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EXISTENCE, UNIQUENESS AND CONTROLLABILITY RESULTS FOR FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES AND DELAY

K. MALAR[1] AND R. ILAVARASI[1]

Abstract. In this paper, we prove the existence, uniqueness and controllability results for fractional neutral integro-differential equation and non-instantaneous impulses in Banach spaces. To obtain the existence and controllability results, we have enforced the concepts of fractional calculus and fixed point theorems. Examples are also given to illustrate the results.

1. Introduction

The theory of fractional differential and integral equations have been demonstrated to be important apparatuses and successful within the modeling of numerous marvels in different areas of building and logical disciplines such as material science, chemistry, science, control theory, flag and picture preparing, blood stream wonders, optimal design and so on. Fractional derivatives give an fabulous instrument for the portrayal of memory and innate properties of different materials and processes. The investigation of both qualitative and quantitative properties of solutions to fractional differential equations is an active and ongoing area of research. For more information on the theory of fractional calculus, one can refer to the monographs of Kilbas et al. [23], Lakshmikanthan et al. [25], Miller and Rose [27] and Podlubny [34], Baleanu et al. [4], as well as the papers by [6, 7, 9, 18–21, 38, 39] along with the reference cited therein.

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Impulsive functional differential and integro-differential systems are particularly advantageous in modeling processes and phenomena that undergo short-time disturbances at the time their evolution. Impulsive differential equations of integer order have been broadly applications in reasonable scientific modeling for a wide extend of commonsense circumstances, counting organic wonders with edges, bursting beat models in medication and science, ideal control models in financial matters, and recurrence tweaked frameworks. For a comprehensive understanding of impulsive differential equations, including relevent developments, [26] and the references therein. Ordinarily, the impulses within the advancement handle depicted by impulsive differential equations are expected to be unexpected and instantaneous. In other words, the annoyances (impulses) begin suddenly and their term is irrelevant in compared to the overall term of the method. However, Hernández et al. [18] introduced the concept of non-instantaneous impulses where the impulses start abruptly at the points t_k and their action continues over a finite time interval $[t_k, s_k]$. This speaks to a circumstance impulsive action that begin suddenly and remains dynamic for a limited period of time. Pierri et al. [35] examined the existence of solutions for a class of first order semilinear abstract impulsive differential equations with non-instantaneous impulses utilizing the hypothesis of analytic semigroup and fractional power of closed operators. Gabeleh et al. [16] investigating a new survey of the theory of measure of noncompactness and their applications. Wang et al. [42] studied the concept of a PC-mild solution to a general new class of noninstantaneous impulsive fractional differential inclusions involving the generalized Caputo derivative with the lower bound at zero in infinite dimensional Banach spaces. One can refer to that references for further details [1–3, 5, 40].

Neutral differential equations arise in many areas of applied mathematics and have received significant attention in recent decades. Good references for ordinary neutral functional differential equations include the books by Graef et al. [15], Benchohra et al. [5], Lakshmikantham et al. [26] and the reference cited therein. Integro-differential equations are imperative for examining issues emerging from common wonders and have been examined from different points of view. In later a long time, this hypothesis has been connected to a wide course of non-linear differential equations in Banach spaces. For more data, see the references therein [11, 12].

Meraj and Pandey [30] examined the existence of mild solutions for fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions by utilizing noncompact semigroup hypothesis and fixed point theorem.

In recent years, fractional calculus has brought almost modern viewpoints within the field of control theory. The primary challenges in control theory, such as post task, stabilization, and ideal control, can be tended to by accepting that the framework is controllable. The concept of controllability was first introduced by Kalmen in 1960 and has been extensively studied. Controllability could be a significant characteristic, both in terms of quantity and quality, in control systems and plays a pivotal role in various control problems, including those in finite and infinite-dimensional spaces.

In recent times, the controllability of fractional dynamical systems has risen as a profoundly dynamic range inside this field. The controllability of linear systems in finite dimensional spaces has been broadly inspected in [24] and controllability of fractional evolution dynamical systems in a dimensional space has been talked about in works such as, [6, 7, 10, 14, 36, 41].

Ji et al. [22] considered the controllability of impulsive differential systems with nonlocal conditions by using Mönch's fixed point technique. Wang et al. [42] established the adequate conditions for nonlocal controllability for fractional evolution systems and the comes about were gotten by utilizing fractional calculus and Mönch's fixed point theorem. Meraj and Pandey [31] considered the existence of mild solutions and approximate controllability for a class of fractional semilinear integrodifferential equations with nonlocal and impulsive conditions where the impulses are not instantaneous. They employed semigroup theory and fixed point theorems to analyze this problem.

This paper deals with the Fractional Neutral Integro-Differential Equations and Non-Instantaneous impulses with infinite delay

$$\begin{aligned} &{}^{C}D_t^r[u(t) - \mathcal{G}(t, u_t)] = \mathfrak{A}u(t) + \mathscr{F}\Big(t, u_t, \int_0^t \mathcal{H}(t, s, u_s)ds\Big), \quad t \in (s_k, t_{k+1}], \\ &\qquad k = 0, 1, 2, \dots, m, \end{aligned} \tag{1.1}$$

$$u(t) = \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \dots, m, \tag{1.2}$$

$$u_0 = \Phi \in \mathscr{B}_h, \quad t \in (-\infty, 0]. \tag{1.3}$$

where ${}^{C}D_t^r$ is the Caputo fractional derivative of order $r \in (0, 1)$ and $\mathcal{J} = [0, \mathcal{T}]$. The operator $\mathfrak{A}$ denotes the infinitesimal generator of an analytic semigroup $\{\mathcal{Q}(t)\}_{t\geq 0}$ in a Banach space $\mathcal{X}$ having norm $\|\cdot\|$, this suggests that we can find $\mathscr{M}_A \geq 1$ to ensure that $\|\mathcal{Q}(t)\| \leq \mathscr{M}_A$, $\mathscr{F} : \mathcal{J} \times \mathscr{B}_h \times \mathcal{X} \to \mathcal{X}$, $\mathcal{G} : \mathcal{J} \times \mathscr{B}_h \to \mathcal{X}$, $\mathcal{H} : \mathfrak{D} \times \mathscr{B}_h \to \mathcal{X}$ are given functions satisfying certain assumptions, $\mathscr{G}_k : (t_k, s_k] \times \mathcal{X} \to \mathcal{X}$, $\mathfrak{I}_k : \mathcal{X} \to \mathcal{X}$ for $k = 1, 2, \dots, m$. $\mathscr{B}_h$ is a phase space characterised in preliminaries. Here $\mathfrak{D} = \{(t, s) \in \mathcal{J} \times \mathcal{J} : 0 \leq s \leq t \leq \mathcal{T}\}$, $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = \mathcal{T}$ are fixed numbers.

The impulses in problem (1.1)–(1.3) start abruptly at the points t_k and their action continues on the interval $[t_k, s_k]$. To be precise, the function u takes an abrupt impulse at t_k and follows different rules in the two subintervals $(t_k, s_k]$ and $(s_k, t_{k+1}]$ of the interval $(t_k, t_{k+1}]$. At the point s_k, the function u is continuous. The term $\mathfrak{I}_k(u(t_k))$ means that the impulses are also related to the value of $u(t_k) = u(t_k^-)$.

We remark that if $t_k = s_k$ and the second equation of (1.1)–(1.3) takes the form of $\Delta u(t_k) = \mathfrak{I}_k(u(t_k)) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+) = \lim_{\varepsilon\to 0^+} u(t_k + \varepsilon)$, $u(t_k^-) = \lim_{\varepsilon\to 0^-} u(t_k - \varepsilon)$ representing the right and left limits of $u(t)$ at $t = t_k$.

For almost every continuous function u defined on $(-\infty, \mathcal{T}]$ and for almost every $t \geq 0$, we designate by u_t the part of $\mathscr{B}_h$ characterized by $u_t(\theta) = u(t + \theta)$ for $\theta \leq 0$. Now $u_t(\cdot)$ refers to the historical backdrop of the state from every $\theta \in (-\infty, 0]$ like the current time t.

Motivated by the above mentioned works, the main aim of this paper is to establish the existence and controllability of impulsive fractional neutral integro-differential system and non-instantaneous impulse with infinite delay using the new definition of the phase space and fixed point theorem of Mӧnch's and the technique of Hausdorff measure of noncompactness. To best of our knowledge there is some new results in this paper.

The paper is organized as follows. In Section 2, we recall some basic definitions, notations and preliminary facts. In Section 3, the existence and uniqueness results for equation (1.1)–(1.3) using fixed point theorems. In Section 4, the controllability results for fractional neutral integro-differential equation and non-intantaneous impulses with delay. In Section 5, we have examples to demonstrate the obtained results.

2. Preliminaries

In this section, we mention notations, definitions, lemmas and preliminary facts needed to establish our main results.

