^{-v_{p_i}(u_\epsilon)}$  and  $v_{p_i}$  is the valuation function associated with the ultra pseudo-seminorms family which makes  $E$  a locally convex space. We have also extended the two theorems of Chattergia and Kannan in the framework of Colombeau algebras. On the other hand,

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our paper contains a very legitimate notion of contraction, which will allow us to cite some fundamental theorems of fixed point in the algebra of generalized functions of Colombeau. We have also introduced a characterization of the continuity notion of  $\tilde{\mathcal{C}}$ -linear maps between Colombeau algebras, based on the sharp topology defined on Colombeau algebras seen as modules on  $\tilde{\mathcal{C}}$ . In this article, we have proved three well-known fixed point theorems in the classical framework, namely Banach's fixed point theorem, Kannan's fixed point theorem, and that of Chatterjea (see [2, 14]). Fixed point theorems are very useful to know if an equation has a solution, finding a solution to a differential equation can then be interpreted as finding a map and proving that it has a fixed point, which will be the solution to the problem.

The present work is formulated in the framework of Colombeau generalized functions. This will allow us to use the powerful tools from this theory to combine generalized function data with nonlinearities and measure regularity. We give some properties in the theory of topological  $\tilde{\mathcal{C}}$ -modules and locally convex topological  $\tilde{\mathcal{C}}$ -modules, and illustrated this with application to an evolution problem. In [8] the authors, shows that in one space dimension, an initial singularity at the origin propagates along the characteristic lines emanating from the origin, as in the linear case. The proof is based on a fixed point theorem in a suitable ultrametric topology on the subset of Colombeau solutions possessing the required regularity. J. A. Marti in [16] and several other authors have found together a lot of results on the existence and uniqueness of generalized fixed point by a method which consists of transforming the problem given in the sense of Colombeau to its equivalence in classic, after they showed that its solutions are moderate and therefore deduce that the generalized solution exists. But our point of view is to use a definition of contraction in the sense of generalized functions to mount existence and uniqueness of fixed point in this frame of Colombeau generalized functions. Starting from the notions of  $\tilde{\mathcal{C}}$ -linear and locally convex  $\tilde{\mathcal{C}}$ -linear topologies which are introduced and described from their neighborhoods, a characterization of the continuity of  $\tilde{\mathcal{C}}$ -linear maps from a locally convex topological  $\tilde{\mathcal{C}}$ -modules into another  $\tilde{\mathcal{C}}$ -modules is given in this work. We are inspired by the analogous statement involving seminormes and locally convex vector spaces, making use of the concept of ultra pseudo-seminorm, for example, the continuity of  $\tilde{\mathcal{C}}$ -modules is studied in this paper. Our main result is to give a necessary and sufficient condition in order to a linear map from a Colombeau algebra to another be continuous, and we focus on proving a new theorems of fixed point in this context.

The present paper is organized as follows. After this introduction, we will recall some basic properties concerning Colombeaus algebra in Section 2. The notion of  $\tilde{\mathcal{C}}$ -modules topology and some properties are presents in Section 3. In Section 4, we are talking about continuity, contraction and fixed point in  $\tilde{\mathcal{C}}$ -modules. Finally, Section 5 is devoted to the existence-uniqueness result of a differential equation.

2. PRELIMINARIES

Before describing our results in more detail, a few words about Colombeau algebras are in order. The elements of Colombeau algebra  $\mathcal{G}$  are equivalence classes of nets of smooth functions satisfying asymptotic conditions in the regularization parameter  $\epsilon$ , for more details [3–6, 12, 13, 17]. We define the set  $\mathcal{E}(\mathbb{R}^n) = (\mathcal{C}^\infty(\mathbb{R}^n))^{(0,1]}$ . The set of moderate functions is given as follows

$$\mathcal{E}_M(\mathbb{R}^n) = \left\{ (\sigma_\epsilon)_\epsilon \in \mathcal{E}(\mathbb{R}^n) \mid \text{for all } K \subset\subset \mathbb{R}^n, \alpha \in \mathbb{N}, \right. \\ \left. \text{there exists } N \in \mathbb{N} : \sup_{x \in K} |D^\alpha \sigma_\epsilon(x)| = O_{\epsilon \rightarrow 0}(\epsilon^{-N}) \right\},$$

where  $D^\alpha$  is the differential operator of order  $\alpha$ . The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \left\{ (\sigma_\epsilon)_\epsilon \in \mathcal{E}(\mathbb{R}^n) \mid \text{for all } K \subset\subset \mathbb{R}^n, \alpha \in \mathbb{N}, \right. \\ \left. \text{for all } q \in \mathbb{N} : \sup_{x \in K} |D^\alpha \sigma_\epsilon(x)| = O_{\epsilon \rightarrow 0}(\epsilon^q) \right\}.$$

The Colombeau algebra is defined as a factor algebra

$$(2.1) \quad \mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n) / \mathcal{N}(\mathbb{R}^n).$$

We denote  $\mathbb{C}$  the field of complex numbers. Let  $\tilde{\mathbb{C}}$  be the ring of complex generalized numbers obtained by factorizing,

$$(2.2) \quad \mathcal{E}_{\mathbb{C}} = \left\{ (r_\epsilon)_\epsilon \in \mathbb{C}^{(0,1]} \mid \text{there exists } N \in \mathbb{N}, |r_\epsilon| = O_{\epsilon \rightarrow 0}(\epsilon^{-N}) \right\},$$

with respect to the ideal

$$(2.3) \quad \mathcal{N}_{\mathbb{C}} = \left\{ (r_\epsilon)_\epsilon \in \mathbb{C}^{(0,1]} \mid \text{for all } q \in \mathbb{N}, |r_\epsilon| = O_{\epsilon \rightarrow 0}(\epsilon^q) \right\}.$$

Also, the algebra of generalized complex numbers is defined as follows

$$(2.4) \quad \tilde{\mathbb{C}} = \mathcal{E}_{\mathbb{C}} / \mathcal{N}_{\mathbb{C}}.$$

3. DEFINITION AND BASIC PROPERTIES OF  $\tilde{\mathbb{C}}$ -MODULE  $\mathcal{G}$

It is clear that  $\tilde{\mathbb{C}}$  is trivially a module over itself and it can be endowed with a structure of a topological ring (see [10, 11]). In the sequel, we need the following function, which is inspired by non-standard analysis [17, 20] and the previous work in this field [1, 18, 19] we define the following function

$$(3.1) \quad v : \begin{cases} \mathcal{E}_M(\mathbb{R}^n) \rightarrow (-\infty, +\infty], \\ (\sigma_\epsilon)_\epsilon \rightarrow \sup \{ l \in \mathbb{R} \mid |\sigma_\epsilon| = O_{\epsilon \rightarrow 0}(\epsilon^l) \}. \end{cases}$$

It satisfies the following conditions:

- (i)  $v((\sigma_\epsilon)_\epsilon) = +\infty$  if and only if  $(\sigma_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^n)$ ;
- (ii)  $v((\sigma_\epsilon)_\epsilon(\varrho_\epsilon)_\epsilon) \geq v((\sigma_\epsilon)_\epsilon) + v((\varrho_\epsilon)_\epsilon)$ , for all  $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{E}_M(\mathbb{R}^n)$ ;

- (ii')  $v((\sigma_\epsilon)_\epsilon(\varrho_\epsilon)_\epsilon) = v((\sigma_\epsilon)_\epsilon) + v((\varrho_\epsilon)_\epsilon)$ , for all  $\varrho_\epsilon = c\epsilon^b$ ,  $c \in \mathbb{C}$ ,  $b \in \mathbb{R}$ ;  
 (iii)  $v((\sigma_\epsilon)_\epsilon + (\varrho_\epsilon)_\epsilon) \geq \min \{v((\sigma_\epsilon)_\epsilon), v((\varrho_\epsilon)_\epsilon)\}$ , for all  $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{E}_M(\mathbb{R}^n)$ .

Note that if  $(\sigma_\epsilon - \sigma'_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^n)$ , (i) combined with (iii) yields  $v((\sigma_\epsilon)_\epsilon) = v((\sigma'_\epsilon)_\epsilon)$ . This means that we can use the function  $v$  to define the valuation

$$(3.2) \quad v_{\tilde{\mathbb{C}}}(\sigma) = v((\sigma_\epsilon)_\epsilon),$$

of a complex generalized number  $\sigma = [(\sigma_\epsilon)_\epsilon]$ , and that all the previous properties hold for elements of  $\tilde{\mathbb{C}}$ . Now let us consider the following map

$$(3.3) \quad |\cdot|_e : \begin{cases} \tilde{\mathbb{C}} \rightarrow [0, +\infty), \\ \sigma \rightarrow |\sigma|_e = e^{-v_{\tilde{\mathbb{C}}}(\sigma)}. \end{cases}$$

**Definition 3.1.** Let  $E$  be a locally convex topological vector space equipped through the family of seminorms  $\{p_i\}_{i \in I}$ . The elements of

$$(3.4) \quad \mathcal{M}_E = \left\{ (\sigma_\epsilon)_\epsilon \in E^{(0,1]} \mid \text{for all } i \in I, \text{ there exists } N \in \mathbb{N}, p_i(\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^{-N}) \right\}$$

and

$$(3.5) \quad \mathcal{N}_E = \left\{ (\sigma_\epsilon)_\epsilon \in E^{(0,1]} \mid \text{for all } i \in I, q \in \mathbb{N}, p_i(\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^q) \right\},$$

are called respectively  $\mathcal{M}_E$ -moderate space and  $\mathcal{N}_E$ -negligible space, respectively.

We introduce the space of generalized functions based on  $E$  as the factor space  $\mathcal{G}_E = \mathcal{E}_E / \mathcal{N}_E$

It is clear that the definition of  $\mathcal{G}_E$  does not depend on the family of seminorms which determines the locally convex topology of  $E$ . We adopte the notation  $\sigma = [(\sigma_\epsilon)_\epsilon]$  for the class  $\sigma$  of  $(\sigma_\epsilon)_\epsilon$  in  $\mathcal{G}_E$ , and  $\mathcal{C}^\infty(\mathbb{R})$  embedded into this algebra via the constant embedding  $f \mapsto [(f)_\epsilon]$ . By the properties of seminorms on  $E$  we may define the product between complex generalized numbers and elements of  $\mathcal{G}_E$  via the map  $\tilde{\mathbb{C}} \times \mathcal{G}_E \rightarrow \mathcal{G}_E$ . It is natural to introduce the  $p_i$ -valuation of  $(\sigma_\epsilon)_\epsilon \in \mathcal{M}_E$  as

$$(3.6) \quad v_{p_i}((\sigma_\epsilon)_\epsilon) = \sup \left\{ b \in \mathbb{R} \mid \text{for all } i \in I, p_i(\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^b) \right\}.$$

Note that  $v_{p_i}((\sigma_\epsilon)_\epsilon) = v((p_i(\sigma_\epsilon))_\epsilon)$ , (where the function  $v$ ) gives the valuation on  $\mathcal{M}_E$ . Clearly  $v_{p_i}$  maps  $\mathcal{M}_E$  into  $(-\infty, +\infty)$  and the following properties hold:

- (i)  $v_{p_i}((\sigma_\epsilon)_\epsilon) = +\infty$  for all  $i \in I$  if and only if  $((\sigma_\epsilon)_\epsilon) \in \mathcal{N}_E$ ;  
 (ii)  $v_{p_i}((\varrho_\epsilon \sigma_\epsilon)_\epsilon) \geq v_{p_i}((\varrho_\epsilon)_\epsilon) + v_{p_i}((\sigma_\epsilon)_\epsilon)$ , for all  $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$ ;  
 (iii)  $v_{p_i}((\lambda_\epsilon \sigma_\epsilon)_\epsilon) = v_{p_i}((\lambda_\epsilon)_\epsilon) + v_{p_i}((\sigma_\epsilon)_\epsilon)$  for all  $\lambda_\epsilon = c\epsilon^b$ ,  $c \in \mathbb{C}$ ,  $b \in \mathbb{R}$ ;  
 (iv)  $v_{p_i}((\sigma_\epsilon)_\epsilon + (\varrho_\epsilon)_\epsilon) \geq \min \{v_{p_i}((\sigma_\epsilon)_\epsilon), v_{p_i}((\varrho_\epsilon)_\epsilon)\}$ , for all  $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$ .

Last assertion (i) combined with (iv) shows that

$$v_{p_i}((\sigma_\epsilon)_\epsilon) = v_{p_i}((\sigma'_\epsilon)_\epsilon), \quad \text{if } (\sigma_\epsilon - \sigma'_\epsilon)_\epsilon \text{ is } \mathcal{N}_E\text{-negligible.}$$

This means that we can use (3.6) for defining the  $p_i$ -valuation of a generalized function  $\sigma = [(\sigma_\epsilon)_\epsilon] \in \mathcal{G}_E$  by

$$v_{p_i}(\sigma) = v_{p_i}((\sigma_\epsilon)_\epsilon).$$

And thus  $\mathcal{P}_i(\sigma) = e^{-v_{p_i}(\sigma)}$  is an ultra pseudo-seminorm on the  $\tilde{\mathbb{C}}$ -module  $\mathcal{G}_E$ . By [10, Theorem 1.10]  $\mathcal{G}_E$  endowed with the topology of ultra pseudo-seminorms  $(\mathcal{P}_i)_{i \in I}$  is a locally convex topological  $\tilde{\mathbb{C}}$ -module. The topology induced by the ultra pseudo-seminorms  $(\mathcal{P}_i)_{i \in I}$  called the sharp topology on  $\mathcal{G}_E$ . A basis of 0-neighbourhood is the set of all balls

$$\mathcal{B}(i, \gamma) = \{ \sigma \in \mathcal{G}_E \mid \mathcal{P}_i(\sigma) < \gamma \}, \quad i \in I \text{ and } \gamma > 0.$$

#### 4. MAIN RESULTS

This section is devoted to the important results of this paper. Let's start with the subsection of continuity.

##### 4.1. Continuity and contraction in $\tilde{\mathbb{C}}$ -modules.

First, we are looking if it is possible to define a  $\tilde{\mathbb{C}}$ -linear map  $\mathcal{S} : \mathcal{G}_E \rightarrow \mathcal{G}_F$  by means of a given family  $(\mathcal{S}_\epsilon)_{\epsilon \in (0,1]}$  of  $\mathbb{C}$ -linear maps  $\mathcal{S}_\epsilon : E \rightarrow F$ , where  $E$  and  $F$  are locally convex topological vector spaces. The general requirement is given in the following.

**Lemma 4.1** ([16]). *Let  $(\mathcal{S}_\epsilon)_{\epsilon \in (0,1]}$  be a given family of  $\mathbb{C}$ -linear maps  $\mathcal{S}_\epsilon : E \rightarrow F$ . Suppose that*

- 1.  $(\sigma_\epsilon)_\epsilon \in \mathcal{M}_E$  implies  $(\mathcal{S}_\epsilon \sigma_\epsilon)_\epsilon \in \mathcal{M}_F$ ;
- 2.  $(\sigma_\epsilon)_\epsilon \in \mathcal{N}_E$  implies  $(\mathcal{S}_\epsilon \sigma_\epsilon)_\epsilon \in \mathcal{N}_F$ .

Then,  $\tilde{\mathbb{C}}$ -linear map  $\mathcal{S} : (\mathcal{G}_E, (\mathcal{P}_i)_{i \in I}) \rightarrow (\mathcal{G}_F, (\mathcal{Q}_j)_{j \in J})$  is well defined by

$$\mathcal{S}\sigma = [(\mathcal{S}_\epsilon \sigma_\epsilon)_\epsilon], \quad \text{for all } \sigma \in \mathcal{G}_E.$$

Now we turn up to the continuity of mappings in Colombeau algebras. A function  $f : \mathcal{G}_E \rightarrow \mathcal{G}_F$  between Colombeau algebras is said to be continuous in  $\sigma_0$  in  $\mathcal{G}_E$ , if for all  $\gamma > 0$  there exists  $\delta_\gamma > 0$  such that  $\sigma - \sigma_0 \in \mathcal{B}(i, \delta_\gamma)$  implies that  $f(\sigma) - f(\sigma_0) \in \mathcal{B}(j, \gamma)$ .

**Definition 4.1.** Let  $\mathcal{S} : \mathcal{G}_E \rightarrow \mathcal{G}_F$  be a map with  $(\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  and  $(\mathcal{G}_F, (\mathcal{Q}_j)_{j \in J})$  are two locally convex topological  $\tilde{\mathbb{C}}$ -modules. We say that  $\mathcal{S}$  is continuous if and only if, for every  $j \in J$ , there exist  $i_0 \in I$ ,  $c > 0$  such that

$$\mathcal{Q}_j(\mathcal{S}\sigma - \mathcal{S}\rho) \leq c\mathcal{P}_{i_0}(\sigma - \rho), \quad \text{for all } \sigma, \rho \in \mathcal{G}_E.$$

*Example 4.1.* Let  $E = \mathcal{C}^\infty(\mathbb{R})$ ,  $\mathcal{P}_i(\cdot) = e^{-v_{p_i}(\cdot)}$  and  $p_i(\cdot) = \|\cdot\|_\infty$  for all  $i \in I$ . The mapping

$$\mathcal{S} : \begin{cases} \mathcal{G}_E \rightarrow \mathcal{G}_E, \\ \sigma \rightarrow \mathcal{S}\sigma = [(\mathcal{S}_\epsilon \sigma_\epsilon)_\epsilon] = [(\frac{1}{\epsilon^2} \sin(\sigma_\epsilon))_\epsilon], \end{cases}$$

is continuous in  $\mathcal{G}_E$ . Indeed, for all  $\sigma, \rho \in \mathcal{G}_E$ , we have

$$|\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \rho_\epsilon| \leq \frac{c}{\epsilon^2} |\sigma_\epsilon - \rho_\epsilon|, \quad c > 0.$$

And thus,  $v_{p_i}(\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \rho_\epsilon) \geq -2 + v_{p_{i_0}}(\sigma_\epsilon - \rho_\epsilon)$ . It follows that,

$$\mathcal{P}_i(\mathcal{S}\sigma - \mathcal{S}\rho) \leq c\mathcal{P}_{i_0}(\sigma - \rho), \quad c = e^2 > 0.$$

In particular for the continuity of a linear map on Colombeau’s algebra we have the following.

**Definition 4.2.** Let  $\mathcal{S} : \mathcal{G}_E \rightarrow \mathcal{G}_F$  be a  $\tilde{\mathbb{C}}$ -linear map with  $(\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  and  $(\mathcal{G}_F, (\mathcal{Q}_j)_{j \in J})$  are two locally convex topological  $\tilde{\mathbb{C}}$ -modules. We say that  $\mathcal{S}$  is continuous if and only if, for every  $j \in J$ , there exist  $i_0 \in I$  and  $c > 0$  such that

$$(4.1) \quad \mathcal{Q}_j(\mathcal{S}\sigma) \leq c\mathcal{P}_{i_0}(\sigma), \quad \text{for all } \sigma.$$

**Theorem 4.1.** Under the some notations above, let  $(\mathcal{S}_\epsilon)_\epsilon$  be a  $\mathbb{C}$ -linear maps family given by the constant family  $(s)_\epsilon$  (i.e,  $\mathcal{S}\sigma = [(s\sigma)_\epsilon]$ ). If  $s : E \rightarrow F$  is continuous, then  $\mathcal{S} : \mathcal{G}_E \rightarrow \mathcal{G}_F$  is a well defined  $\tilde{\mathbb{C}}$ -linear and continuous.

*Proof.* First, show that  $\mathcal{S}$  is well defined. Let  $(\sigma_\epsilon)_\epsilon \in \mathcal{M}_E$ . Then, for any  $i \in I$ , there exists  $N \in \mathbb{N}$  such that  $p_i(\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^{-N})$ , and  $\sigma_\epsilon \in E$ , for all  $\epsilon \in (0, 1]$ . So,  $s\sigma_\epsilon \in F$ . By continuity of  $s$ , for each  $j \in J$ , there exist  $i_0 \in I$ ,  $c > 0$ , such that

$$q_j(s\sigma_\epsilon) \leq cp_i(\sigma_\epsilon) \leq c \times c' \epsilon^{-N} = c_2 \epsilon^{-N}.$$

Then, for  $j \in J$ , there exists  $N \in \mathbb{N}$ , with  $q_j(s\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^{-N})$ . Hence,  $(s\sigma_\epsilon)_\epsilon \in \mathcal{M}_F$ , which implies that  $\mathcal{S}\sigma \in \mathcal{G}_F$ .

Let  $(\varrho_\epsilon)_\epsilon \in \mathcal{N}_E$ , i.e., for each  $i \in I$ ,  $p_i(\varrho_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^q)$ , for all  $q \in \mathbb{N}$  and  $\varrho_\epsilon \in F$ , for all  $\epsilon \in (0, 1]$ . Since  $s$  is linear and continuous, then for any  $j \in J$  there exist  $i_0 \in I$  and  $c > 0$  such that

$$q_j(s\varrho_\epsilon) \leq cp_{i_0}(\varrho_\epsilon) \leq c' \epsilon^q, \quad \text{where } c' > 0.$$

Hence,  $q_j(\mathcal{S}_\epsilon \varrho_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^q)$ , and thus  $(s\sigma_\epsilon)_\epsilon \in \mathcal{N}_F$ . From Lemma 4.1,  $\mathcal{S}$  is well defined.

The continuity. Let  $h = \sup \{b \in \mathbb{R} \mid q_j(s\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^b)\}$ . For every  $j \in J$ , we have

$$v_{q_j}((s\sigma_\epsilon)_\epsilon) = v((q_j(s\sigma_\epsilon))_\epsilon) = \sup \{b \in \mathbb{R} \mid q_j(s\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^b)\} = h.$$

And  $s$  is a linear continuous mapping, then for all  $j \in J$ , there exist  $i_0 \in I, c > 0$  such that  $q_j(s\sigma_\epsilon) \leq cp_{i_0}(\sigma_\epsilon)$ . Now we have to consider

$$d = v_{p_i}((\sigma_\epsilon)_\epsilon) = v((p_i(\sigma_\epsilon))_\epsilon) = \sup \{b \in \mathbb{R} \mid p_i(\sigma_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^b)\}.$$

Then, we have

$$q_j(s\sigma_\epsilon) \leq c \times c'' \epsilon^d = c_0 \epsilon^d, \quad \text{where } c_0, c'' > 0.$$

So,  $d \leq h$ , which implies  $v_{p_{i_0}}((\sigma_\epsilon)_\epsilon) \leq v_{q_j}((s\sigma_\epsilon)_\epsilon)$ . Hence,  $e^{-v_{q_j}((s\sigma_\epsilon)_\epsilon)} \leq e^{-v_{p_{i_0}}((\sigma_\epsilon)_\epsilon)}$ , and we get  $\mathcal{Q}_j(\mathcal{S}\sigma) \leq \mathcal{P}_{i_0}(\sigma)$ . Conclusion  $\mathcal{S}$  is  $\tilde{\mathbb{C}}$ -linear continuous.  $\square$

**Theorem 4.2.** With the previous notations, if the following map

$$\mathcal{S}_\epsilon : \begin{cases} E \rightarrow F, \\ \sigma_\epsilon \rightarrow \mathcal{S}_\epsilon \sigma_\epsilon, \end{cases}$$

is linear and contraction with the constant of contraction  $k_\epsilon = M\epsilon^{k_0}$  and  $k_0, M > 0$ , then  $\mathcal{S} : \mathcal{G}_E \rightarrow \mathcal{G}_F$  is contraction.

*Proof.* For the proof of this result, we will follow the same procedure as in [7]. Let  $\epsilon \in (0, 1]$ , and let the linear mapping  $\mathcal{S}_\epsilon : (E, (p_i)_{i \in I}) \rightarrow (F, (q_j)_{j \in J})$  be contraction. Then for all  $j \in J$  there exist  $i_0 \in I$  and  $k_\epsilon \in (0, 1)$  such that for all  $\sigma_\epsilon, \varrho_\epsilon \in E$ , we have

$$(4.2) \quad q_j(\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon) \leq k_\epsilon p_{i_0}(\sigma_\epsilon - \varrho_\epsilon).$$

Setting

$$\begin{aligned} h &= v_{q_j}((\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon)_\epsilon) = v((q_j(\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon))_\epsilon) \\ &= \sup \{ b \in \mathbb{R} \mid q_j(\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^b) \} \end{aligned}$$

and

$$\begin{aligned} d &= v_{p_{i_0}}((\sigma_\epsilon - \varrho_\epsilon)_\epsilon) = v((p_{i_0}(\sigma_\epsilon - \varrho_\epsilon))_\epsilon) \\ &= \sup \{ b \in \mathbb{R} \mid p_{i_0}(\sigma_\epsilon - \varrho_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^b) \}. \end{aligned}$$

Thank's to (4.2), we get

$$q_j(\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon) \leq k_\epsilon p_{i_0}(\sigma_\epsilon - \varrho_\epsilon) \leq \epsilon^{k_0} M c \epsilon^d = c_1 \epsilon^{d+k_0},$$

where  $c_1 > 0$ . Hence,  $d + k_0 \leq h$ , which implies that

$$\begin{aligned} v_{q_j}((\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon)_\epsilon) &\geq v_{p_{i_0}}((\sigma_\epsilon - \varrho_\epsilon)_\epsilon) + k_0, \\ e^{-v_{q_j}((\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon)_\epsilon)} &\leq e^{-k_0} e^{-v_{p_{i_0}}((\sigma_\epsilon - \varrho_\epsilon)_\epsilon)}, \\ \mathcal{Q}_j(\mathcal{S}\sigma - \mathcal{S}\varrho) &\leq e^{-k_0} \mathcal{P}_{i_0}(\sigma - \varrho). \end{aligned}$$

We conclude that  $\mathcal{S}$  is contraction. □

*Example 4.2.* Let  $\mathcal{G}_E = \tilde{\mathbb{R}}$ ,  $\mathcal{P}_i(\cdot) = e^{-v_{p_i}(\cdot)}$  and  $p_i(\cdot) = |\cdot|$  for all  $i \in I$ . The mapping

$$\mathcal{S} : \begin{cases} \tilde{\mathbb{R}} \rightarrow \tilde{\mathbb{R}}, \\ \sigma \rightarrow \mathcal{S}\sigma = [(\mathcal{S}_\epsilon \sigma_\epsilon)_\epsilon] = [(\epsilon^r e^{-\sigma_\epsilon})_\epsilon], \end{cases}$$

where  $r > 0$ , is continuous in  $\tilde{\mathbb{R}}$ . Indeed, for all  $\sigma, \varrho \in \tilde{\mathbb{R}}$ , we have

$$|\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon| \leq k_\epsilon |\sigma_\epsilon - \varrho_\epsilon|, \quad k_\epsilon = \epsilon^r.$$

And thus,  $v_{p_i}(\mathcal{S}_\epsilon \sigma_\epsilon - \mathcal{S}_\epsilon \varrho_\epsilon) \geq r + v_{p_{i_0}}(\sigma_\epsilon - \varrho_\epsilon)$ . Consequently, we have

$$\mathcal{P}_i(\mathcal{S}\sigma - \mathcal{S}\varrho) \leq c \mathcal{P}_{i_0}(\sigma - \varrho), \quad \text{with } c = e^{-r} \in (0, 1).$$

**Proposition 4.1.** *The space  $(\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  is a separated locally convex topological  $\tilde{\mathbb{C}}$ -module.*

*Proof.* By the definition of the negligible space  $\mathcal{N}_E$ , if  $u \neq 0$  in  $\mathcal{G}_E$ , then  $v_{p_i}((\sigma_\epsilon)_\epsilon) \neq \pm\infty$  for some  $i \in I$ , hence  $\mathcal{P}_i(u) > 0$ , so,  $(\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  is separate. □

A sequence in Colombeau algebra is a map

$$\sigma : \mathbb{N} \rightarrow \mathcal{G}_E, \quad n \mapsto \sigma_n = [(\sigma_{n,\epsilon})_\epsilon],$$

and is denoted by  $(\sigma_n)_{n \in \mathbb{N}}$ . We say that  $(\sigma_n)_{n \in \mathbb{N}}$  converges to  $\sigma \in \mathcal{G}_E$  if for all  $\gamma > 0$ , there is  $n_0 \in \mathbb{N}$ , such that if  $n > n_0$ , then  $\sigma_n - \sigma \in \mathcal{B}(i, \gamma)$ . Such a sequence  $(\sigma_n)_{n \in \mathbb{N}}$  is a Cauchy if for all  $\gamma > 0$  there is  $n_0 \in \mathbb{N}$  such that if  $m, n > n_0$ , then  $\sigma_m - \sigma_n \in \mathcal{B}(i, \gamma)$ . Since the sharp topology is a Hausdorff, then limits are unique whenever they exist. Which is equivalent in terms of families of seminorms to the following definition.

**Definition 4.3.** A sequence  $(\sigma_n)_{n \in \mathbb{N}}$  in locally convex topological  $\tilde{\mathcal{C}}$ -module  $(\mathcal{G}_E, \mathcal{P}_{i \in I})$  is called convergent and converges to  $\sigma \in \mathcal{G}_E$  if,

$$\lim_{n \rightarrow +\infty} \mathcal{P}_i(\sigma_n) = \sigma, \quad \text{for all } i \in I.$$

*Example 4.3.* If we take  $\mathcal{G}_E = \tilde{\mathbb{R}}$ ,  $\mathcal{P}_i(\cdot) = e^{-v_{p_i}(\cdot)}$  and  $p_i(\cdot) = |\cdot|$  for all  $i \in I$ . The sequence  $\sigma_n = [(\sigma_{n,\epsilon})_\epsilon]$  in  $\tilde{\mathbb{R}}$ , where  $\sigma_{n,\epsilon} = \frac{1}{n}$ ,  $n \in \mathbb{N}$  does not converge to 0 in  $\tilde{\mathbb{R}}$  with respect to the sharp topology, because if  $\mathcal{P}_i(\frac{1}{n}) < \gamma$ , for all  $\gamma > 0$ , we obtain  $n < \frac{-1}{\ln(\gamma)}$ , which is absurd when  $n$  is large enough. However, the sequence  $\sigma_n = [(\sigma_{n,\epsilon})_\epsilon]$  in  $\tilde{\mathbb{R}}$ , where  $\sigma_{n,\epsilon} = \frac{r_\epsilon}{n}$  and  $r_\epsilon = \epsilon^{\frac{r^2}{r^4 + \epsilon^4}}$ ,  $r > 0$  converges to 0. Indeed, for all  $\gamma > 0$ , take  $n_0 = \lceil |r_\epsilon| \epsilon^{\ln(\gamma)} \rceil + 1$  and  $\lceil \cdot \rceil$  symbolizes the integer part. If  $n > n_0$ , we have that  $|\frac{r_\epsilon}{n}| < \epsilon^{-\ln(\gamma)}$ , then  $v_{p_i}(\frac{r_\epsilon}{n}) > -\ln(\gamma)$ . And thus  $\mathcal{P}_i(\frac{r_\epsilon}{n}) < \gamma$  which implies that  $\sigma_n \in \mathcal{B}(i, \gamma)$ .

**Definition 4.4.** A sequence  $(\sigma_n)_{n \in \mathbb{N}}$  in locally convex topological  $\tilde{\mathcal{C}}$ -module  $(\mathcal{G}_E, \mathcal{P}_{i \in I})$  is called Cauchy if, for each  $m, n \in \mathbb{N}$ , where  $m > n$  such that

$$\lim_{n \rightarrow +\infty} \mathcal{P}_i(\sigma_n - \sigma_m) = 0, \quad \text{for all } i \in I.$$

The following proposition has been proved in [10].

**Proposition 4.2.** ([10, Proposition 3.4, p. 25]) *The space  $(\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  is complete.*

*Remark 4.1.* Let  $(\sigma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{G}_E$ . Then  $(\sigma_n)_{n \in \mathbb{N}}$  is convergent if and only if, it is a Cauchy sequence if and only if, for all  $\gamma > 0$  there is  $n_0 \in \mathbb{N}$  such that  $n > m > n_0$  implies that  $\sigma_m - \sigma_n \in \mathcal{B}(i, \gamma)$ .

**4.2. Some fixed point theorems in Colombeau algebra.** The second part of this section is dealing with some new theorems of fixed point in the framework of Colombeau algebra based on the locally convex space. In this subsection to simplify the formula we take  $v((\sigma_\epsilon)_\epsilon) = v(\sigma_\epsilon)$ . The idea of this theorem inspired by that of the Banach fixed point theorem in a classical metric space.

**Theorem 4.3.** *Let  $A : (\mathcal{G}_E, (\mathcal{P}_i)_{i \in I}) \rightarrow (\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  be a mapping from an algebra of generalized functions into itself such that*

1. *there exists  $\sigma_0 \in \mathcal{G}_E$ ,  $p_i(A_\epsilon \sigma_{0,\epsilon} - \sigma_{0,\epsilon}) = O_{\epsilon \rightarrow 0}(\epsilon^c)$ , where  $\sigma_0 = [(\sigma_{0,\epsilon})_\epsilon]$ ;*

2. for every  $i \in I$ ,  $p_i(A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon) \leq M\epsilon^{k_0} p_i(\sigma_\epsilon - \varrho_\epsilon)$  for all  $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$ , and  $M, k_0 > 0$ .

Then,  $A$  has a unique fixed point.

