SOME q-ANALOGUES OF GRANVILLE AND SUN'S CONGRUENCES

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ABSTRACT. In this paper, we use q-binomial theorem to establish some new q-analogues of Granville and Sun's congruence:

$$\sum_{k=1}^{p-1} \frac{x^k}{k} \equiv \frac{1 - x^p - (x-1)^p}{p} \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p}\left(\frac{1 + (x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k} \right) \pmod{p},$$

where x is a variable and p is an odd prime.

1. INTRODUCTION

In recent years, q-analogues of congruences involving harmonic numbers have been widely studied by many authors. In 2004, Granville [1] showed that for any prime $p \geq 5$,

$$(1.1) \qquad \sum_{k=1}^{p-1} \frac{x^k}{k} \equiv \frac{1 - x^p - (x-1)^p}{p} \pmod{p}.$$

For any postive integer n, the q-integer is defined as

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

It is easy to see that $\lim_{q \to 1} [n]_q = n$.

Pan [3, (5.1)] showed that Granville [1] confirmed a conjecture of Skula: For any prime $p \geq 5$,

$$(1.2) \qquad \left(\frac{2^{p-1}-1}{p}\right)^2 \equiv -\sum_{j=1}^{p-1} \frac{2^j}{j^2} \pmod{p}.$$

Pan [3, Theorem 5.1] established a q-analogue of (1.2) as follows:

$$\sum_{j=1}^{p-1} \frac{q^j (-q;q)_j}{[j]_q^2} + Q_p(2,q)^2$$

$$\equiv -(p-1) Q_p(2,q)(1-q) - \frac{(7p-5)(p-1)(1-q)^2}{24} \pmod{[p]_q},$$

where $Q_p(2,q) = \frac{(-q;q)_{p-1}-1}{[p]_q}$.

The harmonic numbers are given by

$$H_n = \sum_{k=1}^n \frac{1}{k} \quad \text{and} \quad H_0 = 0.$$

A q-analogues of harmonic numbers H_n is given by

$$H_n(q) = \sum_{k=1}^n \frac{q^k}{[k]_q}.$$

The q-Pochhammer symbol is given by

$$(x;q)_n = (1-x)(1-xq)\cdots\left(1-xq^{n-1}\right), \quad (x;q)_0 = 1.$$

The q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear to see that $\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$.

In [4, Theorem 1], Shi and Pan showed that for prime $p \geq 5$,

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{(p-1)(1-q)}{2} + \frac{(p^2-1)(1-q)^2 [p]_q}{24} \pmod{[p]_q^2}.$$

In [5, Lemma 4.1], Sun established an interesting congruence:

$$\sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p}\left(\frac{1+(x-1)^p-x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k-1}{k}\right) + p\sum_{k=2}^{p-1} \frac{x^k}{k^2} H_{k-1} \pmod{p^2},$$

where x is a real number and p is an odd prime. Thus, it is easy to get the following congruence:

$$(1.3) \qquad \sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p} \left(\frac{1 + (x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k} \right) \quad (\mathrm{mod}\ p).$$

In [6, Theorem 2.1], Tauraso proved that if p is an odd prime, and a, b, d are integers such that $a, d > 0, b \geq 0$ and $\gcd(a, p)=1$, then

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^d} \equiv \frac{(1-q)^d}{p^d} \left((-1)^d p \sum_{s=0}^{d-1} c_s \binom{r_0 + sp}{d} - \sum_{s=0}^{d} (-1)^s \binom{d}{s} \binom{sp}{2d} \right) \quad (\mathrm{mod}\ [p]_q),$$

where

$$r_0 \equiv \frac{-b}{a} \quad (\mathrm{mod}\ p), \quad r_0 \in \{0, 1, \ldots, p-1\},$$

$$c_s = \sum_{k=0}^{s} (-1)^{s-k} \binom{r_0 + kp + d - 1}{d - 1} \binom{d}{s - k}.$$

The nth cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ (n,k)=1}} (q - \zeta^k),$$

where ζ denotes a primitive nth root of unity.

The first aim of the paper is to give two new q-analogues of (1.1).