Let $\mathcal{L}(\mathcal{X}) : \mathcal{X} \to \mathcal{X}$ represents the Banach space of all bounded linear operators, obtain its norm recognized as $\|\cdot\|_{\mathcal{L}(\mathcal{X})}$.

Let $C(\mathcal{J}, \mathcal{X})$ symbolize the space of all continuous functions from $\mathcal{J}$ into $\mathcal{X}$, having norm $\|\cdot\|_{C(\mathcal{J},\mathcal{X})}$. Moreover, $B_r(u, \mathcal{X})$ represents the closed ball in $\mathcal{X}$ with the middle at u and the distance r.

We recall that a measurable function $u : \mathcal{J} \to \mathcal{X}$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. To get extra insights as regards the Bochner integral, refer to the treatise of Yosida [45].

Permit $\mathcal{L}^1(\mathcal{J}, \mathcal{X})$ signifies the Banach space of all measurable functions $u : \mathcal{J} \to \mathcal{X}$ which are Bochner integrable and have the norm

$$\|u\|_{L^1} = \int_0^{\mathcal{J}} \|u(t)\| dt, \quad \text{for all } u \in L^1(\mathcal{J}, \mathcal{X}).$$

Definition 2.1 ([37])**.** Let $\mathfrak{A} : \mathfrak{D} \subseteq \mathcal{X} \to \mathcal{X}$ be a closed linear operator. The operator $\mathfrak{A}$ is considered to be sectorial if we can find $0 < \theta < \frac{\pi}{2}$, $\mathscr{M} > 0$, and $\mu \in \mathbb{R}$ in such a way that the $\rho(\mathfrak{A})$ exists exterior of the segment

$$\mu + S_\theta = \Big\{\mu + \lambda : \lambda \in \tilde{C}, \|\arg(-\lambda)\| < \theta\Big\},$$

$$\Big\|(\lambda I - \mathfrak{A})^{-1}\Big\| \leq \frac{\mathscr{M}}{\lambda - \mu}, \quad \lambda \notin \mu + S_\theta.$$

For short, we say that $\mathfrak{A}$ is sectorial of type $(\mathscr{M}, \theta, \mu)$.

Let $\mathfrak{A}$ defines the infinitesimal generator of an anlyatic semigroup in a Banach space and $0 \in \rho(\mathfrak{A})$, where $\rho(\mathfrak{A})$ is the resolvent set of $\mathfrak{A}$. We characterize the fractional power $\mathfrak{A}^q$ for $0 < q \leq 1$, as a closed linear operator on its domain $\mathfrak{D}(\mathfrak{A}^q)$ with inverse $\mathfrak{A}^{-q}$ (see [33]). The following are basic properties of $\mathfrak{A}^q$.

(i) $\mathfrak{D}(\mathfrak{A}^q)$ is a Banach space with the norm $\|u\|_q = \|\mathfrak{A}^q u\|$ for $u \in \mathfrak{D}(\mathfrak{A}^q)$.

(ii) $\mathcal{Q}(t) : \mathcal{X} \to \mathcal{X}_q$ for $t \geq 0$.
(iii) $\mathfrak{A}^q \mathcal{Q}(t)u = \mathcal{Q}(t)\mathfrak{A}^q u$ for each $u \in \mathfrak{D}(\mathfrak{A}^q)$ and $t \geq 0$.
(iv) For every $t > 0$, $\mathfrak{A}^q \mathcal{Q}(t)$ is bounded on $\mathcal{X}$ and there exists $\mathscr{M}_q > 0$ such that $\|\mathfrak{A}^q \mathcal{Q}(t)\| \leq \mathscr{M}_q t^q e^{-\delta t}$.
(v) For $0 < q \leq 1$ and $u \in \mathfrak{D}(\mathfrak{A}^q)$, we obtain $\|\mathcal{Q}(t)u - u\| \leq C_q t^q \|\mathfrak{A}^q u\|$.

We denote by $PC([0,\mathcal{T}],\mathcal{X})$ the space of piecewise continuous function from $[0,\mathcal{T}]$ into $\mathcal{X}$. $g : PC(\mathcal{J},\mathcal{X}) \to \mathcal{X}$ are given functions satisfying certain assumptions. In particular, we introduce the space PC formed by all piecewise continuous function $u : [0,\mathcal{T}] \to \mathcal{X}$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for $i = 1, 2, \dots, m$. We assume that PC is a Banach space, endowed with the norm $\|u\|_{PC} = \sup_{s\in[0,\mathcal{T}]} \|u(s)\|_{PC}$. It is clear that $(PC, \|\cdot\|_{PC})$ is a Banach space. $PC((0,\mathcal{T}],\mathcal{X}) = \{u : (0,\mathcal{T}] \to \mathcal{X}$ such that $u_i \in C((t_i,t_{i+1}],\mathcal{X})$, $i = 0,1,2,\dots,m$ and there exist $u(t_i^+)$ and $u(t_i^-)$ with norm $u(t_i) = u(t_i^-)$, $i = 0,1,2,\dots,m\}$. We define $C_L(\mathcal{J},\mathcal{X}) = \{v \in PC((0,\mathcal{T}],\mathcal{X}) : \|v(t) - v(s)\| \leq \tilde{I}|t-s|$ for all $t,s \in [0,\mathcal{T}]\}$, where $\tilde{I}$ is some positive constant, is a Banach space endowed with piecewise norm. It should be fixed that, once the delay is infinite, then we need to discuss about the theoretical phase space $\mathscr{B}_h$ in a useful way. In this we consider the phase spaces $\mathscr{B}_h, \mathscr{B}_h'$ which are same as described in [13].

We present the abstract phase space $\mathscr{B}_h$. Suppose $\mathcal{H} : (-\infty,0] \to (0,+\infty)$ is a continuous function with $l = \int_{-\infty}^0 \mathcal{H}(t)dt < +\infty$ and for any $a > 0$, we define $\mathscr{B} = \{\psi : [-a,0] \to \mathcal{X}$ such that $\psi(t)$ is bounded and measurable$\}$ and equip the space $\mathscr{B}$ with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|$ and $\psi \in \mathscr{B}$. Let us define $\mathscr{B}_h = \{\psi : (-\infty,0] \to \mathcal{X}$ for any $c > 0$, $\psi|_{[-c,0]} \in \mathscr{B}$ and $\int_{-\infty}^0 \mathcal{H}(s)\|\psi\|_{[s,0]}ds < +\infty\}$. If $\mathscr{B}_h$ is endowed with the norm $\|\psi\|_{\mathscr{B}_h} = \int_{-\infty}^0 \mathcal{H}(s)\|\psi\|_{[s,0]}ds$, for every $\psi \in \mathscr{B}_h$, then it is clear that $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a Banach space. Now we consider the space $\mathscr{B}_h' = PC((-\infty,\mathcal{T}],\mathcal{X}) = \{u : (-\infty,\mathcal{T}] \to \mathcal{X}$ such that $u|_{\mathcal{J}} \in C(\mathcal{J}_i,\mathcal{X})$ and there exist $u(t_i^+)$ and $u(t_i^-)$ with $u(t_i) = u(t_i^-)$, $u_0 = \phi \in \mathscr{B}_h$, $i = 0,1,2,\dots,m\}$, where u_i is the restriction of u to $\mathcal{J}_i = (t_i,t_{i+1}]$, set $\|\cdot\|_{\mathscr{B}_h'}$ be the seminorm in $\mathscr{B}_h'$ defined by $\|u\|_{\mathscr{B}_h'} = \|\phi\|_{\mathscr{B}_h} + \sup\{\|u(s)\| : s \in [0,\mathcal{T}]\}$, $u \in \mathscr{B}_h'$.

We count on that the phase space $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ could be a semi-normed linear area of function mapping $(-\infty,0]$ into $\mathcal{X}$, and enjoyable the next elementary adages as a results of Hale and Kato (see case in purpose in [17,20]).

If u is continuous function from $(-\infty,\mathcal{T}]$, $\mathcal{T} > 0$, into $\mathcal{X}$, defined on $\mathcal{J}$ and $u_0 \in \mathscr{B}_h$, then for every $t \in \mathcal{J}$ the following situations preserve.

(J1) u_t is in $\mathscr{B}_h$.
(J2) $\|u(t)\|_{\mathcal{X}} \leq \tilde{\mathscr{H}}\|u_t\|_{\mathscr{B}_h}$.
(J3) $\|u_t\|_{\mathscr{B}_h} \leq \mathcal{D}_1(t)\sup\{\|u(s)\|_{\mathcal{X}} : 0 \leq s \leq t\} + \mathcal{D}_2(t)\|u_0\|_{\mathscr{B}_h}$, where $\tilde{\mathscr{H}} > 0$ is a constant and $\mathcal{D}_1(\cdot) : [0,+\infty) \to [0,+\infty)$ is continuous, $\mathcal{D}_2(\cdot) : [0,+\infty) \to [0,+\infty)$ is locally bounded, and $\mathcal{D}_1$, $\mathcal{D}_2$ are independent of $u(\cdot)$. For our convenience, denote $\mathcal{D}_1^* = \sup_{s\in\mathcal{J}} \mathcal{D}_1(s)$, $\mathcal{D}_2^* = \sup_{s\in\mathcal{J}} \mathcal{D}_2(s)$.