*Proof.* We introduce the sequence  $(x_n)_{n \in \mathbb{N}}$  defined by

$$\begin{cases} x_{n+1} = Ax_n, \\ x_0 = \sigma_0, \end{cases} \text{ is equivalent to } \begin{cases} x_{n+1,\epsilon} = Ax_{n,\epsilon} + n_\epsilon \\ x_{0,\epsilon} = \sigma_{0,\epsilon} + m_\epsilon, \end{cases}$$

where  $(n_\epsilon)_\epsilon, (m_\epsilon)_\epsilon \in \mathcal{N}_E$ . We have,

$$e^{-v_{p_i}(x_{n+1,\epsilon} - x_{n,\epsilon})} = e^{-v_{p_i}(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon})}.$$

We set

$$h = v_{p_i}(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}) = \sup \{ b \in \mathbb{R} \mid p_i(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}) = O_{\epsilon \rightarrow 0}(\epsilon^b) \}$$

and

$$d = v_{p_i}(x_{n,\epsilon} - x_{n-1,\epsilon}) = \sup \{ b \in \mathbb{R} \mid p_i(x_{n,\epsilon} - x_{n-1,\epsilon}) = O_{\epsilon \rightarrow 0}(\epsilon^b) \}.$$

By taking  $\sigma_\epsilon = x_{n,\epsilon}$  and  $\varrho_\epsilon = x_{n-1,\epsilon}$  in the second condition, we get

$$p_i(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}) \leq M\epsilon^{k_0} p_i(x_{n,\epsilon} - x_{n-1,\epsilon}) \leq M\epsilon^{k_0} c_1 \epsilon^d = c_1 \epsilon^{d+k_0},$$

where  $c_1 > 0$ . Hence,  $d + k_0 \leq h$  then, it follows that

$$\begin{aligned} v_{p_i}(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}) &\geq v_{p_i}(x_{n,\epsilon} - x_{n-1,\epsilon}) + k_0, \\ e^{-v_{p_i}(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon})} &\leq e^{-k_0} e^{-v_{p_i}(x_{n,\epsilon} - x_{n-1,\epsilon})}. \end{aligned}$$

Therefore,  $e^{-v_{p_i}(x_{n+1,\epsilon} - x_{n,\epsilon})} \leq e^{-k_0} e^{-v_{p_i}(x_{n,\epsilon} - x_{n-1,\epsilon})}$ . By induction, we obtain

$$e^{-v_{p_i}(x_{n+1,\epsilon} - x_{n,\epsilon})} \leq (e^{-k_0})^n e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})}.$$

Now let us prove that  $(x_n)$  is a Cauchy sequence. Let  $m, n \in \mathbb{N}$ ,  $m > n$ . We have

$$\begin{aligned} e^{-v_{p_i}(x_{m,\epsilon} - x_{n,\epsilon})} &= e^{-v_{p_i}(x_{m,\epsilon} - x_{n,\epsilon})} \\ &\leq e^{\sup \{ -v_{p_i}(x_{m,\epsilon} - x_{m-1,\epsilon}), -v_{p_i}(x_{m-1,\epsilon} - x_{n,\epsilon}) \}} \\ &\leq \sup \{ e^{-v_{p_i}(x_{m,\epsilon} - x_{m-1,\epsilon})}, e^{-v_{p_i}(x_{m-1,\epsilon} - x_{n,\epsilon})} \} \\ &\leq e^{-v_{p_i}(x_{m,\epsilon} - x_{m-1,\epsilon})} + e^{-v_{p_i}(x_{m-1,\epsilon} - x_{n,\epsilon})} \\ &\leq (e^{-k_0})^{m-1} p_i(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}) + \dots + (e^{-k_0})^n p_i(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}) \\ &\leq (e^{-k_0})^n (1 + e^{-k_0} + \dots + e^{-k_0(m-n-1)}) p_i(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}) \\ &= (e^{-k_0})^n \left( \frac{1 - e^{-k_0(m-n)}}{1 - e^{-k_0}} \right) e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})} \\ &\leq \frac{(e^{-k_0})^n}{1 - e^{-k_0}} e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})}, \end{aligned}$$

by the first condition, we get  $p_i(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}) = O_{\epsilon \rightarrow 0}(\epsilon^c)$ . Then,

$$\begin{aligned} v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}) &\geq c, \\ -v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}) &\leq -c, \\ e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})} &\leq e^{-c} = \text{const.} \end{aligned}$$

And thus,

$$e^{-v_{p_i}(x_{m,\epsilon} - x_{n,\epsilon})} \leq \frac{(e^{-k_0})^n}{1 - e^{k_0}} \text{const.}$$

Since the right hand side of the last inequality tends to zero as  $n \rightarrow +\infty$ , it follows that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(\mathcal{G}, (\mathcal{P}_i)_{i \in I})$  which is a complete space, by Proposition 4.2, then  $(x_n)_{n \in \mathbb{N}}$  is convergent. Then there exists  $\sigma$  in  $\mathcal{G}_E$  such that  $\sigma = \lim_{n \rightarrow +\infty} x_n$ . On the other hand, we have for any  $\sigma, \varrho \in \mathcal{G}_E$ , we can prove that  $A$  is a continuous mapping. Indeed for any  $\sigma, \varrho \in \mathcal{G}_E$ , we have

$$\mathcal{P}_i(A\sigma - A\varrho) = e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon)} \leq e^{-k_0} e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} \leq e^{-k_0} \mathcal{P}_i(\sigma - \varrho).$$

We deduce that  $A$  is continuous. And we have,  $x_{n+1} = Ax_n \rightarrow \sigma$  as  $n \rightarrow +\infty$  which implies that  $\sigma = A\sigma$ . Therefore,  $A$  has a fixed point. Now, assume that  $A$  has another fixed point  $\varrho \in \mathcal{G}_E$ . Then  $A\varrho = \varrho$ , we can write

$$e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} = \mathcal{P}_i(\sigma - \varrho) = \mathcal{P}_i(A\sigma - A\varrho) = e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon)} \leq e^{-k_0} e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)},$$

this implies  $e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} = 0$ , which signifies that  $v_{p_i}(\sigma_\epsilon - \varrho_\epsilon) = +\infty$ , and thus,  $(\sigma_\epsilon - \varrho_\epsilon)_\epsilon \in \mathcal{N}_E$ . Therefore,  $\sigma = \varrho$  in  $\mathcal{G}_E$ .  $\square$

The theorem below is an extended of Kannan’s fixed point theorem in classical metric space.

**Theorem 4.4.** *Let  $A : (\mathcal{G}_E, (\mathcal{P}_i)_{i \in I}) \rightarrow (\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  be a mapping from an algebra of generalized functions equipped with a family of ultra pseudo-norms  $(\mathcal{P}_i)_{i \in I}$  into itself, and satisfying the two following conditions:*

1. *there exists  $\sigma_0 \in \mathcal{G}_E$ ,  $p_i(A_\epsilon \sigma_{0,\epsilon} - \sigma_{0,\epsilon}) = O_{\epsilon \rightarrow 0}(\epsilon^d)$ , with  $d > 0$ ;*
2. *for every  $i \in I$ ,  $p_i(A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon) \leq M\epsilon^\lambda [p_i(A_\epsilon \sigma_\epsilon - \sigma_\epsilon) + p_i(A_\epsilon \varrho_\epsilon - \varrho_\epsilon)]$ , for all  $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$ ,  $\lambda > \ln(2)$ ,  $M > 0$ .*

*Then  $A$  has a unique fixed point in  $\mathcal{G}_E$ .*

*Proof.* Let us consider the following sequence

$$\begin{cases} y_{n+1} = Ay_n, \\ y_0 = \sigma_0, \end{cases} \text{ is equivalent to } \begin{cases} y_{n+1,\epsilon} = Ay_{n,\epsilon} + n_\epsilon, \\ y_{0,\epsilon} = \sigma_{0,\epsilon} + m_\epsilon, \end{cases}$$

where  $(n_\epsilon)_\epsilon, (m_\epsilon)_\epsilon \in \mathcal{N}_E$ . We have

$$e^{-v_{p_i}(y_{n+1,\epsilon} - y_{n,\epsilon})} = e^{-v_{p_i}(y_{n+1,\epsilon} - y_{n,\epsilon})} = e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})}.$$

By the definition of the valuation function  $v_{p_i}$ , we can write

$$p_i(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon}) \leq M\epsilon^\lambda e^{\min\{v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n,\epsilon}), v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})\}}.$$

Then,

$$v_{p_i}(A_\epsilon y_{n+1,\epsilon} - A_\epsilon y_{n-1,\epsilon}) \geq \lambda + \min\{v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n,\epsilon}), v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})\}.$$

Thus,

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} &\leq e^{-\lambda} e^{-\min\{v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n,\epsilon}), v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})\}} \\ &\leq e^{-\lambda} \max\{e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n,\epsilon})}, e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})}\} \\ &\leq e^{-\lambda} \left( e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n,\epsilon})} + e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})} \right). \end{aligned}$$

It follows,

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} \leq \frac{e^{-\lambda}}{1 - e^{-\lambda}} e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})}.$$

By induction, we can conclude that

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} \leq \left( \frac{e^{-\lambda}}{1 - e^{-\lambda}} \right)^n e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})}.$$

Now we have to prove that  $(y_n)_n$  is a Cauchy sequence. Let  $n, p \in \mathbb{N}$ ,

$$\begin{aligned} e^{-v_{p_i}(x_{n+p,\epsilon} - y_{n,\epsilon})} &= e^{-v_{p_i}(x_{n+p,\epsilon} - y_{n,\epsilon})} \\ &\leq e^{\max\{-v_{p_i}(y_{n+p,\epsilon} - y_{n+p-1,\epsilon}), -v_{p_i}(y_{n+p-1,\epsilon} - y_{n,\epsilon})\}} \\ &\leq \max\{e^{-v_{p_i}(y_{n+p,\epsilon} - y_{n+p-1,\epsilon})}, e^{-v_{p_i}(y_{n+p-1,\epsilon} - y_{n,\epsilon})}\} \\ &\leq e^{-v_{p_i}(y_{n+p,\epsilon} - y_{n+p-1,\epsilon})} + e^{-v_{p_i}(y_{n+p-1,\epsilon} - y_{n,\epsilon})} \\ &\quad \vdots \\ &\leq \left[ \left( \frac{e^{-\lambda}}{1 - e^{-\lambda}} \right)^n + \dots + \left( \frac{e^{-\lambda}}{1 - e^{-\lambda}} \right)^{n+p-1} \right] e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})} \\ &\leq \frac{\left( \frac{e^{-\lambda}}{1 - e^{-\lambda}} \right)^n}{1 - \frac{e^{-\lambda}}{1 - e^{-\lambda}}} e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})}. \end{aligned}$$

From the first property in the theorem we have  $e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})}$  is finite, the right hand side of the last inequality tends to zero as  $n \rightarrow +\infty$ . Then,  $(y_n)_n$  is a Cauchy sequence in  $\mathcal{G}_E$  which is complete by Proposition 4.2, and thus there is  $\sigma$  in  $\mathcal{G}_E$  such that  $y_n \rightarrow 0$  as  $n \rightarrow +\infty$ .

Now, let us prove that  $\sigma$  is a fixed point of the mapping  $A$ , we have

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} &\leq e^{\sup\{-v_{p_i}(A_\epsilon \sigma_\epsilon - y_{n,\epsilon}), -v_{p_i}(y_{n,\epsilon} - \sigma_\epsilon)\}} \\ &\leq e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - y_{n,\epsilon})} + e^{-v_{p_i}(y_{n,\epsilon} - \sigma_\epsilon)}. \end{aligned}$$

On the other hand

$$\begin{aligned} e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} &\leq e^{-\lambda} \left( e^{-\min\{v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon}), v_{p_i}(\sigma_\epsilon-A_\epsilon\sigma_\epsilon)\}} \right) \\ &\leq e^{-\lambda} \sup \left\{ (e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}, e^{-v_{p_i}(\sigma_\epsilon-A_\epsilon\sigma_\epsilon)}) \right\}. \end{aligned}$$

Hence,

$$e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} + e^{-\lambda} e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})} + e^{-\lambda} e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)}.$$

Thus,

$$(1 - e^{-\lambda}) e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} + e^{-\lambda} e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}.$$

By passing to the limit as  $n \rightarrow +\infty$ , we obtain  $e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} = 0$ , this implies

$$v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon) = +\infty,$$

which means that  $(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)_\epsilon \in \mathcal{N}_E$ . Then,  $A\sigma = \sigma$ . It now remains to demonstrate the uniqueness. Assume that there is another fixed point  $\varrho$  of  $A$ ,  $A\varrho = \varrho$ , such that  $\sigma \neq \varrho$ .

So, we can write

$$\begin{aligned} e^{-v_{p_i}(\sigma_\epsilon-\varrho_\epsilon)} &\leq e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-A_\epsilon\varrho_\epsilon)} \\ &\leq e^{-\lambda} e^{\sup\{(-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon), -v_{p_i}(A_\epsilon\varrho_\epsilon-\varrho_\epsilon))\}} \\ &\leq e^{-\lambda} \sup \left\{ e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)}, e^{-v_{p_i}(A_\epsilon\varrho_\epsilon-\varrho_\epsilon)} \right\} \\ &\leq e^{-\lambda} \left( e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} + e^{-v_{p_i}(A_\epsilon\varrho_\epsilon-\varrho_\epsilon)} \right). \end{aligned}$$

Since  $A\sigma = \sigma$  and  $A\varrho = \varrho$ , then

$$v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon) = v_{p_i}(A_\epsilon\varrho_\epsilon - \varrho_\epsilon) = +\infty,$$

which implies that  $e^{-v_{p_i}(\sigma_\epsilon-\varrho_\epsilon)} = 0$ . Thus,  $v_{p_i}(\sigma_\epsilon - \varrho_\epsilon) = +\infty$ . Then,  $(\sigma_\epsilon - \varrho_\epsilon)_\epsilon \in \mathcal{N}_E$ .

Conclusion is  $\sigma = \varrho$  in  $\mathcal{G}_E$ . □

The following theorem is based on the theorem in the classical case, of Chatterjia.

**Theorem 4.5.** *Let  $A : (\mathcal{G}_E, (\mathcal{P}_i)_{i \in I}) \rightarrow (\mathcal{G}_E, (\mathcal{P}_i)_{i \in I})$  be a mapping from an algebra of generalized functions equipped with a family of ultra pseudoseminorms  $(\mathcal{P}_i)_{i \in I}$  into itself, and satisfying the two following conditions:*

1. *there exists  $\sigma_0 \in \mathcal{G}_E$ ,  $p_i(A_\epsilon\sigma_{0,\epsilon} - \sigma_{0,\epsilon}) = O_{\epsilon \rightarrow 0}(\epsilon^k)$ , with  $k > 0$ ;*
2. *for any  $i \in I$ ,  $p_i(A_\epsilon\sigma_\epsilon - A_\epsilon v_\epsilon) \leq M\epsilon^\beta [p_i(A_\epsilon\sigma_\epsilon - \varrho_\epsilon) + p_i(A_\epsilon v_\epsilon - \sigma_\epsilon)]$ , for all  $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$ ,  $\beta > \ln(2)$ ,  $M > 0$ .*

*Then  $A$  has a unique fixed point in  $\mathcal{G}_E$ .*

*Proof.* Let us consider the following sequence

$$\begin{cases} y_{n+1} = Ay_n, \\ y_0 = \sigma_0, \end{cases} \quad \text{is equivalent to} \quad \begin{cases} y_{n+1,\epsilon} = Ay_{n,\epsilon} + n_\epsilon, \\ y_{0,\epsilon} = \sigma_{0,\epsilon} + m_\epsilon, \end{cases}$$

where  $(n_\epsilon)_\epsilon, (m_\epsilon)_\epsilon \in \mathcal{N}_E$ . We have  $e^{-v_{p_i}(y_{n+1,\epsilon} - y_{n,\epsilon})} = e^{-v_{p_i}(A_\epsilon y_{n+1,\epsilon} - A_\epsilon y_{n-1,\epsilon})}$ . By the definition of the valuation function  $v_{p_i}$  we can write

$$p_i(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon}) \leq M\epsilon^\beta \epsilon^{\min\{v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n,\epsilon}), v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})\}}.$$

So,

$$v_{p_i}(A_\epsilon y_{n+1,\epsilon} - A_\epsilon y_{n-1,\epsilon}) \geq \beta + \min\{v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n-1,\epsilon}), v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon})\}.$$

Thus,

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} &\leq e^{-\beta} e^{-\min\{v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n-1,\epsilon}), v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon})\}} \\ &\leq e^{-\beta} \sup\{e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n-1,\epsilon})}, e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon})}\}. \end{aligned}$$

Since,  $(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon})_\epsilon \in \mathcal{N}_E$ , it follows  $v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon}) = +\infty$ . Then,

$$e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon})} = 0.$$

We can write the last inequality as follow

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} \leq e^{-\beta} e^{-v_{p_i}(y_{n+1,\epsilon} - y_{n-1,\epsilon})},$$

and by the properties of the valuation function we can conclude

$$\begin{aligned} v_{p_i}(y_{n+1,\epsilon} - y_{n-1,\epsilon}) &= v_{p_i}((y_{n+1,\epsilon} - y_{n,\epsilon}) + (y_{n,\epsilon} - y_{n-1,\epsilon})) \\ &\geq \min\{v_{p_i}(y_{n+1,\epsilon} - y_{n,\epsilon}), v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon})\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} &\leq e^{-\beta} (e^{-v_{p_i}(y_{n+1,\epsilon} - y_{n,\epsilon})} + e^{-v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon})}) \\ &\leq e^{-\beta} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} + e^{-\beta} e^{-v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon})}. \end{aligned}$$

Then,

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} \leq \left(\frac{e^{-\beta}}{1 - e^{-\beta}}\right) e^{-v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon})}.$$

By induction we can conclude that

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})} \leq \left(\frac{e^{-\beta}}{1 - e^{-\beta}}\right)^n e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})}.$$

Now let us prove that  $(y_n)_n$  is a Cauchy sequence in  $\mathcal{G}_E$

$$\begin{aligned} e^{-v_{p_i}(y_{n+p,\epsilon} - y_{n,\epsilon})} &= e^{-v_{p_i}(y_{n+p,\epsilon} - y_{n,\epsilon})} \\ &\leq e^{\max\{-v_{p_i}(y_{n+p,\epsilon} - y_{n+p-1,\epsilon}), -v_{p_i}(y_{n+p-1,\epsilon} - y_{n,\epsilon})\}} \\ &\leq \max\{e^{-v_{p_i}(y_{n+p,\epsilon} - y_{n+p-1,\epsilon})}, e^{-v_{p_i}(y_{n+p-1,\epsilon} - y_{n,\epsilon})}\} \\ &\leq e^{-v_{p_i}(y_{n+p,\epsilon} - y_{n+p-1,\epsilon})} + e^{-v_{p_i}(y_{n+p-1,\epsilon} - y_{n,\epsilon})} \\ &\vdots \end{aligned}$$

$$\begin{aligned} &\leq \left[ \left( \frac{e^{-\beta}}{1 - e^{-\beta}} \right)^n + \dots + \left( \frac{e^{-\beta}}{1 - e^{-\beta}} \right)^{n+p-1} \right] e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})} \\ &\leq \frac{\left( \frac{e^{-\beta}}{1 - e^{-\beta}} \right)^n}{1 - \left( \frac{e^{-\beta}}{1 - e^{-\beta}} \right)} e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})}. \end{aligned}$$

Thus  $(y_n)_n$  is a Cauchy sequence in  $\mathcal{G}_E$  which is complete by Proposition 4.2, so there is  $\sigma \in \mathcal{G}_E$  such that  $y_n \rightarrow \sigma$  as  $n \rightarrow +\infty$ . We show that  $\sigma$  is a fixed point of the mapping  $A$ . We have

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} &\leq e^{\sup\{-v_{p_i}(A_\epsilon \sigma_\epsilon - y_{n,\epsilon}), -v_{p_i}(y_{n,\epsilon} - \sigma_\epsilon)\}} \\ &\leq e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - y_{n,\epsilon})} + e^{-v_{p_i}(y_{n,\epsilon} - \sigma_\epsilon)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} e^{-v_{p_i}(y_{n,\epsilon} - \sigma_\epsilon)} &\leq e^{-\lambda} \left( e^{-\min\{v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon}), v_{p_i}(\sigma_\epsilon - A_\epsilon \sigma_\epsilon)\}} \right) \\ &\leq e^{-\lambda} \sup\{e^{-v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon})}, e^{-v_{p_i}(\sigma_\epsilon - A_\epsilon \sigma_\epsilon)}\}. \end{aligned}$$

Hence,

$$e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon} - \sigma_\epsilon)} + e^{-\beta} e^{-v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon})} + e^{-\beta} e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)}.$$

So,

$$(1 - e^{-\beta}) e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon} - \sigma_\epsilon)} + e^{-\beta} e^{-v_{p_i}(y_{n,\epsilon} - y_{n-1,\epsilon})}.$$

By passing to the limit as  $n \rightarrow +\infty$  we obtained  $e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} = 0$ , which implies  $v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon) = +\infty$ , and thus  $(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)_\epsilon \in \mathcal{N}_E$ . Then,  $A\sigma = \sigma$ .

Now, let us prove the uniqueness of fixed point. Assume that there is another fixed point  $\varrho$  of  $A$ ,  $A\varrho = \varrho$ , such that  $\sigma \neq \varrho$ . So, we can write

$$\begin{aligned} e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} &= e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon)} \\ &\leq e^{-\beta} e^{\sup\{-v_{p_i}(A_\epsilon \sigma_\epsilon - \varrho_\epsilon), -v_{p_i}(A_\epsilon \varrho_\epsilon - \varrho_\epsilon)\}} \\ &\leq e^{-\beta} \sup\{e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \varrho_\epsilon)}, e^{-v_{p_i}(A_\epsilon \varrho_\epsilon - \varrho_\epsilon)}\} \\ &\leq e^{-\beta} \left( e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} + e^{-\beta} e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} \right) \\ &\leq \left( \frac{e^{-\beta}}{1 - e^{-\beta}} \right) e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)}. \end{aligned}$$

Since,  $A\sigma = \sigma$ . Then,  $v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon) = +\infty$ , that implies  $e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} = 0$ . Thus,  $e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} = 0$ , then  $v_{p_i}(\sigma_\epsilon - \varrho_\epsilon) = +\infty$ , which means that  $(\sigma_\epsilon - \varrho_\epsilon)_\epsilon$  is a negligible element. Finally  $\sigma = \varrho$  in  $\mathcal{G}_E$ . □

5. APPLICATION TO AN EVOLUTION PROBLEM

We consider the standard Cauchy problem

$$(5.1) \quad \begin{cases} \sigma'(t) = f(t, \sigma(t)), & t \in \mathbb{R}^+, \\ \sigma(0) = \sigma_0 \in \tilde{\mathbb{R}}, \end{cases}$$

where  $f : \mathbb{R} \times \tilde{\mathbb{R}} \rightarrow \tilde{\mathbb{R}}$  and  $\sigma \in \mathcal{G}_{\mathbb{R}}$ .

**Proposition 5.1.** *If  $f_\epsilon$  satisfies the following condition*

$$(5.2) \quad |f_\epsilon(s, \sigma_\epsilon) - f_\epsilon(s, \varrho_\epsilon)| \leq k_\epsilon(s) |\sigma_\epsilon(s) - \varrho_\epsilon(s)|, \quad \text{for all } \sigma, \varrho \in \mathcal{G}_{\mathbb{R}},$$

where  $k_\epsilon(s) = M(s)\epsilon^\lambda$  with  $0 < M(s) < 1$  and  $\lambda > 0$ .

Then, the problem (5.1) has a unique generalized solution.

*Proof.*  $\tilde{u}$  is a solution of the problem (5.1) if and only if it is a fixed point of the mapping

$$(5.3) \quad \begin{cases} A : \tilde{\mathbb{R}} \rightarrow \tilde{\mathbb{R}}, \\ x \mapsto A\sigma = \sigma_0 + \int_0^t f(s, \sigma(s))ds. \end{cases}$$

Let

$$(5.4) \quad \begin{cases} A_\epsilon : \mathbb{R} \rightarrow \mathbb{R}, \\ x_\epsilon \mapsto A_\epsilon \sigma_\epsilon = \sigma_{0\epsilon} + \int_0^t f_\epsilon(s, \sigma_\epsilon(s))ds, \end{cases}$$

be a representative of  $A$ . Since  $f$  satisfies condition (5.2) and we defined the following ultra pseudo-seminorms on  $\tilde{\mathbb{R}}$  by  $\mathcal{P}_T(\sigma) = e^{-v_{p_T}(\sigma_\epsilon)}$ , where  $p_T(\sigma_\epsilon) = \sup_{t \in [0, T]} |\sigma_\epsilon(t)|$  and  $T$  is a non negative real number. We have

$$\begin{aligned} |A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon| &= \left| \int_0^t [f_\epsilon(s, \sigma_\epsilon(s)) - f_\epsilon(s, \varrho_\epsilon(s))]ds \right| \\ &\leq \int_0^t |f_\epsilon(s, \sigma_\epsilon(s)) - f_\epsilon(s, \varrho_\epsilon(s))|ds \\ &\leq \int_0^t M \epsilon^\lambda |\sigma_\epsilon(s) - \varrho_\epsilon(s)|ds \quad \left( M = \sup_{t \in [0, T]} |M(t)| \right) \\ &\leq T M \epsilon^\lambda p_T(\sigma_\epsilon - \varrho_\epsilon) \\ &\leq C \epsilon^\lambda p_T(\sigma_\epsilon - \varrho_\epsilon). \end{aligned}$$

So,  $p_T(A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon) \leq C \epsilon^\lambda p_T(\sigma_\epsilon - \varrho_\epsilon)$ . Then, the second condition of first Theorem 4.3 is satisfied. Moreover, we have

$$p_T(A_\epsilon \sigma_{0\epsilon} - \sigma_{0\epsilon}) = p_T \left( \int_0^t f_\epsilon(s, \sigma_{0\epsilon})ds \right) \leq T p_T(f_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^c), \quad c \in \mathbb{R}.$$

According to our first theorem there is a generalized solution for the abstract Cauchy problem. We have to prove now the uniqueness, assume that there is another fixed point  $\varrho$  of  $A$ ,  $A\varrho = \varrho$ , such that  $\sigma \neq \varrho$ . Then we can write

$$e^{-v_{p_T}(\sigma_\epsilon - \varrho_\epsilon)} = \mathcal{P}_T(\sigma - \varrho) = \mathcal{P}_T(A\sigma - A\varrho) = e^{-v_{p_T}(A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon)} \leq e^{-k} e^{-v_{p_T}(\sigma_\epsilon - \varrho_\epsilon)}.$$

Hence,  $e^{-v_{p_T}(\sigma_\epsilon - \varrho_\epsilon)}(1 - e^{-k}) \leq 0$ , which gives,  $e^{-v_{p_T}(\sigma_\epsilon - \varrho_\epsilon)} = 0$ , in other words,  $v(\sigma_\epsilon - \varrho_\epsilon) = +\infty$ . Therefore,  $(\sigma_\epsilon - \varrho_\epsilon)_\epsilon \in \mathcal{N}_E$ . Finally,  $\sigma = \varrho$  in  $\tilde{\mathbb{R}}$ . And the solution is unique in  $\tilde{\mathbb{R}}$ .  $\square$

*Example 5.1.* Let's consider the example that inspired the fixed point theorem in Colombeau algebra. Consider the following problem from [9, 15]:

$$(5.5) \quad \begin{cases} \partial^2\sigma(t) = h(\sigma(t))\delta(t) + g(t), \\ \sigma(-1) = \sigma_0, \\ \sigma'(-1) = \sigma_1, \end{cases}$$

where  $\delta$  the Dirac distribution and  $g, h \in \mathcal{C}^\infty(\mathbb{R})$ .

It is a significant differential equation which comes from physics having a product of the distributions in the first equation, initial conditions are singular generalized numbers  $\sigma_0, \sigma_1$  and does not allow to use the classical tools to have a solution. In the references mentioned above we find proof of moderation, other nontrivial steps implying classical results. Let  $\alpha$  be positive constant,  $L = \int_{-2}^1 \int_{-2}^1 |g(\tau)| d\tau ds$  and  $k$  is a Lipschitz constant of  $h$  on a compact subset of  $\mathbb{R}$  containing  $\Omega = ]-1 - \frac{\alpha}{2}, \frac{\alpha}{2}[$ . The equation (5.5) can be reformulated as the Cauchy problem (5.1) with  $f = [(f)_\epsilon]$ ,  $f(t, \cdot) = h(\cdot)\rho_\epsilon(t) + g(t)$  is a smooth function and  $\delta = [(\delta * \rho)_\epsilon]$  is the embedding of the Dirac measure in  $\mathcal{G}_E$ ,  $E = \mathcal{C}^\infty$ , where  $\rho_\epsilon(t) = \frac{1}{\epsilon}\rho(\frac{t}{\epsilon})$  and  $\rho \in \mathcal{C}^\infty(\mathbb{R})$ ,  $\int_{\mathbb{R}} \rho(t) dt = 1$ ,  $\rho(t) \geq 0$ . We defined the following norm on  $\mathcal{G}_E$  by

$$\mathcal{P}_T(\sigma) = e^{-v_{p_T}(\sigma_\epsilon)},$$

where  $p_T(\sigma_\epsilon) = \sup_{t \in \Omega} |\sigma_\epsilon(t)|$  and let

$$\begin{cases} A_\epsilon : \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R}), \\ \sigma_\epsilon \mapsto A_\epsilon\sigma_\epsilon(t) = \sigma_{0\epsilon} + (t+1)\sigma_{1\epsilon} + \int_{-1}^1 \int_{-1}^s f(\sigma_\epsilon(\tau))\rho_\epsilon(\tau) d\tau ds + \int_{-1}^t \int_{-1}^s g(\tau) d\tau ds. \end{cases}$$

We have

$$\begin{aligned} A_\epsilon\sigma_{0\epsilon} - \sigma_{0\epsilon} &= (t+1)\sigma_{1\epsilon} + \int_{-1}^1 \int_{-1}^s f(\sigma_\epsilon(\tau))\rho_\epsilon(\tau) d\tau ds + \int_{-1}^t \int_{-1}^s g(\tau) d\tau ds \\ &\leq (|t|+1)|\sigma_{1\epsilon}| + \int_{-1}^1 \int_{-1}^s |f(\sigma_\epsilon(\tau))\rho_\epsilon(\tau)| d\tau ds + \int_{-1}^t \int_{-1}^s |g(\tau)| d\tau ds \\ &\leq |\sigma_{1\epsilon}|(\alpha/2+1) + \|h\|_\infty \|\rho_\epsilon\|_\infty \frac{\alpha}{4}(\alpha+2) + L := M_\epsilon. \end{aligned}$$

So,

$$p_T(A_\epsilon\sigma_{0\epsilon} - \sigma_{0\epsilon}) \leq p_T(M_\epsilon) = O_{\epsilon \rightarrow 0}(\epsilon^c), \quad c \in \mathbb{R}.$$

On the other hand, we have

$$\begin{aligned}
 A_\epsilon \sigma_\epsilon(t) - A_\epsilon \varrho_\epsilon(t) &= \int_{-1}^1 \int_{-1}^s f(\sigma_\epsilon(\tau)) \rho_\epsilon(\tau) d\tau ds - \int_{-1}^1 \int_{-1}^s f(\varrho_\epsilon(\tau)) \rho_\epsilon(\tau) d\tau ds \\
 &= \int_{-1}^1 \int_{-1}^s [f(\sigma_\epsilon(\tau)) - f(\varrho_\epsilon(\tau))] \rho_\epsilon(\tau) d\tau ds \\
 &\leq k \|\rho_\epsilon\|_\infty \int_{-1}^1 \int_{-1}^s |\sigma_\epsilon(\tau) - \varrho_\epsilon(\tau)| d\tau ds \\
 &\leq k \|\rho_\epsilon\|_\infty \frac{\alpha}{4} (\alpha + 2) p_T(\sigma_\epsilon - \varrho_\epsilon) \\
 &\leq k_\epsilon p_T(\sigma_\epsilon - \varrho_\epsilon).
 \end{aligned}$$

Thus,  $p_T(A_\epsilon \sigma_\epsilon - A_\epsilon \varrho_\epsilon) \leq k_\epsilon p_T(\sigma_\epsilon - \varrho_\epsilon)$ . It follows that the mapping  $A$  is Lipschitz, since it is continuous in  $\mathcal{G}_E$ . Moreover, from Theorem 4.3,  $A$  has a fixed point which is a solution for (5.5). Once the Theorem 4.3 is applied.

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## SOME RESULTS ON NORMAL ALMOST CONTACT MANIFOLDS WITH B-METRIC

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**ABSTRACT.** In this study, normal almost contact manifolds with B-metric are considered. Almost complex manifolds with Norden metric are obtained by multiplying almost contact manifolds with B-metric by warped product (by using a function on real numbers). New examples of normal almost complex manifolds with Norden metric are derived. Furthermore, curvature properties of the almost complex manifolds with Norden metric obtained from almost contact manifolds with B-metric are investigated.

### 1. INTRODUCTION

In this work, relations between almost contact manifolds with B-metric and almost complex manifolds with Norden metric are investigated. An almost contact manifold with B-metric is obtained from an almost complex manifold with Norden metric using a method similar to that in [4]. The classifications of almost contact manifolds with B-metric and almost complex manifolds with Norden metric are made by using the covariant derivative of their fundamental tensors. Classification of almost contact manifolds with B-metric and almost complex manifolds with Norden metric can be found in [2, 3], respectively. Relations between the almost contact manifolds with B-metric and almost complex manifolds with Norden metric are investigated in [8, 9] with respect to these classifications.

In this manuscript, we study the warped product of almost contact manifolds with B-metric and  $\mathbb{R}$ . After presenting necessary preliminary informations, we obtain an

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*Key words and phrases.* Almost contact manifold with B-metric, almost complex manifold with Norden metric, Einstein manifold, curvature, warped product.

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almost complex structure on the product of an almost contact manifold with B-metric with  $\mathbb{R}$ . Then, we define a metric which is indefinite of signature  $(n + 1, n + 1)$  on the product manifold. The product manifold is an almost complex manifold with Norden metric. We write the covariant derivative of the metric and the almost complex structure of the almost complex manifold with Norden metric in terms of the covariant derivative of the metric of the almost contact manifold with B-metric. We, then, state the relations between normal classes of almost contact manifolds with B-metric and almost complex manifolds with Norden manifolds.

## 2. PRELIMINARIES

A  $(2n + 1)$ -dimensional smooth manifold  $M$  is said to have an almost contact structure  $(\varphi, \xi, \eta)$ , if this manifold admits an endomorphism  $\varphi$  of the tangent bundle, a vector field  $\xi$  and its dual 1-form  $\eta$  such that the conditions

$$(2.1) \quad \varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1,$$

are satisfied for an arbitrary vector field  $X$ . If a manifold admits an almost contact structure, it is called an almost contact manifold. If  $(M, \varphi, \xi, \eta)$  is an almost contact manifold endowed with a pseudo-Riemannian metric  $g$  of signature  $(n + 1, n)$  such that

$$(2.2) \quad g(\varphi(X), \varphi(Y)) = -g(X, Y) + \eta(X)\eta(Y),$$

for all vector fields  $X, Y$ , then  $M$  is called an almost contact manifold with B-metric. From equations (2.1) and (2.2), one can easily see that

$$\eta(\varphi(X)) = 0, \quad \varphi(\xi) = 0, \quad \eta(X) = g(\xi, X), \quad g(\varphi(X), Y) = g(X, \varphi(Y)),$$

for all vector fields  $X, Y$ . The tensor  $\tilde{g}$  given by

$$\tilde{g}(X, Y) = g(X, \varphi(Y)) + \eta(X)\eta(Y),$$

is a B-metric associated with the metric  $g$ . Let  $\nabla$  be the Levi-Civita connection of the pseudo-metric  $g$ . For all vector fields  $X, Y, Z$  on  $M$ , we define the structure tensor  $\alpha$  of type  $(0, 3)$  as

$$\alpha(X, Y, Z) = g((\nabla_X \varphi)(Y), Z).$$

It is not difficult to see that the tensor  $\alpha$  has following properties:

$$(2.3) \quad \begin{aligned} \alpha(X, Y, Z) &= \alpha(X, Z, Y), \\ \alpha(X, \varphi(Y), \varphi(Z)) &= \alpha(X, Y, Z) - \eta(Y)\alpha(X, \xi, Z) - \eta(Z)\alpha(X, Y, \xi), \end{aligned}$$

$$(2.4) \quad \alpha(X, \xi, \xi) = 0.$$

The following 1-forms are defined as

$$\theta(X) = g^{ij}\alpha(E_i, E_j, X), \quad \theta^*(X) = g^{ij}\alpha(E_i, \varphi(E_j), X), \quad w(X) = \alpha(\xi, \xi, X),$$

where  $\{E_1, \dots, E_{2n}, \xi\}$  is a local frame,  $X$  is a vector field and  $(g^{ij})$  is the inverse matrix of  $(g_{ij})$ .