Theorem 1.1. *For any positive integer n and variable x, we have*

$$(1.4) \quad \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} \equiv \frac{1 - (qx)^n - (qx; q)_n}{[n]_q} + \frac{(n-1)(1-q)(x^n + 1)}{2} \quad (\mathrm{mod}\ \Phi_n(q)),$$

and

$$(1.5)$$

$$\sum_{k=1}^{n-1} \frac{q^{-k}(x; q^{-1})_k}{[k]_q}$$

$$\equiv \frac{1 - (qx)^n - (qx; q)_n}{[n]_q} + \frac{(q-1)(x-1)}{x}$$

$$+ \frac{x(n-1)(1-q)(x^n + 1) - 2(1-q)(1 - xq^{1-n})(x; q^{-1})_{n-1}}{2x} \quad (\mathrm{mod}\ \Phi_n(q)).$$

Letting $q \to 1$ and $n = p$ in Theorem 1.1, (1.4) and (1.5) will reduce to (1.1). So (1.4) and (1.5) are q-analogues of (1.1).

When $x = -q$ and $n = p$, we will see that Theorem 1.1 is a q-analogue of [3, (5.5)]:

$$\frac{2^{p-1} - 1}{p} \equiv -\sum_{j=1}^{p-1} \frac{2^{j-1}}{j} \quad (\mathrm{mod}\ p).$$

The second aim of the paper is to examine two q-analogues of (1.3).

Theorem 1.2. *For any odd positive integer n and variable x, we have*

$$
\sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2} \equiv \left(\frac{(n-1)(1-q)}{2} - \frac{1}{[n]_q} \right) \sum_{k=1}^{n-1} \frac{(q^{-1}x; q^{-1})_k - 1}{[k]_q}
$$

(1.6)

$$
+ \frac{t_0 \left(1 - (q^{-1}x; q^{-1})_n \right) - (q^{-1}x)^n}{[n]_q^2} \qquad (\mathrm{mod}\ \Phi_n(q)),
$$

and

$$
\sum_{k=1}^{n-1} \frac{q^k(x; q)_k}{[k]_q^2} \equiv \left(\frac{(n-1)(1-q)}{2} - \frac{1}{[n]_q} \right) \sum_{k=1}^{n-1} \frac{x^k - 1}{[k]_q}
$$

(1.7)

$$
+ \frac{t_0(1 - x^n) - (x; q)_n}{[n]_q^2} \qquad (\mathrm{mod}\ \Phi_n(q)),
$$

where $t_0 = 1 - \frac{(n-1)(1-q^n)}{2} + \frac{(n-1)(n-3)(1-q^n)^2}{8}$.

In particular, it is easy to see that (1.6) is a q-analogue of (1.2) when $x = 2$, and (1.7) is a q-analogue of (1.2) when $x = -q$.

We will prove Theorems 1.1 and 1.2 in Sections 2 and 4. In addition, we establish some generalizations of Theorems 1.1 and 1.2 in Sections 3 and 5.

2. Proof of Theorem 1.1

Firstly, we need to build some lemmas.

Lemma 2.1. *For any positive integer n and variable x, we have*

(2.1)
$$
\sum_{k=1}^{n-1} \frac{x^k - 1}{[k]_q} = \sum_{k=1}^{n-1} \frac{(-q)^k (x; q)_k}{[k]_q} q^{\binom{k+1}{2} - nk} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.
$$

Proof. By [7, (1.1)], for any positive integer j and n

(2.2)
$$
\sum_{k=j}^n q^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_q.
$$

Letting $j \to j-1$, $n \to n-2$ and $q \to q^{-1}$ in (2.2) gives

$$
q^{(j-1)^2 + j - 1} \sum_{k=j-1}^{n-2} q^{-jk} \begin{bmatrix} k \\ j-1 \end{bmatrix}_q = q^{j^2 - (n-1)j} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q.
$$

By q-binomial theorem: for variable x, positive integer n

(2.3)
$$
x^n = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \prod_{j=0}^{i-1} (x - q^j).
$$

Letting $n \to k$ and $q \to q^{-1}$ in (2.3), we have

(2.4)
$$
x^k - 1 = \sum_{i=1}^k \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^i (x; q)_i\, q^{\binom{i+1}{2} - ki}.
$$