Let us recall the following known definitions. For more details see [4,23,27].

Definition 2.2 ([23])**.** The fractional integral of order r with the lower limit zero for a function f is defined as

$$\mathscr{I}^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, r > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 ([23])**.** The Riemann-Liouville derivative of order r with the lower limit zero for a function $f : [0, +\infty) \to \mathbb{R}$ can be written as

$$^L D^r f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d^n}{dt^n} \right) \int_0^t (t-s)^{n-r-1} f(s) ds, \quad n-1 < r < n, t > 0.$$

Definition 2.4 ([23, 34])**.** The Caputo derivative of order r for a function $f : [0, +\infty) \to \mathbb{R}$ can be written as

$$^C D^r f(t) = {}^L D^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n-1 < r < n.$$

Remark 2.1 ([23]). (1) If $f(t) \in C^n[0, +\infty)$, then

$$^C D^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} f^{(n)} ds = \mathscr{I}^{n-r} f^{(n)}(t), \quad t > 0.$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If f is an abstract function with values in $\mathfrak{X}$, then integrals which appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

Definition 2.5. A function $u \in C(\mathfrak{I}, \mathfrak{X})$ is said to be a mild solution of the following problem:

$$\begin{cases} ^C D^r u(t) = \mathfrak{A} u(t) + v(t), & t \in (0, \mathfrak{T}], \\ u(0) = u_0, \end{cases}$$

if it satisfies the integral equation

$$u(t) = \mathcal{Q}_r(t) u_0 + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) v(s) ds.$$

Here

$$\begin{aligned} \mathcal{Q}_r(t) &= \int_0^{+\infty} \wp_r(\theta) T(t^r \theta) d\theta, \quad \mathcal{P}_r(t) = r \int_0^{+\infty} \theta \wp_r(\theta) T(t^r \theta) d\theta, \\ \wp_r(\theta) &= \frac{1}{r} \theta^{-1-\frac{1}{r}} \sigma_r\left(\theta^{-\frac{1}{r}}\right) \geq 0, \\ \sigma_r(\theta) &= \frac{1}{\pi} \sum_{n=1}^{+\infty} (-1)^{n-1} \theta^{-nr-1} \frac{\Gamma(nr+1)}{n!} \sin(n\pi r), \quad \theta \in (0, +\infty), \end{aligned}$$

and $\wp_r$ is a probability density function defined on $(0, +\infty)$, that is,

$$\wp_r(\theta) \geq 0, \quad \theta \in (0, +\infty), \quad \int_0^{+\infty} \wp_r(\theta) d\theta = 1.$$

It is not difficult to verify that

$$\int_0^{+\infty} \theta \wp_r(\theta) d\theta = \frac{1}{\Gamma(1+r)}.$$

We make the following assumption H(A1) in the whole paper.

H(A1): The operator $\mathfrak{A}$ generators a strongly continuous semigroup $\{\mathcal{Q}_r(t) : t \geq 0\}$ in $\mathcal{X}$, and there is a constant, $\mathscr{M}_A \geq 1$ such that $\sup_{t\in[0,\infty)} \|\mathcal{Q}_r(t)\|_{L(\mathcal{X})} \leq \mathscr{M}_A$. For any $t > 0$, $\mathcal{Q}_r(t)$ is compact.

Lemma 2.1 ([41, 46]). *Let* H(A1) *hold. Then, the operators $\mathcal{Q}_r$ and $\mathcal{P}_r$ have the following properties.*

(1) *For any fixed $t \geq 0$, $\mathcal{Q}_r(t)$ and $\mathcal{P}_r(t)$ are linear and bounded operators, and for any $u \in \mathcal{X}$,*

$$\left\|\mathcal{Q}_r(t)u\right\| \leq \mathscr{M}_A\|u\|, \quad \left\|\mathcal{P}_r(t)u\right\| \leq \frac{r\mathscr{M}_A}{\Gamma(1+r)}\|u\|.$$

(2) *$\{\mathcal{Q}_r(t) : t \geq 0\}$ and $\{\mathcal{P}_r(t) : t \geq 0\}$ are strongly continuous.*
(3) *For every $t > 0$, $\mathcal{Q}_r(t)$ and $\mathcal{P}_r(t)$ are compact operators.*

We define the following definiton of the mild solution for the problem (1.1)–(1.3).

Definition 2.6. A function $u \in PC(\mathcal{I}, \mathcal{X})$ is said to be a PC mild solution of problem (1.1)–(1.3) if it satisfies the following equation:

$$u(t) = \begin{cases} \mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0, \Phi(0))] - \mathcal{G}(t, u(t)) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\right) ds, \quad t \in [0, t_1], \\ k = 0, 1, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ \mathcal{Q}_r(t-s_k)\mathscr{D}_k - \mathcal{G}(t, u(t)) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\right) ds, \quad t \in (s_k, t_{k+1}], \\ k = 1, 2, \ldots, m, \end{cases}$$

where

$$\begin{aligned} \mathscr{D}_k =& \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, u(s_k)) + \mathcal{G}(s_k, u(s_k)) \\ &- \int_0^{s_k} (s_k - s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k - s)\mathcal{G}(s, u(s_k))ds \\ &- \int_0^{s_k} (s_k - s)^{r-1}\mathcal{P}_r(s_k - s)\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big)ds, \quad k = 1, 2, \ldots, m. \end{aligned} \tag{2.1}$$

Now, we introduce the Hausdorff measure of noncompactness $\hbar_{\mathcal{Y}}$ defined by

$$\hbar_{\mathcal{Y}}(\mathcal{B}) = \inf\{\epsilon > 0 : \mathcal{B} \text{ has a finite } \epsilon\text{-net in } \mathcal{X}\},$$

for a bounded set $\mathcal{B}$ in any Banach space $\mathcal{Y}$. Some basic properties of $\hbar_{\mathcal{Y}}(\cdot)$ are given in the following definition and lemmas.

Definition 2.7 ([8]). Let $\mathcal{Y}$ be a real Banach space and $\mathcal{B}, C \subseteq \mathcal{Y}$ be bounded, and the following properties are satisfied:

(1) $\mathcal{B}$ is pre-compact if and only if $\hbar_{\mathcal{Y}}(\mathcal{B}) = 0$;
(2) $\hbar_{\mathcal{Y}}(\mathcal{B}) = \hbar_{\mathcal{Y}}(\bar{\mathcal{B}}) = \hbar\mathcal{Y}(\operatorname{conv}\mathcal{B})$, where $\bar{\mathcal{B}}$ and $\operatorname{conv}\mathcal{B}$ are the closure and the convex hull of $\mathcal{B}$, respectively;
(3) $\hbar_{\mathcal{Y}}(\mathcal{B}) \leq \hbar_{\mathcal{Y}}(C)$, when $\mathcal{B} \subseteq C$;
(4) $\hbar_{\mathcal{Y}}(\mathcal{B} + C) \leq \hbar_{\mathcal{Y}}(\mathcal{B}) + \hbar_{\mathcal{Y}} C$, where $\mathcal{B} + C = \{x + y : x \in \mathcal{B}, y \in C\}$;
(5) $\hbar_{\mathcal{Y}}(\mathcal{B} \cup C) = \max\{\hbar_{\mathcal{Y}}(\mathcal{B}), \hbar_{\mathcal{Y}}(C)\}$;
(6) $\hbar_{\mathcal{Y}}(\lambda\mathcal{B}) \leq |\lambda|_{\hbar_{\mathcal{Y}}}(\mathcal{B})$ for any $\lambda \in \mathbb{R}$;
(7) if the map $\Phi : \mathcal{D}(\Phi) \subseteq \mathcal{Y} \to \mathcal{Z}$ is Lipschitz continuous with constant κ then $\hbar_{\mathcal{Y}}(\Phi\mathcal{B}) \leq \kappa_{\hbar_{\mathcal{Y}}}(\mathcal{B})$ for any bounded subset $\mathcal{B} \subseteq \mathcal{D}(\Phi)$, where $\mathcal{Z}$ is a Banach space;
(8) if $\{\mathcal{W}_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of $\mathcal{Y}$ and $\lim_{n\to+\infty\hbar_{\mathcal{Y}}}(\mathcal{W}_n) = 0$, then $\cap_{n=1}^{+\infty}\mathcal{W}_n$ is nonempty and compact in $\mathcal{Y}$.