Using the properties above, the space of covariant derivatives of the endomorphism  $\varphi$  are defined as

$$\mathcal{F} = \left\{ \alpha \in \otimes_3^0 M : \alpha(X, Y, Z) = \alpha(X, Z, Y), \right. \\ (2.5) \quad \left. \alpha(X, \varphi(Y), \varphi(Z)) = \alpha(X, Y, Z) - \eta(Y)\alpha(X, \xi, Z) - \eta(Z)\alpha(X, Y, \xi) \right\}$$

The space  $\mathcal{F}$  decomposes into eleven subspaces

$$\mathcal{F} = \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_{11},$$

which are orthogonal and invariant under the action of  $G \times I$  where  $I$  is the identity on  $\text{Span}\{\xi\}$  and  $G = GL(n, \mathbb{C}) \cap O(n, n)$  [2]. An almost contact manifold  $M$  is called normal if the corresponding almost complex structure  $J$  on the even dimensional product manifold  $M \times \mathbb{R}$  is integrable, i.e., the Nijenhuis torsion  $[J, J]$  is identically zero [1, 7], or equivalently  $N = [\varphi, \varphi] + d\eta \otimes \xi = 0$ , or equivalently

$$(2.6) \quad \alpha(X, Y, \xi) = \alpha(Y, X, \xi),$$

$$(2.7) \quad \mathfrak{S}_{X,Y,Z} \{ \alpha(X, Y, \varphi(Z)) - \alpha(X, \varphi(Y), \xi)\eta(Z) \} = 0,$$

see [10]. In this study, we consider only the classes of normal almost contact manifolds with B-metric. The class of the normal contact manifolds with B-metric is  $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$  [6], and defining relations of this subspaces are:

$$(2.8) \quad \mathcal{F}_1 : \alpha(X, Y, Z) = \frac{1}{2n} \left\{ g(X, \varphi(Y))\theta(\varphi(Z)) + g(X, \varphi(Z))\theta(\varphi(Y)) \right. \\ \left. + g(\varphi(X), \varphi(Y))\theta(\varphi^2(Z)) + g(\varphi(X), \varphi(Z))\theta(\varphi^2(Y)) \right\},$$

$$(2.9) \quad \mathcal{F}_2 : \alpha(\xi, Y, Z) = \alpha(X, \xi, Z) = 0, \quad \theta = 0, \\ \alpha(X, Y, \varphi(Z)) + \alpha(Y, Z, \varphi(X)) + \alpha(Z, X, \varphi(Y)) = 0,$$

$$(2.10) \quad \mathcal{F}_4 : \alpha(X, Y, Z) = -\frac{\theta(\xi)}{2n} \{ g(\varphi(X), \varphi(Y))\eta(Z) + g(\varphi(X), \varphi(Z))\eta(Y) \},$$

$$(2.11) \quad \mathcal{F}_5 : \alpha(X, Y, Z) = -\frac{\theta^*(\xi)}{2n} \{ g(X, \varphi(Y))\eta(Z) + g(X, \varphi(Z))\eta(Y) \},$$

$$(2.12) \quad \mathcal{F}_6 : \alpha(X, Y, Z) = \alpha(X, Y, \xi)\eta(Z) + \alpha(X, Z, \xi)\eta(Y), \\ \alpha(X, Y, \xi) = \alpha(Y, X, \xi) = -\alpha(\varphi(X), \varphi(Y), \xi), \quad \theta(\xi) = \theta^*(\xi) = 0.$$

If a smooth manifold  $N$  has a tensor field  $J$  (almost complex structure) and a pseudo-Riemannian metric  $h$  satisfying the conditions

- $J^2(X) = -X,$
- $h(J(X), J(Y)) = -h(X, Y),$

for all vector fields  $X, Y$  on  $N$ , then the manifold  $N$  is called an almost complex manifold with a Norden metric [3]. The metric  $h$  is necessarily indefinite of signature  $(n, n)$ . An almost complex manifold with a Norden metric has even dimension ( $\dim N = 2n$ ) and the structure group of the tangent bundle reduces to the group

$GL(n, \mathbb{C}) \cap O(n, n)$ . The tensor  $\tilde{h}$  given by  $\tilde{h}(X, Y) = h(J(X), Y)$  for any vector field  $X, Y$  is symmetric:

$$h(J(X), Y) = -h(J^2(X), J(Y)) = h(X, J(Y)).$$

The structure tensor  $F$  of type  $(0, 3)$  on  $M$  is defined as

$$F(X, Y, Z) = h((\nabla_X J)(Y), Z).$$

The tensor  $F$  has the following properties:

$$\begin{aligned} F(X, Y, Z) &= F(X, Z, Y), \\ F(X, J(Y), J(Z)) &= F(X, Y, Z). \end{aligned}$$

In addition, for any vector field  $X$  on  $N$  the 1-form  $\tilde{\theta}$  associated with  $F$  is defined as

$$\tilde{\theta}(X) = h^{ij} F(E_i, E_j, X),$$

where  $\{E_1, E_2, \dots, E_{2n}\}$  is a frame field on  $N$  and  $h^{ij}$  is the inverse matrix of  $h$ .

Then the subspace  $W$  of  $\otimes_3^0 N$  is defined as follows:

$$W := \left\{ \alpha \in \otimes_3^0 N \mid \alpha(X, Y, Z) = \alpha(X, J(Y), J(Z)) = \alpha(X, Z, Y) \right\},$$

where  $X, Y, Z$  are vector fields on  $N$ . According to the symmetries of  $W$ , this space splits into the direct sum  $W = W_1 \oplus W_2 \oplus W_3$ . The subspaces  $W_i$  are invariant and irreducible under the group  $GL(n, \mathbb{C}) \cap O(n, n)$ . The defining relations for invariant subspaces are the following.

(a) Kaehler manifolds with a Norden metric:

$$(2.13) \quad F(X, Y, Z) = 0.$$

(b) Class  $W_1$ : Conformally Kaehlerian manifolds with a Norden metric:

$$(2.14) \quad F(X, Y, Z) = \frac{1}{2n} \left\{ h(X, Y)\tilde{\theta}(Z) + h(X, Z)\tilde{\theta}(Y) + h(X, J(Y))\tilde{\theta}(J(Z)) + h(X, J(Z))\tilde{\theta}(J(Y)) \right\}.$$

(c) Class  $W_2$ : Special complex manifolds with a Norden metric

$$(2.15) \quad F(X, Y, J(Z)) + F(Y, Z, J(X)) + F(Z, X, J(Y)) = 0,$$

and  $\tilde{\theta} = 0$ .

(d) Class  $W_3$ : Quasi-Kaehlerian manifolds with Norden metric

$$(2.16) \quad F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0.$$

(e) Class  $W_1 \oplus W_2$ : Complex manifolds with Norden metric

$$F(X, Y, J(Z)) + F(Y, Z, J(X)) + F(Z, X, J(Y)) = 0,$$

or equivalently  $N = 0$ .

(f) Class  $W_2 \oplus W_3$ : Semi-Kaehlerian manifolds with Norden metric

$$(2.17) \quad \tilde{\theta} = 0.$$

(g) Class  $W_1 \oplus W_3$ :

$$\begin{aligned}
 F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = & \frac{1}{n} \left\{ h(X, Y)\tilde{\theta}(Z) + h(X, Z)\tilde{\theta}(Y) \right. \\
 & + h(Y, Z)\tilde{\theta}(X) + h(X, J(Y))\tilde{\theta}(J(Z)) \\
 & + h(X, J(Z))\tilde{\theta}(J(Y)) \\
 & \left. + h(Y, J(Z))\tilde{\theta}(J(X)) \right\}.
 \end{aligned}
 \tag{2.18}$$

(h) Class  $W_1 \oplus W_2 \oplus W_3$ : No relation.

### 3. ALMOST COMPLEX MANIFOLDS WITH A NORDEN METRIC FROM ALMOST CONTACT MANIFOLDS WITH B-METRIC

In this section, first, we define an almost complex structure on the product of an almost contact manifold with B-metric with  $\mathbb{R}$ . We write a metric on the product manifold depending on a function  $\sigma$  where  $\sigma : M \times \mathbb{R} \rightarrow \mathbb{R}$  only depends on  $t$ . Then, we obtain an almost complex manifold with Norden metric and we give the relations between covariant derivatives.

Let  $(M, \varphi, \xi, \eta, g)$  be a  $(2n + 1)$ -dimensional almost contact manifold with B-metric and consider the product manifold  $M \times \mathbb{R}$ . A vector field on the manifold  $M \times \mathbb{R}$  is of the form  $\left(X, a \frac{d}{dt}\right)$  where  $t$  is the coordinate of  $\mathbb{R}$  and  $a$  is a smooth function on  $M \times \mathbb{R}$ . The almost complex structure  $J$  on  $M \times \mathbb{R}$  is defined by

$$J \left( X, a \frac{d}{dt} \right) = \left( \varphi(X) - ae^{-\sigma}\xi, e^\sigma \eta(X) \frac{d}{dt} \right).
 \tag{3.1}$$

Then  $J^2 = -I$ . In addition, we define a pseudo-Riemannian metric on  $M \times \mathbb{R}$  with signature  $(n + 1, n + 1)$  by

$$h \left( \left( X, a \frac{d}{dt} \right), \left( Y, b \frac{d}{dt} \right) \right) := e^{2\sigma} g(X, Y) - ab.
 \tag{3.2}$$

One can easily see that

$$h \left( J \left( X, a \frac{d}{dt} \right), J \left( Y, b \frac{d}{dt} \right) \right) = -h \left( \left( X, a \frac{d}{dt} \right), \left( Y, b \frac{d}{dt} \right) \right).
 \tag{3.3}$$

Hence  $(M \times \mathbb{R}, J, h)$  is an almost complex manifold with Norden metric. Let  $\nabla$  be the Levi-Civita covariant derivative of the pseudo-Riemannian metric  $g$  on  $M$ . Levi-Civita covariant derivative of the metric  $h$  on  $M \times \mathbb{R}$  is obtained using the Kozsul formula as

$$\nabla_{\left(X, a \frac{d}{dt}\right)} \left( Y, b \frac{d}{dt} \right) = \left( \nabla_X Y + \frac{d\sigma}{dt}(aY + bX), \left\{ X[b] + a \frac{db}{dt} + e^{2\sigma} \frac{d\sigma}{dt} g(X, Y) \right\} \frac{d}{dt} \right).$$

Note that the covariant derivative on the product manifold  $M \times \mathbb{R}$  will also be denoted with the same symbol  $\nabla$ . Also, covariant derivative of the almost complex structure

$J$  is calculated as

$$\begin{aligned} \left(\nabla_{(X, a \frac{d}{dt})} J\right) \left(Y, b \frac{d}{dt}\right) &= \left( (\nabla_X \varphi)(Y) - be^{-\sigma} \nabla_X \xi - b \frac{d\sigma}{dt} \varphi(X) \right. \\ &\quad \left. + e^\sigma \frac{d\sigma}{dt} (\eta(Y)X + g(X, Y)\xi), \right. \\ &\quad \left. \left\{ -2be^\sigma \frac{d\sigma}{dt} \eta(X) + e^\sigma (\nabla_X \eta)(Y) + e^{2\sigma} \frac{d\sigma}{dt} g(X, \varphi(Y)) \right\} \frac{d}{dt} \right), \end{aligned}$$

for any vector field  $(X, a \frac{d}{dt})$ ,  $(Y, b \frac{d}{dt})$  and  $(Z, c \frac{d}{dt})$  on  $M \times \mathbb{R}$ . It follows that

$$\begin{aligned} (3.4) \quad F(\tilde{X}, \tilde{Y}, \tilde{Z}) &= h\left((\nabla_{\tilde{X}} J)(\tilde{Y}), \tilde{Z}\right) \\ &= e^{2\sigma} \alpha(X, Y, Z) + 2bce^\sigma \frac{d\sigma}{dt} \eta(X) \\ &\quad - e^\sigma \{bg(\nabla_X \xi, Z) + cg(\nabla_X \xi, Y)\} \\ &\quad + e^{3\sigma} \frac{d\sigma}{dt} \{\eta(Y)g(X, Z) + \eta(Z)g(X, Y)\} \\ &\quad - e^{2\sigma} \frac{d\sigma}{dt} \{bg(X, \varphi(Z)) + cg(X, \varphi(Y))\}, \end{aligned}$$

is obtained. If we take  $\tilde{X} = (\xi, 0)$ ,  $\tilde{Y} = \tilde{Z} = (0, \frac{d}{dt})$ , then  $F(\tilde{X}, \tilde{Y}, \tilde{Z}) = 2e^\sigma \frac{d\sigma}{dt}$  is different than zero for non-constant  $\sigma$ . Thus,  $F$  is not equal to zero for any function  $\sigma$ . Since

$$\nabla_{(X, a \frac{d}{dt})}(\xi, 0) = \left( \nabla_X \xi + \frac{d\sigma}{dt} a\xi, e^{2\sigma} \frac{d\sigma}{dt} \eta(X) \frac{d}{dt} \right) \neq 0,$$

$(\xi, 0)$  is not parallel even if  $\xi$  is parallel. In addition, if  $\xi$  is Killing,  $(\xi, 0)$  is also Killing:

$$\begin{aligned} (3.5) \quad h\left(\nabla_{(X, a \frac{d}{dt})}(\xi, 0), \left(Y, b \frac{d}{dt}\right)\right) &= e^{2\sigma} g(\nabla_X \xi, Y) + ae^{2\sigma} \frac{d\sigma}{dt} \eta(Y) - be^{2\sigma} \frac{d\sigma}{dt} \eta(Y) \\ &= -h\left(\nabla_{(Y, b \frac{d}{dt})}(\xi, 0), \left(X, a \frac{d}{dt}\right)\right). \end{aligned}$$

Note that

$$h\left(\nabla_{(X, a \frac{d}{dt})} \left(0, \frac{d}{dt}\right), \left(Y, b \frac{d}{dt}\right)\right) = e^\sigma \frac{d\sigma}{dt} g(X, Y) = g\left(\nabla_{(Y, b \frac{d}{dt})} \left(0, \frac{d}{dt}\right), \left(X, a \frac{d}{dt}\right)\right).$$

Let  $\{e_1, \dots, e_{2n}, \xi\}$  be a local pseudo-orthonormal frame field on  $M$ . Then one can obtain an orthonormal frame field on  $M \times \mathbb{R}$  as follows:

$$\left\{ (e^{-\sigma} e_1, 0), \dots, (e^{-\sigma} e_{2n}, 0), (e^{-\sigma} \xi, 0), \left(0, \frac{d}{dt}\right) \right\}.$$

Using this frame, the 1-form  $\tilde{\theta}$ , associated with  $F$  given in [3], is evaluated as

$$(3.6) \quad \tilde{\theta} \left(X, a \frac{d}{dt}\right) = \theta(X) - ae^{-\sigma} \theta^*(\xi) + w(X) + 2(n+1)e^\sigma \frac{d\sigma}{dt} \eta(X).$$

In addition, we write the curvature tensor  $\tilde{R}$  on the product manifold  $M \times \mathbb{R}$  with respect to the curvature tensor  $R$  on  $M$ . Let  $\tilde{X} = (X, a\frac{d}{dt})$ ,  $\tilde{Y} = (Y, b\frac{d}{dt})$ ,  $\tilde{Z} = (Z, c\frac{d}{dt})$  be vector fields on the product manifold  $M \times \mathbb{R}$ , then we have

$$(3.7) \quad \begin{aligned} \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = & \left( R(X, Y)Z + c \left( \left( \frac{d\sigma}{dt} \right)^2 + \frac{d^2\sigma}{dt^2} \right) (aY - bX) \right. \\ & + e^{2\sigma} \left( \frac{d\sigma}{dt} \right)^2 (g(Y, Z)X - g(X, Z)Y), \\ & \left. e^{2\sigma} \left( \left( \frac{d\sigma}{dt} \right)^2 + \frac{d^2\sigma}{dt^2} \right) g(aY - bX, Z) \frac{d}{dt} \right). \end{aligned}$$

As a result, Ricci curvature can be calculated as

$$(3.8) \quad \begin{aligned} \tilde{Q}(\tilde{X}, \tilde{Y}) = & Q(X, Y) - ab(2n + 1) \left( \left( \frac{d\sigma}{dt} \right)^2 + \frac{d^2\sigma}{dt^2} \right) \\ & + e^{2\sigma} \left( \frac{d\sigma}{dt} \right)^2 (2n + 1)g(X, Y) + e^{2\sigma} \frac{d^2\sigma}{dt^2} g(X, Y). \end{aligned}$$

In addition, we can evaluate the scalar curvature as

$$(3.9) \quad \tilde{s} = e^{-2\sigma}s + (2n + 1)(2n + 2) \left( \frac{d\sigma}{dt} \right)^2 + 2(2n + 1) \frac{d^2\sigma}{dt^2}.$$

Let  $M$  be an almost contact manifold with B-metric with zero scalar curvature. Then we can construct an almost complex manifold with Norden metric with scalar curvature  $k > 0$  with the appropriate choice of the function  $\sigma$ , see Example (3.1). If we take  $s = 0$ , then the solution of the differential equation

$$(3.10) \quad k = (2n + 1)(2n + 2) \left( \frac{d\sigma}{dt} \right)^2 + 2(2n + 1) \frac{d^2\sigma}{dt^2},$$

takes the form

$$(3.11) \quad \sigma(t) = \frac{1}{n + 1} \ln \left[ \cosh \left( \sqrt{2k} \left( \frac{\sqrt{n + 1}}{2\sqrt{2n + 1}} t - \sqrt{(2n + 1)(n + 1)} c_1 \right) \right) \right] + c_2,$$

where  $c_1, c_2 \in \mathbb{R}$ .

If the almost contact manifold with B-metric  $M$  is Einstein, that is  $Q(X, Y) = \lambda g(X, Y)$ , then the almost complex manifold with Norden metric  $M \times \mathbb{R}$  is Einstein if and only if

$$(3.12) \quad \frac{\lambda}{2n} = e^{2\sigma} \frac{d^2\sigma}{dt^2}.$$

If  $K = (2n + 1) \left( \left( \frac{d\sigma}{dt} \right)^2 + \frac{d^2\sigma}{dt^2} \right)$ , then we have  $\tilde{Q}(\tilde{X}, \tilde{Y}) = Kh(\tilde{X}, \tilde{Y})$ .

Differential equation (3.12) has the solution

$$\sigma(t) = \ln \left( \frac{1}{2} e^{-\sqrt{c_1}(t+c_2)} \lambda + \frac{e^{\sqrt{c_1}(t+c_2)}}{4nc_1} \right),$$

where  $c_1, c_2 \in \mathbb{R}$  and  $c_1 > 0$ . If Einstein constant  $\lambda > 0$ , domain of the function  $\sigma$  is the set of all real numbers. Hence the product manifold  $M \times \mathbb{R}$  is Einstein with Einstein constant  $K = (2n+1)c_1 > 0$ , since  $c_1 = \left( \left( \frac{d\sigma}{dt} \right)^2 + \frac{d^2\sigma}{dt^2} \right)$ .

If  $\lambda < 0$ , it can be easily seen that domain of the function  $\sigma$  is  $(t_0, +\infty)$ , where  $t_0 = \frac{1}{2\sqrt{c_1}} \ln(-2nc_1\lambda) - c_2$ . Then the product manifold  $M \times (t_0, +\infty)$  is Einstein with Einstein constant  $K = (2n+1)c_1 > 0$ . If  $\lambda = 0$ , then solution of the equation (3.12) is  $\sigma(t) = c_1 t + c_2$ , where  $c_1, c_2 \in \mathbb{R}$ . In this case,  $K = (2n+1)c_1^2$  is obtained. Hence, for all cases, we obtain Einstein product manifold with positive Einstein constant.

Now the relations between classes of the product manifold  $M \times \mathbb{R}$  and the classes of the almost contact manifold with B-metric  $M$  are investigated.

**Theorem 3.1.** *If  $(M, \varphi, \xi, \eta, g)$  is a cosymplectic almost contact B-metric manifold, then the product manifold  $M \times \mathbb{R}$  is of the class  $W_1$  for all non-constant  $\sigma$  functions.*

*Proof.* Since  $(M, \varphi, \xi, \eta, g)$  is a cosymplectic almost contact B-metric manifold we have

$$\tilde{\theta}(\tilde{X}) = 2(n+1)e^\sigma \frac{d\sigma}{dt} \eta(X), \quad \tilde{\theta}(J(\tilde{X})) = -2a(n+1) \frac{d\sigma}{dt}.$$

Implying that  $\tilde{\theta} \neq 0$  and the product manifold is not of the class  $W_2$ . Since  $\alpha(X, Y, Z) = 0$  for all vector fields  $X, Y, Z$  from (3.4), we have

$$\begin{aligned} F(\tilde{X}, \tilde{Y}, \tilde{Z}) &= e^{3\sigma} \frac{d\sigma}{dt} \{ \eta(Y)g(X, Z) + \eta(Z)g(X, Y) \} \\ &\quad - e^{2\sigma} \frac{d\sigma}{dt} \{ bg(X, \varphi(Z)) + cg(X, \varphi(Y)) \} + 2bce^\sigma \frac{d\sigma}{dt} \eta(X), \end{aligned}$$

which is equivalent to the right hand side of the defining relation (2.14) of a conformally Kaehlerian manifold by direct calculation. Thus for a non-constant function  $\sigma$ , the product manifold is a conformally Kaehlerian manifold with Norden metric (class  $W_1$ ).  $\square$

**Theorem 3.2.** *If  $(M, \varphi, \xi, \eta, g)$  is of the class  $\mathcal{F}_1$ , then the product manifold  $M \times \mathbb{R}$  is of the class  $W_1 \oplus W_2$  for all non-constant  $\sigma$  functions.*

*Proof.* Since  $(M, \varphi, \xi, \eta, g)$  is of the class  $\mathcal{F}_1$ , we have

$$\theta(\xi) = 0, \quad \theta^*(\xi) = 0,$$

for all vector fields  $X$  on  $M$  and  $\xi$  is parallel [6]. Then, we have

$$\tilde{\theta} \left( X, a \frac{d}{dt} \right) = \theta(X) + 2(n+1)e^\sigma \frac{d\sigma}{dt} \eta(X).$$

Hence we say that  $\tilde{\theta} \neq 0$  and the product manifold  $M \times \mathbb{R}$  is not of the class  $W_2$ .

When  $M$  is of the class  $\mathcal{F}_1$ , if we take  $\tilde{X} = (0, \frac{d}{dt})$ ,  $\tilde{Y} = (Y, 0)$  and  $\tilde{Z} = (\xi, 0)$  in equation (2.14), the left hand side of (2.14) becomes  $F(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$ , whereas the right hand side of (2.14) is

$$-\frac{1}{2(n+1)}e^{\sigma}\theta(\varphi(Y)).$$

Thus, the equation (2.14) is not satisfied and the product manifold  $M \times \mathbb{R}$  is not in the class  $W_1$ . Since

$$\begin{aligned} (3.13) \quad F(\tilde{X}, \tilde{Y}, J(\tilde{Z})) &= e^{2\sigma}\alpha(X, Y, \varphi(Z)) - e^{2\sigma}\eta(Z)g(\nabla_X\xi, Y) \\ &\quad - e^{\sigma}\{c\alpha(X, Y, \xi) - b\alpha(X, Z, \xi)\} \\ &\quad + e^{3\sigma}\frac{d\sigma}{dt}\{\eta(Y)g(X, \varphi(Z)) - \eta(Z)g(X, \varphi(Y))\} \\ &\quad - e^{2\sigma}\frac{d\sigma}{dt}\{bg(\varphi(X), \varphi(Z)) - cg(\varphi(X), \varphi(Y))\} \\ &\quad + 2e^{2\sigma}\frac{d\sigma}{dt}\eta(X)\{b\eta(Z) - c\eta(Y)\}, \end{aligned}$$

it can be checked that the normality condition (2.15) is satisfied when the manifold  $M$  belongs to class  $\mathcal{F}_1$ . As a result the product manifold  $M \times \mathbb{R}$  is of the class  $W_1 \oplus W_2$ . □

**Theorem 3.3.** *If  $(M, \varphi, \xi, \eta, g)$  is in  $\mathcal{F}_2$ , then the product manifold  $M \times \mathbb{R}$  is in  $W_1 \oplus W_2$  for all non-constant  $\sigma$  functions.*

*Proof.* Since  $(M, \varphi, \xi, \eta, g)$  is of the class  $\mathcal{F}_2$ , the equation (2.9) yields

$$\theta(X) = 0, \quad \theta^*(X) = 0,$$

for all vector fields  $X$  on  $M$  and  $\xi$  is parallel [6]. Then,

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X).$$

Hence,  $\tilde{\theta} \neq 0$ , as a result  $M \times \mathbb{R}$  is not in  $W_2$ .

When  $M$  is of the class  $\mathcal{F}_2$ , if we take  $\tilde{X} = \tilde{Y} = (0, \frac{d}{dt})$  and  $\tilde{Z} = (\xi, 0)$  in equation (2.14), the left hand side of equation (2.14) is  $F(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$ , and the right hand side of the equation (2.14) is  $-e^{\sigma}\frac{d\sigma}{dt}$ . Thus, the equation (2.14) is satisfied if and only if  $\sigma$  is constant. So,  $M \times \mathbb{R}$  does not belong to  $W_1$  for non-constant  $\sigma$ . In addition, one can easily check that normality condition (2.15) is satisfied when the manifold  $M$  is of the class  $\mathcal{F}_2$ . To sum up  $M \times \mathbb{R}$  is in  $W_1 \oplus W_2$  for non-constant  $\sigma$ . □

**Theorem 3.4.** *If  $(M, \varphi, \xi, \eta, g)$  is of the class  $\mathcal{F}_4$ , then the product manifold  $M \times \mathbb{R}$  is of the class  $W_1 \oplus W_2$  for all non-constant  $\sigma$  functions.*

*Proof.* Since  $(M, \varphi, \xi, \eta, g)$  is of the class  $\mathcal{F}_4$ , the equation (2.10) gives

$$(3.14) \quad \theta(X) = \eta(X)\theta(\xi), \quad \theta^*(\xi) = 0, \quad \omega(X) = 0, \quad \nabla_X \xi = \frac{\theta(\xi)}{2n}.$$

From equations (3.4) and (3.14), we get

$$\begin{aligned} F(\tilde{X}, \tilde{Y}, \tilde{Z}) &= -e^{2\sigma} \left( \frac{\theta(\xi)}{2n} + e^\sigma \frac{d\sigma}{dt} \right) \{ \eta(Y)g(\varphi(X), \varphi(Z)) + \eta(Z)g(\varphi(X), \varphi(Y)) \} \\ &\quad - e^\sigma \left( \frac{\theta(\xi)}{2n} + e^\sigma \frac{d\sigma}{dt} \right) \{ bg(X, \varphi(Z)) + cg(X, \varphi(Y)) \} \\ &\quad + 2e^{3\sigma} \frac{d\sigma}{dt} \eta(X)\eta(Y)\eta(Z) + 2bce^\sigma \frac{d\sigma}{dt} \eta(X) \end{aligned}$$

and

$$\begin{aligned} F(\tilde{X}, \tilde{Y}, J(\tilde{Z})) &= e^{2\sigma} \left( \frac{\theta(\xi)}{2n} + e^\sigma \frac{d\sigma}{dt} \right) \{ \eta(Y)g(X, \varphi(Z)) - \eta(Z)g(X, \varphi(Y)) \} \\ &\quad - e^\sigma \left( \frac{\theta(\xi)}{2n} + e^\sigma \frac{d\sigma}{dt} \right) \{ bg(\varphi(X), \varphi(Z)) - cg(\varphi(X), \varphi(Y)) \} \\ &\quad - 2e^{2\sigma} \frac{d\sigma}{dt} \eta(X) (c\eta(Y) - b\eta(Z)) \eta(Y). \end{aligned}$$

One can see that the normality condition (2.15) is satisfied. Hence the product manifold  $M \times \mathbb{R}$  is of the class  $W_1 \oplus W_2$ . In addition, we have

$$\tilde{\theta} \left( X, a \frac{d}{dt} \right) = \eta(X) \left( \theta(\xi) + 2(n+1)e^\sigma \frac{d\sigma}{dt} \right).$$

If we take  $\tilde{X} = (X, 0)$ ,  $\tilde{Y} = \left( Y, \frac{d}{dt} \right)$  and  $\tilde{Z} = \left( Z, \frac{d}{dt} \right)$ , then the equation (2.14) is not satisfied. Hence  $M \times \mathbb{R}$  is not of the class  $W_1$ . If  $\theta(\xi)$  is constant and the function  $\sigma$  has the property that

$$e^\sigma \frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2(n+1)},$$

then the product manifold  $M \times \mathbb{R}$  is of the class  $W_2$ . □

**Theorem 3.5.** *If  $(M, \varphi, \xi, \eta, g)$  is of the class  $\mathcal{F}_5$ , then the product manifold  $M \times \mathbb{R}$  is of the class  $W_1 \oplus W_2$  for all non-constant  $\sigma$  functions.*

*Proof.* If  $(M, \varphi, \xi, \eta, g)$  belongs to  $\mathcal{F}_5$ , then from (2.11) we have

$$\theta(X) = 0, \quad \theta^*(X) = \theta^*(\xi)\eta(X), \quad w(X) = 0,$$

for all vector fields  $X$  on  $M$  [6]. Therefore,

$$\tilde{\theta} \left( X, a \frac{d}{dt} \right) = -e^{-\sigma} a \theta^*(\xi) + 2(n+1)e^\sigma \frac{d\sigma}{dt} \eta(X) \neq 0,$$

since  $\tilde{\theta}\left(0, \frac{d}{dt}\right) = -e^{-\sigma}\theta^*(\xi)$ . Therefore  $M \times \mathbb{R}$  is not in  $W_2$ . Replacing  $\tilde{X} = \left(0, \frac{d}{dt}\right)$ ,  $\tilde{Y} = (\xi, 0)$  and  $\tilde{Z} = (\xi, 0)$  in defining relation (2.14) of the class  $W_1$ , we have

$$\theta^*(\xi) = 0.$$

This is a contradiction since  $\theta^*(\xi) \neq 0$  is in the class  $\mathcal{F}_5$  for non-constant function  $\sigma$ . In addition, if the manifold  $M$  is of the class  $\mathcal{F}_5$ , then one can easily check that equation (2.15) is satisfied on  $M \times \mathbb{R}$ . Hence, if the manifold  $M$  is of the class  $\mathcal{F}_5$ , the product manifold is of the class  $W_1 \oplus W_2$ .  $\square$

**Theorem 3.6.** *If  $(M, \varphi, \xi, \eta, g)$  is in  $\mathcal{F}_6$ , then the product manifold  $M \times \mathbb{R}$  is in  $W_1 \oplus W_2$  for all non-constant  $\sigma$  functions.*

*Proof.* Since  $(M, \varphi, \xi, \eta, g)$  is of the class  $\mathcal{F}_6$ , by (2.12) we obtain

$$\theta(X) = 0, \quad \theta^*(X) = 0, \quad w(X) = 0,$$

for all vector fields  $X$  on  $M$  [6]. Then,

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = 2(n+1)e^\sigma \frac{d\sigma}{dt} \eta(X) \neq 0,$$

since, for instance,  $\tilde{\theta}(\xi, 0) = 2(n+1)e^\sigma \frac{d\sigma}{dt}$  is not equal to zero for non-constant function  $\sigma$ . Choosing  $\tilde{X} = \left(0, \frac{d}{dt}\right)$ ,  $\tilde{Y} = (Y, 0)$  and  $\tilde{Z} = \left(0, \frac{d}{dt}\right)$  in the defining relation (2.14) of the class  $W_1$ , we have

$$e^\sigma \frac{d\sigma}{dt} \eta(Y) = 0.$$

This is a contradiction for a non-constant function  $\sigma$ . One can also check that if the manifold  $M$  is of the class  $\mathcal{F}_6$ , then the equation (2.15) is satisfied on  $M \times \mathbb{R}$ . Hence, if the manifold  $M$  is of the class  $\mathcal{F}_6$ , the product manifold is of the class  $W_1 \oplus W_2$ .  $\square$

Now we show that if the product manifold  $M \times \mathbb{R}$  is normal, then so is  $M$ .

**Theorem 3.7.** *If the product manifold  $M \times \mathbb{R}$  is of the class  $W_1$ , then the almost contact manifold with B-metric is cosymplectic.*

*Proof.* If the product manifold  $M \times \mathbb{R}$  is of the class  $W_1$ , the defining relation (2.14) is satisfied for all vector fields  $\tilde{X}, \tilde{Y}$  and  $\tilde{Z}$ . Take  $\tilde{X} = \left(0, \frac{d}{dt}\right)$  and  $\tilde{Y} = \tilde{Z} = (\xi, 0)$ , we get  $\theta^*(\xi) = 0$ . Replace  $\tilde{X} = \left(0, \frac{d}{dt}\right)$  and  $\tilde{Y} = (\xi, 0)$  and  $\tilde{Z} = (Z, 0)$  in (2.14) to get  $\theta(\varphi(Z)) = -w(\varphi(Z))$ , and hence  $\theta(Z) + w(Z) = \eta(Z)\theta(\xi)$ . For  $\tilde{X} = (X, 0)$ ,  $\tilde{Y} = (Y, 0)$  and  $\tilde{Z} = (Z, 0)$ , we get

$$(3.15) \quad \alpha(X, Y, Z) = \frac{\theta(\xi)}{2(n+1)} \{ \eta(Y)g(X, Z) + \eta(Z)g(X, Y) \}.$$

Note that  $\alpha(X, \varphi(Y), \varphi(Z)) = 0$ . From the equation (3.15) we have

$$\theta(X) = \frac{n}{n+1} \theta(\xi) \eta(X),$$

for all vector field on  $M$ . Then, we obtain  $\theta(\xi) = 0$ . Hence, we get  $\alpha(X, Y, Z) = 0$ .  $\square$

**Theorem 3.8.** *If the product manifold  $M \times \mathbb{R}$  is of the class  $W_2$ , then the almost contact manifold with B-metric is of the class  $\mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_6$ .*

*Proof.* Since the product manifold  $M \times \mathbb{R}$  is of the class  $W_2$ ,  $\tilde{\theta}(X, a \frac{d}{dt}) = 0$  for all vector fields  $(X, a \frac{d}{dt})$ . We, thus, obtain

$$\tilde{\theta} \left( 0, \frac{d}{dt} \right) = -e^{-\sigma} \theta^*(\xi) = 0.$$

Hence,  $\theta^*(\xi) = 0$  and

$$\tilde{\theta}(\xi, 0) = \theta(\xi) + 2(n + 1)e^\sigma \frac{d\sigma}{dt} = 0.$$

Since equation (2.15) is satisfied in the class  $W_2$ , if one takes  $\tilde{X} = (0, \frac{d}{dt})$ ,  $\tilde{Y} = (Y, 0)$  and  $\tilde{Z} = (Z, 0)$ , then the following equation is obtained:

$$\alpha(Y, Z, \xi) = \alpha(Z, Y, \xi).$$

Since  $0 = \alpha(X, \xi, \xi) = \alpha(\xi, X, \xi) = \alpha(\xi, \xi, X)$ , we have  $\nabla_\xi \xi = 0$ . In addition, in the class  $W_2$  we get

$$0 = \tilde{\theta} \left( X, \frac{d}{dt} \right) = \theta(X) + (2n + 1)e^\sigma \frac{d\sigma}{dt} \eta(X),$$

and  $\theta(\varphi(X)) = 0$ . Moreover, taking  $\tilde{X} = (X, 0)$ ,  $\tilde{Y} = (Y, 0)$ ,  $\tilde{Z} = (Z, 0)$  in equation (2.15), we obtain  $\mathfrak{S}_{XYZ} \{ \alpha(X, Y, \varphi(Z)) - \alpha(X, \varphi(Y), \xi) \eta(Z) \} = 0$ . Thus, equations (2.6) and (2.7) hold and the manifold  $M$  is normal  $(\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6)$  [10]. Since  $\theta^*(\xi) = 0$  and  $\theta(\varphi(X)) = 0$ ,  $M$  is in  $\mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_6$ . □

**Theorem 3.9.** *If the product manifold  $M \times \mathbb{R}$  is of the class  $W_1 \oplus W_2$ , then the manifold  $M$  is normal almost contact manifold with B-metric (the class  $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$ ).*

*Proof.* If the product manifold  $M \times \mathbb{R}$  is of the class  $W_1 \oplus W_2$ , the defining relation (2.15) is satisfied for all vector fields  $\tilde{X}, \tilde{Y}$  and  $\tilde{Z}$ . If we take  $\tilde{X} = (X, 0)$ ,  $\tilde{Y} = (Y, 0)$  and  $\tilde{Z} = (Z, 0)$ , we obtain

$$\mathfrak{S}_{XYZ} (\alpha(X, Y, \varphi(Z)) - \alpha(X, \varphi(Y), \xi) \eta(Z)) = 0.$$

In addition, if we take  $\tilde{X} = (0, \frac{d}{dt})$ ,  $\tilde{Y} = (Y, 0)$  and  $\tilde{Z} = (Z, 0)$ , we get

$$\alpha(Y, Z, \xi) = \alpha(Z, Y, \xi).$$

Hence, manifold  $M$  is normal almost contact manifold with B-metric from (2.6) and (2.7) [10]. □

*Example 3.1.* Consider the Lie group  $G$  of dimension 5 with a basis of left-invariant vector fields  $\{e_1, e_2, e_3, e_4, e_5\}$  defined by the non-zero brackets

$$\begin{aligned} [e_1, e_5] &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_1 e_3 + \lambda_4 e_4, \\ [e_2, e_5] &= -\lambda_2 e_1 - \lambda_1 e_2 - \lambda_4 e_3 - \lambda_1 e_4, \\ [e_3, e_5] &= -\lambda_1 e_1 - \lambda_4 e_2 + \lambda_1 e_3 + \lambda_2 e_4, \\ [e_4, e_5] &= \lambda_4 e_1 + \lambda_1 e_2 - \lambda_2 e_3 - \lambda_1 e_4. \end{aligned}$$

One can define an invariant almost contact structure with B-metric on  $G$  as

$$\begin{aligned} g(e_0, e_0) &= g(e_1, e_1) = g(e_2, e_2) = 1, \\ g(e_3, e_3) &= g(e_4, e_4) = -1, \quad g(e_i, e_j) = 0, \quad i \neq j, \\ e_5 &= \xi, \quad \varphi(e_1) = e_3, \quad \varphi(e_2) = e_4. \end{aligned}$$

This almost contact structure with B-metric on  $G$  has zero scalar curvature [5]. Then we can construct an almost complex structure with Norden metric on  $G \times \mathbb{R}$  having any scalar curvature  $k > 0$  from the equation (3.9).