Therefore, we have

$$
\begin{aligned}
\sum_{k=1}^{n-1} \frac{x^k - 1}{[k]_q} &= \sum_{k=1}^{n-1} \frac{1}{[k]_q} \sum_{i=1}^{k} (-1)^i (x;q)_i \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i+1}{2} - ik} \\
&= \sum_{i=1}^{n-1} (-1)^i (x;q)_i q^{\binom{i+1}{2}} \sum_{k=i}^{n-1} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i \end{bmatrix}_q \\
&= \sum_{i=1}^{n-1} (-1)^i (x;q)_i q^{\binom{i+1}{2} - i} \sum_{k=i-1}^{n-2} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q \\
&= \sum_{i=1}^{n-1} \frac{(-q)^i (x;q)_i}{[i]_q} q^{\binom{i+1}{2} - ni} \begin{bmatrix} n-1 \\ i \end{bmatrix}_q. \qquad \square
\end{aligned}
$$

As Shi and Pan [4, Theorem 1] mentioned, the following q-congruence lemma can be proved by the same method. We replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [4, Theorem 1].

Lemma 2.2. *For any positive integer n, we have*

$$
(2.5) \qquad \sum_{k=1}^{n-1} \frac{1}{[k]_q} \equiv \frac{(n-1)(1-q)}{2} + \frac{(n^2-1)(1-q)^2 [n]_q}{24} \qquad (\mathrm{mod}\ \Phi_n(q)^2).
$$

Lemma 2.3. *For any positive integer n and variable x, we have*

$$
(2.6) \qquad \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \equiv \frac{1 - x^n - (x;q)_n}{[n]_q} + x^n \frac{(n-1)(1-q)}{2} \qquad (\mathrm{mod}\ \Phi_n(q)).
$$

Proof. Note that

$$
(2.7) \qquad \begin{bmatrix} n-1 \\ i \end{bmatrix}_q = (-1)^i q^{-\binom{i+1}{2}} \prod_{j=1}^{i} \left(1 - \frac{[n]_q}{[j]_q}\right) \equiv (-1)^i q^{-\binom{i+1}{2}} \qquad (\mathrm{mod}\ \Phi_n(q)),
$$

and

$$
(2.8) \qquad
\begin{aligned}
\sum_{k=1}^{n-1} \frac{q^k (x;q)_k}{[k]_q} &\equiv \sum_{k=1}^{n-1} \frac{q^{n(n-1)/2 - nk + k} (x;q)_k}{[k]_q} \\
&\equiv -\frac{1}{[n]_q} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n}{2} - nk + \binom{k}{2}} (x;q)_k \\
&= -\frac{1}{[n]_q} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (x;q)_k \\
&= -\frac{1}{[n]_q} \left(q^{\binom{n}{2}} x^n - q^{\binom{n}{2}} - (x;q)_n \right) \qquad (\mathrm{mod}\ \Phi_n(q)).
\end{aligned}
$$

With the help of (2.1) and (2.7), we have

$$
(2.9) \qquad \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \equiv \sum_{k=1}^{n-1} \frac{q^k (x;q)_k}{[k]_q} + \sum_{k=1}^{n-1} \frac{1}{[k]_q} \qquad (\mathrm{mod}\ \Phi_n(q)).
$$

Since

$$q^{tn} = 1 - (1 - q^{tn})$$

$$= 1 - (1 - q^n) \left(1 + q^n + q^{2n} + \cdots + q^{(t-1)n}\right)$$

$$\equiv 1 - t \left(1 - q^n\right) \qquad (\mathrm{mod}\ \Phi_n(q)^2),$$

we have

$$(2.10) \qquad q^{\binom{n}{2}} \equiv 1 - \frac{(n-1)(1-q^n)}{2} \qquad (\mathrm{mod}\ \Phi_n(q)^2).$$

Combining (2.8), (2.9) and (2.10), we get Lemma 2.3. $\qquad\square$

Lemma 2.4. *For any positive integer n and variable x, we have*

$$(2.11) \qquad \sum_{k=1}^{n-1} \frac{(x;q^{-1})_k - 1}{[k]_q} = \sum_{k=1}^{n-1} \frac{(-qx)^k}{[k]_q} q^{\binom{k+1}{2}-nk} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