Lemma 2.2 ([8]). *If $\mathcal{W} \subset C([a,b],\mathcal{X})$ is bounded and equicontinuous, then $\hbar(\mathcal{W}(t))$ is continuous for $t \in [a,b]$ and $\hbar(\mathcal{W}) = \sup\{\hbar(\mathcal{W}(t)) : t \in [a,b]\}$, where $\mathcal{W}(t) = \{u(t) : u \in \mathcal{W}\} \subseteq \mathcal{X}$.*

Theorem 2.1 ([32, 41]). *If $\{x_n\}_{n=1}^{+\infty}$ is a sequence of Bochner integrable functions from $\mathcal{I}$ into $\mathcal{X}$ with the estimation $\|x_n(t)\| \leq \mu(t)$ for almost all $t \in \mathcal{I}$ and every $n \geq 1$, where $\mu \in L^1(\mathcal{I},\mathbb{R})$, then the function $\chi(t) = \hbar(\{x_n(t) : n \geq 1\})$ belongs to $L^1(\mathcal{I},\mathbb{R})$ and satisfies $\hbar(\{\int_0^t \chi(s)ds : n \geq 1\} \leq 2\int_0^t \chi(s)ds$.*

Lemma 2.3 ([8] Darbo-Sadovskii). *If $\mathcal{W} \subseteq \mathcal{Y}$ is bounded, closed and convex, the continuous map $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ is an $\hbar$-contraction, then the map $\mathcal{F}$ has at least one fixed point in $\mathcal{W}$.*

The following fixed point thoerem, a nonlinear alternative of Mönch's type, plays a key role in our proof of system (1.1)–(1.3).

Lemma 2.4 ([28], Theorem 2.2). *Let $\mathcal{D}$ be a closed convex subset of a Banach space $\mathcal{X}$ and $0 \in \mathcal{D}$. Assume that $\mathcal{F} : \mathcal{D} \to \mathcal{X}$ is a continuous map which satisfies Mönch's condition, that is ($M \subseteq \mathcal{D}$ is countable, $M \subseteq \bar{co}(\{0\} \cup \mathcal{F}(M))$ implies $\bar{M}$ is compact). Then, $\mathcal{F}$ has a fixed point in $\mathcal{D}$.*

3. Main Results

In this section, we present and prove the existence results for problem (1.1)–(1.3). In order to prove the main theorem of this section, we assume the following hypotheses.

H(A2): (i) A genertates a strongly continuous semigroup $\{\mathcal{Q}_r(t) : t \geq 0\}$ in $\mathcal{X}$.
(ii) For all bounded subsets $\mathcal{D} \subset \mathcal{X}$ and $u \in \mathcal{D}$, $\|\mathcal{Q}_r(t_2^r\theta)u - \mathcal{Q}_r(t_1^r\theta)u\| \to 0$ as $t_1 \to t_2$ for each fixed $\theta \in (0,+\infty)$.

H(A3): The function $\mathcal{G} : \mathcal{I} \times \mathscr{B}_h \to \mathcal{X}$ is continuous and we can find constants $\beta \in (0, 1)$. $\mathscr{C}_1 > 0$, and $\mathscr{C}_2 > 0$ in such a way that $\mathcal{G}$ is $\mathcal{X}_\beta$-valued and fulfills the subsequent and assumptions:

$$\left\|\mathfrak{A}^\beta \mathcal{G}(t, u_1) - \mathfrak{A}^\beta \mathscr{G}(t, u_2)\right\|_{\mathcal{X}} \leq \mathscr{C}_1 \left\|u_1 - u_2\right\|_{\mathscr{B}_h}, \quad t \in \mathcal{I}, u_1, u_2 \in \mathscr{B}_h,$$

$$\left\|\mathfrak{A}^\beta \mathcal{G}(t, u)\right\|_{\mathcal{X}} \leq \mathscr{C}_1 \|u\|_{\mathscr{B}_h} + \mathscr{C}_2, \quad t \in \mathcal{I}, u \in \mathscr{B}_h.$$

H(A4): The function $\mathcal{H} : \mathcal{D} \times \mathscr{B}_h \to \mathcal{X}$ satisfies the following.

(i) For every $(t, s) \in \mathcal{D}$, the function $\mathcal{H}(t, s, \cdot) : \mathscr{B}_h \to \mathcal{X}$ is continuous and for each $u \in \mathscr{B}_h$, the function $\mathcal{H}(\cdot, \cdot, u) : \mathcal{D} \to \mathcal{X}$ is strongly measurable.

(ii) There exist a function $\nu \in L^1(\mathcal{I}, \mathbb{R}^+)$ and a continuous non-decreasing function $\omega : \mathbb{R}^+ \to (0, \infty)$ to ensure that

$$\|\mathcal{H}(t, s, u)\|_{\mathcal{X}} \leq \nu(s)\omega(\|u\|_{\mathscr{B}_h}), \quad \text{for a.e. } t, s \in \mathcal{I}, u \in \mathscr{B}_h.$$

(iii) There exists $\Theta \in L^1(\mathcal{I} \times \mathcal{I}, \mathbb{R}^+)$ to ensure that

$$\hbar\left(\mathcal{H}(t, s, D)\right) \leq \zeta(t, s)\Big[\sup_{-\infty<\theta\leq 0} \hbar(D(\theta))\Big], \quad \text{for a.e. } t, s \in \mathcal{I},$$

where $D(\theta) = \{x(\theta) : x \in \mathcal{X}\}$ and $\hbar$ is the Hausdorff measures of non-compactness.

H(A5): The function $\mathscr{F} : \mathcal{I} \times \mathscr{B}_h \times \mathcal{X} \to \mathcal{X}$ satisfies the following.

(i) For a.e. $t \in \mathcal{X}$, $(\phi, u) \mapsto \mathscr{F}(t, \phi, u)$ is continuous and for all $(\phi, u) \in \mathscr{B}_h \times \mathcal{X}$, $t \mapsto \mathscr{F}(t, \phi, u)$ is strongly measurable.

(ii) There exists a function $m \in L^1(\mathcal{I}, \mathbb{R}^+)$ and a continuous non-decreasing function $\Omega : \mathbb{R}^+ \to (0, +\infty)$ to ensure that

$$\left\|\mathscr{F}(t, \phi, u)\right\|_{\mathcal{X}} \leq m(t)\Omega(\|\phi\|_{\mathscr{B}_h} + \|u\|), \quad (t, \phi, u) \in \mathcal{X} \times \mathscr{B}_h \times \mathcal{X}.$$

(iii) For every bounded sets $D \subset \mathscr{B}_h$, $F^* \subset \mathcal{X}$, there exists a positive function $\eta \in L^1(\mathcal{X}, \mathbb{R}^+)$ is such a way that

$$\hbar\Big(\mathscr{F}(t, D, F^*)\Big) \leq \eta(t)\Big[\sup_{-\infty<\theta\leq 0} \hbar(D(\theta)) + \hbar(F^*)\Big], \quad \text{for a.e. } t \in \mathcal{I},$$

where $D(\theta) = \{v(\theta) : v \in D\}$.

H(A6): The function $\mathscr{G}_k : (t_k, s_k] \times \mathscr{B}_h \to \mathcal{X}$, $k = 1, 2, \ldots, m$ are continuous, and satisfies the following conditions.

(i) There exist constants $\mathcal{C}_i, \bar{\mathcal{C}}_i > 0$, $i = 1, 2, \ldots, m$, in such a way that

$$\left\|\mathscr{G}_k(t, \phi)\right\|_{\mathcal{X}} \leq \mathcal{C}_i \|\phi\|_{\mathscr{B}_h} + \bar{\mathcal{C}}_i, \quad t \in (t_k, s_k], \phi \in \mathscr{B}_h.$$

(ii) There exists constants $\tilde{\nu}_i > 0$ such that, for each bounded $D \subset \mathscr{B}_h$.

$$\hbar(\mathscr{G}_k(t, D)) \leq \tilde{\nu}_i\Big[\sup_{-\infty<\theta\leq 0} \hbar(D(\theta))\Big], \quad \text{for a.e. } t \in (t_k, s_k], k = 1, 2, \ldots, m,$$

where $D(\theta) = \{x(\theta) : x \in D\}$.

H(A7): For $k = 1, 2, \ldots, m$, $\mathfrak{I}_K \in C(\mathcal{X}, \mathcal{X})$ and there is a constant $\mathcal{L}_f > 0$ such that

$$\left\|\mathfrak{I}_k(u) - \mathfrak{I}_k(v)\right\| \leq \mathcal{L}_f \|u - v\|, \quad \text{for all } u, v \in \mathcal{X}.$$

H(A8): For $k = 1, 2, \ldots, m$, the functions $\mathscr{G}_k \in C([t_k, s_k] \times \mathcal{X}; \mathcal{X})$ and there exists $\mathcal{L}_g \in C(\mathcal{I}, \mathbb{R}^+)$ such that

$$\left\|\mathscr{G}_k(t, u) - \mathscr{G}_k(t, v)\right\| \leq \mathcal{L}_g(t)\|u - v\|, \quad \text{for all } u, v \in \mathcal{X} \text{ and } t \in [t_k, s_k].$$

H(A9): For every bounded set $\chi \subset \mathscr{B}_h$, the set $\{t \mapsto \mathscr{G}_k(t, u_t) : u_t \in \chi\}$, $k = 1, 2, \ldots, m$, is equicontinuous in $\mathscr{B}_h$.