For example, from (3.11), the function

$$\sigma(t) = \frac{1}{3} \ln \left( \cosh \left( \frac{\sqrt{6}}{2\sqrt{5}} t \right) \right),$$

satisfies the differential equation (3.10):

$$1 = 30 \left( \frac{d\sigma}{dt} \right)^2 + 10 \frac{d^2\sigma}{dt^2}.$$

Thus the scalar curvature  $k$  of the product manifold  $G \times \mathbb{R}$  is  $k = 1$  from (3.9).

Similarly for

$$\sigma(t) = \frac{1}{3} \ln \left( \cosh \left( \frac{\sqrt{3}}{\sqrt{5}} t \right) \right),$$

the scalar curvature of  $G \times \mathbb{R}$  is 2.

*Example 3.2.* Let  $\mathbb{R}^{2n+1} = \{(u^1, \dots, u^n, v^1, \dots, v^n, t) \mid u^i, v^i, t \in \mathbb{R}\}$ . Consider the cosymplectic almost contact structure with B-metric given in [2] by

$$\begin{aligned} \xi &= \frac{\partial}{\partial t}, \quad \eta = dt, \\ \varphi \left( \frac{\partial}{\partial u^i} \right) &= \frac{\partial}{\partial v^i}, \quad \varphi \left( \frac{\partial}{\partial v^i} \right) = -\frac{\partial}{\partial u^i}, \quad \varphi \left( \frac{\partial}{\partial t} \right) = 0, \\ g(X, X) &= -\delta_{ij} \lambda^i \lambda^j + \delta_{ij} \mu^i \mu^j + \nu^2, \end{aligned}$$

where  $X = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i} + \nu \frac{\partial}{\partial t}$  and  $\delta_{ij}$  are the Kronecker's symbols. We obtain infinitely many conformally Kaehlerian structure with Norden metric on  $\mathbb{R}^{2n+1} \times \mathbb{R}$

from Theorem 3.1 for any non-constant function  $\sigma$ . For instance, choose  $\sigma(t) = t$ . Then, for

$$\begin{aligned} X &= x_i \frac{\partial}{\partial u_i} + \tilde{x}_i \frac{\partial}{\partial v_i} + \bar{x} \frac{d}{dt}, \\ Y &= y_i \frac{\partial}{\partial u_i} + \tilde{y}_i \frac{\partial}{\partial v_i} + \bar{y} \frac{d}{dt}, \\ Z &= z_i \frac{\partial}{\partial u_i} + \tilde{z}_i \frac{\partial}{\partial v_i} + \bar{z} \frac{d}{dt} \end{aligned}$$

and  $\tilde{X} = (X, a \frac{d}{dt})$ ,  $\tilde{Y} = (Y, b \frac{d}{dt})$ ,  $\tilde{Z} = (Z, c \frac{d}{dt})$ , from the proof of Theorem 3.1, we have

$$\begin{aligned} F(\tilde{X}, \tilde{Y}, \tilde{Z}) &= e^{3t} \{ \bar{y}g(X, Z) + \bar{z}g(X, Y) \} \\ &\quad - e^{2t} \frac{d\sigma}{dt} \{ bg(X, \varphi(Z)) + cg(X, \varphi(Y)) \} + 2bce^t \bar{x}, \end{aligned}$$

and  $\tilde{\theta}(\tilde{X}) = 2(n+1)e^t \bar{x}$ ,  $\tilde{\theta}(J(\tilde{X})) = -2a(n+1)$ . Theorem 3.1 implies that  $F$  satisfies the defining relation (2.14) of  $W_1$ .

*Example 3.3.* Let  $\mathbb{R}^{2n+2} = \{(u^1, \dots, u^{n+1}; v^1, \dots, v^{n+1}) \mid u^i, v^i \in \mathbb{R}\}$  and consider  $\mathbb{R}^{2n+2}$  as a complex Riemannian manifold with the canonical complex structure  $J$  and the metric  $g$  defined by

$$g(x, x) = -\delta_{ij} \lambda^i \lambda^j + \delta_{ij} \mu^i \mu^j,$$

where  $x = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i}$ . Let  $Z$  denote the position vector of the point  $p$ .

We consider the unit time-like sphere  $S^{2n+1} : g(Z, Z) = -1$  of the metric  $g$  given in [2]. The characteristic vector field  $\xi$  on  $S^{2n+1}$  is given by

$$\xi = \lambda Z + \mu JZ, \quad g(Z, \xi) = 0, \quad g(\xi, \xi) = 1.$$

For each  $p$  in  $S^{2n+1}$ , setting  $g(J\xi, Z) = \tan t$ ,  $t \in (-\pi/2, \pi/2)$ , it is obtained that

$$\xi = (\sin t)Z + (\cos t)JZ, \quad J\xi = -(\cos t)Z + (\sin t)JZ.$$

For any  $x \in T_p S^{2n+1}$ , let  $\varphi x$  be the projection of the vector  $Jx$  into  $T_p S^{2n+1}$  with respect to  $J\xi$ . Then, one has the unique decomposition  $Jx = \varphi x + \eta(x)J\xi$ , where  $\eta$  is a 1-form in  $T_p S^{2n+1}$ . It is shown that  $(S^{2n+1}, \varphi, \xi, \eta, g)$  is an almost contact manifold with B-metric in the class  $\mathcal{F}_4 \oplus \mathcal{F}_5$ , that is

$$\begin{aligned} \alpha(X, Y, Z) &= -\frac{\theta(\xi)}{2n} \{ g(\varphi(X), \varphi(Y))\eta(Z) + g(\varphi(X), \varphi(Z))\eta(Y) \} \\ &\quad - \frac{\theta^*(\xi)}{2n} \{ g(X, \varphi(Y))\eta(Z) + g(X, \varphi(Z))\eta(Y) \}. \end{aligned}$$

For any choice of a non-constant function  $\sigma$ , we obtain infinitely many almost complex manifolds with Norden metrics on  $S^{2n+1} \times \mathbb{R}$  of the class  $W_1 \oplus W_2$  from Theorem 3.4 and Theorem 3.5.

For instance, let  $\sigma(t) = t$ . Since  $(S^{2n+1}, \varphi, \xi, \eta, g)$  is in  $\mathcal{F}_4 \oplus \mathcal{F}_5$ , we have  $\theta(X) = \eta(X)\theta(\xi)$ , and from equation (3.6), we get

$$\tilde{\theta} \left( 0, \frac{d}{dt} \right) = -\frac{\theta^*(\xi)}{e^t} \neq 0,$$

which implies that the structure is not in  $W_2$ . Similar to the proof of Theorem (3.4), by direct calculation, it can be seen that the defining relation (2.14) of the class  $W_1$  is not satisfied and the defining relation of  $W_1 \oplus W_2$  holds. Thus the product manifold  $S^{2n+1} \times \mathbb{R}$  is in the class  $W_1 \oplus W_2$ .

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**SOME NOVEL RESULTS ON THE EXISTENCE AND  
UNIQUENESS OF A POSITIVE SOLUTION TO A KIND OF  
NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS**

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**ABSTRACT.** This work investigates a fractional boundary value problem in the sense of Riemann-Liouville derivative and integral. We derive some novel results for the necessary and sufficient conditions for the existence and uniqueness of the positive solution. In this regard, some fixed-point theorems on cones are used. Also, a convergent successive sequence to find the solution to the problem is introduced. We derive the numerical scheme for the proposed problems. The correctness of the proposed results is verified with some illustrative examples.

1. INTRODUCTION

Fractional Calculus, which extends integer order calculus to arbitrary order calculus, has garnered attention from scientists recently. Fractional differential equations represent physical processes in science and engineering [20, 28, 29, 33]. Some recent applications of fractional-order operators can be seen in epidemiology [8, 18, 22, 31, 35], ecology [23], mechanics [17], psychology [25], chemical reactor theory [16], etc. Particularly, Fractional-order Boundary Value Problems (FBVPs) have been used to describe various real-world problems. In [14], the authors derived a Caputo-type boundary value problem representing a corneal shape model. The authors in [24] proposed a heat conduction model of fractional-order in terms of the Caputo-type boundary value problem. In [6], the authors proposed a boundary value problem related to the dynamics of glucose graph.

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*Key words and phrases.* Fractional boundary value problem, Riemann-Liouville derivative and integral, Green's function, fixed point theorem.

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Numerous publications addressing the existence, uniqueness, and multiplicity of positive solutions to fractional initial and boundary value issues have been written in recent decades (see [2–4, 7, 9, 10, 13, 30, 32, 40]). Bai in [5] derived positive solutions of a nonlocal fractional boundary value problem. In [1], the authors derived some novel simulations on the existence of a unique positive solution for FBVPs. In [34], an analysis of the existence and uniqueness of positive solutions to a coupled system of nonlinear FBVPs with anti-periodic boundary conditions has been given. In [26], the authors produced some novel findings for the existence and uniqueness of positive solutions to  $m$ -point FBVPs. In [21], the same results are produced for multi-point FBVPs with  $p$ -Laplacian operator. There have been some theoretical improvements in [36] on the existence of a unique positive solution for a class of nonlinear FBVPs with mixed-type boundary conditions. The positive solution of a nonlinear fractional  $q$ -difference equation with integral boundary conditions has been studied for existence and uniqueness in [19]. Some existence and stability results for nonlocal FBVPs were derived by Erturk et al. [15]. Bekri et al. [11] investigated some novel findings on the existence and uniqueness of a nonlinear  $q$ -difference FBVP. In [12], the analyses of existence and uniqueness on two Caputo-type FBVPs have been given.

In this study, we address the existence of positive solutions for the following FBVP and the uniqueness of each of those solutions.

$$(1.1) \quad \begin{aligned} \mathfrak{D}_{0+}^{\delta}(u(t) + \mathfrak{I}_{0+}^{\epsilon}\Psi(t, u(t))) + \Phi(t, u(t)) &= 0, \\ \lim_{t \rightarrow 0} t^{\delta-3}u(t) = \lim_{t \rightarrow 0} t^{\delta-3}u'(t) = u'(1) &= 0, \end{aligned}$$

where  $2 < \delta \leq \epsilon \leq 3$ ,  $t \in [0, 1]$  and  $\mathfrak{D}_{0+}^{\delta}$  is the standard Riemann-Liouville (R-L) fractional derivative of order  $\delta$  and  $\mathfrak{I}_{0+}^{\epsilon}$  is the R-L fractional integral of order  $\epsilon$ . Also, the functions  $\Phi$  and  $\Psi$  have some properties which will be presented later.

This article is put together in the following way. In section 2, some necessary definitions are presented. Section 3 calculates the Green function of the problem and presents some properties of this function. In section 4, the main results about the existence and uniqueness of positive solutions of the proposed FBVP (1.1) are obtained. Section 5 gives illustrative examples verifying main results with numerical solutions. In section 6, we conclude our findings.

## 2. FUNDAMENTALS

Here we recall some fundamentals [20, 27–29, 33] used throughout the study.

**Definition 2.1.** The Riemann-Liouville (R-L) fractional integral is given by

$$\mathfrak{I}_{a+}^{\delta}f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1}f(s)ds,$$

where  $\Gamma$  denotes the Gamma function and  $a$  is an arbitrary fixed initial point. The function  $f$  is considered locally integrable and  $\delta$  is a real or complex number  $\text{Re}(\delta) > 0$ .

**Definition 2.2.** The R-L fractional derivative of order  $\delta > 0$  of a continuous function  $f : (0, +\infty) \rightarrow \mathbb{R}$  is given by

$$\mathfrak{D}_{a^+}^\delta f(t) = \frac{1}{\Gamma(n - \delta)} \cdot \frac{d^n}{dt^n} \int_a^t (t - s)^{n-\delta-1} f(s) ds,$$

where  $n = [\delta] + 1$ , considering right-hand is point-wise defined on  $(0, +\infty)$ .

**Lemma 2.1** ([27]). *Let  $u \in C(0, 1) \cap L(0, 1)$  with a fractional derivative of order  $\delta > 0$  that belongs to  $C(0, 1) \cap L(0, 1)$ . Then,*

$$\mathfrak{I}_{0^+}^\delta \mathfrak{D}_{0^+}^\delta u(t) = u(t) + C_1 t^{\delta-1} + C_2 t^{\delta-2} + \dots + C_n t^{\delta-n},$$

where  $n = [\delta] + 1$ .

Throughout the paper, let  $(E, \|\cdot\|)$  be a real Banach space and  $\theta$  be a zero of  $E$ . A nonempty closed convex set  $P$  is a cone if satisfies the following conditions

- i)  $u \in P, \lambda \geq 0$  implies  $\lambda u \in P$ ;
- ii)  $u_1 \leq u_2 \Leftrightarrow u_2 - u_1 \in P$ .

Also, cone  $P$  is a normal cone if there exists  $N \in \mathbb{R}$  such that for all  $u_1, u_2 \in P$  with  $\theta \leq u_1 \leq u_2$  we have  $\|u_1\| \leq N\|u_2\|$  and  $N$  is called the normality constant.

For all  $u_1, u_2 \in E$ , write  $u_1 \sim u_2$  (we say  $u_1$  is equivalent with  $u_2$ ) if there exist constants  $\lambda, \mu > 0$  such that  $\lambda u_1 \leq u_2 \leq \mu u_1$ . If  $h > \theta$ , then  $P_h = \{u \in P : u \sim h\}$ . It is clear that  $P_h \subset P$ .

**Definition 2.3.** Let  $\eta \in (0, 1)$ . An operator  $T : P \rightarrow P$  is called  $\eta$ -concave if for all  $\lambda \in (0, 1)$  and  $u \in P$  we have  $T(\lambda u) \geq \lambda^\eta T(u)$ . Also an operator  $T : P \rightarrow P$  is called sub-homogeneous if for all  $\lambda > 0$  and  $u \in P$  we have  $T(\lambda u) \geq \lambda T(u)$ .

Now we recall some fixed point theorems.

**Theorem 2.1** ([39]). *Let  $P$  be a normal cone in a real Banach space  $E, T_1, T_2 : P \rightarrow P$  be an increasing  $\eta$ -concave operator and an increasing sub-homogeneous operator, respectively. If*

- i) *for some  $h > \theta$  we have  $T_1 h \in P_h$  and  $T_2 h \in P_h$ ;*
- ii) *for some constant  $\rho_0$  and all  $u \in P$  we have  $T_1 u \geq \rho_0 T_2 u$ ,*

*then the operator  $T = T_1 + T_2$  has unique fixed point. In the other words, the operator equation  $u = T_1 u + T_2 u$  has unique solution  $u^* \in P_h$ . Moreover, for any initial value  $u_0$ , the successive sequence  $u_{n+1} = T_1 u_n + T_2 u_n$ , for  $n = 0, 1, 2, \dots$  converges to the  $u^*$ .*

**Theorem 2.2** ([37]). *Let  $P$  be a normal cone in real Banach space  $E, T_1, T_2 : P \rightarrow P$  are respectively increasing and decreasing operator. Assume*

- i) *for any  $u \in P$  and  $\lambda \in (0, 1)$ , there exist  $\varphi_i(\lambda) \in (\lambda, 1), i = 1, 2$  such that*

$$T_1(\lambda u) \geq \varphi_1(\lambda) T_1(u), \quad T_2(\lambda u) \leq \frac{1}{\varphi_2(\lambda)} T_2(u),$$

- ii) *there exists  $h_0 \in P_h$  such that  $T_1 h_0 + T_2 h_0 \in P_h$ .*

Then, the operator equation  $u = T_1u + T_2u$  has unique solution  $u^* \in P_h$ . Moreover, for any initial values  $v_0, u_0$  successive sequences

$$u_{n+1} = T_1u_n + T_2v_n, \quad v_{n+1} = T_1v_n + T_2u_n, \quad n = 0, 1, 2, \dots$$

converge to  $u^* \in P_h$ .

### 3. GREEN FUNCTION AND BOUNDS

We need to calculate the Green function of a desired operator for applying the fixed point theorems. In this section, in addition to calculate Green function, we also outline some properties of it which is used throughout this paper.

**Lemma 3.1.** *Suppose  $g, h : [0, 1] \rightarrow [0, +\infty)$  be continuous functions, then the solution of the FBVP*

$$(3.1) \quad \begin{aligned} \mathfrak{D}_{0+}^\delta [u(t) + \mathfrak{I}_{0+}^\epsilon g(t)] + h(t) &= 0, \\ \lim_{t \rightarrow 0} t^{\delta-3} u(t) = \lim_{t \rightarrow 0} t^{\delta-3} u'(t) &= u'(1) = 0, \end{aligned}$$

is expressed by

$$(3.2) \quad u(t) = \int_0^1 G_1(t, s)h(s)ds + \int_0^1 G_2(t, s)g(s)ds,$$

where

$$(3.3) \quad G_1(t, s) = \begin{cases} \frac{t^{\delta-1}(1-s)^{\delta-2} - (t-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \leq s \leq t < 1, \\ \frac{t^{\delta-1}(1-s)^{\delta-2}}{\Gamma(\delta)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$(3.4) \quad G_2(t, s) = \begin{cases} \frac{(\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2} - (\delta-1)(t-s)^{\epsilon-1}}{(\delta-1)\Gamma(\epsilon)}, & 0 \leq s \leq t < 1, \\ \frac{(\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

*Proof.* Integrating the first equation of (3.1), follows

$$u(t) + \mathfrak{I}_{0+}^\epsilon g(t) = -\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} h(s)ds + c_1 t^{\delta-1} + c_2 t^{\delta-2} + c_3 t^{\delta-3}.$$

One can easily check that from the boundary conditions  $\lim_{t \rightarrow 0} t^{\delta-3} u(t) = \lim_{t \rightarrow 0} t^{\delta-3} u'(t) = 0$ , we have  $c_2 = c_3 = 0$ . By derivation from the above relation, we have

$$u'(t) = -\frac{\epsilon-1}{\Gamma(\epsilon)} \int_0^t (t-s)^{\epsilon-2} g(s)ds - \frac{\delta-1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-2} h(s)ds + c_1(\delta-1)t^{\delta-2}.$$

Now from the third boundary condition, we have

$$c_1 = \frac{\epsilon-1}{(\delta-1)\Gamma(\epsilon)} \int_0^1 (1-s)^{\epsilon-2} g(s)ds + \frac{1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-2} h(s)ds.$$

Hence,

$$u(t) = -\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} h(s)ds + \frac{1}{\Gamma(\delta)} \int_0^1 t^{\delta-1} (1-s)^{\delta-2} h(s)ds$$

$$\begin{aligned}
 & -\frac{1}{\Gamma(\epsilon)} \int_0^t (t-s)^{\epsilon-1} g(s) ds + \frac{\epsilon-1}{(\delta-1)\Gamma(\epsilon)} \int_0^1 t^{\delta-1} (1-s)^{\epsilon-2} g(s) ds \\
 & = \int_0^1 G_1(t,s) h(s) ds + \int_0^1 G_2(t,s) g(s) ds. \quad \square
 \end{aligned}$$

**Corollary 3.1.** *Let  $\Phi, \Psi \in C([0, 1] \times [0, +\infty))$ . Then  $u$  is a solution of problem (1.1) if and only if  $u$  is a solution of integral equation*

$$(3.5) \quad u(t) = \int_0^1 G_1(t,s)\Phi(s, u(s)) ds + \int_0^1 G_2(t,s)\Psi(s, u(s)) ds.$$

**Lemma 3.2.** *The functions  $G_1(t, s), G_2(t, s)$  defined by (3.3) and (3.4) have the following properties:*

- (a)  $t^{\delta-1}G_1(1, s) \leq G_1(t, s) \leq \frac{t^{\delta-1}(1-s)^{\delta-2}}{\Gamma(\delta)}$ ;
- (b)  $t^{\delta-1}G_2(1, s) \leq G_2(t, s) \leq \frac{t^{\delta-1}(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}$ .

*Proof.* Statement (a) concluded from [38]. We prove the statement (b). For  $s \leq t$  we have

$$\begin{aligned}
 (\delta-1)\Gamma(\epsilon)G_2(t, s) & = (\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2} - (\delta-1)(t-s)^{\epsilon-1} \\
 & \geq (\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2} - (\delta-1)(t-s)(t-st)^{\epsilon-2} \\
 & = (\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2} - (\delta-1)(t-s)t^{\epsilon-2}(1-s)^{\epsilon-2} \\
 & \geq t^{\delta-1} [(\epsilon-1)(1-s)^{\epsilon-2} - (\delta-1)(t-s)(1-s)^{\epsilon-2}] \\
 & \geq t^{\delta-1} [(\epsilon-1)(1-s)^{\epsilon-2} - (\delta-1)(1-s)^{\epsilon-1}] \\
 & = t^{\delta-1}(\delta-1)\Gamma(\epsilon)G_2(1, s).
 \end{aligned}$$

Thus, for  $s \leq t$ , we have  $G_2(t, s) \geq t^{\delta-1}G_2(1, s)$ . On the other hand, for  $s > t$ , we have

$$\frac{G_2(t, s)}{G_2(1, s)} = \frac{(\epsilon-1)t^{\delta-1}(t-s)^{\epsilon-2}}{(\epsilon-1)(1-s)^{\epsilon-2}} = t^{\delta-1}.$$

So, for all  $(t, s) \in [0, 1] \times [0, 1]$ , we have

$$G_2(t, s) \geq t^{\delta-1}G_2(1, s).$$

The other side of the inequality in the statement (b) is clearly established. □

#### 4. MAIN RESULTS

In this section, by using Theorem 2.1 and Theorem 2.2, we prove some existence and uniqueness results for the FBVP (1.1). For convenience, we list the following hypothesis:

- (H1)  $\Phi, \Psi \in C([0, 1] \times [0, +\infty))$  and they are increasing functions with respect to the second variable, also  $\Psi(t, 0) \not\equiv 0$ ;
- (H2) for  $0 < \mu < 1, (t, u) \in [0, 1] \times [0, +\infty)$ , we have  $\Psi(t, \mu u) \geq \mu\Psi(t, u)$ ;
- (H3) for  $0 < \mu, \eta < 1, (t, u) \in [0, 1] \times [0, +\infty)$ , we have  $\Phi(t, \mu u) \geq \mu^\eta\Phi(t, u)$ ;

(H4) there exists a constant  $\rho_0 > 0$  such that  $\Phi(t, u) \geq \rho_0 \Psi(t, u)$ ,  $t \in [0, 1]$ ,  $u \geq 0$ .

Now we set

$$A_1 = \int_0^1 G_1(1, s)\Phi(s, 0)ds, \quad A_2 = \int_0^1 \frac{(1-s)^{\delta-2}}{\Gamma(s)}\Phi(s, 1)ds,$$

$$B_1 = \int_0^1 G_2(1, s)\Psi(s, 0)ds, \quad B_2 = \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(s)}\Psi(s, 1)ds.$$

**Theorem 4.1.** *Assume that (H1)-(H4) hold. Then, fractional boundary value problem (1.1) has unique positive solution. In fact, the problem has unique solution  $u$  in  $P_h$ , with  $h(t) = t^{\delta-1}$ ,  $t \in [0, 1]$ . Also, for any initial value  $u_0 \in P_h$ , the successive sequence*

$$u_{n+1}(t) = \int_0^1 G_1(t, s)\Phi(s, u_n(s))ds + \int_0^1 G_2(t, s)\Psi(s, u_n(s))ds, \quad n = 0, 1, \dots,$$

*converges to the solution  $u^*$ .*

*Proof.* Let  $P$  be the cone of all positive functions and  $P_h \subset P$ . From Corollary 3.1 we know that problem (1.1) has an integral formulation given by

$$(4.1) \quad u(t) = \int_0^1 G_1(t, s)\Phi(s, u(s))ds + \int_0^1 G_2(t, s)\Psi(s, u(s))ds,$$

where  $G_1, G_2$  are defined by (3.3) and (3.4). We define two operators  $T_1, T_2 : P \rightarrow E$  by

$$(4.2) \quad (T_1u)(t) = \int_0^1 G_1(t, s)\Phi(s, u(s))ds, \quad (T_2u)(t) = \int_0^1 G_2(t, s)\Psi(s, u(s))ds.$$

It is clear that  $u$  is the solution of FBVP (1.1) if and only if  $u = T_1u + T_2u$ . From (H1)-(H2), we know that  $T_1 : P \rightarrow P$  and  $T_2 : P \rightarrow P$ . In the following, we check that  $T_1, T_2$  satisfy all assumptions of Theorem 2.1. This will be done in the following steps.

Step 1.  $T_1$  and  $T_2$  are increasing operators.

Let  $u_1, u_2 \in P$  and  $u_1 \leq u_2$ , then for all  $t \in [0, 1]$  we have  $u_1(t) \leq u_2(t)$ . So, by (H1),

$$(T_1u_1)(t) = \int_0^1 G_1(t, s)\Phi(s, u_1(s))ds \leq \int_0^1 G_1(t, s)\Phi(s, u_2(s))ds = (T_1u_2)(t).$$

By a similar way one can show  $(T_2u_1)(t) \leq (T_2u_2)(t)$ .

Step 2.  $T_1$  is a  $\eta$ -concave and  $T_2$  is a sub-homogeneous operator. Let  $\mu \in (0, 1)$  and  $u \in P$ , then from (H3), we have

$$(T_1(\mu u))(t) = \int_0^1 G_1(t, s)\Phi(s, \mu u(s))ds \geq \mu^\eta \int_0^1 G_1(t, s)\Phi(s, u(s))ds = \mu^\eta (T_1u)(t).$$

So,  $T_1$  is a  $\eta$ -concave operator. Now, from (H2) and the same properties for  $\mu$ , we get

$$(T_2(\mu u))(t) = \int_0^1 G_2(t, s)\Psi(s, \mu u(s))ds \geq \mu \int_0^1 G_2(t, s)\Psi(s, u(s))ds = \mu (T_2u)(t).$$

Hence, we can conclude that  $T_2$  is a sub-homogeneous operator.

Step 3. For  $h(t) = t^{\delta-1}$  we have  $T_1h, T_2h \in P_h$ .

In view of (H1) and Lemma 3.2, we get

$$(T_1h)(t) = \int_0^1 G_1(t, s)\Phi(s, s^{\delta-1})ds \leq \frac{t^{\delta-1}}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-2}\Phi(s, 1)ds,$$

$$(T_1h)(t) = \int_0^1 G_1(t, s)\Phi(s, s^{\delta-1})ds \geq t^{\delta-1} \int_0^1 G_1(1, s)\Phi(s, 0)ds.$$

Since  $A_2 \geq A_1 > 0$ , we can conclude  $A_1h(t) \leq T_1h(t) \leq A_2h(t)$ . Thus,  $T_1h \in P_h$ . By a similar way

$$(T_2h)(t) = \int_0^1 G_2(t, s)\Psi(s, s^{\delta-1})ds \leq t^{\delta-1} \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}\Psi(s, 1)ds,$$

$$(T_2h)(t) = \int_0^1 G_2(t, s)\Psi(s, s^{\delta-1})ds \geq t^{\delta-1} \int_0^1 G_2(1, s)\Psi(s, 0)ds.$$

So,  $T_2 \in P_h$ .

Step 4. For some  $\lambda > 0$  and all  $u \in P$ ,  $T_1u \geq \lambda T_2u$ .

Let  $u \in P$ . Since both  $G_1$  and  $G_2$  are positive continuous and bounded functions, there exists a constant such that  $G_1(t, s) \geq \kappa G_2(t, s)$ . Hence, by (H4), we have

$$(T_1u)(t) = \int_0^1 G_1(t, s)\Phi(s, u(s))ds \geq \kappa \int_0^1 G_2(t, s)\Phi(s, u(s))ds$$

$$\geq \kappa \rho_0 \int_0^1 G_2(t, s)\Psi(s, u(s))ds = \lambda \int_0^1 G_2(t, s)\Psi(s, u(s))ds$$

$$= \lambda(T_2u)(t),$$

where  $\lambda = \kappa \rho_0$ .

Thus, from Step 1-4. we conclude that all conditions of Theorem 2.1 are satisfied and the operator

$$(4.3) \quad Tu = T_1u + T_2u$$

has a unique fixed-point that is the unique positive solution of the FBVP (1.1). Also, from Theorem 2.1, we know for any initial value  $u_0 \in P_h$ , the successive sequence  $u_n = T_1u_{n-1} + T_2u_{n-1}$ ,  $n = 1, 2, \dots$  converges to  $u^* \in P_h$ . In another words, the successive sequence

$$u_{n+1}(t) = \int_0^1 G_1(t, s)\Phi(s, u_n(s))ds + \int_0^1 G_2(t, s)\Psi(t, u_n(s))ds \rightarrow u^*, \quad n = 1, 2, \dots,$$

as  $n \rightarrow +\infty$ . □

Our second result is based on Theorem 2.2. Let us add the following hypothesis to the previous hypothesis (H1)-(H4).

(H'1)  $\Phi, \Psi : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  are respectively increasing and decreasing function with respect to the second variable and  $\Phi(t, 0) \neq 0, \Psi(t, 1) \neq 0$ .

(H5) For any  $\mu \in (0, 1)$ , there exist  $f(\mu), g(\mu) \in (\mu, 1)$  such that for all  $t \in [0, 1]$  we have

$$\Phi(t, \mu u) \geq f(\mu)\Phi(t, u), \quad \Psi(t, \mu u) \leq \frac{1}{g(\mu)}\Psi(t, u).$$

**Theorem 4.2.** *Assume (H'1) and (H5) hold, then FBVP (1.1) has unique solution  $u^*$  in  $P_h$  with  $h(t) = t^{\delta-1}$ ,  $t \in [0, 1]$ . Also, for any initial value problem  $u_0$  and  $v_0$  in  $P_h$  constructing successively the sequences*

$$u_{n+1}(t) = \int_0^1 G_1(t, s)\Phi(s, u_n(s))ds + \int_0^1 G_2(t, s)\Psi(s, v_n(s))ds, \quad n = 0, 1, \dots,$$

$$v_{n+1}(t) = \int_0^1 G_1(t, s)\Phi(s, v_n(s))ds + \int_0^1 G_2(t, s)\Psi(s, u_n(s))ds, \quad n = 0, 1, \dots,$$

we have  $u_n(t) \rightarrow u^*(t)$ ,  $v_n(t) \rightarrow u^*(t)$  as  $n \rightarrow +\infty$ , where  $G_1(t, s)$  and  $G_2(t, s)$  are given in (3.3) and (3.4).

*Proof.* Again, we consider the operators defined in (4.3), from (H'1), (H5), and similar to the proof of previous theorem, one can show  $T_1$  and  $T_2$  satisfy the first condition of Theorem 2.2. So, we need only to verify the second condition of Theorem 2.2. Let us set

$$A_3 = \int_0^1 G_1(1, s)\Phi(s, 0)ds + \int_0^1 G_2(1, s)\Psi(s, 1)ds,$$

$$B_3 = \int_0^1 \frac{(1-s)^{\delta-2}}{\Gamma(\delta)}\Phi(s, 1)ds + \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}\Psi(s, 0)ds.$$

In view of Lemma 3.2 and (H'1), (H5), we have

$$\begin{aligned} (T_1h)(t) + (T_2h)(t) &= \int_0^1 G_1(t, s)\Phi(s, s^{\delta-1})ds + \int_0^1 G_2(t, s)\Psi(s, s^{\delta-1})ds \\ &\geq t^{\delta-1} \left[ \int_0^1 G_1(1, s)\Phi(s, 0)ds + \int_0^1 G_2(1, s)\Psi(s, 1)ds \right] \\ &= t^{\delta-1}A_3, \end{aligned}$$

$$\begin{aligned} (T_1h)(t) + (T_2h)(t) &= \int_0^1 G_1(t, s)\Phi(s, s^{\delta-1})ds + \int_0^1 G_2(t, s)\Psi(s, s^{\delta-1})ds \\ &\leq t^{\delta-1} \left[ \int_0^1 \frac{(1-s)^{\delta-2}}{\gamma(\delta)}\Phi(s, 1)ds + \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}\Psi(s, 0)ds \right] \\ &= t^{\delta-1}B_3. \end{aligned}$$

Therefore,  $A_3h(t) \leq (T_1h)(t) + (T_2h)(t) \leq B_3h(t)$  and  $(T_1h)(t) + (T_2h)(t) \in P_h$ . Thus, all conditions of Theorem 2.2 are satisfied and for any initial values  $v_0$  and  $u_0$  in  $P_h$ , constructing successively the sequences

$$u_{n+1}(t) = \int_0^1 G_1(t, s)\Phi(s, u_n(s))ds + \int_0^1 G_2(t, s)\Psi(s, v_n(s))ds, \quad n = 0, 1, \dots,$$

$$v_{n+1}(t) = \int_0^1 G_1(t, s)\Phi(s, v_n(s))ds + \int_0^1 G_2(t, s)\Psi(s, u_n(s))ds, \quad n = 0, 1, \dots,$$

we have  $u_n(t) \rightarrow u^*(t)$ ,  $v_n(t) \rightarrow u^*(t)$  as  $n \rightarrow +\infty$ . □

5. EXAMPLES

*Example 5.1.* Let us consider the following FBVP

$$(5.1) \quad \begin{aligned} D_{0+}^{\frac{5}{2}}(u(t) + I_{0+}^{\frac{8}{3}}\Psi(t, u(t)) + \Phi(t, u(t)) &= 0, \\ \lim_{t \rightarrow 0} t^{-\frac{1}{2}}u(t) = \lim_{t \rightarrow 0} t^{-\frac{1}{2}}u'(t) = u'(1) &= 0, \end{aligned}$$

where  $\Phi(t, u) = u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})}$ ,  $\Psi(t, u) = \frac{u}{1+u}e^t + \frac{(b-a)t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})}$ , with  $b > a > 0$ . Now

$$\begin{aligned} \Phi(t, \mu u) &= \mu^{\frac{1}{3}}u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})} \geq \mu^{\frac{1}{3}}\left(u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})}\right) = \mu^\eta\Phi(t, u), \\ \Psi(t, \mu u) &= \frac{\mu u}{1 + \mu u}e^t + \frac{(b-a)t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})} \geq \mu\left[\left(\frac{u}{1+u}\right)e^t + \frac{b-a}{\Gamma(\frac{7}{2})}t^{\frac{7}{2}}\right]. \end{aligned}$$

If we set  $\rho_0 \in [0, a/(e + b - a)]$ , then

$$\begin{aligned} \Phi(t, u) &= u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})} \geq a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})(e + b - a)}(e + b - a) \\ &\geq \rho_0\left[\frac{u}{1+u}e^t + \frac{b-a}{\Gamma(\frac{7}{2})}t^{\frac{7}{2}}\right] = \rho_0\Psi(t, u). \end{aligned}$$

So, all conditions of Theorem 4.1 are satisfied. Therefore, the problem (5.1) with  $\Phi$ ,  $\Psi$  has positive solution.