Proof. The q-binomial theorem [2, (3.3.6)]: For variable x, positive integer n,

$$(2.12) \qquad (x;q)_n = \sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i,$$

letting $q \to q^{-1}$ in (2.12), we have

$$(2.13) \qquad (x;q^{-1})_n = \sum_{i=0}^{n} (-1)^i q^{\binom{i+1}{2}-ni} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i.$$

Using (2.2) and (2.13), we get

$$\sum_{k=1}^{n-1} \frac{(x;q^{-1})_k - 1}{[k]_q} = \sum_{k=1}^{n-1} \frac{1}{[k]_q} \sum_{i=1}^{k} (-x)^i \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i+1}{2}-ik}$$

$$= \sum_{i=1}^{n-1} (-x)^i q^{\binom{i+1}{2}} \sum_{k=i}^{n-1} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i \end{bmatrix}_q$$

$$= \sum_{i=1}^{n-1} (-x)^i q^{\binom{i+1}{2}-i} \sum_{k=i-1}^{n-2} \frac{q^{-ik}}{[k]_q} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q$$

$$= \sum_{i=1}^{n-1} \frac{(-qx)^i}{[i]_q} q^{\binom{i+1}{2}-ni} \begin{bmatrix} n-1 \\ i \end{bmatrix}_q. \qquad\square$$

Next we will proof (1.4).

By (2.11) we have

$$\sum_{k=1}^{n-1} \frac{(x;q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1} \frac{(-qx)^k}{[k]_q} q^{\binom{k+1}{2}-nk} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \sum_{k=1}^{n-1} \frac{1}{[k]_q}.$$

Using (2.7), we get

$$\sum_{k=1}^{n-1} \frac{(x;q^{-1})_k}{[k]_q} \equiv \sum_{k=1}^{n-1} \frac{(xq)^k}{[k]_q} + \sum_{k=1}^{n-1} \frac{1}{[k]_q} \qquad (\mathrm{mod}\ \Phi_n(q)),$$

where we have used the fact that $q^{kn} \equiv 1 \pmod{\Phi_n(q)}$. Finally, use (2.5) and (2.6), letting $x \to qx$ in (2.6), we have

$$\sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} \equiv \frac{1 - (qx)^n - (qx; q)_n}{[n]_q} + \frac{(n-1)(1-q)(x^n+1)}{2} \pmod{\Phi_n(q)}.$$

Next we will give the proof of (1.5).

Let

$$M(k, x, q) = \frac{(x; q^{-1})_k}{[k]_q}.$$

For $k > 1$, it is easy to check that

(2.14) $$(1 - q^k)M(k, x, q) = \left(1 - q^{k-1}\right)\left(1 - xq^{1-k}\right) M(k-1, x, q).$$

Summing both sides of (2.14) over k from 1 to $n-1$, we have

$$\sum_{k=1}^{n-1} \left(1 - q^k\right) M(k, x, q) = \sum_{k=1}^{n-1} (1 - q^{k-1})\left(1 - xq^{1-k}\right) M(k-1, x, q).$$

Then

$$\sum_{k=0}^{n-1} \left(1 - q^k\right) M(k, x, q) = \sum_{k=0}^{n-2} \left(1 - q^k\right)(1 - xq^{-k})M(k, x, q).$$

After simplifying, we get

(2.15)
$$\sum_{k=1}^{n-1} \frac{q^{-k}(x; q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1} \frac{(x; q^{-1})_k}{[k]_q} - \frac{(1-q)(1-xq^{1-n})(x; q^{-1})_{n-1} - (x-1)(q-1)}{x}.$$

Finally, combining (1.4), we get (1.5).

So we complete the proof of Theorem 1.1.

3. COROLLARY 1

Corollary 3.1. *For any positive integer n and variable x, we have*

(3.1)
$$\sum_{1 \le k \le j \le n-1} q^{-j}\frac{(x; q^{-1})_k}{[k]_q} \equiv q^{1-n}\left(1 - (xq; q)_n\right) - q\left(x^n - 1\right)$$

$$+ \frac{2q(1 - xq^{1-n})(x; q^{-1})_{n-1} - x(q^{1-n} - q)(x^n + 1)(n-1) - 2q}{2x} \pmod{\Phi_n(q)}.$$

When $q \to 1$, $x = 2$ and $n = p$, the corollary reduce to the following congruence:

$$\sum_{k=1}^{n-1} \sum_{j=1}^{k} \frac{(-1)^j}{j} \equiv -2q_p(2) \pmod{p},$$

where $q_p(2) = \frac{2^{p-1}-1}{p}$.