H(A9*): The following inequalities hold:

$$\left[\mathscr{M}_A(\mathcal{L}_k + \tilde{\nu}_i) + \frac{2\mathscr{M}_A^2 \mathscr{M}_6 r \mathcal{T}^r}{\Gamma(1+r)} \int_0^t \eta(s) ds + \frac{2\mathscr{M}_A \mathscr{M}_6 r \mathcal{T}^r}{\Gamma(1+r)} \int_0^t \eta(s) ds\right] \hbar_{PC}(\tilde{\mathcal{W}}(\tau)) < 1,$$

$$\tilde{\mathscr{L}} = (\mathscr{M}_A + 1)\left[\left(\mathscr{M}_0 + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{T^{r\beta}}{\beta}\right)\mathscr{C}_1 \mathcal{D}_1\right] < 1,$$

and

$$\iota = \max_{1 \leq k \leq m} (\mathscr{M}_A + 1)\left\{(\mathcal{L}_k + \tilde{\nu}_i) + \frac{\mathscr{M}_A(1+\zeta^*)T^r}{\Gamma(r+1)} \int_0^T \tilde{\eta}(s) ds\right\} < 1.$$

For our convenience, let us take

$$\mathcal{K}_i := \left[\left(\frac{1-q_i}{q-q_i}\right) b^{\frac{q-q_i}{1-q_i}}\right]^{1-q_i}, \quad i = 0, 1, 2, \quad \mathscr{M}_4 := \mathcal{K}_1 \|m\|_{L^{\frac{1}{q_1}}(\mathscr{I}, R^+)}$$

and $\mathscr{M}_6 := \mathcal{K}_2 \|\eta\|_{L^{\frac{1}{q_2}}(\mathscr{I}, R^+)}$.

Theorem 3.1. *Assume the hypotheses* H(A1)-H(A9*) *are satisfied, then the problem* (1.1)–(1.3) *has at least one mild solution on* $[0, \mathcal{T}]$ *provided that*

$$\max_{1 \leq k \leq m} \mathcal{D}_1 \left[(\mathscr{M}_A(\mathcal{L}_k + \tilde{\nu}_i)) + \mathscr{M}_0 \mathscr{C}_1 + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r+1)} \mathscr{C}_1 \frac{T^{r\beta}}{\beta}\right] < 1. \tag{3.1}$$

Proof. We will transform the system (1.1)–(1.3) into a fixed point problem. Let the operator $\Upsilon : \mathscr{B}_h' \to \mathscr{B}_h'$ be defined by

$$(\Upsilon u)(t) = \begin{cases} \Phi(t), \quad t \in (-\infty, 0], \\ \mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0, \Phi(0))] - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1} \mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s) ds \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\right) ds, \quad t \in [0, t_1], \\ k = 0, 1, 2, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1} \mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s) ds \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\right) ds, \quad t \in (s_k, t_{k+1}], \\ k = 1, 2, \ldots, m, \end{cases}$$

with $\mathscr{D}_k$, $k = 1, 2, \ldots, m$, defined by (2.1).

Obviously the fixed points of the operator Υ are mild solutions of the model (1.1)–(1.3). The function $\tilde{y}(\cdot) : (-\infty, \mathfrak{T}] \to \mathfrak{X}$ is defined by

$$\tilde{y}(t) = \begin{cases} \Phi(t), & t \in (-\infty, 0], \\ \mathcal{Q}_r(t)\Phi(0), & t \in \mathfrak{J}. \end{cases}$$

Then, $\tilde{y}_0 = \Phi$. For every function $\tilde{z} \in C(\mathfrak{J}, \mathbb{R})$ with $\tilde{z}(0) = 0$, we take as z is defined by

$$z(t) = \begin{cases} 0, & t \le 0, \\ \tilde{z}(t), & t \in \mathfrak{J}. \end{cases}$$

If $u(\cdot)$ fulfils (2.1), we are able to split it as $u(t) = \tilde{y}(t) + \tilde{z}(t)$, $t \in \mathfrak{J}$, which suggests $u_t = \tilde{y}_t + \tilde{z}_t$, for each $t \in \mathfrak{J}$ and also the function $\tilde{z}(\cdot)$ fulfills

$$\tilde{z}(t) = \begin{cases} \mathcal{Q}_r(t)\mathcal{G}(0,\Phi) - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\right)ds, \quad t \in [0, t_1], \\ k = 0, 1, 2, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, \tilde{z}_t + \tilde{y}_t), \quad t \in (t_k, s_k], \quad k = 1, 2, 3, \ldots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\right)ds, \\ t \in (s_k, t_{k+1}], \quad k = 1, 2, \ldots, m, \end{cases}$$

where

$$\begin{aligned} \mathscr{D}_k =& \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k}) + \mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k}) \\ &- \int_0^{s_k} (s_k - s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k - s)\mathcal{G}(s, \tilde{z}_{s_k} + \tilde{y}_{s_k})ds \\ &- \int_0^{s_k} (s_k - s)^{r-1}\mathcal{P}_r(s_k - s)\mathscr{F}\left(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\right)ds, \end{aligned} \tag{3.2}$$

$k = 1, 2, 3, \ldots, m$. Let $\mathscr{B}_h'' = \{\tilde{z} \in \mathscr{B}_h' : \tilde{z}_0 = 0 \in \mathscr{B}_h\}$. For any $\tilde{z} \in \mathscr{B}_h''$,

$$\|\tilde{z}\|_{\mathscr{B}_h''} = \sup_{t \in \mathfrak{J}} \|\tilde{z}(t)\|_{\mathfrak{X}} + \|\tilde{z}_0\|_{\mathscr{B}_h} = \sup_{t \in \mathfrak{J}} \|\tilde{z}(t)\|_{\mathfrak{X}}, \quad \tilde{z} \in \mathscr{B}_h'',$$

as a result $(\mathscr{B}_h'', \|\cdot\|_{\mathscr{B}_h''})$ is a Banach space. Consider $B_q = \{\tilde{z} \in \mathscr{B}_h'' : \|\tilde{z}\|_{\mathfrak{X}} \le q\}$ for some $q \ge 0$. Then for each $B_q \subset \mathscr{B}_h''$ is uniformly bounded, and for $\tilde{z} \in B_q$. We have the phase space axioms (J1)-(J2),

$$\begin{aligned} \|\tilde{z}_s + \tilde{y}_s\|_{\mathscr{B}_h} &\le \|\tilde{z}_s\|_{\mathscr{B}_h} + \|\tilde{y}_s\|_{\mathscr{B}_h} \\ &\le \mathcal{D}_1 \sup_{(0 \le \tau \le \tilde{z}_s + \tilde{y}_s)} \|\tilde{z}(\tau)\|_{\mathfrak{X}} + \mathcal{D}_2\|\tilde{z}_0\|_{\mathscr{B}_h} + \mathcal{D}_1 \sup_{(0 \le \tau \le \tilde{z} + \tilde{y})} \|\tilde{y}(\tau)\| + \mathcal{D}_2\|\tilde{y}_0\|_{\mathscr{B}_h} \\ &\le \mathcal{D}_1 \sup_{(0 \le \tau \le s)} \|\tilde{z}(\tau)\|_{\mathfrak{X}} + \mathcal{D}_1\|\mathcal{Q}_r(t)\|_{\mathcal{L}(\mathfrak{X})}\|\Phi(0)\|_{\mathscr{B}_h} + \mathcal{D}_2\|\Phi\|_{\mathscr{B}_h} \\ &\le \mathcal{D}_1\|\tilde{z}\|_{\mathfrak{X}} + (\mathcal{D}_1\mathscr{M}_1 + \mathcal{D}_2)\|\Phi\|_{\mathscr{B}_h} \\ &\le \mathcal{D}_1 q + \tilde{c}_n. \end{aligned}$$

In the event that $\|\tilde{z}\|_{\mathfrak{X}} < q,\ q > 0$,

$$\|\tilde{z}_s + \tilde{y}_s\|_{\mathscr{B}_h} \leq \mathcal{D}_1 q + \tilde{c}_n, \tag{3.3}$$

where $\tilde{c}_n = (\mathcal{D}_1\mathscr{M}_1 + \mathcal{D}_2)\|\Phi\|_{\mathscr{B}_h}$. Define the operator $\tilde{\Upsilon} : \mathscr{B}_h'' \to \mathscr{B}_h''$ by

$$(\tilde{\Upsilon}\tilde{z})(t) = \begin{cases} \mathcal{Q}_r(t)\mathcal{G}(0,\Phi) - \mathcal{G}(t,\tilde{z}_t + \tilde{y}_t) \\ + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s) \\ \times \mathscr{F}\left(s,\tilde{z}_s+\tilde{y}_s,\int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\right)ds, \quad t\in[0,t_1], \\ k=0,1,2,\ldots,m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t), \quad t\in(t_k,s_k], \quad k=1,2,3,\ldots,m, \\ \mathcal{Q}_r(t-s_k)\mathscr{D}_k - \mathcal{G}(t,\tilde{z}_t+\tilde{y}_t) \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s,\tilde{z}_s+\tilde{y}_s,\int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\right)ds, \\ t\in(s_k,t_{k+1}], \end{cases}$$

with $\mathscr{D}_k$, $k=1,2,3,\ldots,m$, defined by (3.2).