*Example 5.2.* Again we consider FBVP (5.1) with  $\Phi(t, u) = u^{\frac{1}{3}}e^t + \alpha$ ,  $\alpha > 0$  and  $\Psi(t, u) = \frac{e^t}{1+u^{\frac{1}{4}}}$ . It is clear,  $\Phi : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous and increasing with respect to the second variable and  $\Phi(t, 0) = \alpha > 0$ ,  $\Psi : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous and decreasing with respect to the second variable and  $\Psi(t, 1) = \frac{e^t}{2} \neq 0$ . Now, if we set  $f_1(\mu) = \mu^{\frac{1}{3}}$  and  $f_2(\mu) = \mu^{\frac{1}{4}}$ , then  $f_1(\mu), f_2(\mu) \in (\mu, 1)$  for all  $\mu \in (0, 1)$  and

$$\begin{aligned} \Phi(t, \mu u) &= \mu^{\frac{1}{3}}u^{\frac{1}{3}}e^t + \alpha \geq \mu^{\frac{1}{3}}(u^{\frac{1}{3}}e^t + \alpha) = f_1(\mu)\Phi(t, u), \\ \Psi(t, \mu u) &= \frac{e^t}{1 + (\mu u)^{\frac{1}{4}}} \leq \frac{e^t}{\mu^{\frac{1}{4}}(1 + u^{\frac{1}{4}})} = \frac{1}{f_2(\mu)}\Psi(t, u(t)). \end{aligned}$$

Consequently, all conditions of Theorem 4.2 are satisfied. So, problem (5.1) has unique positive solution in  $P_h$  with  $h(t) = t^{\frac{3}{2}}$ ,  $t \in [0, 1]$ .

*Example 5.3.* Consider the fractional boundary value problem

$$(5.2) \quad \begin{aligned} D_{0+}^{2.3}(u(t) + I_{0+}^{2.2}\Psi(t, u(t)) + \Phi(t, u(t)) &= 0, \\ \lim_{t \rightarrow 0} t^{-0.7}u(t) = \lim_{t \rightarrow 0} t^{-0.8}u'(t) = u'(1) &= 0, \end{aligned}$$

where  $\Phi(t, u) = \sqrt[3]{3u^2(t) + t^3 + 3}$  and  $\Psi(t, u) = \frac{3 \cos^2 t}{\sqrt{5u^2(t) + \sin^2 t + 1}}$ .

Clearly,  $\Phi : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous and increasing with respect to the second variable and  $\Phi(t, 0) = \sqrt[3]{3} > 0$ ,  $\Psi : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous and decreasing with respect to the second variable and  $\Psi(t, 1) \neq 0$ . Let  $f_1(\mu) = \mu^{\frac{2}{3}}$  and  $f_2(\mu) = \mu^{\frac{1}{2}} \in (\mu, 1)$  for all  $\mu \in (0, 1)$  and

$$\begin{aligned}\Phi(t, \mu u(t)) &= \sqrt[3]{3\mu^2 u^2(t) + 3} \geq \sqrt[3]{\mu^2(3u^2(t) + t^3 + 3)} = \mu^{\frac{2}{3}} \Phi(t, u(t)), \\ \Psi(t, \mu u(t)) &= \frac{3 \cos^2 t}{\sqrt{5\mu^2 u^2(t) + \sin^2 t}} \leq \frac{3 \cos^2 t}{\mu^{\frac{1}{2}} \sqrt{5u^2(t) + \sin^2 t}} = \frac{1}{f_2(\mu)} \Psi(t, u(t)).\end{aligned}$$

Consequently, all conditions of Theorem 4.2 are satisfied. So, problem (5.2) has unique positive solution in  $P_h$  with  $h(t) = t^{1.3}$ ,  $t \in [0, 1]$ .

## 6. NUMERICAL SOLUTION

Since we have already established the existence and uniqueness of a solution to (1.1), our focus here will be on its numerical solution. The method is straightforward to some degree by recalling Theorems 4.1 and 4.2, and the recurrence relation formula is the equation given in Theorem 4.1, which comes from operator (4.1). The formula for the recurrence relation can be employed without much difficulty by the initial trial solution, say, for example,  $u_0(t) \equiv 0$ , and then the programme iterates to find sequential  $u_n(t)$  stopping when the maximum difference in two successive iterations drops below a given tolerance value. The computer algebra system Mathematica is used to execute this iterative scheme.

The passing from an iteration to the next one is done symbolically and numerically. The latter happens when, to approximate the integral appearing in the equation given in Theorem 4.1, cubic spline interpolation is used.

Firstly, we consider Example 5.1 to confirm the validity of the presented numerical method.

Using the Green's function method, we have following algorithm.

Step 1. The node points  $t_0, t_1, \dots, t_M$  are considered for adequately large number of  $M$ .

Step 2. Cubic spline interpolation is used to obtain  $u_n(s)$ 's.

Step 3. The following approximate solution is obtained by the numerical integration:

$$\begin{aligned}u_{n+1}(t_j) &= \frac{1}{\Gamma(\delta)} \int_0^{t_j} \left[ t_j^{\delta-1} (1-s)^{\delta-2} - (t_j-s)^{\delta-1} \right] \left[ u_n^{1/3}(s) + a \frac{s^{7/2}}{\Gamma(7/2)} \right] ds \\ &\quad + \frac{t_j^{\delta-1}}{\Gamma(\delta)} \int_{t_j}^1 (1-s)^{\delta-2} \left[ u_n^{1/3}(s) + a \frac{s^{7/2}}{\Gamma(7/2)} \right] ds \\ &\quad + \frac{1}{(\delta-1)\Gamma(\epsilon)} \int_0^t \left[ (\epsilon-1)t_j^{\delta-1} (1-s)^{\epsilon-2} - (\delta-1)(t_j-s)^{\epsilon-1} \right]\end{aligned}$$

$$\begin{aligned} & \times \left[ \frac{u_n(s)}{1 + u_n(s)} e^s + \frac{(b - a)s^{7/2}}{\Gamma(7/2)} \right] ds \\ & + \frac{(\epsilon - 1)t_j^{\delta-1}}{(\delta - 1)\Gamma(\epsilon)} \int_{t_j}^1 (1 - s)^{\epsilon-2} \left[ \frac{u_n(s)}{1 + u_n(s)} e^s + \frac{(b - a)s^{7/2}}{\Gamma(7/2)} \right] ds, \quad n = 0, 1, \dots \end{aligned}$$

Step 4. Steps 1, 2, 3 are iterated to find consecutive  $u_n(u)$  stopping when  $|u_{n+1} - u_n| < TOL$ .

The exact solution is unknown infact, but the iteration stopping criteria used is set  $|u_{n+1} - u_n| < 10^{-10}$ , and then, the numerical solution is obtained. For the step size of the node points,  $h = 0.05$ , the number of iterations,  $M=20$ , and  $TOL = 10^{-10}$ , the errors are of order  $10^{-10}$ . The solution curve  $u(t)$  is shown graphically in Figure 1 for  $\delta = 2.5$  and  $\epsilon = 2.66$  when  $a = 1$  and  $b = 2$ . For other graphical simulations,  $(\delta, \epsilon)$ 's are taken as  $(2.1, 2.5)$ ,  $(2.5, 2.5)$ ,  $(2.5, 2.9)$ ,  $(2.8, 2.9)$ , and  $(3, 3)$ . The solution curves  $u(t)$ 's are displayed in Figures 2-6, respectively. For  $\delta = 2.5$  and  $\epsilon = 2.66$  when  $a = 1$  and  $b = 2$ , the convergence is plotted in Figure 7, and the error is plotted in Figure 8.

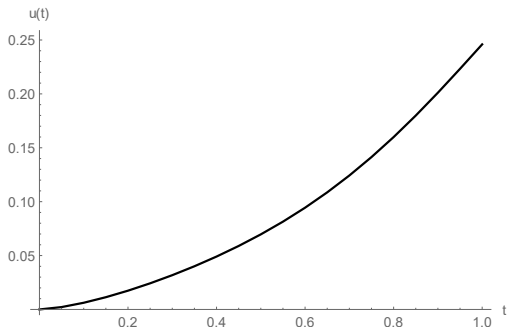


FIGURE 1. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.66$ .

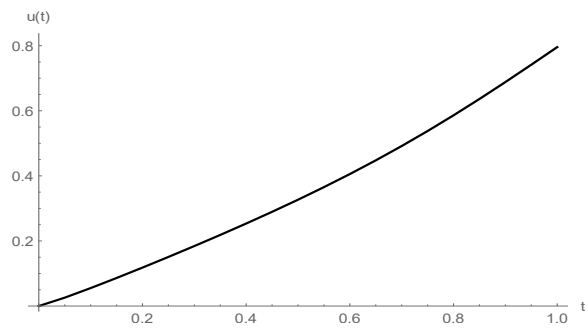


FIGURE 2. Solution curve  $u(t)$  for  $\delta = 2.1$  and  $\epsilon = 2.5$ .

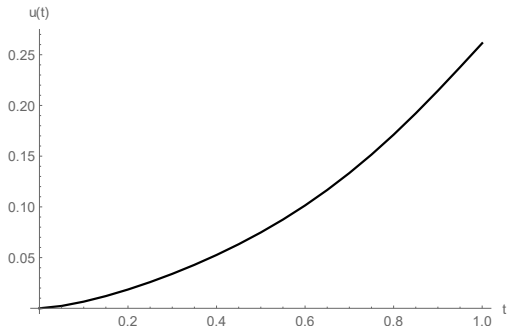


FIGURE 3. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.5$ .

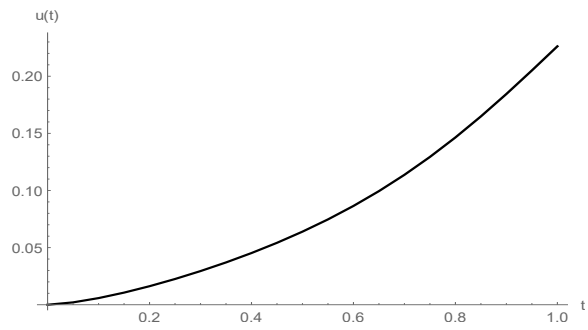


FIGURE 4. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.9$ .

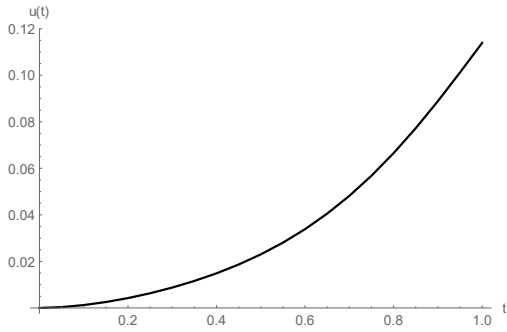


FIGURE 5. Solution curve  $u(t)$  for  $\delta = 2.8$  and  $\epsilon = 2.9$ .

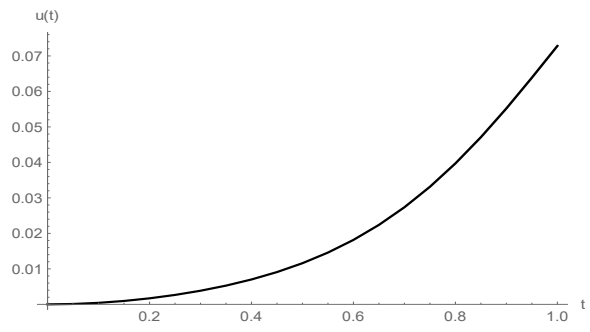


FIGURE 6. Solution curve  $u(t)$  for  $\delta = 3$  and  $\epsilon = 3$ .

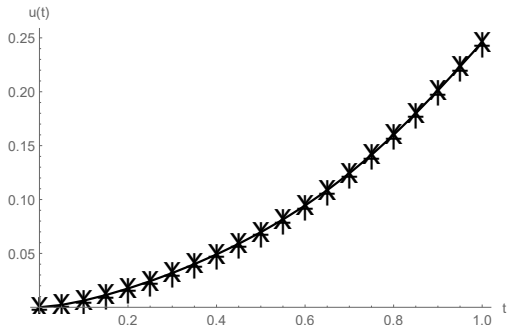


FIGURE 7. Convergence curve  $n = 5$  (horizontal bar),  $n = 10$  (vertical bar),  $n = 15$  (x) and  $n = 20$  (solid) for  $\delta = 2.5$  and  $\epsilon = 2.66$ .

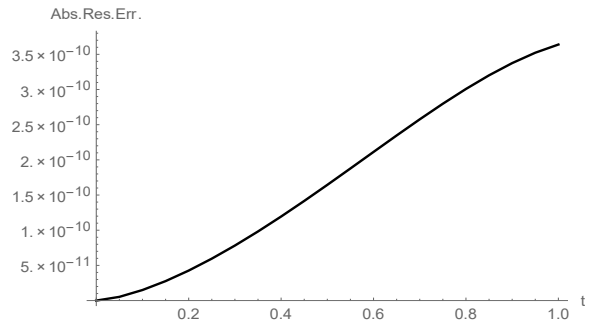


FIGURE 8. Error curve for  $\delta = 2.5$  and  $\epsilon = 2.66$ .

Now, let us consider Example 5.2 to confirm the validity of the presented numerical method. Similar to the previous algorithm, the following solution is obtained:

$$\begin{aligned}
 u_{(n+1)}(t_j) = & \frac{1}{\Gamma(\delta)} \int_0^{t_j} [t_j^{\delta-1}(1-s)^{\delta-2} - (t_j-s)^{\delta-1}] [u_n^{1/3}(s)e^s + \alpha] ds \\
 & + \frac{t_j^{\delta-1}}{\Gamma(\delta)} \int_{t_j}^1 (1-s)^{\delta-2} [u_n^{1/3}(s)e^s + \alpha] ds + \frac{1}{(\delta-1)\Gamma(\epsilon)} \\
 & \times \int_0^t [(\epsilon-1)t_j^{\delta-1}(1-s)^{\epsilon-2} - (\delta-1)(t_j-s)^{\epsilon-1}] \left[ \frac{e^s}{1+u_n^{1/4}(s)} \right] ds \\
 & + \frac{(\epsilon-1)t_j^{\delta-1}}{(\delta-1)\Gamma(\epsilon)} \int_{t_j}^1 (1-s)^{\epsilon-2} \left[ \frac{e^s}{1+u_n^{1/4}(s)} \right] ds, \quad n = 0, 1, \dots
 \end{aligned}$$

For the step size of the node points,  $h = 0.05$ , the number of iterations,  $M=15$ , and  $TOL = 10^{-10}$ , the errors is of order  $10^{-16}$ . The solution curve  $u(t)$  is shown graphically in Figure 9 for  $\delta = 2.5$  and  $\epsilon = 2.66$  when  $\alpha = 1$ . For other graphical

simulations,  $(\delta, \epsilon)$ 's are taken as  $(2.1, 2.5)$ ,  $(2.5, 2.5)$ ,  $(2.5, 2.9)$ ,  $(2.8, 2.9)$ , and  $(3, 3)$ . The solution curves  $u(t)$ 's are displayed in Figures 10–14, respectively. For  $\delta = 2.5$  and  $\epsilon = 2.66$  when  $\alpha = 1$ , the convergence is plotted in Figure 15, and the error is plotted in Figure 16.

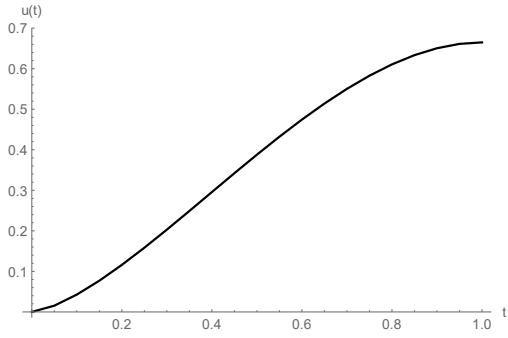


FIGURE 9. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.66$ .

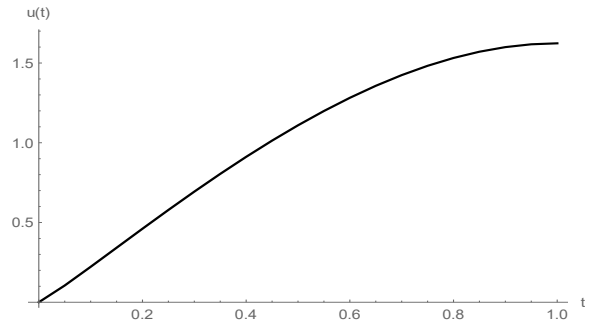


FIGURE 10. Solution curve  $u(t)$  for  $\delta = 2.1$  and  $\epsilon = 2.5$ .

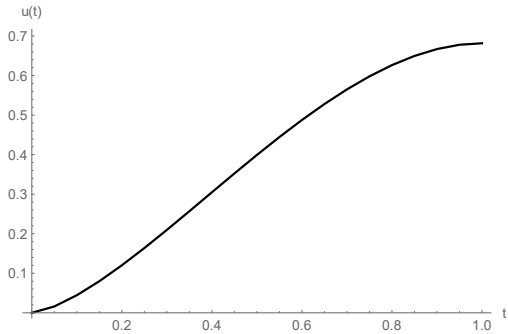


FIGURE 11. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.5$ .

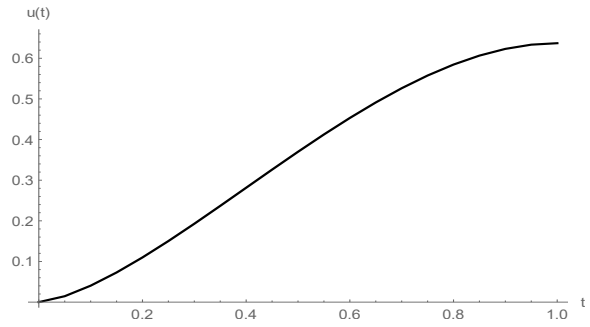


FIGURE 12. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.9$ .

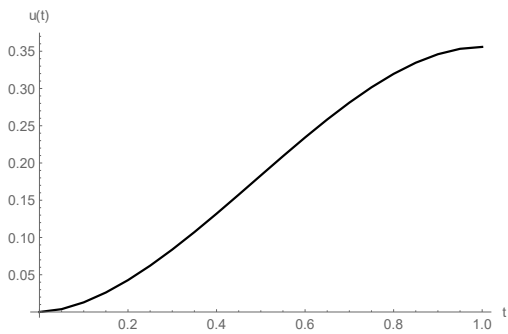


FIGURE 13. Solution curve  $u(t)$  for  $\delta = 2.8$  and  $\epsilon = 2.9$ .

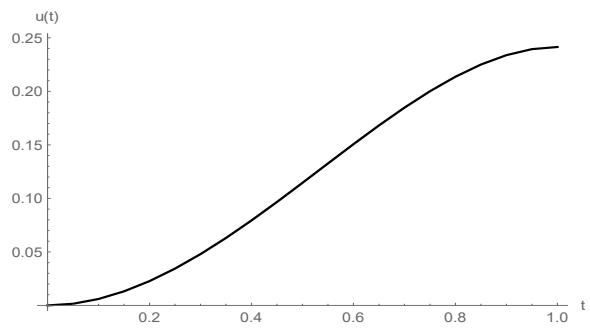


FIGURE 14. Solution curve  $u(t)$  for  $\delta = 3$  and  $\epsilon = 3$ .

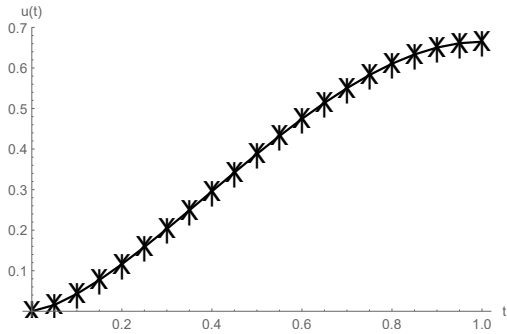


FIGURE 15. Convergence curve  $n = 5$  (horizontal bar),  $n = 10$  (vertical bar),  $n = 15$  (x) and  $n = 20$  (solid) for  $\delta = 2.5$  and  $\epsilon = 2.66$ .

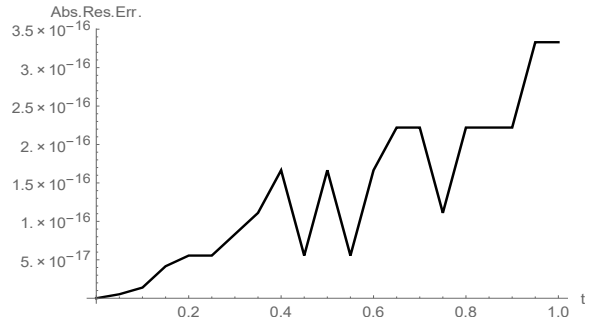


FIGURE 16. Error curve for  $\delta = 2.5$  and  $\epsilon = 2.66$ .

Finally, let us consider Example 5.3 to confirm the validity of the presented numerical method. Similar to the current algorithm, the following solution is obtained:

$$\begin{aligned}
 u_{n+1}(t_j) = & \frac{1}{\Gamma(\delta)} \int_0^{t_j} [t_j^{\delta-1}(1-s)^{\delta-2} - (t_j-s)^{\delta-1}] \sqrt[3]{3u_n^2(s) + s^3 + 3} ds \\
 & + \frac{t_j^{\delta-1}}{\Gamma(\delta)} \int_{t_j}^1 (1-s)^{\delta-2} \sqrt[3]{3u_n^2(s) + s^3 + 3} ds \\
 & + \frac{1}{(\delta-1)\Gamma(\epsilon)} \int_0^t [(\epsilon-1)t_j^{\delta-1}(1-s)^{\epsilon-2} - (\delta-1)(t_j-s)^{\epsilon-1}] \\
 & \times \left( \frac{3 \cos^2 s}{\sqrt{5u_n^2(s) + \sin^2 s + 1}} \right) ds \\
 & + \frac{(\epsilon-1)t_j^{\delta-1}}{(\delta-1)\Gamma(\epsilon)} \int_{t_j}^1 (1-s)^{\epsilon-2} \left( \frac{3 \cos^2 s}{\sqrt{5u_n^2(s) + \sin^2 s + 1}} \right) ds, \quad n = 0, 1, \dots
 \end{aligned}$$

For the step size of the node points,  $h = 0.05$ , the number of iterations,  $M = 10$ , and  $TOL = 10^{-10}$ , the order of errors is of around  $10^{-10}$ . The solution curve  $u(t)$  is shown graphically in Figure 17 for  $\delta = 2.3$  and  $\epsilon = 2.2$ . For other graphical simulations,  $(\delta, \epsilon)$ 's are taken as  $(2.1, 2.5)$ ,  $(2.5, 2.5)$ ,  $(2.5, 2.9)$ ,  $(2.8, 2.9)$ , and  $(3, 3)$ . The solution curves  $u(t)$ 's are displayed in Figures 18-22, respectively. For  $\delta = 2.3$  and  $\epsilon = 2.2$ , the convergence is plotted in Figure 23, and the error is plotted in Figure 24.

Table 1 shows the numerical results and absolute residual errors of the present method for  $M = 10$ ,  $\delta = 2.3$  and  $\epsilon = 2.2$ .

Table 1. Numerical solution and absolute residual error of Example 5.3 for  $M = 10$ ,  $\delta = 2.3$  and  $\epsilon = 2.2$ .

$t$	Numerical solution	Absolute residual error
0.0	0	
0.1	0.1145872311	$4.57641 \times 10^{-11}$
0.2	0.2543086295	$1.02926 \times 10^{-10}$
0.3	0.3838093159	$1.47172 \times 10^{-10}$
0.4	0.4917811004	$1.67472 \times 10^{-10}$
0.5	0.5736563020	$1.63345 \times 10^{-10}$
0.6	0.6285251227	$1.40516 \times 10^{-10}$
0.7	0.6578210989	$1.06733 \times 10^{-10}$
0.8	0.6644649486	$6.91701 \times 10^{-11}$
0.9	0.6521485611	$3.31993 \times 10^{-11}$
1.0	0.6246825709	$2.08899 \times 10^{-12}$

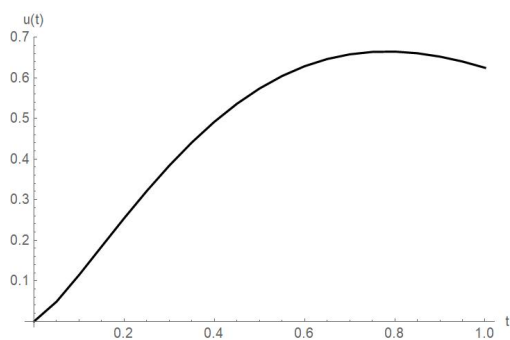


FIGURE 17. Solution curve  $u(t)$  for  $\delta = 2.3$  and  $\epsilon = 2.2$ .

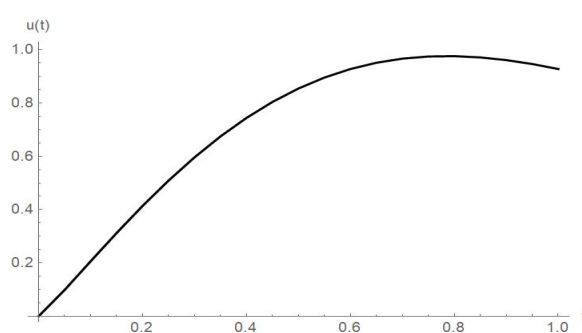


FIGURE 18. Solution curve  $u(t)$  for  $\delta = 2.1$  and  $\epsilon = 2.5$ .

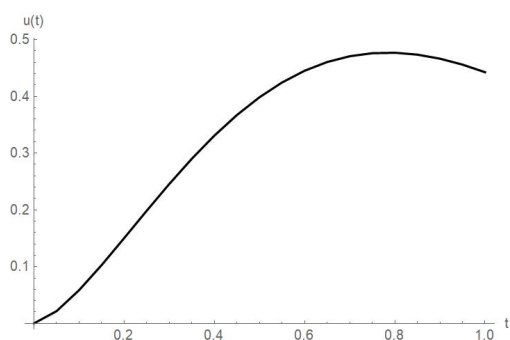


FIGURE 19. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.5$ .

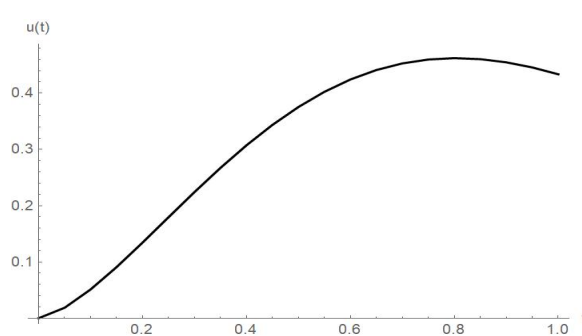


FIGURE 20. Solution curve  $u(t)$  for  $\delta = 2.5$  and  $\epsilon = 2.9$ .

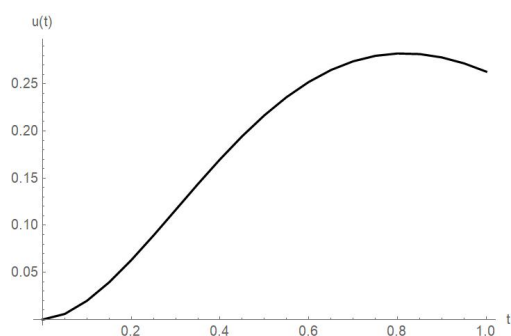


FIGURE 21. Solution curve  $u(t)$  for  $\delta = 2.8$  and  $\epsilon = 2.9$ .

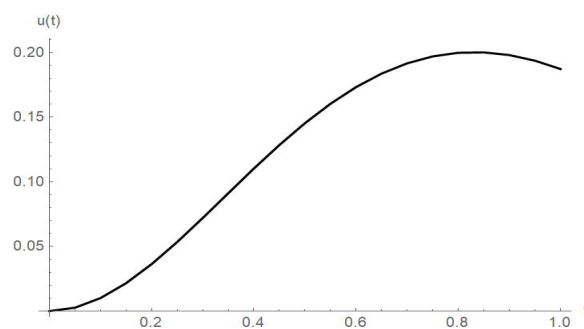


FIGURE 22. Solution curve  $u(t)$  for  $\delta = 3$  and  $\epsilon = 3$ .

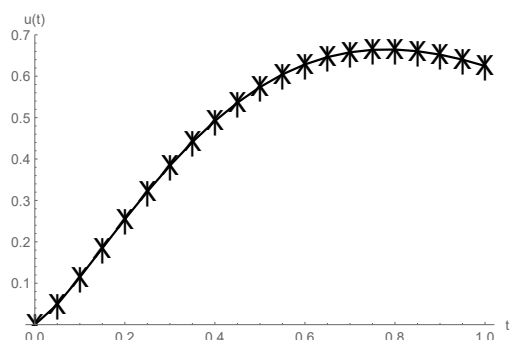


FIGURE 23. Convergence curve  $n = 5$  (vertical bar),  $n = 10$  (x) and  $n = 15$  (solid) for  $\delta = 2.3$  and  $\epsilon = 2.2$ .

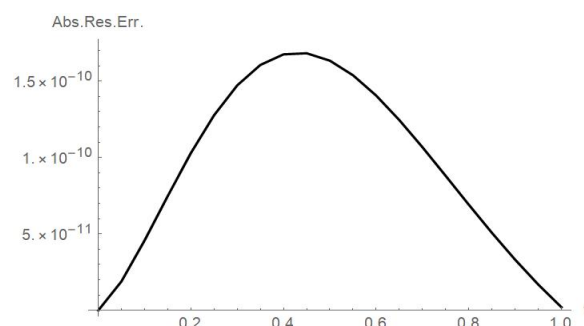


FIGURE 24. Error curve for  $\delta = 2.3$  and  $\epsilon = 2.2$ .

## 7. CONCLUSION

In this article, we have considered a class of FBVPs with a Riemann-Liouville derivative and integral for deriving some novel, necessary, and sufficient conditions for the existence and uniqueness of the positive solution. We have utilised some fixed-point theorems on cones. A convergent successive sequence for finding the solution of the proposed FBVP has been derived. We have verified the validity of the proposed results by implementing some problems with the derivation of numerical methodology. The obtained results will be beneficial in proving the existence and uniqueness of positive solutions while dealing with the proposed FBVPs. In the future, the researchers can try to model real-life problems using the given fractional boundary value problem along with its qualitative and quantitative analyses.

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## COMMON FIXED POINT RESULTS FOR INTERPOLATIVE KANNAN TYPE CONTRACTION OVER $m$ -METRIC SPACES

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ABSTRACT. The objective of this paper is to derive common fixed point results in  $m$ -metric spaces by using the interpolative condition proposed by Karapınar. We discuss three distinct scenarios: when the sum of the “interpolative exponents” is less than, equal to, or greater than 1. The validity of each result is supported by illustrative examples.

### 1. INTRODUCTION

Following Banach’s famous fixed point (FP) theorem [2], FP theory has flourished across multiple dimensions and has assumed a pivotal role in various mathematical domains. In recent times, a considerable amount of research has been dedicated to developing techniques for proving FP results concerning interpolative Kannan type contractions (IKTCs). For instance, Karapınar [6] demonstrated a FP result for IKTC. Similarly, Gabba et al. [4] established this result in scenarios where the sum of “interpolative exponents” is less than 1. Moreover, Errai et al. [3] achieved such a result for the case in which the sum of “interpolative exponents” is greater than or equal to 1. Notably, all these outcomes have been proven within the realm of standard metric spaces (MSs). Furthermore, Safeer et al. [8] delve into FP outcomes concerning IKTCs within the framework of  $m$ -metric spaces ( $m$ -MSs). The concept of  $m$ -MSs was initially introduced by Asadi et al. in [1], constituting as an extension of the partial metric space ( $p$ -MS).

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*Key words and phrases.* Common fixed point,  $m$ -interpolative Kannan type contraction (IKTC),  $m$ -metric space ( $m$ -MS).

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On the other hand Noorwali [10] initiate the study of common FP for IKTC, after that Gaba and Karapinar [5] proved the common FP results for the case when the sum of the “interpolative exponents” is less than 1.

This paper introduces a study on the existence of common FP for a pair of IKTCs within the framework of  $m$ -MSs. We explore all potential scenarios characterized by “interpolative exponents”. The first section provides necessary definitions and fundamental results concerning common FPs,  $m$ -MSs, and IKTCs. In the second section, we establish three distinct results regarding common FPs for  $m$ -MSs, each under different conditions on the “interpolative exponents”. Furthermore, we illustrate each result with examples in  $m$ -MSs.

Moreover, we examine our examples in standard MSs and elaborate on how the corresponding mappings fail to yield common fixed points. This underscores the significance of our established results. Additionally, we investigate similar outcomes in  $p$ -MSs which arise as specific instances of our results for  $m$ -MSs, yet represent novel discoveries in their own regard. Finally we note that our results generalize results of [5, 10].