Proof. Note that

$$\sum_{1\le k\le j\le n-1} q^{-j}\frac{(x;q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q}\sum_{j=k}^{n-1}q^{-j}$$

$$= \sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q}\cdot\frac{q^{-k}-q^{-n}}{1-q^{-1}}$$

$$= \frac{q}{q-1}\sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q} - \frac{q}{(q-1)q^n}\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q}.$$

Then combining Theorem 1.1, we get (3.1). $\qquad\square$

Corollary 3.2. *For any positive integer n, and variable x, we have*

$$\sum_{1\le k\le j\le n-1} q^{j-k}\frac{(x;q^{-1})_k}{[k]_q} \equiv \frac{2q^n\left((x;q^{-1})_{n-1}-1\right)+x(x^n+1)(n-1)(1-q^n)}{2x}$$

$$- (xq;q)_n - q(x;q^{-1})_{n-1} - q^n\left(x^n-1\right)+1 \pmod{\Phi_n(q)^2}.$$

When $q\to 1$, $x=2$ and $n=p$, the corollary reduce to the following congruence:

$$\sum_{k=1}^{n-1}\sum_{j=1}^{k}\frac{(-1)^j}{j} \equiv -2q_p(2) + q_p(2)^2 p \pmod{p^2}.$$

Proof.

$$\sum_{1\le k\le j\le n-1} q^{j-k}\frac{(x;q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q}\sum_{j=k}^{n-1}q^{j}$$

$$= \sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q}\cdot\frac{q^{k}-q^{n}}{1-q}$$

$$= \frac{1}{1-q}\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q} - \frac{q^n}{1-q}\sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q}.$$

Combining (1.4) and (2.15), we get Corollary 3.2, where we have used the fact that $q^n \equiv 1 \pmod{\Phi_n(q)}$, so the result can modulo $\Phi_n(q)^2$. $\qquad\square$

4. Proof Of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemma.

Lemma 4.1. *For any positive integer n, variable x, we have*

$$(4.1) \qquad \sum_{k=1}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}\frac{x^k}{[k]_q} = q^{\binom{n}{2}}\sum_{k=1}^{n}\frac{(x;q^{-1})_k-1}{[k]_q}$$

and

$$(4.2) \qquad \sum_{k=1}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}\frac{(x;q)_k}{[k]_q} = q^{\binom{n}{2}}\sum_{k=1}^{n}\frac{x^k-1}{[k]_q}.$$

Proof. The q-binomial coefficients satisfy the following recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

It is easy to see that

$$(4.3) \qquad q^{\binom{n-k}{2}} = q^{\binom{n}{2} + \binom{k+1}{2} - nk}.$$

So, we have

$$(4.4) \qquad \begin{aligned} &\sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} \\ &= \sum_{k=1}^n (-1)^k \left(q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} \\ &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{1}{[n]_q} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} x^k. \end{aligned}$$

By induction and using (2.13), (4.3) and (4.4), we have

$$\begin{aligned} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{q^{\binom{n}{2}}}{[n]_q} ((x; q^{-1})_n - 1) \\ &= q^{\binom{n}{2}} \sum_{k=1}^n \frac{(x; q^{-1})_k - 1}{[k]_q}. \end{aligned}$$

Using (2.4), and (4.3), we have

$$(4.5) \qquad q^{\binom{n}{2}} x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (-1)^k (x; q)_k.$$

So, we have

$$(4.6) \qquad \begin{aligned} &\sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(x; q)_k}{[k]_q} \\ &= \sum_{k=1}^n (-1)^k \left(q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) q^{\binom{n-k}{2}} \frac{(x; q)_k}{[k]_q} \\ &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{(x; q)_k}{[k]_q} + \frac{1}{[n]_q} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (x; q)_k. \end{aligned}$$

By induction and using (4.5) and (4.6), we have

$$\begin{aligned} \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(x; q)_k}{[k]_q} &= q^{n-1} \sum_{k=1}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{q^{\binom{n}{2}}}{[n]_q} (x^n - 1) \\ &= q^{\binom{n}{2}} \sum_{k=1}^n \frac{x^k - 1}{[k]_q}. \end{aligned} \qquad \square$$

Next we will proof Theorem 1.2.