Thus, the operator Υ has a fixed point if and only if $\tilde{\Upsilon}$ has a fixed point. Now, first we calculate the following estimations.

Remark 3.1. By utilizing the above equation H(A1)-H(A9), we obtain

$$P_1 = \|\mathcal{Q}_r(t)\mathcal{G}(0,\Phi)\|_{\mathfrak{X}} = \mathscr{M}_A\|\mathfrak{A}^{-\beta}\|\left\|\mathfrak{A}^{-\beta}\mathcal{G}(0,\Phi)\right\|_{\mathfrak{X}} \leq \mathscr{M}_A\mathscr{M}_0\left[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2\right],$$

where $\mathscr{M}_0 = \|\mathfrak{A}^{-\beta}\|$,

$$\begin{aligned} P_2 &= \|\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathfrak{X}} = \left\|\mathfrak{A}^{-\beta}(\mathfrak{A}^{\beta})\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\right\|_{\mathfrak{X}} \leq \mathscr{M}_0\left[\mathscr{C}_1\left\|\tilde{z}_t+\tilde{y}_t\right\|_{\mathscr{B}_h} + \mathscr{C}_2\right] \\ &\leq \mathscr{M}_0\left[\mathscr{C}_1\left(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n\right) + \mathscr{C}_2\right] \leq \mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{M}_0\mathscr{C}_2, \\ P_3 &= \|\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathfrak{X}} = \|\mathfrak{A}^{-\beta}(\mathfrak{A}^{\beta})\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathfrak{X}} \\ &\leq \mathscr{M}_0\left[\mathscr{C}_1\|\tilde{z}_{s_k}+\tilde{y}_{s_k}\|_{\mathscr{B}_h} + \mathscr{C}_2\right] \leq \mathscr{M}_0\left[\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k} + \tilde{c}_n) + \mathscr{C}_2\right] \\ &\leq \mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k} + \tilde{c}_n) + \mathscr{M}_0\mathscr{C}_2, \quad t\in(s_k,t_{k+1}], \\ P_4 &= \left\|\int_0^t (t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\right\|_{\mathfrak{X}} \\ &\leq \frac{r\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}\int_0^t (t-s)^{r\beta-1}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n) + \mathscr{C}_2)ds \\ &\leq \frac{r\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2)\int_0^t (t-s)^{r\beta-1}ds \\ &\leq \frac{\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2)\frac{(t_1)^{r\beta}}{\beta}, \quad t\in[0,t_1], \\ P_5 &= \left\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(s_k-s)(\mathfrak{A}^{\beta})\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\right\|_{\mathfrak{X}} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{(s_k)^{r\beta}}{\beta}, \quad t\in(s_k,t_{k+1}],\\
P_6 &= \Big\|\int_0^t (t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}\\
&\leq \frac{\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{T^{r\beta}}{\beta}, \quad t\in[0,\mathcal{T}],\\
P_7 &= \|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}\leq \mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i, \quad t\in(t_k,s_k],\\
P_8 &= \|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\
&\leq \mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n]+\bar{\mathcal{C}}_i, \qquad t\in(s_k,t_{k+1}],\\
P_9 &= \Big\|\int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\
&\leq \frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n)ds\\
&\leq \frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(1+\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds\\
&\leq \frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s), \quad t\in[0,t_1],\\
P_{10} &= \Big\|\int_0^{s_k} (s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\
&\leq \frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s), \quad t\in(s_k,t_{k+1}],\\
P_{11} &\leq \Big\|\int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\
&\leq \frac{\mathscr{M}_A T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s), \quad t\in[0,\mathcal{T}],\\
P_{12} &\leq \|\mathcal{G}(t,\tilde{z}_t^n+\tilde{y}_s)\|_{\mathcal{X}}\leq \Big\|(\mathfrak{A})^{\beta}(\mathfrak{A})^{-\beta}\left[\mathcal{G}(t,\tilde{z}_t^n+\tilde{y}_t)-\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\right]\Big\|_{\mathcal{X}}\\
&\leq \mathscr{M}_0\Big\|(\mathfrak{A})^{\beta}\left[\mathcal{G}(t,\tilde{z}_t^n+\tilde{y}_t)-\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\right]\Big\|_{\mathcal{X}},\\
P_{13} &\leq \Big\|\int_0^t (t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]ds\Big\|_{\mathcal{X}}\\
&\leq \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\frac{t_1^{r\beta}}{\beta}\int_0^t \Big\|\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]\Big\|_{\mathcal{X}}ds, \quad t\in[0,t_1],\\
P_{14} &\leq \Big\|\int_0^{s_k} (s_k-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(s_k-s)\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]ds\Big\|_{\mathcal{X}}\\
&\leq \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\frac{(s_k)^{r\beta}}{\beta}\int_0^{s_k} \Big\|\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]\Big\|_{\mathcal{X}}ds,\\
&\quad t\in(s_k,t_{k+1}],
\end{aligned}
$$

$$
\begin{aligned}
P_{15} \leq & \Big\| \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big] \Big\|_{\mathcal{X}} ds \\
\leq & \frac{\mathscr{M}_A t_1^r}{\Gamma(r+1)} \int_0^t \Big\| \mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big] \Big\|_{\mathcal{X}} ds, \quad t \in [0,t_1], \\
P_{16} \leq & \Big\| \int_0^{s_k} (s_k-s)^{r-1} \mathcal{P}_r(s_k-s) \Big[\mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big] \Big\|_{\mathcal{X}} ds \\
\leq & \frac{\mathscr{M}_A (s_k)^r}{\Gamma(r+1)} \int_0^{s_k} \Big\| \mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big] \Big\|_{\mathcal{X}} ds, \quad t \in (s_k, t_{k+1}].
\end{aligned}
$$

Now, we define $(\tilde{\Upsilon}_2 \tilde{z})(t)$ as

$$
(\tilde{\Upsilon}_2 \tilde{z})(t) = \begin{cases}
\mathcal{Q}_r(t)\mathcal{G}(0,\Phi) - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) + \int_0^t (t-s)^{r-1} \mathfrak{A} \mathcal{P}_r(t-s) \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) ds, \\
t \in [0,t_1], \quad k = 0,1,2,\dots,m, \\
0, \quad t \in (t_k, s_k], \quad k = 1,2,\dots,m, \\
\mathcal{Q}_r(t)(t-s_k)\Big[\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k}) - \int_0^{s_k} (s_k-s)^{r-1} \mathfrak{A} \mathcal{P}_r(s_k-s) \\
\times \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) ds\Big] - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) \\
+ \int_0^t (t-s)^{r-1} \mathfrak{A} \mathcal{P}_r(t-s) \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) ds, \quad t \in (s_k, t_{k+1}], \\
k = 1,2,3,\dots,m.
\end{cases}
$$