## 2. PRELIMINARIES

**Definition 2.1** ([9]). A partial metric on a nonempty set  $\Upsilon$  is a function  $p : \Upsilon \times \Upsilon \rightarrow \mathbb{R}^+$  such that for all  $\varrho_1, \varrho_2, \varrho_3 \in \Upsilon$

- (p<sub>1</sub>)  $p(\varrho_1, \varrho_2) = p(\varrho_1, \varrho_1) = p(\varrho_2, \varrho_2) \Leftrightarrow \varrho_1 = \varrho_2$ ;
- (p<sub>2</sub>)  $p(\varrho_1, \varrho_1) \leq p(\varrho_1, \varrho_2)$ ;
- (p<sub>3</sub>)  $p(\varrho_1, \varrho_2) = p(\varrho_2, \varrho_1)$ ;
- (p<sub>4</sub>)  $p(\varrho_1, \varrho_2) \leq p(\varrho_1, \varrho_3) + p(\varrho_3, \varrho_2) - p(\varrho_3, \varrho_3)$ .

A partial MS is a pair  $(\Upsilon, p)$  such that  $\Upsilon$  is nonempty set and  $p$  is a partial metric on  $\Upsilon$ .

**Definition 2.2** ([1]). Let  $\Upsilon$  be a nonempty set. Then  $m$ -metric is a function  $m : \Upsilon \times \Upsilon \rightarrow \mathbb{R}^+$  satisfying the following conditions:

- (m<sub>1</sub>)  $m(\varrho_1, \varrho_2) = m(\varrho_1, \varrho_1) = m(\varrho_2, \varrho_2) \Leftrightarrow \varrho_1 = \varrho_2$ ;
- (m<sub>2</sub>)  $m_{\varrho_1\varrho_2} \leq m(\varrho_1, \varrho_2)$  where  $m_{\varrho_1\varrho_2} := \min\{m(\varrho_1, \varrho_1), m(\varrho_2, \varrho_2)\}$ ;
- (m<sub>3</sub>)  $m(\varrho_1, \varrho_2) = m(\varrho_2, \varrho_1)$ ;
- (m<sub>4</sub>)  $(m(\varrho_1, \varrho_2) - m_{\varrho_1\varrho_2}) \leq (m(\varrho_1, \varrho_3) - m_{\varrho_1\varrho_3}) + (m(\varrho_3, \varrho_2) - m_{\varrho_3\varrho_2})$

for all  $\varrho_1, \varrho_2, \varrho_3 \in \Upsilon$ . The pair  $(\Upsilon, m)$  is called  $m$ -MS.

**Lemma 2.1** ([1]). *Every  $p$ -MS  $(\Upsilon, p)$  is a  $m$ -MS.*

The converse of the above result may not hold, as we can see in Example 6 provided by Karapinar et al. in [7].

**Definition 2.3** ([1]). Let  $(\Upsilon, m)$  be a  $m$ -MS. Then

1. a sequence  $(\varrho_n)$  in  $(\Upsilon, m)$  converges to a point  $\varrho \in \Upsilon$  if and only if

$$\lim_{n \rightarrow +\infty} (m(\varrho_n, \varrho) - m_{\varrho_n, \varrho}) = 0;$$

2. a sequence  $(\varrho_n)$  in  $(\Upsilon, m)$  is called  $m$ -Cauchy sequence if

$$\lim_{n,j \rightarrow +\infty} (m(\varrho_n, \varrho_j) - m_{\varrho_n, \varrho_j})$$

and

$$\lim_{n,j \rightarrow +\infty} (M_{\varrho_n, \varrho_j} - m_{\varrho_n, \varrho_j}),$$

exists (and are finite), where  $M_{\varrho_n, \varrho_j} = \max\{m(\varrho_n, \varrho_n), m(\varrho_j, \varrho_j)\}$ ;

3. The space  $(\Upsilon, m)$  is said to be complete if every  $m$ -Cauchy sequence  $(\varrho_n)$  in  $\Upsilon$  converges to a point in  $\Upsilon$ .

**Lemma 2.2** ([1]). *Assume that  $\varrho_n \rightarrow \varrho$  and  $\kappa_n \rightarrow \kappa$  as  $n \rightarrow +\infty$  in a  $m$ -MS  $(\Upsilon, m)$ . Then*

$$\lim_{n \rightarrow +\infty} (m(\varrho_n, \kappa_n) - m_{\varrho_n, \kappa_n}) = m(\varrho, \kappa) - m_{\varrho, \kappa}.$$

In [6], Karapınar introduce the following IKTC.

**Definition 2.4** ([6]). Let  $(\Upsilon, d)$  be a MS. A self mapping  $T : \Upsilon \rightarrow \Upsilon$  is said to be an interpolative Kannan type contraction (IKTC), if there exist  $\lambda \in [0, 1)$  and  $\alpha \in (0, 1)$  such that

$$d(T\varrho, T\kappa) \leq \lambda d(\varrho, T\varrho)^\alpha d(\kappa, T\kappa)^{1-\alpha},$$

for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq T\varrho$ ,  $\kappa \neq T\kappa$ .

We term  $\alpha$  as an “interpolative exponent”.

The following result by Karapınar is proved in [6].

**Theorem 2.1** ([6]). *Let  $(\Upsilon, d)$  be a complete MS and  $T$  be an IKTC. Then  $T$  has a unique FP .*

In [4], Gabba et al. defined the following IKTC.

**Definition 2.5.** Let  $(\Upsilon, d)$  be a MS, a self mapping  $T : \Upsilon \rightarrow \Upsilon$  is called  $(\lambda, \alpha, \beta)$ -IKTC if there exist  $\lambda \in [0, 1)$  and  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta < 1$ , such that

$$d(T\varrho, T\kappa) \leq \lambda d(\varrho, T\varrho)^\alpha d(\kappa, T\kappa)^\beta,$$

for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq T\varrho$ ,  $\kappa \neq T\kappa$ .

Moreover, they proved the following FP theorem.

**Theorem 2.2** ([4]). *Let  $(\Upsilon, d)$  be a complete MS such that  $d(\varrho, \kappa) \geq 1$  for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq \kappa$ . Let  $T : \Upsilon \rightarrow \Upsilon$  be a  $(\lambda, \alpha, \beta)$ -IKTC. Then  $T$  has a FP.*

Errai et al. [3] proved the following FP result for IKTC for the case  $\alpha + \beta > 1$  with  $\alpha, \beta \in (0, 1)$ .

**Theorem 2.3** ([3]). *Let  $(\Upsilon, d)$  be a complete MS and  $T$  a self mapping on  $\Upsilon$  such that*

$$d(T\varrho, T\kappa) \leq \lambda d(\varrho, T\varrho)^\alpha d(\kappa, T\kappa)^\beta,$$

for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq T\varrho$  and  $\kappa \neq T\kappa$ , and where  $\lambda \in (0, 1)$  and  $\alpha, \beta \in (0, 1)$  such that  $\alpha + \beta \geq 1$ . If there exists  $\varrho \in \Upsilon$  such that  $d(\varrho, T\varrho) \leq 1$ , then  $T$  has a FP in  $\Upsilon$ .

Note that all above results of interpolative contractions have been proved in ordinary MS  $(\Upsilon, d)$ . Also Safeer et al. [8] extend these results into the structure of  $m$ -MSs.

On the other hand, the common FP of any two self mappings  $R, T$  is a point  $\varrho \in \Upsilon$  such that  $R\varrho = \varrho = T\varrho$ . The Noorwali [10] initiate the study of common FP for IKTC and proved the following result.

**Theorem 2.4** ([10]). *Let  $(\Upsilon, d)$  be a complete MS,  $R, T : \Upsilon \rightarrow \Upsilon$  be two self mappings. Assume that there are some  $\lambda \in [0, 1)$ ,  $\alpha \in (0, 1)$  such that the condition*

$$d(R\varrho, T\kappa) \leq \lambda d(\varrho, R\varrho)^\alpha d(\kappa, T\kappa)^{1-\alpha}$$

*is satisfied for all  $\varrho, \kappa \in \Upsilon$  such that  $\varrho \neq R\varrho, \kappa \neq T\kappa$ . Then  $R$  and  $T$  have a common FP.*

Moreover, Gabba and Karapinar [5] proved the common FP result for the case when the sum of the “interpolative exponents” is less than one and their result is elaborated as follows.

**Theorem 2.5** ([5]). *Let  $(\Upsilon, d)$  be a complete MS and  $(R, T)$  be a  $(\lambda, \alpha, \beta)$ -IKTC pair. Then  $R$  and  $T$  have a common FP in  $\Upsilon$ .*

The  $(\lambda, \alpha, \beta)$ -IKTC pair is defined as follows.

**Definition 2.6** ([5]). Let  $(\Upsilon, d)$  be a MS and  $R, T : \Upsilon \rightarrow \Upsilon$  be two self mappings. We shall call  $(R, T)$  a  $(\lambda, \alpha, \beta)$ -IKTC pair, if there exist  $\lambda \in [0, 1)$ ,  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta < 1$  such that

$$d(R\varrho, T\kappa) \leq \lambda d(\varrho, R\varrho)^\alpha d(\kappa, T\kappa)^\beta,$$

for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq R\varrho, \kappa \neq T\kappa$ .

### 3. MAIN RESULTS

**Definition 3.1.** Let  $(\Upsilon, m)$  be a  $m$ -MS,  $R, T : \Upsilon \rightarrow \Upsilon$  be two self mappings on  $\Upsilon$ . We call  $(R, T)$  a  $m$ -IKTC pair. If there exists  $\lambda \in [0, 1)$  and  $\alpha \in (0, 1)$  such that

$$(3.1) \quad m(R\varrho, T\kappa) \leq \lambda m(\varrho, R\varrho)^\alpha m(\kappa, T\kappa)^{1-\alpha}$$

holds for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq R\varrho, \kappa \neq T\kappa$  and  $m(\varrho, R\varrho) \neq 0, m(\kappa, T\kappa) \neq 0$ .

**Theorem 3.1.** *Let  $(\Upsilon, m)$  be a complete  $m$ -MS and  $(R, T)$  be a  $m$ -IKTC pair. Then  $R$  and  $T$  have a common FP in  $\Upsilon$ .*

*Proof.* Let  $\varrho_0 \in \Upsilon$ , define a sequence  $(\varrho_n)$  in  $\Upsilon$  such that  $\varrho_{2n+1} = R\varrho_{2n}$  and  $\varrho_{2n+2} = T\varrho_{2n+1}$ . If there exists a natural number  $n_0$  such that  $\varrho_{n_0} = \varrho_{n_0+1} = \varrho_{n_0+2}$ , then  $\varrho_{n_0}$  is the common FP of  $R$  and  $T$ . Consider there does not exist any three identical terms in the sequence  $(\varrho_n)$ . Then by (3.1),

$$\begin{aligned} m(\varrho_{2n+1}, \varrho_{2n+2}) &= m(R\varrho_{2n}, T\varrho_{2n+1}) \\ &\leq \lambda m(\varrho_{2n}, R\varrho_{2n})^\alpha m(\varrho_{2n+1}, T\varrho_{2n+1})^{1-\alpha} \\ &= \lambda m(\varrho_{2n}, \varrho_{2n+1})^\alpha m(\varrho_{2n+1}, \varrho_{2n+2})^{1-\alpha}, \end{aligned}$$

and  $m(\varrho_{2n+1}, \varrho_{2n+2})^\alpha \leq \lambda m(\varrho_{2n}, \varrho_{2n+1})^\alpha$ , i.e.,

$$m(\varrho_{2n+1}, \varrho_{2n+2}) \leq \lambda^{1/\alpha} m(\varrho_{2n}, \varrho_{2n+1}) \leq \lambda m(\varrho_{2n}, \varrho_{2n+1}).$$

Therefore,

$$(3.2) \quad m(\varrho_{2n+1}, \varrho_{2n+2}) \leq \lambda m(\varrho_{2n}, \varrho_{2n+1}).$$

Consequently, for all  $n \in \mathbb{N}$  we have

$$(3.3) \quad m(\varrho_n, \varrho_{n+1}) \leq \lambda m(\varrho_{n-1}, \varrho_n).$$

So,

$$m(\varrho_n, \varrho_{n+1}) \leq \lambda m(\varrho_{n-1}, \varrho_n) \leq \lambda^2 m(\varrho_{n-2}, \varrho_{n-1}) \leq \cdots \leq \lambda^n m(\varrho_0, \varrho_1).$$

Thus,

$$m(\varrho_n, \varrho_{n+1}) \leq \lambda^n m(\varrho_0, \varrho_1),$$

by taking limit as  $n \rightarrow +\infty$ ,

$$\limsup_{n \rightarrow +\infty} m(\varrho_n, \varrho_{n+1}) \leq \limsup_{n \rightarrow +\infty} \lambda^n m(\varrho_0, \varrho_1) = 0.$$

Hence,  $\lim_{n \rightarrow +\infty} m(\varrho_n, \varrho_{n+1}) = 0$ . By definition of  $m$ -metric

$$\lim_{n \rightarrow +\infty} m_{\varrho_n, \varrho_{n+1}} \leq \lim_{n \rightarrow +\infty} m(\varrho_n, \varrho_{n+1}) = 0,$$

thus  $\lim_{n \rightarrow +\infty} m_{\varrho_n, \varrho_{n+1}} = \min\{m(\varrho_n, \varrho_n), m(\varrho_{n+1}, \varrho_{n+1})\} = 0$ . As a result

$$\lim_{n \rightarrow +\infty} m(\varrho_n, \varrho_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} m(\varrho_{n+1}, \varrho_{n+1}) = 0.$$

Thus, for any  $n, j \in \mathbb{N}$  with  $n \geq j$

$$\lim_{n, j \rightarrow +\infty} (M_{\varrho_n, \varrho_j} - m_{\varrho_n, \varrho_j}) = 0$$

and by triangular inequality of  $m$ -metric

$$\lim_{n, j \rightarrow +\infty} (m(\varrho_n, \varrho_j) - m_{\varrho_n, \varrho_j}) = 0.$$

Thus, by definition  $(\varrho_n)$  is a Cauchy sequence in  $m$ -MS  $\Upsilon$ , since  $\Upsilon$  is  $m$ -complete so there exists  $\varrho \in \Upsilon$  such that  $(\varrho_n)$  converges to  $\varrho$  in  $\Upsilon$  w.r.t. the convergence of  $m$ -metric. Thus, by definition

$$\lim_{n \rightarrow +\infty} (m(\varrho_n, \varrho) - m_{\varrho_n, \varrho}) = 0.$$

Also,  $(\varrho_{2n+1})$  and  $(\varrho_{2n+2})$  converge to the same limit  $\varrho$ . Now for any  $n \in \mathbb{N}$  and by using the relation (3.1) for  $R = T$ , we get

$$\begin{aligned} m(\varrho_{2n+1}, R\varrho) &= m(R\varrho_{2n}, R\varrho) \\ &\leq \lambda m(\varrho_{2n}, R\varrho_{2n})^\alpha m(\varrho, R\varrho)^{1-\alpha} \\ &= \lambda m(\varrho_{2n}, \varrho_{2n+1})^\alpha m(\varrho, R\varrho)^{1-\alpha}. \end{aligned}$$

By taking limit as  $n \rightarrow +\infty$  on both sides and using the  $m_2$  condition of  $m$ -metric, we get

$$\lim_{n \rightarrow +\infty} (m(\varrho_{2n+1}, R\varrho) - m_{\varrho_{2n+1}, R\varrho}) = 0.$$

So,  $(\varrho_{2n+1})$  converges to  $R\varrho$  in  $m$ -metric, i.e.  $\varrho_{2n+1} = R\varrho_{2n} \rightarrow R\varrho$ . Also

$$\begin{aligned} m(\varrho_{2n+2}, R\varrho) &= m(T\varrho_{2n+1}, R\varrho) \\ &\leq \lambda m(\varrho_{2n+1}, T\varrho_{2n+1})^\alpha m(\varrho, R\varrho)^{1-\alpha} \\ &= \lambda m(\varrho_{2n+1}, \varrho_{2n+2})^\alpha m(\varrho, R\varrho)^{1-\alpha}. \end{aligned}$$

By taking limit as  $n \rightarrow +\infty$  on both sides and using the  $m_2$  condition of  $m$ -metric, we get

$$\lim_{n \rightarrow +\infty} (m(\varrho_{2n+2}, R\varrho) - m_{\varrho_{2n+2}, R\varrho}) = 0.$$

So,  $(\varrho_{2n+2})$  converges to  $R\varrho$  in  $m$ -metric, i.e.,  $\varrho_{2n+2} = T\varrho_{2n+1} \rightarrow R\varrho$ . Thus,  $(\varrho_n)$  converges to  $R\varrho$  as well.

**Case I.** If  $n$  is even, then  $\varrho_{2n+2} = T\varrho_{2n+1} \rightarrow R\varrho$  and  $\varrho_{2n+2} \rightarrow \varrho$ , so  $(\varrho_n)$  converges to both  $R\varrho$  and  $\varrho$ . Thus, by using Lemma 2.2,

$$0 = \lim_{n \rightarrow +\infty} (m(\varrho_{2n+2}, \varrho_{2n+2}) - m_{\varrho_{2n+2}, \varrho_{2n+2}}) = m(\varrho, R\varrho) - m_{\varrho, R\varrho}.$$

Also, we have  $\lim_{n \rightarrow +\infty} m(\varrho_{2n+2}, \varrho_{2n+2}) = 0$ , because  $\lim_{n \rightarrow +\infty} m(\varrho_n, \varrho_n) = 0$  and

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} (m(\varrho_{2n+2}, \varrho_{2n+2}) - m_{\varrho_{2n+2}, \varrho_{2n+2}}) \\ &= \lim_{n \rightarrow +\infty} (m(T\varrho_{2n+1}, T\varrho_{2n+1}) - m_{\varrho_{2n+2}, T\varrho_{2n+1}}) \\ &= m(R\varrho, R\varrho) - m_{\varrho, R\varrho}. \end{aligned}$$

Moreover,

$$0 = \lim_{n \rightarrow +\infty} (m(\varrho_{2n+2}, \varrho_{2n+2}) - m_{\varrho_{2n+2}, T\varrho_{2n+1}}) = m(\varrho, \varrho) - m_{\varrho, R\varrho}.$$

Thus by combining, we have

$$m(\varrho, \varrho) = m(\varrho, R\varrho) = m(R\varrho, R\varrho) = m_{\varrho, R\varrho},$$

by  $m_1$  condition of  $m$ -metric we have  $\varrho = R\varrho$ .

**Case II.** If  $n$  is odd, then  $\varrho_{2n+1} = R\varrho_{2n} \rightarrow R\varrho$  and  $\varrho_{2n+1} \rightarrow \varrho$ , we have

$$0 = \lim_{n \rightarrow +\infty} (m(\varrho_{2n+1}, \varrho_{2n+1}) - m_{\varrho_{2n+1}, \varrho_{2n+1}}) = m(\varrho, R\varrho) - m_{\varrho, R\varrho}.$$

Also,

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} (m(\varrho_{2n+1}, \varrho_{2n+1}) - m_{\varrho_{2n+1}, \varrho_{2n+1}}) \\ &= \lim_{n \rightarrow +\infty} (m(R\varrho_{2n}, R\varrho_{2n}) - m_{\varrho_{2n+1}, R\varrho_{2n}}) \\ &= m(R\varrho, R\varrho) - m_{\varrho, R\varrho}. \end{aligned}$$

Moreover,  $0 = \lim_{n \rightarrow +\infty} (m(\varrho_{2n+1}, \varrho_{2n+1}) - m_{\varrho_{2n+1}, R\varrho_{2n}}) = m(\varrho, \varrho) - m_{\varrho, R\varrho}$ . Thus, by combining, we have  $m(\varrho, \varrho) = m(\varrho, R\varrho) = m(R\varrho, R\varrho) = m_{\varrho, R\varrho}$ , by  $m_1$  condition of  $m$ -metric we have  $\varrho = R\varrho$ . Consequently,  $\varrho = R\varrho$ . By using similar steps we can get  $\varrho = T\varrho$ , thus  $\varrho$  is the common FP for  $T$  and  $R$ .  $\square$

**Corollary 3.1.** *If we take  $R = T$ , then Theorem 3.2 of [8] becomes the special case of our result in Theorem 3.1.*

*Example 3.1.* Let  $\Upsilon = [1/8, 4]$  and the  $m$ -metric on  $\Upsilon$  is defined as follows:

$$(3.4) \quad m(\varrho, \kappa) = \begin{cases} \varrho, & \varrho = \kappa, \\ \varrho + \kappa, & \varrho \neq \kappa. \end{cases}$$

Let  $R, T : \Upsilon \rightarrow \Upsilon$  be self mappings, such that

$$R\varrho = \begin{cases} 1/2, & \varrho \in [1/8, 2], \\ 1/(\varrho + 3), & \varrho \in (2, 4], \end{cases} \quad T\varrho = \begin{cases} 1/2, & \varrho \in [1/8, 2], \\ 1/2\varrho, & \varrho \in (2, 4]. \end{cases}$$

We discuss the following cases for  $\alpha = 1/2$  and  $\lambda = 17/18$ .

Case 1. If  $\varrho, \kappa \in [1/8, 2]$ , then for  $\varrho \neq 1/2$  and  $\kappa \neq 1/2$ , we have,

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/2, 1/2) = 1/2 \leq (17/18)(1/8 + 1/2) \\ &\leq \lambda(\varrho + 1/2)^{1/2}(\kappa + 1/2)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 2. If  $\varrho \in [1/8, 2]$  and  $\kappa \in (2, 4]$ , then for  $\varrho \neq 1/2$ , we have

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/2, 1/2\kappa) \leq 1/2 + 1/4 \leq (17/18)(1/8 + 1/2)^{1/2}(2 + 1/4)^{1/2} \\ &\leq \lambda(\varrho + 1/2)^{1/2}(\kappa + 1/2\kappa)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 3. If  $\varrho \in (2, 4]$  and  $\kappa \in [1/8, 2]$  then for  $\kappa \neq 1/2$ , we have

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/(\varrho + 3), 1/2) \leq 1/2 + 1/5 \leq (17/18)(2 + 1/5)^{1/2}(1/8 + 1/2)^{1/2} \\ &\leq \lambda(\varrho + 1/(\varrho + 3))^{1/2}(\kappa + 1/2)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 4. If  $\varrho, \kappa \in (2, 4]$ , then

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/2\varrho, 1/(\kappa + 3)) \leq 1/4 + 1/5 \leq (17/18)(2 + 1/4)^{1/2}(2 + 1/5)^{1/2} \\ &\leq (17/18)(\varrho + 1/2\varrho)^{1/2}(\kappa + 1/(\kappa + 3))^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Hence,  $(R, T)$  is a  $m$ -IKTC, so by Theorem 3.1,  $R$  and  $T$  have a common FP and it is actually  $\varrho = 1/2$ .

*Remark 3.1.* If we use the standard metric  $d(\varrho, \kappa) = |\varrho - \kappa|$  instead of the  $m$ -metric (3.4), then Case 2 and 3 of the above example do not satisfy the required IKTC for the pair  $(R, T)$  across many different combinations of  $\varrho$  and  $\kappa$ . One combination where Case 2 fails to satisfy IKTC is when  $\kappa = 3 \in (2, 4]$  and  $\varrho \in (4481/9826, 5345/9826) \subset [1/8, 2]$ . Therefore, the common fixed point results of the standard MS, as elaborated in [10], do not apply to our given pair  $(R, T)$ .

The following example asserts that the common fixed point is not always unique.

*Example 3.2.* Let  $\Upsilon = [0, +\infty)$  and the mapping  $m : \Upsilon \times \Upsilon \rightarrow \mathbb{R}^+$  be defined as  $m(\varrho, \kappa) = |\varrho - \kappa| + a$ , where “ $a$ ” is any non-negative real number. Let  $R, T$  be the self mappings defined on  $\Upsilon$  as follows:

$$R\varrho = \begin{cases} 1, & \varrho \in [0, 1/2), \\ \varrho, & \varrho \in [1/2, 200), \\ 1/\varrho^2, & \varrho \in [200, +\infty). \end{cases} \quad T\varrho = \begin{cases} 1, & \varrho \in [0, 1/2), \\ \varrho, & \varrho \in [1/2, 200), \\ e^{-2\varrho}, & \varrho \in [200, +\infty). \end{cases}$$

Now we discuss following cases to prove that  $(R, T)$  is  $m$ -IKTC for  $\alpha = 1/2$  and  $\lambda = 3/4$ .

Case 1. If  $\varrho, \kappa \in [0, 1/2)$ , then for all  $a \in [0, 3/2]$  following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= a \leq (3/4)(a + 1/2) \\ &\leq \lambda(|\varrho - 1| + a)^{1/2}(|\kappa - 1| + a)^{1/2} \\ &= \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 2. If  $\varrho \in [0, 1/2)$  and  $\kappa \in [200, +\infty)$ , then for all  $0 \leq a \leq 253$ , the following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= |1 - e^{-2\kappa}| + a \leq 1 + a \leq (3/4)(1/2 + a)^{1/2}(200 - e^{-400} + a)^{1/2} \\ &\leq \lambda(|\varrho - 1| + a)^{1/2}(|\kappa - e^{-2\kappa}| + a)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 3. If  $\varrho \in [200, +\infty)$  and  $\kappa \in [0, 1/2)$ , then for all  $0 \leq a \leq 253$ , the following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= |1/\varrho^2 - 1| + a \leq 1 + a \leq (3/4)(200 - (1/200^2) + a)^{1/2}(1/2 + a)^{1/2} \\ &\leq \lambda(|\varrho - 1/\varrho^2| + a)^{1/2}(|\kappa - 1| + a)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 4. If  $\varrho, \kappa \in [200, +\infty)$ , then for all  $0 \leq a \leq 600$ , the following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= |1/\varrho^2 - e^{-2\kappa}| + a \leq (1/200^2) + a \\ &\leq (3/4)(200 - (1/200^2) + a)^{1/2}(200 - e^{-400} + a)^{1/2} \\ &\leq (3/4)(|\varrho - 1/\varrho^2| + a)^{1/2}(|\kappa - e^{-2\kappa}| + a)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Hence, from all the above cases we conclude that the interpolative condition of Definition 3.1 holds when  $a \in [0, 3/2]$ . Thus for such values of  $a$ , by Theorem 3.1,  $R$  and  $T$  have common FPs and they actually are all the points in interval  $[1/2, 200)$ .

*Remark 3.2.* Given that our previous example remains valid for  $a \in [0, 3/2]$ , when  $a = 0$ , the corresponding  $m$ -metric aligns with the standard metric on the real line. However, for  $a \neq 0$ , the results derived in [10] do not apply to our specified pair  $(R, T)$ , as they were established solely for standard metric spaces. In such instances, our results concerning the  $m$ -metric will prove effective for identifying common fixed points.

**Definition 3.2.** Let  $(\Upsilon, m)$  be a  $m$ -MS and  $R, T : \Upsilon \rightarrow \Upsilon$  be two self mappings. We call  $(R, T)$  a  $(\lambda, \alpha, \beta)$ - $m$ -IKTC, if there exist  $\lambda \in [0, 1)$  and  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta < 1$  such that

$$(3.5) \quad m(R\varrho, T\kappa) \leq \lambda m(\varrho, R\varrho)^\alpha m(\kappa, T\kappa)^\beta,$$

for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq R\varrho$ ,  $\kappa \neq T\kappa$  and  $m(\varrho, R\varrho) \geq 1$ ,  $m(\kappa, T\kappa) \neq 0$ .

**Theorem 3.2.** Let  $(\Upsilon, m)$  be a complete  $m$ -MS and  $(R, T)$  be  $(\lambda, \alpha, \beta)$ - $m$ -IKTC. Then  $R$  and  $T$  have a common FP.

*Proof.* Let  $\varrho_0 \in \Upsilon$ , we construct the iterating sequence  $(\varrho_n)$  such that  $\varrho_{2n+1} = R\varrho_{2n}$  and  $\varrho_{2n+2} = T\varrho_{2n+1}$ . Thus,

$$\begin{aligned} m(\varrho_{2n+1}, \varrho_{2n+2}) &= m(R\varrho_{2n}, T\varrho_{2n+1}) \\ &\leq \lambda m(\varrho_{2n}, R\varrho_{2n})^\alpha m(\varrho_{2n+1}, T\varrho_{2n+1})^\beta \\ &= \lambda m(\varrho_{2n}, \varrho_{2n+1})^\alpha m(\varrho_{2n+1}, \varrho_{2n+2})^\beta, \\ m(\varrho_{2n+1}, \varrho_{2n+2})^{1-\beta} &\leq \lambda m(\varrho_{2n}, \varrho_{2n+1})^\alpha, \end{aligned}$$

since  $\alpha < 1 - \beta$  and  $m(\varrho_{2n}, \varrho_{2n+1}) \geq 1$ , so we have

$$\begin{aligned} m(\varrho_{2n+1}, \varrho_{2n+2})^{1-\beta} &\leq \lambda m(\varrho_{2n}, \varrho_{2n+1})^{1-\beta}, \\ m(\varrho_{2n+1}, \varrho_{2n+2}) &\leq \lambda m(\varrho_{2n}, \varrho_{2n+1}). \end{aligned}$$

The rest of the proof follows the similar procedure as in Theorem 3.1. To avoid the repetition, we leave it for the interested reader to dig out the details.  $\square$

**Corollary 3.2.** *If we take  $R = T$ , then Theorem 3.6 of [8] becomes the special case of our result in Theorem 3.2.*

*Example 3.3.* Let  $\Upsilon = [0, +\infty)$  and  $m$ -metric on  $\Upsilon$  be defined as in (3.4), define self mappings  $R, T : \Upsilon \rightarrow \Upsilon$  as follows:

$$R\varrho = \begin{cases} \varrho, & \varrho \in [0, 5], \\ 1/\varrho, & \varrho \in (5, +\infty), \end{cases} \quad T\varrho = \begin{cases} \varrho, & \varrho \in [0, 5], \\ 1/\ln \varrho, & \varrho \in (5, +\infty). \end{cases}$$

We discuss the required case to confirm that  $(R, T)$  is  $(2/3, 1/2, 1/4)$ - $m$ -IKTC used in Theorem 3.2. For any  $\varrho, \kappa \in (5, +\infty)$ , we have

$$\begin{aligned} m(R\varrho, T\kappa) &\leq (1/5 + 1/\ln 5) \leq (2/3)(5 + 1/5)^{1/2}(5 + 1/\ln 5)^{1/4} \\ &\leq (2/3)(\varrho + 1/\varrho)^{1/2}(\kappa + 1/\ln \kappa)^{1/4} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/4}. \end{aligned}$$

Consequently,  $(R, T)$  satisfies the required  $m$ -IKTC of Theorem 3.2, so every  $\varrho \in [0, 5]$  is the common FP of  $R$  and  $T$ .

*Remark 3.3.* In the case of the discrete metric  $d(\varrho, \kappa) = 1$  if  $\varrho \neq \kappa$  and zero if  $\varrho = \kappa$ , the IKTC in the above example is not satisfied for the pair  $(R, T)$ .

**Theorem 3.3.** *Let  $(\Upsilon, m)$  be a complete  $m$ -MS,  $R, T : \Upsilon \rightarrow \Upsilon$  be two self mappings and let there exists  $\lambda \in [0, 1)$  and  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta > 1$  such that*

$$(3.6) \quad m(R\varrho, T\kappa) \leq \lambda m(\varrho, R\varrho)^\alpha m(\kappa, T\kappa)^\beta,$$

for all  $\varrho, \kappa \in \Upsilon$  with  $\varrho \neq R\varrho$ ,  $\kappa \neq T\kappa$  and  $m(\varrho, R\varrho) \neq 0$ ,  $m(\kappa, T\kappa) \neq 0$ . If there exist  $\varrho_0 \in \Upsilon$  such that  $m(\varrho_0, R\varrho_0) \leq 1$ , then  $R$  and  $T$  have common FP in  $\Upsilon$ .

*Proof.* Since  $\varrho_0 \in \Upsilon$  such that  $m(\varrho_0, R\varrho_0) \leq 1$ , we construct a sequence  $(\varrho_n)$  in  $\Upsilon$  such that  $\varrho_{2n+1} = R\varrho_{2n}$  and  $\varrho_{2n+2} = T\varrho_{2n+1}$ . So,

$$\begin{aligned} m(\varrho_1, \varrho_2) &= m(R\varrho_0, T\varrho_1) \leq \lambda m(\varrho_0, R\varrho_0)^\alpha m(\varrho_1, T\varrho_1)^\beta = \lambda m(\varrho_0, R\varrho_0)^\alpha m(\varrho_1, \varrho_2)^\beta, \\ m(\varrho_1, \varrho_2)^{1-\beta} &\leq \lambda m(\varrho_0, R\varrho_0)^\alpha, \\ m(\varrho_1, \varrho_2) &\leq \lambda^{1/1-\beta} m(\varrho_0, R\varrho_0)^{\alpha/1-\beta} \leq \lambda, \end{aligned}$$

because  $\alpha/(1-\beta) > 1$  and  $m(\varrho_0, R\varrho_0) \leq 1$ . Similarly, by mathematical induction, the relation  $m(\varrho_n, \varrho_{n+1}) \leq \lambda^n$  holds for all natural numbers  $n \in \mathbb{N}$ . Thus, by taking limit we get  $\lim_{n \rightarrow +\infty} m(\varrho_n, \varrho_{n+1}) = 0$ . Also, by  $m_2$  condition of  $m$ -metric, we have

$$\lim_{n \rightarrow +\infty} m_{\varrho_n, \varrho_{n+1}} = 0,$$

and thus

$$\lim_{n \rightarrow +\infty} m(\varrho_n, \varrho_n) = 0, \quad \lim_{n \rightarrow +\infty} m(\varrho_{n+1}, \varrho_{n+1}) = 0.$$

Moreover, for any  $n, j \in \mathbb{N}$  with  $n \geq j$ , we have

$$\lim_{n, j \rightarrow +\infty} (M_{\varrho_n, \varrho_j} - m_{\varrho_n, \varrho_j}) = 0,$$

by triangular inequality of  $m$ -metric

$$\lim_{n, j \rightarrow +\infty} (m(\varrho_n, \varrho_j) - m_{\varrho_n, \varrho_j}) = 0.$$

Thus  $(\varrho_n)$  is a  $m$ -Cauchy sequence in  $\Upsilon$ , since  $\Upsilon$  is complete so it converges to some  $\varrho \in \Upsilon$ . Now

$$\begin{aligned} m(\varrho_{2n+1}, R\varrho) &\leq \lambda m(\varrho_{2n}, R\varrho_{2n})^\alpha m(\varrho, R\varrho)^\beta \\ &= \lambda m(\varrho_{2n}, \varrho_{2n+1})^\alpha m(\varrho, R\varrho)^\beta, \\ &\leq \lambda^{1+\alpha 2n} m(\varrho, R\varrho), \end{aligned}$$

thus by applying limit, we get  $\lim_{n \rightarrow +\infty} m(\varrho_{2n+1}, R\varrho) = 0$  and then by  $m_2$  condition of  $m$ -metric we have

$$\lim_{n \rightarrow +\infty} (m(\varrho_{2n+1}, R\varrho) - m_{\varrho_{2n+1}, R\varrho}) = 0,$$

by definition  $(\varrho_{2n+1})$  converges to  $R\varrho$ . On similar steps,  $(\varrho_{2n+2})$  converges to  $R\varrho$ , thus by combining both the arguments, we get the sequence  $(\varrho_n)$  also converges to  $R\varrho$ . Moreover, by using the similar arguments as in Case I and Case II of Theorem 3.1, we get  $\varrho = R\varrho$ .