For any odd positive integer n, variable x, we have

(4.7)
$$\sum_{i=1}^{n-1} \frac{(xq)^i}{[i]_q^2} \equiv \sum_{i=1}^{n-1} \frac{q^{n(n-1)/2-ni}(xq)^i}{[i]_q^2}$$

$$\equiv -\frac{1}{[n]_q} \sum_{i=1}^{n-1} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{n-i}{2}} \frac{x^i}{[i]_q}$$

$$= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^{n} \frac{(x;q^{-1})_i - 1}{[i]_q} - \frac{x^n}{[n]_q^2}$$

$$= -\frac{q^{\binom{n}{2}}}{[n]_q} \left(\sum_{i=1}^{n-1} \frac{(x;q^{-1})_i - 1}{[i]_q} + \frac{(x;q^{-1})_n - 1}{[n]_q} \right) - \frac{x^n}{[n]_q^2}$$

$$= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^{n-1} \frac{(x;q^{-1})_i - 1}{[i]_q} + \frac{q^{\binom{n}{2}} (1 - (x;q^{-1})_n) - x^n}{[n]_q^2} \qquad (\text{mod } \Phi_n(q)).$$

In the first step we need n to be an odd positive integer, otherwise it doesn't work, where in the third step we used (4.1).

Obviously, we have,

$$q^{tn} = 1 - (1 - q^n) \left(1 + q^n + q^{2n} + \cdots + q^{(t-1)n} \right)$$

$$\equiv 1 - (1 - q^n) \left(t - \frac{t(t-1)(1-q^n)}{2} \right) \qquad (\text{mod } \Phi_n(q)^3).$$

Note that

(4.8) $\quad q^{\binom{n}{2}} \equiv 1 - \frac{(n-1)(1-q^n)}{2} + \frac{(n-1)(n-3)(1-q^n)^2}{8} \qquad (\text{mod } \Phi_n(q)^3),$

where we let $t_0 = 1 - \frac{(n-1)(1-q^n)}{2} + \frac{(n-1)(n-3)(1-q^n)^2}{8}$. Then, letting $x \to q^{-1}x$ in (4.7) and combining (2.10), (4.7) and (4.8), we get (1.6).

Furthermore,

$$\sum_{i=1}^{n-1} \frac{q^i (x;q)_i}{[i]_q^2} \equiv \sum_{i=1}^{n-1} \frac{q^{n(n-1)/2-ni} (x;q)_i}{[i]_q^2}$$

$$\equiv -\frac{1}{[n]_q} \sum_{i=1}^{n-1} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{n-i}{2}} \frac{(x;q)_i}{[i]_q}$$

(4.9)

$$= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^{n} \frac{x^i - 1}{[i]_q} - \frac{(x;q)_n}{[n]_q^2}$$

$$= -\frac{q^{\binom{n}{2}}}{[n]_q} \sum_{i=1}^{n-1} \frac{x^i - 1}{[i]_q} + \frac{q^{\binom{n}{2}} (1 - x^n) - (x;q)_n}{[n]_q^2} \qquad (\text{mod } \Phi_n(q)),$$

where in the third step we have used (4.2). Combining (2.10), (4.8) and (4.9), we get (1.7).