We obtain

$$
\begin{aligned}
\|(\tilde{\Upsilon}_2 \tilde{z})(t) - (\tilde{\Upsilon}_2 \tilde{\bar{z}})(t)\|_{\mathcal{X}} \leq & \|(\mathfrak{A})^\beta\| \left\|(\mathfrak{A})^\beta \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)_{\mathcal{X}} - (\mathfrak{A})^\beta \mathcal{G}(t, \tilde{\bar{z}}_t, \tilde{y}_t)\right\|_{\mathcal{X}} \\
& + \Big\| \int_0^t (t-s)^{r-1} (\mathfrak{A})^{1-\beta} \mathcal{P}_r(t-s) \\
& \times \left[(\mathfrak{A})^\beta \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) - (\mathfrak{A}^\beta) \mathcal{G}(s, \tilde{\bar{z}}_s + \tilde{y}_s)\right] ds \Big\|_{\mathcal{X}} \\
\leq & \mathscr{M}_0 \mathscr{C}_1 \left\| \tilde{z}_t - \tilde{\bar{z}}_t \right\|_{\mathscr{B}_h} + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{t_1^{r\beta}}{\beta} \mathscr{C}_1 \left\| \tilde{z}_t - \tilde{\bar{z}}_t \right\|_{\mathscr{B}_h},
\end{aligned}
$$

since

$$
\begin{aligned}
\left\| \tilde{z}_t - \tilde{\bar{z}}_t \right\|_{\mathscr{B}_h} \leq & \mathcal{D}_1 \|\tilde{z}(t)\|_{\mathcal{X}} + (\mathcal{D}_2 + \mathcal{J}^\Phi)\|z_0\|_{\mathscr{B}_h} - \mathcal{D}_1 \|\tilde{\bar{z}}(t)\|_{\mathcal{X}} - (\mathcal{D}_2 + \mathcal{J}^\Phi)\|\tilde{\bar{z}}_0\|_{\mathscr{B}_h} \\
\leq & \mathcal{D}_1 \|\tilde{z}(t) - \tilde{\bar{z}}(t)\|_{\mathcal{X}} \leq \mathcal{D}_1 \|\tilde{z} - \tilde{\bar{z}}\|_{\mathscr{B}_h''},
\end{aligned}
$$

we have

$$\Big\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\bar{z}})(t)\Big\|_{\mathcal{X}} \leq \mathscr{M}_0\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''}+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{t_1^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''}$$
$$\leq\Big[\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{t_1^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\Big]\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''},$$
$$t\in[0,t_1],$$

$$\Big\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\bar{z}})(t)\Big\|_{\mathcal{X}}$$
$$\leq\|\mathcal{Q}_r(t-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|(\mathfrak{A})^{-\beta}\|\|(\mathfrak{A})^{\beta}\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})-(\mathfrak{A})^{\beta}\mathcal{G}(s_k,\tilde{\bar{z}}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}$$
$$+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(s_k-s)\Big[\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s-\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{\bar{z}}_s+\tilde{y}_s)\Big]ds\Big\|_{\mathcal{X}}\Big]$$
$$+\|\mathfrak{A}^{-\beta}\|\|\mathfrak{A}^{\beta}\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)-\mathfrak{A}^{\beta}\mathcal{G}(t,\tilde{\bar{z}}_t+\tilde{y}_t)\|_{\mathcal{X}}$$
$$+\Big\|\int_0^{t}(t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\Big[\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)-\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{\bar{z}}_s+\tilde{y}_s)\Big]ds\Big\|_{\mathcal{X}}$$
$$\leq\mathscr{M}_A\Big[\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\bar{z}}\|_{\mathcal{B}_h''}+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{(s_k)^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''}\Big]$$
$$\times\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''}$$
$$+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{(t_{k+1})^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''}$$
$$\leq\Big[(\mathscr{M}_A+1)\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\mathcal{D}_1\Big\{\frac{(t_{k+1})^{r\beta}}{\beta}+\frac{\mathscr{M}_A(s_k)^{r\beta}}{\beta}\Big\}\Big]\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''},$$
$$t\in(s_k,t_{k+1}],$$

and

$$\Big\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\bar{z}})(t)\Big\|_{\mathcal{X}}$$
$$\leq(\mathscr{M}_A+1)\Big[\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{T^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\Big]\|\tilde{z}-\tilde{\bar{z}}\|_{\mathscr{B}_h''},\quad t\in\mathcal{J}.$$

Presently, let us demostrate that Υ has a fixed point. Subsequently we will prove that $\tilde{\Upsilon}$ has a fixed point by using Lemma 2.4.

Step 1. We show that there exist some $q>0$ such that $\tilde{\Upsilon}(\mathscr{B}_q)\subseteq\mathscr{B}_q$. If it is not true, then for each positive number q, there exists a function $\tilde{z}^q(.)\in\mathscr{B}_q$ and some $t\in\mathcal{J}$ such that $\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|>q$.

On the other hand, from hypotheses the H(A2)-H(A9), Lemma 2.1 (1) and Hölder's inequality, for $t\in[0,t_1]$,

$$q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\leq\|\mathcal{Q}_r(t)\|_{\mathcal{L}(\mathcal{X})}\|\mathcal{G}(0,\Phi)\|_{\mathcal{X}}+\|\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}$$
$$+\Big\|\int_0^{t}(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}$$

$$
\begin{aligned}
&+\Big\|\int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\
\leq&\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\
&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{t_1^{r\beta}}{\beta}\\
&+\frac{\mathscr{M}_A t_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}} m(s)\\
=&\mathcal{J}_1.
\end{aligned}
$$

For any $t\in(t_k,s_k]$, $k=1,2,\ldots,m$, we have

$$
\begin{aligned}
q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\leq\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}\leq\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i\\
=\mathcal{J}_2.
\end{aligned}
$$

In the same way, for any $t\in(s_k,t_{k+1}]$, $k=1,2,\ldots,m$,

$$
\begin{aligned}
q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\leq&\|\mathcal{Q}_r(t-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|+\|\mathscr{G}_k(s_k,\tilde{z}_k+\tilde{y}_{s_k})\|_{\mathcal{X}}\\
&+\|\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\
&+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|\\
&+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\
&\times\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\Big]\\
&+\|\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}+\Big\|\int_0^t(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}\\
&+\Big\|\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\\
&\times\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\
\leq&\mathscr{M}_A\Big[\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n]+\bar{\mathcal{C}}_i+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\
&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{C}_2)\frac{(s_k)^{r\beta}}{\beta}\\
&+\frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}} m(s)\Big]\\
&+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\
&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{(t_{k+1})^{r\beta}}{\beta}\\
&+\frac{\mathscr{M}_A(t_{k+1})^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}} m(s).
\end{aligned}
$$

Then, for all $t \in \mathfrak{I}$, we find that

$$\begin{aligned}\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq & \mathfrak{C}^* + (\mathscr{M}_A+1)\Big[(\mathcal{L}_k + \mathcal{C}_i + \mathscr{M}_0\mathscr{C}_1) + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big] \\ & \times (\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) \\ & + \frac{\mathscr{M}_A(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{C}^* = \max_{1\leq k\leq m} & \Big\{\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2] + (\mathscr{M}_A+1)\Big(\mathscr{M}_0\mathscr{C}_2 + \bar{\mathcal{C}}_i \\ & + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_2\frac{T^{r\beta}}{\beta}\Big)\Big\} \\ & + \frac{\mathscr{M}_A T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s).\end{aligned}$$

Combining the above equations

$$\begin{aligned}q < \|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq & \mathfrak{C}^* + (\mathscr{M}_A+1)\Big[(\mathcal{L}_k + \mathcal{C}_i + \mathscr{M}_0\mathscr{C}_1) + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big] \\ & \times (\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) \\ & + \frac{\mathscr{M}_A(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s).\end{aligned}$$

Dividing both sides by q and letting $q \to +\infty$, we obtain

$$\begin{aligned}1 \leq \|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq & \mathfrak{C}^* + (\mathscr{M}_A+1)\Big[(\mathcal{L}_k + \mathcal{C}_i + \mathscr{M}_0\mathscr{C}_1) + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big] \\ & \times (\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) \\ & + \frac{\mathscr{M}_A(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s).\end{aligned}$$

Then, by hypothesis, we get $1 \leq 0$. This is a contradiction.

Hence, for some positive integer $\tilde{\Upsilon}(\mathscr{B}_q) \subseteq \mathscr{B}_q$.

Step 2. $\tilde{\Upsilon} : \mathscr{B}_h^{''} \to \mathscr{B}_h^{''}$ is continuous. For this purpose let $\{\tilde{z}^{(n)}\}_{n=0}^{+\infty} \subseteq \mathscr{B}_h^{''}$ with $\tilde{z}^{(n)} \to \tilde{z}$ in $\mathscr{B}_h^{''}$. Then there is a number $c' > 0$ such that $\|\tilde{z}^{(n)}(t)\| \leq c'$ for all n and a.e. $t \in \mathfrak{I}$, so $\tilde{z}^{(n)} \in \mathscr{B}_{c'} = \{\tilde{z} \in \mathscr{B}_h^{''} : \|\tilde{z}\|_{\mathscr{B}_h^{''}} \leq c'\} \subseteq \mathscr{B}_h^{''}$ and $\tilde{z} \in \mathscr{B}_{c'}$. From remark, we have $\|\tilde{z}_t + \tilde{y}_t\|_{\mathscr{B}_h} \leq c^{''}$, $t \in \mathfrak{I}$.