Also, for  $T\varrho$  by following the similar procedure as mentioned above for  $\varrho = R\varrho$ , we get  $\varrho = T\varrho$ . Consequently,  $\varrho$  is the common FP for  $R$  and  $T$ .  $\square$

**Corollary 3.3.** *If we take  $R = T$ , then Theorem 3.8 of [8] becomes the special case of our result in Theorem 3.3.*

*Example 3.4.* Let  $\Upsilon = [0, 2]$  and  $m$ -metric on  $\Upsilon$  be defined as in (3.4) and define self mappings  $R, T : \Upsilon \rightarrow \Upsilon$  as follows:

$$R\varrho = \begin{cases} \varrho, & \varrho \in [0, 1), \\ e^{-\varrho}, & \varrho \in [1, 2], \end{cases} \quad T\varrho = \begin{cases} \varrho, & \varrho \in [0, 1), \\ 1/\varrho^2, & \varrho \in [1, 2]. \end{cases}$$

We discuss the following cases to confirm that for  $\alpha = 1/2, \beta = 3/4$  and  $\lambda = 3/4$  the pair  $(R, T)$  is  $m$ -IKTC pair used in Theorem 3.3. For any  $\varrho, \kappa \in [1, 2]$ , we have

$$\begin{aligned} (R\varrho, T\kappa) &\leq e^{-1} + 1 \leq (17/18)(1 + e^{-1})^{1/2}(1 + 1)^{3/4} \\ &\leq \lambda(\varrho + e^{-\varrho})^{1/2}(\kappa + 1/\kappa^2)^{3/4} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{3/4}. \end{aligned}$$

Moreover,  $e^{-\varrho} + 1/\kappa^2 \leq (17/18)(1 + e^{-1})^{3/4}(1 + 1)^{1/2}$ . Thus by Theorem 3.3, the self mappings  $R$  and  $T$  have common FPs for all  $\varrho \in [0, 1)$ .

Furthermore, in the case of the standard MS with  $d(\varrho, \kappa) = |\varrho - \kappa|$ , the IKTC for the pair  $(R, T)$  does not work when  $\kappa = 1$ . Therefore, our results in the  $m$ -MS are the ones applicable for such pairs to determine the common FP.

*Remark 3.4.* By Lemma 2.1, every  $p$ -MS is also a  $m$ -MS. Consequently, similar results of common FPs (Theorem 3.1, Theorem 3.2 and Theorem 3.3) for  $p$ -MSs naturally hold across all possible scenarios: when the sum of the 'interpolative exponents' is equal to 1, less than 1, and greater than 1.

*Remark 3.5.* Since every ordinary metric  $d$  is a  $p$ -metric, our Theorem 3.1 and Theorem 3.2 generalize the corresponding results of [5, 10], respectively.

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## COLORED TVERBERG THEOREMS FOR NON-PRIME POWERS

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ABSTRACT. We prove a relative of both the *original* and the *optimal (Type B)* version of the Colored Tverberg theorem of Živaljević and Vrećica (Theorems 2.2 and 2.3), which modifies these results in two different ways.

(1) We extend the original theorems beyond the prime powers by showing that the theorem is valid if the number of rainbow faces is  $q = p^n - 1$ .

(2) The size of some rainbow simplices may be smaller than in the original theorems. More precisely  $|C_i| \in \{2q - 2, 2q + 1\}$  while (for comparison) in the original theorems it is  $|C_i| = 2q - 1$ .

The proof relies on equivariant index theory and a result of Volovikov [17] about partial coincidences of maps  $f : X \rightarrow \mathbb{R}^d$ , from a  $G$ -space into the Euclidean space.

### 1. INTRODUCTION

Let  $K \subset 2^{[m]}$  be a simplicial complex (with  $m$  vertices). A continuous map  $f : K \rightarrow \mathbb{R}^d$  is called an *almost  $r$ -embedding* if  $f(\Delta_1) \cap \cdots \cap f(\Delta_r) = \emptyset$  for each collection  $\{\Delta_i\}_{i=1}^r$  of pairwise disjoint faces of  $K$ . If an almost  $r$ -embedding of  $K$  in  $\mathbb{R}^d$  does not exist we say that  $K$  is *not almost  $r$ -embeddable* in  $\mathbb{R}^d$ . The general *Tverberg problem* is to describe interesting classes of simplicial complexes which are or are not almost  $r$ -embeddable in  $\mathbb{R}^d$ . Historically the case of an  $N$ -dimensional simplex  $K = \Delta_N$  was studied first. It is still one of the central research themes, side by side with the case when  $K = R_{(C_1, C_2, \dots, C_{k+1})} := C_1 * \cdots * C_{k+1}$  is the join of 0-dimensional complexes (the Colored Tverberg problem).

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*Key words and phrases.* Colored Tverberg theorem, Volovikov index, connectedness, chessboard complex, deleted join, deleted product.

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**1.1. Almost  $r$ -embedding for non prime powers.** It is known [7] that almost  $r$ -embeddability (or non-embeddability) of a simplicial complex is critically dependent on the arithmetical properties of  $r$ . More precisely  $r$  is assumed to be a prime power  $r = p^n$  in the majority of results of this type.

What if  $r$  is not a prime power? For example, if  $K = \Delta_N$  is an  $N$ -dimensional simplex, then, as documented in the following results, the non-prime power case holds only if we substantially increase the dimension of the simplex.

- <sub>1</sub>  $\Delta_N$  is not almost  $r$ -embeddable in  $\mathbb{R}^d$  if  $r = p^\nu$  is a prime power,  $d \geq 1$ , and  $N = (r - 1)(d + 1)$  (I. Bárány, S. B. Shlosman, A. Szűcs 1981 [4]; M. Özaydin 1987 (unpublished); A. Y. Volovikov 1996 [15]; etc.).

- <sub>2</sub>  $\Delta_{r(d+1)-1}$  is not almost  $r$ -embeddable in  $\mathbb{R}^d$  for all  $r \geq 2$  and  $d \geq 1$  (F. Frick and P. Soberon 2020 in the preprint “*The topological Tverberg problem beyond prime powers*”).

- <sub>3</sub>  $\Delta_{(r-1)(d+1)}$  is almost  $r$ -embeddable in  $\mathbb{R}^d$  if  $r = p^\nu$  is not a prime power and  $d \geq 2r + 1$  ([2, 5, 7, 13, 14]).

- <sub>4</sub>  $\Delta_N$  is almost  $r$ -embeddable in  $\mathbb{R}^d$  if  $r$  is not a prime power and  $N = (d + 1)r - r \left\lceil \frac{d+2}{r+1} \right\rceil - 2$  (S. Avvakumov, R. Karasev and A. Skopenkov [1]).

All these results are instances of the following general problem: Determine integers  $a$  and  $d$  such that there exists (or there does not exist) an almost  $r$ -embedding  $\Delta_a \rightarrow \mathbb{R}^d$ . All of them illustrate the fact that the case when  $r$  is not prime power is more subtle and currently in the mainstream of research in this area.

In the same vein it is quite natural to explore the possibilities of extending the *Colored Tverberg problem* [19] to non-prime powers. More explicitly, we want to study the almost  $r$ -non embeddability of “rainbow complexes”

$$K = R_{(C_1, C_2, \dots, C_{k+1})} := C_1 * C_2 * \dots * C_{k+1},$$

if  $r$  is not a prime power.

Our Theorem 4.2 is an example of such an extension where:

- (1) the number of intersecting rainbow faces is  $q = p^n - 1$ ;
- (2)  $|C_1| = |C_2| = \dots = |C_m| = 2q + 1$ ,  $|C_{m+1}| = \dots = |C_{k+1-m}| = 2q - 2$ , under the condition

$$(1.1) \quad m \geq (d - k)(p^n - 1) = (d - k)q.$$

If  $k = d$  the condition (1.1) disappears and we observe (Corollary 4.1) that the result is valid if  $m = 0$ . This is a slight improvement over Theorem 2.2, where

$$|C_1| = |C_2| = \dots = |C_{d+1}| = 2r - 1.$$

(Note however that neither Theorem 2.2 nor Theorem 2.3 is formal consequence of Theorem 4.2.)

Examples 4.1 and 4.2 illustrate some special, low-dimensional cases of Theorem 4.2 which indicate that this result should be often close to the optimal in the case when the number of rainbow simplices is  $p^n - 1$ .

2. AN OVERVIEW OF TOPOLOGICAL TVERBERG TYPE RESULTS

The following result is known as the topological Tverberg theorem.

**Theorem 2.1** (Topological Tverberg theorem, [4, 15] and M. Özaydin 1987 (unpublished)). *Let  $d \geq 1$ ,  $r \geq 2$ , and  $N = (r - 1)(d + 1)$  be integers. If  $r$  is a prime power, then for any continuous map  $f : \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

Interesting problems and (conjectured) extensions and relatives of the Topological Tverberg theorem have emerged over the years. In particular, motivated by questions from discrete and computational geometry, Bárány and Larman [3] formulated in 1992 the *colored Tverberg problem*.

**Definition 2.1** (Coloring). Let  $N \geq 1$  be an integer and let  $V(\Delta_N)$  be the set of vertices of the simplex  $\Delta_N$ . A *coloring* of vertices of  $V(\Delta_N)$  by  $l$  colors is a partition  $(C_1, \dots, C_l)$  of  $V(\Delta_N)$ , that is  $V(\Delta_N) = C_1 \cup \dots \cup C_l$ , with  $C_i \cap C_j = \emptyset$ , for  $1 \leq i < j \leq l$ . The elements of the partition  $(C_1, \dots, C_l)$  are called *color classes*.

**Definition 2.2** (Rainbow face). Let  $(C_1, \dots, C_l)$  be the coloring of  $V(\Delta_N)$  by  $l$  colors. A face  $\sigma$  of the simplex  $\Delta_N$  is a *rainbow face* if  $|\sigma \cap C_i| \leq 1$ , for all  $1 \leq i \leq l$ .

*Problem 2.1* (Bárány-Larman colored Tverberg problem). Let  $d \geq 1$  and  $r \geq 2$  be integers. Determine the smallest number  $n = n(d, r)$  such that for every map  $f : \Delta_{n-1} \rightarrow \mathbb{R}^d$ , and every coloring  $(C_1, \dots, C_{d+1})$  of the vertex set  $V(\Delta_{n-1})$  of the simplex  $\Delta_{n-1}$  by  $d + 1$  colors, with each color of size at least  $r$ , there exist  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_{n-1}$  satisfying  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .

A modified colored Tverberg problem was presented by Živaljević and Vrećica in [20].

*Problem 2.2* (Živaljević-Vrećica colored Tverberg problem). Let  $d \geq 1$  and  $r \geq 2$  be integers. Determine the smallest number  $t = t(d, r)$  such that for every affine (or continuous) map  $f : \Delta \rightarrow \mathbb{R}^d$ , and every coloring  $(C_1, \dots, C_{d+1})$  of the the vertex set  $V(\Delta)$  by  $d + 1$  colors, with each color of size at least  $t$ , there exist  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta$  satisfying  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .

For  $r \geq 2$  a prime power, Živaljević and Vrećica proved that  $t(d, r) \leq 2r - 1$ . This result is known as the (*original*) *Colored Tverberg theorem of Živaljević and Vrećica*.

**Theorem 2.2** (*Colored Tverberg theorem of Živaljević and Vrećica* [20]). *Let  $d \geq 1$  be an integer, and let  $r = p^n \geq 2$  be a prime power. For every continuous map  $f : \Delta \rightarrow \mathbb{R}^d$ , and every coloring  $(C_1, \dots, C_{d+1})$  of the the vertex set  $V(\Delta)$  by  $d + 1$  colors, with each color of size at least  $2r - 1$ , there exist  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta$  satisfying  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

The following result is known as *Optimal (Type B) Colored Tverberg theorem of Živaljević and Vrećica*, see [18, 19].

**Theorem 2.3** (Optimal (Type B) Colored Tverberg theorem of Živaljević and Vrećica). *Assume that  $r = p^\nu$  is a prime power,  $d \geq 1$ , and let  $k$  be an integer such that  $\frac{r-1}{r}d \leq k < d$ . Then, the complex*

$$R_{(C_0, C_1, \dots, C_k)} := C_0 * \dots * C_k$$

*is not almost  $r$ -embeddable in  $\mathbb{R}^d$ , if  $|C_i| \geq 2r - 1$  for all  $i$ .*

### 3. TOPOLOGICAL PRELIMINARIES

In this section we collect central definitions and results needed for the proof of our main theorem.

**3.1. Configuration spaces.** *Deleted joins* and *deleted products* are the standard *configuration spaces* used, in the framework of the *configuration space/test map scheme* [8, 10, 19], in applications of topological methods to problems of combinatorics and discrete and computational geometry.

**Definition 3.1** (Deleted join). Let  $K$  be a simplicial complex, let  $n \geq 2$ ,  $k \geq 2$  be integers, and let  $[n] = \{1, \dots, n\}$ . The  $n$ -fold  $k$ -wise deleted join of the simplicial complex  $K$  is the simplicial complex:

$$K_{\Delta(k)}^{*n} = \left\{ \lambda_1 x_1 + \dots + \lambda_n x_n \in \sigma_1 * \dots * \sigma_n \subset K^{*n} \mid (\forall I \subset [n]) |I| \geq k \Rightarrow \bigcap_{i \in I} \sigma_i = \emptyset \right\},$$

where  $\sigma_1, \dots, \sigma_n$  are faces of  $K$ , including the empty face. The symmetric group  $\mathcal{G}_n = \text{Sym}(n)$  acts on  $K_{\Delta(k)}^{*n}$  by:

$$\pi \cdot (\lambda_1 x_1 + \dots + \lambda_n x_n) = \lambda_{\pi^{-1}(1)} x_{\pi^{-1}(1)} + \dots + \lambda_{\pi^{-1}(n)} x_{\pi^{-1}(n)},$$

for  $\pi \in \mathcal{G}_n$  and  $\lambda_1 x_1 + \dots + \lambda_n x_n \in K_{\Delta(k)}^{*n}$ .

**Definition 3.2** (Deleted product). Let  $K$  be a simplicial complex, let  $n \geq 2$ ,  $k \geq 2$  be integers, and let  $[n] = \{1, \dots, n\}$ . The  $n$ -fold  $k$ -wise deleted product of the simplicial complex  $K$  is the cell complex:

$$K_{\Delta(k)}^{\times n} = \left\{ (x_1, \dots, x_n) \in \sigma_1 \times \dots \times \sigma_n \subset K^{\times n} \mid (\forall I \subset [n]) |I| \geq k \Rightarrow \bigcap_{i \in I} \sigma_i = \emptyset \right\},$$

where  $\sigma_1, \dots, \sigma_n$  are non-empty faces of  $K$ . The symmetric group  $\mathcal{G}_n = \text{Sym}(n)$  acts on  $K_{\Delta(k)}^{\times n}$  by:

$$\pi \cdot (x_1, \dots, x_n) = (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}),$$

for  $\pi \in \mathcal{G}_n$  and  $(x_1, \dots, x_n) \in K_{\Delta(k)}^{\times n}$ .

**Definition 3.3** (Chessboard complex). The  $m \times n$  chessboard complex  $\Delta_{m,n}$  is the simplicial complex whose vertex set is  $[m] \times [n]$ , and the simplexes of  $\Delta_{m,n}$  are the subsets  $\{(i_0, j_0), \dots, (i_k, j_k)\} \subset [m] \times [n]$ , where  $i_s \neq i_{s'}, 1 \leq s < s' \leq k$ , and  $j_t \neq j_{t'}, 1 \leq t < t' \leq k$ .

**Definition 3.4** (Rainbow subcomplex). Let  $\Delta$  be a simplex with a coloring  $\mathcal{C} = (C_1, \dots, C_{d+1})$  by  $(d + 1)$  colors. We define the *rainbow subcomplex*  $R_{(C_1, \dots, C_{d+1})} \subset \Delta$  as follows:

$$R_{(C_1, \dots, C_{d+1})} \cong C_1 * \dots * C_{d+1},$$

where  $C_i$  is a discrete set of points, for every  $i \in [d + 1]$ .

**3.2. Volovikov index.** The following fundamental result of cohomology theory is used in the definition the Volovikov index of a  $G$ -space  $X$ .

**Theorem 3.1** (The cohomology Leray-Serre Spectral sequence [12, Theorem 5.2]). *Let  $R$  be a commutative ring with the unity. Given a fibration  $F \hookrightarrow E \xrightarrow{p} B$ , where  $B$  is a path-wise connected space, there is a first quadrant spectral sequence of algebras  $\{E_r^{*,*}, d_r\}$ , with*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F; R)),$$

*the cohomology of  $B$ , with local coefficients in the cohomology of  $F$ , the fiber of  $p$ , and converging to  $H^*(E; R)$  as an algebra. Furthermore, this spectral sequence is natural with the respect to fiber-preserving maps of fibrations.*

We continue with the definition of the Volovikov index [16]. It is defined as a function on  $G$ -spaces (where  $G$  is a compact Lie group) whose values are either positive integers or  $\infty$ . For our application it is sufficient to assume that  $G$  is a  $p$ -torus  $G = (\mathbb{Z}_p)^n$ , where  $p$  a prime number.

**Definition 3.5** (Volovikov index). Let  $G$  be a compact Lie group and let  $X$  be a Hausdorff paracompact  $G$ -space. The definition of the *Volovikov index of  $X$* , denoted by  $i(X)$ , uses the spectral sequence of the bundle  $p_X : X_G \rightarrow BG$ , with fibre  $X$  (the Borel construction), given in Theorem 3.1. This spectral sequence converges to the equivariant cohomology  $H^*(X_G; \mathbb{Z}_p)$ . Let  $\Lambda^*$  be the equivariant cohomology algebra of a point  $H^*(pt_G; \mathbb{Z}_p) = H^*(BG; \mathbb{Z}_p)$ . Suppose that  $X$  is path connected. Then  $E_2^{*,0} = \Lambda^*$ . Assume that  $E_2^{*,0} = \dots = E_s^{*,0} \neq E_{s+1}^{*,0}$ . Then, by definition,  $i(X) = s$ . If  $E_2^{*,0} = \dots = E_\infty^{*,0}$  then, by definition,  $i(X) = \infty$ . Let  $i'(X)$  be the least number  $r$  such that the kernel of the natural homomorphism  $\Lambda^* \rightarrow E_{r+1}^{*,0}$  contains an element which is not a zero divisor in  $\Lambda^*$ .

The following theorem describes some of the most important properties of the Volovikov index.

**Theorem 3.2** ([16]). (1) *If there exists an equivariant map of  $G$ -spaces  $X \rightarrow Y$ , then  $i(X) \leq i(Y)$  and  $i'(X) \leq i'(Y)$ .*

(2) *If  $X$  is a compact or finite-dimensional cohomological sphere (over the the field  $\mathbb{Z}_p$ ), i.e.,  $H^*(X) = H^*(S^n)$ , and if  $G$  acts with no fixed points on  $X$ , then  $i(X) = i'(X) = n + 1$ .*

(3) *If  $\tilde{H}^j(X; \mathbb{Z}_p) = 0$ , for all  $j < n$ , then  $i(X) \geq n + 1$ .*

(4) *If  $X = A \cup B$ , where  $A$  and  $B$  are closed (or open)  $G$ -invariant subspaces,  $i(X) \leq i'(A) + i(B)$ . In particular,  $i(X * Y) \leq i'(X) + i(Y)$ .*

**3.3. Connectedness.** Here we review the definition and some basic properties of the *connectedness* of topological spaces, including a key result which relates the connectedness to the Volovikov index.

**Definition 3.6** ([10], Definition 4.3.2). Let  $n \geq -1$  be an integer. A topological space  $X$  is  $n$ -connected if any continuous map  $f : S^k \rightarrow X$ , where  $-1 \leq k \leq n$ , can be continuously extended to a continuous map  $g : B^{k+1} \rightarrow X$ , that is  $g|_{\partial B^{k+1}=S^k} = f$  (here  $B^{k+1}$  denotes a  $(k + 1)$ -dimensional closed ball whose boundary is the sphere  $S^k$ ). A topological space is  $(-1)$ -connected if it is non-empty. If the space  $X$  is  $n$ -connected, but not  $(n + 1)$ -connected, we write  $\text{conn}(X) = n$ .

**Theorem 3.3** ([8], p. 332). *Let  $X$  and  $Y$  be topological spaces. Then,*

$$\text{conn}(X * Y) \geq \text{conn}(X) + \text{conn}(Y) + 2.$$

**Theorem 3.4** ([10], Theorem 4.4.1). *Let  $X$  be a nonempty topological space and let  $k \geq 1$ . Then  $X$  is  $k$ -connected if and only if it is simply connected (i.e., the fundamental group  $\pi_1(X)$  is trivial) and  $\tilde{H}_i(X) = 0$ , for all  $i = 0, 1, \dots, k$ .*

**Theorem 3.5.** *Let  $X$  topological space. Then,  $i(X) \geq \text{conn}(X) + 2$ .*

*Proof.* It is a consequence of Theorem 3.4 and Theorem 3.2 (3). □

**Theorem 3.6** ([6]). *Let  $m, n \geq 1$  be integers. Then,*

$$\text{conn}(\Delta_{m,n}) = \min \left\{ m, n, \left\lfloor \frac{m + n + 1}{3} \right\rfloor \right\} - 2.$$

#### 4. COLORED TVERBERG THEOREM WITH $p^n - 1$ FACES

In this section we prove the main result of the paper (Theorem 4.2). First, we state and prove two lemmas that are needed for the proof.

**Definition 4.1** ([17]). Let  $X$  be a  $G$ -space, where  $G$  is a finite group, and let  $f : X \rightarrow Y$  be a continuous map. For  $2 \leq y \leq |G|$ , we set

$$A(f, y) = \{x \in X \mid f(g_1x) = \dots = f(g_yx), \text{ for some distinct } g_i \in G\}.$$

**Lemma 4.1.** *Let  $r = p^n \geq 2$  be a prime power and let  $d \geq 1$ ,  $1 \leq k \leq d$ ,  $2 \leq q \leq r$  be integers. For every continuous map  $f : \Delta \rightarrow \mathbb{R}^d$  and every coloring  $(C_1, \dots, C_{k+1})$  of the vertex set  $V(\Delta)$  by  $(k + 1)$  colors, define the continuous map as follows:*

$$h : (R_{(C_1, \dots, C_{k+1})}_{\Delta(2)})^{\times p^n} \rightarrow \mathbb{R}^d, \quad \text{where } h(x_1, \dots, x_{p^n}) = f(x_1).$$

*If  $A(h, q) \neq \emptyset$ , then there exists  $q$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_q$  of  $\Delta$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_q) \neq \emptyset$ .*

*Proof.* Choose  $(x_1, \dots, x_{p^n}) \in A(h, q) \neq \emptyset$ .

Then, there exist distincts elements  $g_1, \dots, g_q \in (\mathbb{Z}_p)^n$  such that

$$h(g_1(x_1, \dots, x_{p^n})) = \dots = h(g_q(x_1, \dots, x_{p^n})).$$

Therefore, there exist  $q$  elements  $x_{i_1}, \dots, x_{i_q} \in \{x_1, \dots, x_{p^n}\}$  such that  $f(x_{i_1}) = \dots = f(x_{i_q})$ , where  $x_{i_1} \in \sigma_{i_1}, \dots, x_{i_q} \in \sigma_{i_q}$  ( $\sigma_{i_m}$  is a support of  $x_{i_m}$ , for every  $m \in [q]$ ).

By the definition of the configuration space  $(R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}$ , there exist  $q$  pairwise disjoint, non-empty rainbow faces  $\sigma_{i_1}, \dots, \sigma_{i_q}$  such that  $f(\sigma_{i_1}) \cap \dots \cap f(\sigma_{i_q}) \neq \emptyset$ .  $\square$

**Lemma 4.2.** *Let  $d \geq 1, 1 \leq k \leq d, 0 \leq m \leq k + 1$  be integers and let  $r = p^n \geq 2$  be a prime power. Let  $(C_1, \dots, C_{k+1})$  be a coloring of the vertex set  $V(\Delta)$  by  $(k + 1)$  colors, where we have  $|C_i| \geq 2r - 1$ , for all  $i = 1, \dots, m, |C_i| \geq 2r - 4$ , for all  $i = m + 1, \dots, k + 1$  and  $m \geq (d - k)(r - 1)$ . Then,*

$$i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}) \geq d(p^n - 1).$$

*Proof.* Note that

$$(R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n} = A \cup B,$$

where

$$A = \left\{ \lambda_1 x_1 + \dots + \lambda_{p^n} x_{p^n} \in (R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n} \mid (\exists i \in [p^n]) \lambda_i \neq \frac{1}{p^n} \right\},$$

$$B = \left\{ \lambda_1 x_1 + \dots + \lambda_{p^n} x_{p^n} \in (R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n} \mid \lambda_1 = \dots = \lambda_{p^n} = \frac{1}{p^n} \right\}.$$

It is not difficult to see that  $B$  is isomorphic to  $(R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}$ . It follows from Theorem 3.2 (4) that

$$i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}) \leq i'(A) + i(B) = i'(A) + i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}).$$

We want to estimate the indices  $i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n})$  and  $i'(A)$ . In light of the isomorphism

$$(R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times r} \cong \Delta_{|C_1|, r} * \dots * \Delta_{|C_m|, r} * \Delta_{|C_{m+1}|, r} * \dots * \Delta_{|C_{k+1}|, r},$$

we obtain, as a consequence of Theorem 3.3 and Theorem 3.6,

$$\text{conn}((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times r}) \geq m(r - 2) + (k + 1 - m)(r - 3) + 2k.$$

It follows from Theorem 3.5 that

$$i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times r}) \geq [m(r - 2) + (k + 1 - m)(r - 3) + 2k] + 2 = (k + 1)(r - 1) + m.$$

Since  $m \geq (d - k)(r - 1)$ , we have

$$i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times r}) \geq (d + 1)(r - 1).$$

In order to find a bound for  $i'(A)$  let us consider the following  $(\mathbb{Z}_p)^n$ -equivariant map

$$\phi : A \rightarrow \mathbb{R}^{p^n} \setminus \Delta(\mathbb{R}^{p^n}), \quad \phi(\lambda_1 x_1 + \dots + \lambda_{p^n} x_{p^n}) = (\lambda_1, \dots, \lambda_{p^n}),$$

and

$$\Pi : \mathbb{R}^{p^n} \setminus \Delta(\mathbb{R}^{p^n}) \rightarrow (\Delta(\mathbb{R}^{p^n}))^\perp \setminus \{0\} \rightarrow S((\Delta(\mathbb{R}^{p^n}))^\perp),$$

where  $\Pi$  is a composition of the projection and deformation retraction.

Here  $\Delta(\mathbb{R}^{p^n}) = \{(x_1, \dots, x_{p^n}) \in \mathbb{R}^{p^n} \mid x_1 = \dots = x_{p^n}\}$  is the diagonal subspace of  $\mathbb{R}^{p^n}$ , while  $S(V)$  is the unit sphere in the real vector space  $V$ .

It follows that the composition

$$\Pi \circ \phi : A \rightarrow S((\Delta(\mathbb{R}^{p^n}))^\perp) \cong S^{p^n-2},$$

is a  $(\mathbb{Z}_p)^n$ -equivariant map. By Theorem 3.2 ((1) and (2)) we conclude that

$$i'(A) \leq i'(S^{p^n-2}) = p^n - 1,$$

and as an immediate consequence,  $i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}) \geq d(p^n - 1)$ . □

**Theorem 4.1** ([17], Theorem 4). *Let  $X$  be a connected  $G$ -space, where  $G = (\mathbb{Z}_p)^n$  is a  $p$ -torus and  $2 \leq y \leq p^n$ ,  $y \neq 3$ . Assume the inequality  $i(X) \geq (m - 1)(p^n - 1) + y$ . Then,  $A(f, y) \neq \emptyset$  for any continuous map  $f : X \rightarrow \mathbb{R}^m$ .*

*Remark 4.1.* Theorem 4.1 is also true for  $y = 3$  and  $r = 3, 4, 5$ .

**Theorem 4.2.** *Let  $d \geq 1$ ,  $1 \leq k \leq d$ ,  $0 \leq m \leq k + 1$  be integers, and let  $r = p^n \geq 2$  be a prime power. For every continuous map  $f : \Delta \rightarrow \mathbb{R}^d$ , and every coloring  $(C_1, \dots, C_{k+1})$  of the vertex set  $V(\Delta)$  by  $(k + 1)$  colors, such that  $|C_i| \geq 2r - 1$ , for all  $i = 1, \dots, m$ ,  $|C_i| \geq 2r - 4$ , for all  $i = m + 1, \dots, k + 1$  and  $m \geq (d - k)(r - 1)$ , there exist  $q = r - 1 = p^n - 1$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_q$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_q) \neq \emptyset$ .*

*Proof.* It follows from Lemma 4.1 that if  $A(h, q)$  is non-empty then there exist  $q$  pairwise disjoint, rainbow, non-empty faces  $\sigma_1, \dots, \sigma_q$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_q) \neq \emptyset$ .

Therefore, it remains to be shown that  $A(h, q) \neq \emptyset$ .

On the other hand this is an immediate consequence of Theorem 4.1, applied to the  $G$ -space  $X = (R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}$  and the map  $h : (R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n} \rightarrow \mathbb{R}^d$  (as in Lemma 4.1), where  $y = q$ . Indeed,  $(R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}$  is connected and  $i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{\times p^n}) \geq d(p^n - 1) = (m - 1)(p^n - 1) + y$  (by Lemma 4.2). This observation completes the proof of the theorem. □

**Corollary 4.1.** *Let  $d \geq 1$  be an integer, and let  $r = p^n \geq 2$  be a prime power. For every continuous map  $f : \Delta \rightarrow \mathbb{R}^d$ , and every coloring  $(C_1, \dots, C_{d+1})$  of the vertex set  $V(\Delta)$  by  $(d+1)$  colors, with each color of size at least  $2r - 4$ , there exist  $q = r - 1 = p^n - 1$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_q$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_q) \neq \emptyset$ .*

*Proof.* Apply Theorem 4.2 to the case  $k = d$  and  $m = 0$ . □

*Remark 4.2.* Note that if  $m > (d - k)(r - 1)$ . Then,

$$i((R_{(C_1, \dots, C_{k+1})})_{\Delta(2)}^{*r}) \geq (d + 1)(r - 1) + 1,$$

and there exist  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ . This means that the interesting case of Theorem 4.2 is when  $m = (d - k)(r - 1)$ .

*Example 4.1.* Let  $r = 7$ ,  $d = 8$ ,  $k = 7$  and  $m = 6$ . Then we have  $k + 1 = 8$  colors  $C_1, \dots, C_8$  where  $|C_i| \geq 2r - 1 = 13$ , for  $i = 1, \dots, 6$  and  $|C_7|, |C_8| \geq 2r - 4 = 10$ . Note that the condition  $m \geq (d - k)(r - 1)$  follows (more specifically we have an equality). Then, there exist  $q = r - 1 = 6$  pairwise disjoint rainbow faces  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$  and  $\sigma_6$  such that  $f(\sigma_1) \cap f(\sigma_2) \cap f(\sigma_3) \cap f(\sigma_4) \cap f(\sigma_5) \cap f(\sigma_6) \neq \emptyset$ .

The following example illustrates the Corollary 4.1.

*Example 4.2.* Let  $d = 2$ ,  $r = 7$  and  $\mathcal{C} = (C_1, C_2, C_3)$  be a coloring of vertex set  $V(\Delta)$ , with  $|C_1| = |C_2| = |C_3| = 2r - 4 = 10$ . Let  $f : \Delta \rightarrow \mathbb{R}^2$  be a continuous map.

By Corollary 4.1, there exist 6 pairwise disjoint rainbow faces  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$  and  $\sigma_6$  such that  $f(\sigma_1) \cap f(\sigma_2) \cap f(\sigma_3) \cap f(\sigma_4) \cap f(\sigma_5) \cap f(\sigma_6) \neq \emptyset$ .

## 5. CONCLUDING REMARKS

We used throughout the paper versions of Volovikov’s index (Section 3.2). A more elementary and less technical alternative is to use “elementary equivariant index theory” ( $G$ -genus), as presented in [9].

The methods used in the paper are cohomological. Typically, they allow us to conclude that the zero-set of some (equivariant) test map is non-empty, which is sufficient for many applications.

However, the cohomological approach may sometimes lead to a conclusion that the zero-set is “big” in some stronger sense, for example it may support a non-trivial (co)homology class, it may have a high genus (Lusternik-Schnirelmann category), etc.

This point of view is vividly illustrated by “parameterized index theory”, see [11] for examples and a guide to the literature.

We believe that cohomological methods have a great potential for new applications in discrete geometry and combinatorics, including the Tverberg type problems and their relatives.

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**INITIAL COEFFICIENT ESTIMATES FOR A CERTAIN FAMILIES  
OF BI-UNIVALENT FUNCTIONS RELATED TO BAZILEVIČ AND  
 $\lambda$ -PSEUDO FUNCTIONS**

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ABSTRACT. In this article, we define new families of normalized holomorphic and bi-univalent functions  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  and  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  which involve the Bazilevič functions and the  $\lambda$ -pseudo functions defined in the unit disk  $U$ . We determine the coefficient estimates for the initial Taylor-Maclaurin coefficients  $|a_2|$  and  $|a_3|$  and resolve the Fekete-Szegő type inequalities for these families. In addition, we point out several special cases and consequences of our results.

1. INTRODUCTION

Denote by  $\mathcal{A}$  the family of all holomorphic functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{+\infty} a_n z^n$$

in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ . We also denote by  $\mathcal{S}$  the subfamily of  $\mathcal{A}$  consisting of functions which are also univalent in  $U$ .

The famous Koebe one-quarter theorem [11] ensure that the image of  $U$  under each univalent function  $f \in \mathcal{A}$  contain a disk of radius  $\frac{1}{4}$ . Furthermore, each function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$  defined by  $f^{-1}(f(z)) = z$  and

$$f^{-1}(f(w)) = w, \quad |w| < r_0(f), r_0(f) \geq \frac{1}{4},$$

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where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

A function  $f \in \mathcal{A}$  is named bi-univalent in  $U$  if both  $f$  and  $f^{-1}$  are univalent in  $U$ . The family of all bi-univalent functions in  $U$  denoted by  $\Sigma$ .

In fact, Srivastava et al. [31] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [13], Ali et al. [1], Bulut et al. [7] and others (see, for example, [2, 8, 9, 14, 15, 25–28, 32, 35]). From the work of Srivastava et al. [31], we choose to recall the following examples of functions in the family  $\Sigma$ :

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2} \log\left(\frac{1+z}{1-z}\right).$$

We notice that the family  $\Sigma$  is not empty. However, the Koebe function is not a member of  $\Sigma$ .

The problem to obtain the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N}, n \geq 3,$$

for functions  $f \in \Sigma$  is still not completely addressed for many of the subfamilies of  $\Sigma$  (see, for example, [32]). The Fekete-Szegő functional  $|a_3 - \eta a_2^2|$  for  $f \in \mathcal{S}$  is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegő [12] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in the study of many subclasses of the family of univalent functions. This topic has become of considerable interest among researchers in Geometric Function Theory (see, for example, [5, 17, 22, 29, 30]).