5. COROLLARY 2

Corollary 5.1. *For any odd positive integer n and variable x, we have*

(5.1)
$$\sum_{j=1}^{n-1} \frac{x^j}{[j]_q} \sum_{k=1}^{j} \frac{x^k}{[k]_q} \equiv \frac{1}{2} \left(\frac{1 - x^n - (x;q)_n}{[n]_q} + \frac{x^n(n-1)(1-q)}{2} \right)^2$$
$$+ \frac{t_0 \left(1 - (q^{-1}x^2; q^{-1})_n\right) - (q^{-1}x^2)^n}{2[n]_q^2}$$
$$- \left(\frac{1}{2[n]_q} - \frac{(n-1)(1-q)}{4} \right) \sum_{k=1}^{n-1} \frac{(q^{-1}x^2; q^{-1})_k - 1}{[k]_q} \pmod{\Phi_n(q)}.$$

Proof. Note that

$$\sum_{j=1}^{n-1} \frac{x^j}{[j]_q} \sum_{k=1}^{j} \frac{x^k}{[k]_q} = \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \sum_{j=k}^{n-1} \frac{x^j}{[j]_q}$$
$$= \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \left(\sum_{j=1}^{n-1} \frac{x^j}{[j]_q} - \sum_{j=1}^{k} \frac{x^j}{[j]_q} + \frac{x^k}{[k]_q} \right)$$
$$= \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \sum_{j=1}^{n-1} \frac{x^j}{[j]_q} - \sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \sum_{j=1}^{k} \frac{x^j}{[j]_q} + \sum_{k=1}^{n-1} \frac{x^{2k}}{[k]_q^2}.$$

Then,

(5.2)
$$\sum_{j=1}^{n-1} \frac{x^j}{[j]_q} \sum_{k=1}^{j} \frac{x^k}{[k]_q} = \frac{1}{2} \left(\sum_{k=1}^{n-1} \frac{x^k}{[k]_q} \right)^2 + \frac{1}{2} \sum_{k=1}^{n-1} \frac{x^{2k}}{[k]_q^2}.$$

By (1.6), letting $x \to x^2$, we get,

(5.3)
$$\sum_{k=1}^{n-1} \frac{x^{2k}}{[k]_q^2} \equiv \left(\frac{(n-1)(1-q)}{2} - \frac{1}{[n]_q} \right) \sum_{k=1}^{n-1} \frac{(q^{-1}x^2; q^{-1})_k - 1}{[k]_q}$$
$$+ \frac{q^{\binom{n}{2}}(1 - (q^{-1}x^2; q^{-1})_n) - (q^{-1}x^2)^n}{[n]_q^2} \pmod{\Phi_n(q)}.$$

Then combining (2.6), (2.10), (4.8), (5.2) and (5.3), we get Corollary 5.1. □

For example, let $x = -1$. We see that for any odd positive integer n,

(5.4)
$$\sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \sum_{j=1}^{k} \frac{(-1)^j}{[j]_q} \equiv 2Q_n^2(2,q) + (n-1)(1-q)$$
$$+ \frac{(n^2-1)(1-q)^2}{12} \pmod{\Phi_n(q)},$$

where $Q_n(2,q) = \frac{(-q;q)_{n-1}-1}{[n]_q}$. The case $n = p \geq 5$ in (5.4) is Theorem 1.4 in [8].

We replace x with -1 in (5.2), and obtain

$$(5.5) \qquad \sum_{j=1}^{n-1} \frac{(-1)^j}{[j]_q} \sum_{k=1}^{j} \frac{(-1)^k}{[k]_q} = \frac{1}{2}\left(\sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q}\right)^2 + \frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{[k]_q^2}.$$

As Shi and Pan [4, (5)] mentioned, the following q-congruence can be proved by the same method. We replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [4, (5)]:

$$(5.6) \qquad \sum_{k=1}^{n-1} \frac{1}{[k]_q^2} \equiv -\frac{(n-1)(n-5)(1-q)^2}{12} \qquad (\text{mod } \Phi_n(q)).$$

Meanwhile, as B. He [8, (1.7)] mentioned, We replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [8, (1.7)]:

$$(5.7) \qquad \sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \equiv -2Q_n(2,q) - \frac{(n-1)(1-q)}{2} \qquad (\text{mod } \Phi_n(q)).$$

Finally, combining (5.5), (5.6) and (5.7), we get (5.4). Meanwhile, let $x = -1$ in (5.1), we also can simplify to get (5.4), so we omit this part. We replace n with p and $\Phi_n(q)$ with $[p]_q$ in (5.4). This consequence will be [8, Theorem 1.4].