With a view to recalling the principle of subordination between holomorphic functions, let the functions  $f$  and  $g$  be holomorphic in  $U$ . We name the function  $f$  is subordinate to  $g$ , if there exists a Schwarz function  $\omega$ , which is analytic in  $U$  with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1, \quad z \in U,$$

such that

$$f(z) = g(\omega(z)).$$

This subordination is denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in U.$$

It is well known that (see [19]), if the function  $g$  is univalent in  $U$ , then

$$f \prec g \quad (z \in U) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subseteq g(U).$$

A function  $f \in \mathcal{A}$  is called Bazilevič function in  $U$  if (see [24])

$$\operatorname{Re} \left\{ \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} \right\} > 0, \quad z \in U, \gamma \geq 0.$$

On the other hand, a function  $f \in \mathcal{A}$  is called a  $\lambda$ -pseudo-starlike function in  $U$  if (see [3])

$$\operatorname{Re} \left\{ \frac{z (f'(z))^\lambda}{f(z)} \right\} > 0, \quad z \in U, \lambda \geq 1.$$

Recently, several authors introduced and studied different subfamilies associated with Bazilevič and  $\lambda$ -pseudo functions (see, for example, [6, 10, 16, 21, 33, 34, 36–38]).

We shall need the following lemma in our investigation.

**Lemma 1.1** ([20]). *Let the function  $\mathfrak{p} \in \mathfrak{P}$  be given by the following series:*

$$\mathfrak{p}(z) = 1 + \mathfrak{p}_1 z + \mathfrak{p}_2 z^2 + \dots, \quad z \in \mathfrak{U}.$$

*The sharp estimate given by  $|\mathfrak{p}_n| \leq 2$ ,  $n \in \mathbb{N}$ , holds true.*

## 2. MAIN RESULTS

Denote by  $\vartheta(z)$  the holomorphic function with positive real part in  $U$  such that

$$\vartheta(0) = 1, \quad \vartheta'(0) > 0,$$

and  $\vartheta(z)$  is symmetric with respect to real axis, which is of the type:

$$(2.1) \quad \vartheta(z) = 1 + \mathfrak{B}_1 z + \mathfrak{B}_2 z^2 + \mathfrak{B}_3 z^3 + \dots,$$

where  $\mathfrak{B}_1 > 0$ .

Using the subordinations, we now provide the following subfamilies of holomorphic and bi-univalent functions.

**Definition 2.1.** For  $0 \leq \mu \leq 1$ ,  $\gamma \geq 0$  and  $\lambda \geq 1$ , a function  $f \in \Sigma$  is said to be in the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  if it fulfills the subordinations:

$$(1 - \mu) \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \mu \frac{z (f'(z))^\lambda}{f(z)} \prec \vartheta(z)$$

and

$$(1 - \mu) \frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \mu \frac{w (g'(w))^\lambda}{g(w)} \prec \vartheta(w),$$

where  $g(w) = f^{-1}(w)$ .

**Definition 2.2.** For  $0 \leq \mu \leq 1$ ,  $\gamma \geq 0$  and  $\lambda \geq 1$ , a function  $f \in \Sigma$  is said to be in the family  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  if it fulfills the subordinations:

$$(1 - \mu) \left( 1 + \frac{z^{2-\gamma} f''(z)}{(z f'(z))^{1-\gamma}} \right) + \mu \frac{((z f'(z))')^\lambda}{f'(z)} \prec \vartheta(z)$$

and

$$(1 - \mu) \left( 1 + \frac{w^{2-\gamma} g''(w)}{(w g'(w))^{1-\gamma}} \right) + \mu \frac{((w g'(w))')^\lambda}{g'(w)} \prec \vartheta(w),$$

where  $g(w) = f^{-1}(w)$ .

*Remark 2.1.* The families  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  and  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  are a generalization of several known families studied in earlier investigations which are being recalled below.

- (a) For  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ,  $0 < \alpha \leq 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduce to the family  $\mathcal{T}_\Sigma(\mu, \gamma, \lambda; \alpha)$  which was considered by Srivastava et al. [34].
- (b) For  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{T}_\Sigma^*(\mu, \gamma, \lambda; \beta)$  which was studied by Srivastava et al. [34].
- (c) For  $\vartheta(z) = \frac{a+(b-ap)rz}{1-prz-qz^2} + 1 - a$ ,  $r \in \mathbb{R}$ ,  $a, b, p$  and  $q$  are real constant, the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{T}_\Sigma(\mu, \gamma, \lambda, r)$  which was investigated by Wanas et al. [39].
- (d) For  $\vartheta(z) = \frac{2-M(x)z}{1-M(x)z-N(x)z^2}$ ,  $M(x)$  and  $N(x)$  are polynomials with real coefficients, the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{L}_{MN}(\mu, \gamma, \lambda; x)$  which was defined by Wanas et al. [38].
- (e) For  $\mu = 0$  and  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ,  $0 < \alpha \leq 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $P_\Sigma(\alpha, \gamma)$  which was studied by Prema and Keerthi [21].
- (f) For  $\mu = 0$  and  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $P_\Sigma(\beta, \gamma)$  which was investigated by Prema and Keerthi [21].
- (g) For  $\mu = 1$  and  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ,  $0 < \alpha \leq 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{LB}_\Sigma^\lambda(\alpha)$  which was considered by Joshi et al. [16].
- (h) For  $\mu = 1$  and  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{LB}_\Sigma(\lambda, \beta)$  which was introduced by Joshi et al. [16].
- (i) For  $\mu = \gamma = 0$  and  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ,  $0 < \alpha \leq 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $S_\Sigma^*(\alpha)$  which was considered by Brannan and Taha [4].
- (j) For  $\mu = \gamma = 0$  and  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $S_\Sigma^*(\beta)$  which was investigated by Brannan and Taha [4].
- (k) For  $\mu = \gamma = 0$  and  $\vartheta(z) = \frac{a+(b-ap)rz}{1-prz-qz^2} + 1 - a$ ,  $r \in \mathbb{R}$ ,  $a, b, p$  and  $q$  are real constants, the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{W}_\Sigma(r)$  which was defined by Srivastava et al. [25].
- (l) For  $\mu = \gamma = 0$  and  $\vartheta(z) = \frac{1}{1-2tz+z^2}$ ,  $t \in (\frac{1}{2}, 1]$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $S_\Sigma^*(t)$  which was introduced by Bulut et al. [7].
- (m) For  $\mu = 0$ ,  $\gamma = 1$  and  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ,  $0 < \alpha \leq 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{H}_\Sigma(\alpha)$  which was investigated by Srivastava et al. [31].
- (n) For  $\mu = 0$ ,  $\gamma = 1$  and  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$ , the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{H}_\Sigma(\beta)$  which was defined by Srivastava et al. [31].
- (o) For  $\mu = 0$  and  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ,  $0 < \alpha \leq 1$ , the family  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{B}_\Sigma(\gamma; \alpha)$  which was investigated by Sakar and Wanas [23].
- (p) For  $\mu = 0$  and  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$ , the family  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{B}_\Sigma^*(\gamma; \beta)$  which was defined by Sakar and Wanas [23].

(q) For  $\mu = \gamma = 0$  and  $\vartheta(z) = \frac{a+(b-ap)rz}{1-prz-qz^2} + 1 - a$ ,  $r, a, b, p, q \in \mathbb{R}$ , the family  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  reduces to the family  $\mathcal{K}_\Sigma(r)$  which was introduced by Magesh et al. [18].

**Theorem 2.1.** *Let  $f$ , given by (1.1), be in the family  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ . Then,*

$$|a_2| \leq \min \left\{ \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+1) + \mu(2\lambda-1)}, \frac{\sqrt{2}\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda(2\lambda-1)] + 2[(1-\mu)(\gamma+1) + \mu(2\lambda-1)]^2(\mathfrak{B}_1 - \mathfrak{B}_2)|}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2) + \mu(3\lambda-1)} + \frac{2\mathfrak{B}_2}{(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda(2\lambda-1)}, \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2) + \mu(3\lambda-1)} + \frac{\mathfrak{B}_1^2}{[(1-\mu)(\gamma+1) + \mu(2\lambda-1)]^2} \right\},$$

where the coefficients  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  are defined as in (2.1).

*Proof.* Let  $f \in \mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  and  $g = f^{-1}$ . Then, there are holomorphic functions  $\mathfrak{S}, \mathfrak{T} : U \rightarrow U$  with  $\mathfrak{S}(0) = \mathfrak{T}(0) = 0$ , fulfills the following conditions:

$$(2.2) \quad (1-\mu) \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \mu \frac{z(f'(z))^\lambda}{f(z)} = \vartheta(\mathfrak{S}(z)), \quad z \in U,$$

and

$$(2.3) \quad (1-\mu) \frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \mu \frac{w(g'(w))^\lambda}{g(w)} = \vartheta(\mathfrak{T}(w)), \quad w \in U.$$

Define the functions  $x$  and  $y$  by

$$x(z) = \frac{1 + \mathfrak{S}(z)}{1 - \mathfrak{S}(z)} = 1 + x_1 z + x_2 z^2 + \dots$$

and

$$y(z) = \frac{1 + \mathfrak{T}(z)}{1 - \mathfrak{T}(z)} = 1 + y_1 z + y_2 z^2 + \dots$$

Then,  $x$  and  $y$  are analytic in  $U$  with  $x(0) = y(0) = 1$ . Since we have  $\mathfrak{S}, \mathfrak{T} : U \rightarrow U$ , each of the functions  $x$  and  $y$  has a positive real part in  $U$ .

Solving for  $\mathfrak{S}(z)$  and  $\mathfrak{T}(z)$ , we have

$$(2.4) \quad \mathfrak{S}(z) = \frac{x(z) - 1}{x(z) + 1} = \frac{1}{2} \left[ x_1 z + \left( x_2 - \frac{x_1^2}{2} \right) z^2 \right] + \dots, \quad z \in U,$$

and

$$(2.5) \quad \mathfrak{F}(z) = \frac{y(z) - 1}{y(z) + 1} = \frac{1}{2} \left[ y_1 z + \left( y_2 - \frac{y_1^2}{2} \right) z^2 \right] + \dots, \quad z \in U.$$

By substituting (2.4) and (2.5) into (2.2) and (2.3) and applying (2.1), we get

$$(2.6) \quad (1 - \mu) \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \mu \frac{z(f'(z))^\lambda}{f(z)} \\ = \vartheta \left( \frac{x(z) - 1}{x(z) + 1} \right) = 1 + \frac{1}{2} \mathfrak{B}_1 x_1 z + \left[ \frac{1}{2} \mathfrak{B}_1 \left( x_2 - \frac{x_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 x_1^2 \right] z^2 + \dots$$

and

$$(2.7) \quad (1 - \mu) \frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \mu \frac{w(g'(w))^\lambda}{g(w)} \\ = \vartheta \left( \frac{y(w) - 1}{y(w) + 1} \right) = 1 + \frac{1}{2} \mathfrak{B}_1 y_1 w + \left[ \frac{1}{2} \mathfrak{B}_1 \left( y_2 - \frac{y_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 y_1^2 \right] w^2 + \dots$$

Equating the coefficients in (2.6) and (2.7), yields

$$(2.8) \quad [(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)] a_2 = \frac{1}{2} \mathfrak{B}_1 x_1,$$

$$(2.9) \quad [(1 - \mu)(\gamma + 2) + \mu(3\lambda - 1)] a_3 \\ + \left[ \frac{1}{2} (1 - \mu)(\gamma + 2)(\gamma - 1) + \mu(2\lambda(\lambda - 2) + 1) \right] a_2^2 \\ = \frac{1}{2} \mathfrak{B}_1 \left( x_2 - \frac{x_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 x_1^2,$$

$$(2.10) \quad - [(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)] a_2 = \frac{1}{2} \mathfrak{B}_1 y_1$$

and

$$(2.11) \quad [(1 - \mu)(\gamma + 2) + \mu(3\lambda - 1)] (2a_2^2 - a_3) \\ + \left[ \frac{1}{2} (1 - \mu)(\gamma + 2)(\gamma - 1) + \mu(2\lambda(\lambda - 2) + 1) \right] a_2^2 \\ = \frac{1}{2} \mathfrak{B}_1 \left( y_2 - \frac{y_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 y_1^2.$$

From (2.8) and (2.10), we have

$$(2.12) \quad x_1 = -y_1$$

and

$$(2.13) \quad 2 [(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)]^2 a_2^2 = \frac{1}{4} \mathfrak{B}_1^2 (x_1^2 + y_1^2).$$

If we add (2.9) to (2.11), we obtain

(2.14)

$$[(1 - \mu)(\gamma + 2)(\gamma + 1) + 2\mu\lambda(2\lambda - 1)]a_2^2 = \frac{1}{2}\mathfrak{B}_1 \left[ x_2 + y_2 - \frac{x_1^2 + y_1^2}{2} \right] + \frac{1}{4}\mathfrak{B}_2[x_1^2 + y_1^2].$$

Substituting the value of  $x_1^2 + y_1^2$  from (2.13) in the right hand side of (2.14), we deduce that

$$(2.15) \quad a_2^2 = \frac{\mathfrak{B}_1^3(x_2 + y_2)}{2(\mathfrak{B}_1^2[(1 - \mu)(\gamma + 2)(\gamma + 1) + 2\mu\lambda(2\lambda - 1)] + 2[(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)]^2(\mathfrak{B}_1 - \mathfrak{B}_2))}.$$

Applying Lemma 1.1 for the coefficients  $x_1, x_2, y_1, y_2$  in (2.13) and (2.15), we get

$$|a_2| \leq \frac{\sqrt{2}\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{|\mathfrak{B}_1^2[(1 - \mu)(\gamma + 2)(\gamma + 1) + 2\mu\lambda(2\lambda - 1)] + 2[(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)]^2(\mathfrak{B}_1 - \mathfrak{B}_2)|}},$$

i.e.,

$$|a_2| \leq \frac{\mathfrak{B}_1}{(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)},$$

which gives the estimates of  $|a_2|$ .

Furthermore, in order to find the bound on  $|a_3|$ , we subtract (2.11) from (2.9) and also applying (2.12), we obtain  $x_1^2 = y_1^2$ , hence

$$(2.16) \quad 2[(1 - \mu)(\gamma + 2) + \mu(3\lambda - 1)](a_3 - a_2^2) = \frac{1}{2}\mathfrak{B}_1(x_2 - y_2).$$

Then, by substituting of the value of  $a_2^2$  from (2.13) into (2.16), gives

$$a_3 = \frac{\mathfrak{B}_1(x_2 - y_2)}{4[(1 - \mu)(\gamma + 2) + \mu(3\lambda - 1)]} + \frac{\mathfrak{B}_1^2(x_1^2 + y_1^2)}{8[(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)]^2}.$$

So, we have

$$|a_3| \leq \frac{\mathfrak{B}_1}{(1 - \mu)(\gamma + 2) + \mu(3\lambda - 1)} + \frac{\mathfrak{B}_1^2}{[(1 - \mu)(\gamma + 1) + \mu(2\lambda - 1)]^2}.$$

Also, substituting the value of  $a_2^2$  from (2.14) into (2.16), we get

$$a_3 = \frac{\mathfrak{B}_1(x_2 - y_2)}{4[(1 - \mu)(\gamma + 2) + \mu(3\lambda - 1)]} + \frac{\mathfrak{B}_1(x_2 + y_2) + \frac{1}{2}(x_1^2 + y_1^2)(\mathfrak{B}_2 - \mathfrak{B}_1)}{2[(1 - \mu)(\gamma + 2)(\gamma + 1) + 2\mu\lambda(2\lambda - 1)]},$$

and we have

$$|a_3| \leq \frac{\mathfrak{B}_1}{(1 - \mu)(\gamma + 2) + \mu(3\lambda - 1)} + \frac{2\mathfrak{B}_2}{(1 - \mu)(\gamma + 2)(\gamma + 1) + 2\mu\lambda(2\lambda - 1)},$$

which gives us the desired estimates on the coefficient  $|a_3|$ . □

Taking  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots$ ,  $0 < \alpha \leq 1$ , in Theorem 2.1, we obtain the next corollary.

**Corollary 2.1.** *Let  $f$ , given by (1.1), be in the family  $\mathcal{T}_\Sigma(\mu, \gamma, \lambda; \alpha)$ , where  $0 < \alpha \leq 1$ . Then,*

$$|a_2| \leq \min \left\{ \frac{2\alpha}{(1-\mu)(\gamma+1) + \mu(2\lambda-1)}, \frac{2\alpha\sqrt{\alpha}}{\sqrt{\alpha^2 [(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda(2\lambda-1)] + \alpha(1-\alpha) [(1-\mu)(\gamma+1) + \mu(2\lambda-1)]^2}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{2\alpha}{(1-\mu)(\gamma+2) + \mu(3\lambda-1)} + \frac{4\alpha^2}{(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda(2\lambda-1)}, \frac{2\alpha}{(1-\mu)(\gamma+2) + \mu(3\lambda-1)} + \frac{4\alpha^2}{[(1-\mu)(\gamma+1) + \mu(2\lambda-1)]^2} \right\}.$$

Taking  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \dots$ ,  $0 \leq \beta < 1$ , in Theorem 2.1, we obtain the next corollary.

**Corollary 2.2.** *Let  $f$ , given by (1.1), be in the family  $\mathcal{T}_\Sigma^*(\mu, \gamma, \lambda; \beta)$ , where  $0 \leq \beta < 1$ . Then,*

$$|a_2| \leq \min \left\{ \frac{2(1-\beta)}{(1-\mu)(\gamma+1) + \mu(2\lambda-1)}, \frac{2\sqrt{1-\beta}}{\sqrt{|(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda(2\lambda-1)|}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{2(1-\beta)}{(1-\mu)(\gamma+2) + \mu(3\lambda-1)} + \frac{4(1-\beta)}{(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda(2\lambda-1)}, \frac{2(1-\beta)}{(1-\mu)(\gamma+2) + \mu(3\lambda-1)} + \frac{4(1-\beta)^2}{[(1-\mu)(\gamma+1) + \mu(2\lambda-1)]^2} \right\}.$$

The families  $\mathcal{T}_\Sigma(\mu, \gamma, \lambda; \alpha)$  and  $\mathcal{T}_\Sigma^*(\mu, \gamma, \lambda; \beta)$  were given by Srivastava et al. [34] and defined as follows.

**Definition 2.3** ([34]). For  $0 < \alpha \leq 1$ ,  $0 \leq \mu \leq 1$ ,  $\gamma \geq 0$  and  $\lambda \geq 1$ , a function  $f \in \Sigma$  is said to be in the family  $\mathcal{T}_\Sigma(\mu, \gamma, \lambda; \alpha)$  if it fulfills the subordinations:

$$\left| \arg \left( (1-\mu) \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \mu \frac{z(f'(z))^\lambda}{f(z)} \right) \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left( (1-\mu) \frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \mu \frac{w(g'(w))^\lambda}{g(w)} \right) \right| < \frac{\alpha\pi}{2},$$

where  $g(w) = f^{-1}(w)$ .

**Definition 2.4** ([34]). For  $0 \leq \beta < 1$ ,  $0 \leq \mu \leq 1$ ,  $\gamma \geq 0$  and  $\lambda \geq 1$ , a function  $f \in \Sigma$  is said to be in the family  $\mathcal{T}_\Sigma^*(\mu, \gamma, \lambda; \beta)$  if it fulfills the subordinations:

$$\operatorname{Re} \left( (1 - \mu) \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \mu \frac{z(f'(z))^\lambda}{f(z)} \right) > \beta$$

and

$$\operatorname{Re} \left( (1 - \mu) \frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \mu \frac{w(g'(w))^\lambda}{g(w)} \right) > \beta,$$

where  $g(w) = f^{-1}(w)$ .

**Theorem 2.2.** Let  $f$ , given by (1.1), be in the family  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ . Then,

$$|a_2| \leq \min \left\{ \frac{\mathfrak{B}_1}{2(2\mu(\lambda - 1) + 1)}, \frac{\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{|\mathfrak{B}_1^2(2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)) + 4(2\mu(\lambda - 1) + 1)^2(\mathfrak{B}_1 - \mathfrak{B}_2)|}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{\mathfrak{B}_1}{3(3\mu(\lambda - 1) + 2)} + \frac{\mathfrak{B}_2}{2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)}, \frac{\mathfrak{B}_1}{3(3\mu(\lambda - 1) + 2)} + \frac{\mathfrak{B}_1^2}{4(2\mu(\lambda - 1) + 1)^2} \right\},$$

where the coefficients  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  are defined as in (2.1).

*Proof.* Let  $f \in \mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  and  $g = f^{-1}$ . Then, there are holomorphic functions  $\mathfrak{S}, \mathfrak{T} : U \rightarrow U$  such that

$$(2.17) \quad (1 - \mu) \left( 1 + \frac{z^{2-\gamma} f''(z)}{(zf'(z))^{1-\gamma}} \right) + \mu \frac{((zf'(z))')^\lambda}{f'(z)} = \vartheta(\mathfrak{S}(z)), \quad z \in U,$$

and

$$(2.18) \quad (1 - \mu) \left( 1 + \frac{w^{2-\gamma} g''(w)}{(wg'(w))^{1-\gamma}} \right) + \mu \frac{((wg'(w))')^\lambda}{g'(w)} = \vartheta(\mathfrak{T}(w)), \quad w \in U,$$

where  $\mathfrak{S}(z)$  and  $\mathfrak{T}(z)$  have the forms (2.4) and (2.5). From (2.17), (2.18) and (2.1), we deduce that

$$(2.19) \quad (1 - \mu) \left( 1 + \frac{z^{2-\gamma} f''(z)}{(z f'(z))^{1-\gamma}} \right) + \mu \frac{((z f'(z))')^\lambda}{f'(z)} \\ = \vartheta \left( \frac{x(z) - 1}{x(z) + 1} \right) = 1 + \frac{1}{2} \mathfrak{B}_1 x_1 z + \left[ \frac{1}{2} \mathfrak{B}_1 \left( x_2 - \frac{x_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 x_1^2 \right] z^2 + \dots$$

and

$$(2.20) \quad (1 - \mu) \left( 1 + \frac{w^{2-\gamma} g''(w)}{(w g'(w))^{1-\gamma}} \right) + \mu \frac{((w g'(w))')^\lambda}{g'(w)} \\ = \vartheta \left( \frac{y(w) - 1}{y(w) + 1} \right) = 1 + \frac{1}{2} \mathfrak{B}_1 y_1 w + \left[ \frac{1}{2} \mathfrak{B}_1 \left( y_2 - \frac{y_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 y_1^2 \right] w^2 + \dots$$

Equating the coefficients in (2.19) and (2.20), yields

$$(2.21) \quad 2(2\mu(\lambda - 1) + 1) a_2 = \frac{1}{2} \mathfrak{B}_1 x_1,$$

$$(2.22) \quad 3(3\mu(\lambda - 1) + 2) a_3 + 4[2\lambda\mu(\lambda - 2) + \mu(2 - \gamma) + \gamma - 1] a_2^2 \\ = \frac{1}{2} \mathfrak{B}_1 \left( x_2 - \frac{x_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 x_1^2,$$

$$(2.23) \quad -2(2\mu(\lambda - 1) + 1) a_2 = \frac{1}{2} \mathfrak{B}_1 y_1$$

and

$$(2.24) \quad 3(3\mu(\lambda - 1) + 2) (2a_2^2 - a_3) + 4[2\lambda\mu(\lambda - 2) + \mu(2 - \gamma) + \gamma - 1] a_2^2 \\ = \frac{1}{2} \mathfrak{B}_1 \left( y_2 - \frac{y_1^2}{2} \right) + \frac{1}{4} \mathfrak{B}_2 y_1^2.$$

From (2.21) and (2.23), we have

$$(2.25) \quad x_1 = -y_1$$

and

$$(2.26) \quad 8(2\mu(\lambda - 1) + 1)^2 a_2^2 = \frac{1}{4} \mathfrak{B}_1^2 (x_1^2 + y_1^2).$$

If we add (2.22) to (2.24), we obtain

$$(2.27) \quad 2[2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)] a_2^2 \\ = \frac{1}{2} \mathfrak{B}_1 \left[ x_2 + y_2 - \left( \frac{x_1^2 + y_1^2}{2} \right) \right] + \frac{1}{4} \mathfrak{B}_2 [x_1^2 + y_1^2].$$

Substituting the value of  $x_1^2 + y_1^2$  from (2.26) in the right hand side of (2.27), we deduce that

$$(2.28) \quad a_2^2 = \frac{\mathfrak{B}_1^3(x_2 + y_2)}{4 [\mathfrak{B}_1^2 [2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)] + 4(2\mu(\lambda - 1) + 1)^2 (\mathfrak{B}_1 - \mathfrak{B}_2)]}.$$

Applying Lemma 1.1 for the coefficients  $x_1, x_2, y_1, y_2$  in (2.26) and (2.28), we get

$$|a_2| \leq \frac{\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{|\mathfrak{B}_1^2 [2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)] + 4(2\mu(\lambda - 1) + 1)^2 (\mathfrak{B}_1 - \mathfrak{B}_2)|}},$$

i.e.,

$$|a_2| \leq \frac{\mathfrak{B}_1}{2(2\mu(\lambda - 1) + 1)},$$

which gives the estimates of  $|a_2|$ .

Furthermore, in order to find the bound on  $|b_3|$ , we subtract (2.24) from (2.22) and also applying (2.25), we obtain  $x_1^2 = y_1^2$ , hence

$$(2.29) \quad 6(3\mu(\lambda - 1) + 2)(a_3 - a_2^2) = \frac{1}{2}\mathfrak{B}_1(x_2 - y_2).$$

Then, by substituting of the value of  $a_2^2$  from (2.26) into (2.29), gives

$$a_3 = \frac{\mathfrak{B}_1(x_2 - y_2)}{12(3\mu(\lambda - 1) + 2)} + \frac{\mathfrak{B}_1^2(x_1^2 + y_1^2)}{32(2\mu(\lambda - 1) + 1)^2}.$$

So, we have

$$|a_3| \leq \frac{\mathfrak{B}_1}{3(3\mu(\lambda - 1) + 2)} + \frac{\mathfrak{B}_1^2}{4(2\mu(\lambda - 1) + 1)^2}.$$

Also, substituting the value of  $a_2^2$  from (2.27) into (2.29), we get

$$a_3 = \frac{\mathfrak{B}_1(x_2 - y_2)}{12(3\mu(\lambda - 1) + 2)} + \frac{\mathfrak{B}_1(x_2 + y_2) + \frac{1}{2}(x_1^2 + y_1^2)(\mathfrak{B}_2 - \mathfrak{B}_1)}{4[2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)]},$$

and we have

$$|a_3| \leq \frac{\mathfrak{B}_1}{3(3\mu(\lambda - 1) + 2)} + \frac{\mathfrak{B}_2}{2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)},$$

which gives us the desired estimates on the coefficient  $|a_3|$ . □

Taking  $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots$ ,  $0 < \alpha \leq 1$ , in Theorem 2.2, we obtain the next corollary.

**Corollary 2.3.** *Let  $f$  given by (1.1) be in the family  $\mathcal{M}_\Sigma(\mu, \gamma, \lambda; \alpha)$ , where  $0 < \alpha \leq 1$ . Then,*

$$|a_2| \leq \min \left\{ \frac{\alpha}{2\mu(\lambda - 1) + 1}, \frac{\alpha\sqrt{2\alpha}}{\sqrt{|\alpha^2 [2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)] + 2\alpha(1 - \alpha)(2\mu(\lambda - 1) + 1)^2|}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{2\alpha}{3(3\mu(\lambda - 1) + 2)} + \frac{2\alpha^2}{2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)}, \right. \\ \left. \frac{2\alpha}{3(3\mu(\lambda - 1) + 2)} + \frac{\alpha^2}{(2\mu(\lambda - 1) + 1)^2} \right\}.$$

Taking  $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\dots$ ,  $0 \leq \beta < 1$ , in Theorem 2.2, we obtain the next corollary.

**Corollary 2.4.** *Let  $f$ , given by (1.1), be in the family  $\mathcal{M}_\Sigma^*(\mu, \gamma, \lambda; \beta)$ , where  $0 \leq \beta < 1$ . Then,*

$$|a_2| \leq \min \left\{ \frac{1 - \beta}{(2\mu(\lambda - 1) + 1)}, \sqrt{\frac{2(1 - \beta)}{|2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)|}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{2(1 - \beta)}{3(3\mu(\lambda - 1) + 2)} + \frac{2(1 - \beta)}{2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)}, \right. \\ \left. \frac{2(1 - \beta)}{3(3\mu(\lambda - 1) + 2)} + \frac{(1 - \beta)^2}{(2\mu(\lambda - 1) + 1)^2} \right\}.$$

The families  $\mathcal{M}_\Sigma(\mu, \gamma, \lambda; \alpha)$  and  $\mathcal{M}_\Sigma^*(\mu, \gamma, \lambda; \beta)$  are defined as follows:

**Definition 2.5.** For  $0 < \alpha \leq 1$ ,  $0 \leq \mu \leq 1$ ,  $\gamma \geq 0$  and  $\lambda \geq 1$ , a function  $f \in \Sigma$  is said to be in the family  $\mathcal{T}_\Sigma(\mu, \gamma, \lambda; \alpha)$  if it fulfills the subordinations:

$$\left| \arg \left( (1 - \mu) \left( 1 + \frac{z^{2-\gamma} f''(z)}{(z f'(z))^{1-\gamma}} \right) + \mu \frac{((z f'(z))')^\lambda}{f'(z)} \right) \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left( (1 - \mu) \left( 1 + \frac{w^{2-\gamma} g''(w)}{(w g'(w))^{1-\gamma}} \right) + \mu \frac{((w g'(w))')^\lambda}{g'(w)} \right) \right| < \frac{\alpha\pi}{2},$$

where  $g(w) = f^{-1}(w)$ .

**Definition 2.6.** For  $0 \leq \beta < 1$ ,  $0 \leq \mu \leq 1$ ,  $\gamma \geq 0$  and  $\lambda \geq 1$ , a function  $f \in \Sigma$  is said to be in the family  $\mathcal{M}_\Sigma^*(\mu, \gamma, \lambda; \beta)$  if it fulfills the subordinations:

$$\operatorname{Re} \left( (1 - \mu) \left( 1 + \frac{z^{2-\gamma} f''(z)}{(z f'(z))^{1-\gamma}} \right) + \mu \frac{((z f'(z))')^\lambda}{f'(z)} \right) > \beta$$

and

$$\operatorname{Re} \left( (1 - \mu) \left( 1 + \frac{w^{2-\gamma} g''(w)}{(w g'(w))^{1-\gamma}} \right) + \mu \frac{((w g'(w))')^\lambda}{g'(w)} \right) > \beta,$$

where  $g(w) = f^{-1}(w)$ .

In the next theorems, we provide the Fekete-Szegő type inequalities for the functions of the families  $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  and  $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ .

**Theorem 2.3.** For  $\eta \in \mathbb{R}$ , let  $f \in \mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  be of the form (1.1). Then,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}; \\ |\eta - 1| \leq \frac{|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}, \\ \frac{2\mathfrak{B}_1^3|\eta-1|}{|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)|}; \\ |\eta - 1| \geq \frac{|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}. \end{cases}$$

*Proof.* It follows from (2.15) and (2.16) that

$$\begin{aligned} & a_3 - \eta a_2^2 \\ &= \frac{\mathfrak{B}_1(x_2 - y_2)}{4[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]} + (1-\eta)a_2^2 \\ &= \frac{\mathfrak{B}_1(x_2 - y_2)}{4[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]} \\ &+ \frac{\mathfrak{B}_1^3(x_2 + y_2)(1-\eta)}{2(\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2))} \\ &= \frac{\mathfrak{B}_1}{2} \left[ \left( \Upsilon(\eta) + \frac{1}{2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]} \right) x_2 \right. \\ & \quad \left. + \left( \Upsilon(\eta) - \frac{1}{2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]} \right) y_2 \right], \end{aligned}$$

where

$$\Upsilon(\eta) = \frac{\mathfrak{B}_1^2(1-\eta)}{\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)}.$$

According to Lemma 1.1 and (2.1), we find that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}, & 0 \leq |\Upsilon(\eta)| \leq \frac{1}{2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}, \\ 2\mathfrak{B}_1 |\Upsilon(\eta)|, & |\Upsilon(\eta)| \geq \frac{1}{2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}. \end{cases}$$

After some computations, we obtain

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}; \\ |\eta - 1| \leq \frac{|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}, \\ \frac{2\mathfrak{B}_1^3|\eta-1|}{|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)|}; \\ |\eta - 1| \geq \frac{|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}. \end{cases}$$

□

Putting  $\eta = 1$  in Theorem 2.3, we obtain the following result.

**Corollary 2.5.** *If  $f \in \mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  is of the form (1.1), then*

$$|a_3 - a_2^2| \leq \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}.$$

**Theorem 2.4.** *For  $\eta \in \mathbb{R}$ , let  $f \in \mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  be of the form (1.1). Then,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\mathfrak{B}_1}{3(3\mu(\lambda-1)+2)}; \\ |\eta - 1| \leq \frac{|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}, \\ \frac{\mathfrak{B}_1^3|\eta-1|}{|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)|}; \\ |\eta - 1| \geq \frac{|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}. \end{cases}$$

*Proof.* It follows from (2.28) and (2.29) that

$$\begin{aligned} & a_3 - \eta a_2^2 \\ &= \frac{\mathfrak{B}_1(x_2 - y_2)}{12(3\mu(\lambda - 1) + 2)} + (1 - \eta) a_2^2 \\ &= \frac{\mathfrak{B}_1(x_2 - y_2)}{12(3\mu(\lambda - 1) + 2)} \\ &\quad + \frac{\mathfrak{B}_1^3(x_2 + y_2)(1 - \eta)}{4 \left[ \mathfrak{B}_1^2 [2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)] + 4(2\mu(\lambda - 1) + 1)^2 (\mathfrak{B}_1 - \mathfrak{B}_2) \right]} \\ &= \frac{\mathfrak{B}_1}{4} \left[ \left( \Omega(\eta) + \frac{1}{3(3\mu(\lambda - 1) + 2)} \right) x_2 + \left( \Omega(\eta) - \frac{1}{3(3\mu(\lambda - 1) + 2)} \right) y_2 \right], \end{aligned}$$

where

$$\Omega(\eta) = \frac{\mathfrak{B}_1^2(1 - \eta)}{\mathfrak{B}_1^2 [2 + 4\gamma + 9\mu(\lambda - 1) + 8\lambda\mu(\lambda - 2) + 4\mu(2 - \gamma)] + 4(2\mu(\lambda - 1) + 1)^2 (\mathfrak{B}_1 - \mathfrak{B}_2)}.$$

According to Lemma 1.1 and (2.1), we find that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\mathfrak{B}_1}{3(3\mu(\lambda-1)+2)}, & 0 \leq |\Omega(\eta)| \leq \frac{1}{3(3\mu(\lambda-1)+2)}, \\ \mathfrak{B}_1 |\Omega(\eta)|, & |\Omega(\eta)| \geq \frac{1}{3(3\mu(\lambda-1)+2)}. \end{cases}$$

After some computations, we obtain

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\mathfrak{B}_1}{3(3\mu(\lambda-1)+2)}; \\ |\eta - 1| \leq \frac{|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}, \\ \frac{\mathfrak{B}_1^3|\eta-1|}{|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)|}; \\ |\eta - 1| \geq \frac{|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}. \end{cases} \quad \square$$

Putting  $\eta = 1$  in Theorem 2.4, we obtain the following result.

**Corollary 2.6.** *If  $f \in \mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$  is of the form (1.1), then*

$$|a_3 - a_2^2| \leq \frac{\mathfrak{B}_1}{3(3\mu(\lambda-1)+2)}.$$

### 3. CONCLUSION

This work has introduced a new families of bi-univalent functions associated with the Bazilevič functions and the  $\lambda$ -pseudo functions. For these families, coefficient bounds and Fekete-Szegő inequalities have been investigated.

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