Corollary 5.2. *For any odd positive integer n and variable x,*

(5.8)
$$\sum_{1 \leq k \leq j \leq n-1} \frac{q^j x^k}{[k]_q^2} \equiv \frac{t_0\left(1 - (q^{-1}x; q^{-1})_n\right) - (q^{-1}x)^n}{[n]_q} - \sum_{k=1}^{n-1} \frac{x^k}{[k]_q}$$
$$- \frac{2 - (n-1)(1-q^n)}{2} \sum_{k=1}^{n-1} \frac{(q^{-1}x; q^{-1})_k - 1}{[k]_q} \qquad (\text{mod } \Phi_n(q)^2).$$

Proof. Note that

$$\sum_{1 \leq k \leq j \leq n-1} \frac{q^j x^k}{[k]_q^2} = \sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2} \sum_{j=k}^{n-1} q^j$$

$$(5.9) \qquad\qquad = \sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2}\left(\frac{1-q^n}{1-q} - \frac{1-q^k}{1-q}\right)$$

$$= \frac{1-q^n}{1-q}\sum_{k=1}^{n-1} \frac{x^k}{[k]_q^2} - \sum_{k=1}^{n-1} \frac{x^k}{[k]_q}.$$

Finally, combining (1.6) and (5.9), we get the desired result. $\qquad\square$

For example, when $x = 1$, $x = q^k$, $x = (-1)^k$ in (5.8), we will get

(5.10)
$$\sum_{j=1}^{n-1} q^j \sum_{k=1}^{j} \frac{1}{[k]_q^2} \equiv -\frac{(n-1)(1-q)}{2} - \frac{(n-1)(n-3)(1-q)^2[n]_q}{8} \pmod{\Phi_n(q)^2},$$

(5.11)
$$\sum_{j=1}^{n-1} q^j \sum_{k=1}^{j} \frac{q^k}{[k]_q^2} \equiv \frac{(n-1)(1-q)}{2} - \frac{(n^2-1)(1-q)^2[n]_q}{8} \pmod{\Phi_n(q)^2},$$

and

(5.12)
$$\sum_{j=1}^{n-1} q^j \sum_{k=1}^{j} \frac{(-1)^k}{[k]_q^2}$$
$$\equiv 2Q_n(2,n) + \frac{(n-1)(1-q)}{2}$$
$$- \left(Q_n(2,q)^2 + 3Q_n(2,q)(1-q) + \frac{(n+7)(n-1)(1-q)^2}{12}\right)[n]_q \pmod{\Phi_n(q)^2},$$

where n is an odd positive integer.

As B. He [8, Theorem 1.2] mentioned, we replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [8, Theorem 1.2]:

(5.13)
$$\sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \equiv -2Q_n(2,q) - \frac{(n-1)(1-q)}{2}$$
$$+ \left(Q_n(2,q)^2 + Q_n(2,q)(1-q) + \frac{(n^2-1)(1-q)}{12}\right)[n]_q \pmod{\Phi_n(q)^2}$$

and

(5.14)
$$\sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q^2} \equiv -2Q_n(2,q)(1-q) + \frac{(1-n)(1-q)^2}{2} \pmod{\Phi_n(q)},$$

Furthermore, note that

(5.15)
$$\sum_{k=1}^{n-1} \frac{1}{[k]_q} = \sum_{k=1}^{n-1} \frac{1-q^k+q^k}{[k]_q} = (1-q)(n-1) + \sum_{k=1}^{n-1} \frac{q^k}{[k]_q}$$

and

(5.16)
$$\sum_{k=1}^{n-1} \frac{1}{[k]_q^2} = (1-q) \sum_{k=1}^{n-1} \frac{1}{[k]_q} + \sum_{k=1}^{n-1} \frac{q^k}{[k]_q^2}.$$

Then combining (2.6), (5.6),(5.9), (5.13), (5.14), (5.15) and (5.16), it is not difficult for us to get (5.10), (5.11) and (5.12). Meanwhile, letting $x = 1$, $x = q$ and $x = -1$ in (5.8), we also can simplify to get (5.10), (5.11) and (5.12), so we omit this part. We

replace n with p and $\Phi_n(q)$ with $[p]_q$ in (5.10), (5.11) and (5.12). These consequences will be [8, (1.3)], [8, (1.4)] and [8, Theorem 1.3].

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