By H(A4), H(A5), Remark $P_{12}, P_{13}, P_{14}, P_{15}$, and Lebesgue's dominated convergence theroem, we obtain, for $t \in [0, t_1]$,

$$\begin{aligned}& \|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathfrak{X}} \\ \leq & \mathscr{M}_0\Big\|(\mathfrak{A})^\beta\Big[\mathscr{G}(t, \tilde{z}_t^n + \tilde{y}_t) - \mathscr{G}(t, \tilde{z}_t + \tilde{y}_t)\Big]\Big\|_{\mathfrak{X}} + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(t_1)^{r\beta}}{\beta}\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^t \|(\mathfrak{A})^{\beta}[\mathcal{G}(s, \tilde{z}_s^n + \tilde{y}_s) - \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)]\|_{\mathcal{X}} ds \\
&+ \frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)} \int_0^t \Big\| \mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau)d\tau\Big) \\
&- \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)\Big\|_{\mathcal{X}} ds \to 0, \quad \text{as } n \to +\infty.
\end{aligned}
$$

For any $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we obtain $\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}} = 0$. In the same way, for any $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
&\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}} \\
\leq &\|\mathcal{Q}_r(t-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}} + \|\mathscr{G}_k(s_k, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}) \\
&- \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\|_{\mathcal{X}} + \Big\|(\mathfrak{A})^{-\beta}\Big\| \cdot \Big\|(\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}) \\
&- (\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big\|_{\mathcal{X}} + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(s_k)^{r\beta}}{\beta} \\
&\times \int_0^{s_k} \Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s_k, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}) - \mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big]\Big\|_{\mathcal{X}} ds \\
&+ \frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)} \int_0^{s_k} \Big\| \mathscr{F}\Big(s, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau)d\tau\Big) \\
&- \mathscr{F}\Big(s, \tilde{z}_{s_k} + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)\Big\|_{\mathcal{X}} ds\Big] \\
&+ \Big\|(\mathfrak{A})^{-\beta}\Big\| \cdot \Big\|(\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}_t^n + \tilde{y}_t) - (\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)\Big\|_{\mathcal{X}} \\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(t_{k+1})^{r\beta}}{\beta} \int_0^t \Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s, \tilde{z}_s^n + \tilde{y}_s) - (\mathfrak{A})^{\beta}\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)\Big]\Big\|_{\mathcal{X}} ds \\
&+ \frac{\mathscr{M}_A(t_{k+1})^r}{\Gamma(r+1)} \int_0^t \Big\| \mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau)d\tau\Big) \\
&- \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)\Big\|_{\mathcal{X}} ds \to 0, \quad \text{as } n \to +\infty.
\end{aligned}
$$

It is simple to see that $\lim_{n\to+\infty} \Big\|(\tilde{\Upsilon}\tilde{z}^n) - (\tilde{\Upsilon}\tilde{z})\Big\|_{\mathscr{B}_h''} = 0$. Thus, $\tilde{\Upsilon}$ is continuous.

Step 3. $\tilde{\Upsilon}$ is $\hbar$ - contraction.

To demonstrate this we separate $\tilde{\Upsilon}$ as $\tilde{\Upsilon}_2 + \tilde{\Upsilon}_3$ for $t \in \mathfrak{J}$, where $(\tilde{\Upsilon}_2 \tilde{z})(t)$ is defined in axioms and

$$(\tilde{\Upsilon}_3 \tilde{z})(t) = \begin{cases} \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) ds, \\ t \in [0, t_1], \quad k = 0,1,2,\ldots,m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, \tilde{z}_t + \tilde{y}_t), \quad t \in (t_k, s_k], \quad k = 1,2,3,\ldots,m, \\ \mathcal{Q}_r(t-s_k)\Big[\mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k}) \\ - \int_0^{s_k} (s_k - s)^{r-1} \mathcal{P}_r(s_k - s) \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) ds\Big] \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) ds, \\ t \in (s_k, t_{k+1}], \quad k = 1,2,3,\ldots,m. \end{cases}$$

First, we show that $\tilde{\Upsilon}_2$ is Lipschitz continuous on $\mathscr{B}_h''$. In fact $\tilde{z}, \tilde{\tilde{z}} \in \mathscr{B}_h''$ then from axioms, we have, for all $t \in [0, t_1]$,

$$\|(\tilde{\Upsilon}_2 \tilde{z})(t) - (\tilde{\Upsilon}_2 \tilde{\tilde{z}})(t)\|_{\mathfrak{X}} \leq \Big[\Big(\mathscr{M}_0 + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{t_1^{r\beta}}{\beta}\Big)\mathscr{C}_1 \mathcal{D}_1\Big] \|\tilde{z} - \tilde{\tilde{z}}\|_{\mathscr{B}_h''}.$$

In the same way, for any $t \in (s_k, t_{k+1}]$, $k = 1,2,\ldots,m$, we obtain

$$\|(\tilde{\Upsilon}_2 \tilde{z})(t) - (\tilde{\Upsilon}_2 \tilde{\tilde{z}})(t)\|_{\mathfrak{X}} \leq (\mathscr{M}_A + 1)\Big[\Big(\mathscr{M}_0 + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(s_k)^{r\beta}}{\beta}\Big)\mathscr{C}_1 \mathcal{D}_1\Big] \|\tilde{z} - \tilde{\tilde{z}}\|_{\mathscr{B}_h''}.$$

Then, for all $t \in \mathfrak{J}$, we get

$$\begin{aligned} \|(\tilde{\Upsilon}_2 \tilde{z})(t) - (\tilde{\Upsilon}_2 \tilde{\tilde{z}})(t)\| &\leq (\mathscr{M}_A + 1)\Big[\Big(\mathscr{M}_0 + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{T^{r\beta}}{\beta}\Big)\mathscr{C}_1 \mathcal{D}_1\Big] \|\tilde{z} - \tilde{\tilde{z}}\|_{\mathscr{B}_h''} \\ &\leq \tilde{\mathscr{L}} \|\tilde{z} - \tilde{\tilde{z}}\|_{\mathscr{B}_h''}. \end{aligned}$$

From the assupmtion H(A9*), we observe that $\tilde{\mathscr{L}} < 1$. Hence, $\tilde{\Upsilon}_2$ is Lipschitz continuous.

Next, we prove that $\tilde{\Upsilon}_3$ maps bounded sets into equicontinuous sets of $\mathscr{B}_h''$.

Let $0 < \tau_1 < \tau_2 \leq t_1$. For each $u \in \mathscr{B}_{c'}$, we have

$$\begin{aligned} &\Big\|(\tilde{\Upsilon}_3 \tilde{z})(\tau_2) - (\tilde{\Upsilon}_3 \tilde{z})(\tau_1)\Big\|_{\mathfrak{X}} \\ \leq &\Big\| \int_0^{\tau_2} (\tau_2 - s)^{r-1} \mathcal{P}_r(\tau_2 - s) \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) ds \\ &- \int_0^{\tau_1} (\tau_1 - s)^{r-1} \mathcal{P}_r(\tau_1 - s) \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_\tau, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) ds \Big\|_{\mathfrak{X}} \\ \leq &\Big\| \int_0^{\tau_1} [(\tau_2 - s)^{r-1} \mathcal{P}_r(\tau_2 - s) - \mathcal{P}_r(\tau_1 - s)] \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) ds \Big\|_{\mathfrak{X}} \\ &+ \Big\| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{r-1} \mathcal{P}_r(\tau_2 - s) \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) ds \Big\|_{\mathfrak{X}} \\ \leq &\int_0^{\tau_1} \Big\|(\tau_2 - s)^{r-1} \mathcal{P}_r(\tau_2 - s) - (\tau_1 - s)^{r-1} \mathcal{P}_r(\tau_1 - s)\Big\|_{\mathcal{L}(\mathfrak{X})} m(s) \Omega(\tilde{z}_s + \tilde{y}_s) ds \end{aligned}$$

$$+\frac{\mathscr{M}_A}{\Gamma(r+1)}\int_{\tau_1}^{\tau_2}(\tau_2-1)^{r-1}m(s)(1+b(\tilde{z}_s+\tilde{y}_s))ds.$$

For any $t\in(t_k,s_k]$, $\tau_1<\tau_2$, $k=1,2,\dots,m$, we have

$$\begin{aligned}\left\|(\tilde{\Upsilon}_3\tilde{z})(\tau_2)-(\tilde{\Upsilon}_3\tilde{z})(\tau_1)\right\|_{\mathcal{X}}=&\|\mathfrak{I}_k(u(\tau_2))-(u(\tau_1))\|\\&+\left\|\mathscr{G}_k(\tau_2,\tilde{z}_{\tau_2}+\tilde{y}_{\tau_2})-\mathscr{G}_k(\tau_1,\tilde{z}_{\tau_1}+\tilde{y}_{\tau_1})\right\|_{\mathcal{X}}.\end{aligned}$$

Similarly, for any $\tau_1,\tau_2\in(s_k,t_{k+1}]$, $\tau_1<\tau_2$, $k=1,2,\dots,m$, we get

$$\begin{aligned}&\left\|(\tilde{\Upsilon}_3\tilde{z})(\tau_2)-(\tilde{\Upsilon}_3\tilde{z})(\tau_1)\right\|_{\mathcal{X}}\\
\le&\left\|\Big[\mathcal{Q}_r(\tau_2-s_k)-\mathcal{Q}_r(\tau_1-s_k)\Big]\mathfrak{I}_k(u(t_k))+\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\right\|_{\mathcal{X}}\\
&+\left\|\Big[\mathcal{Q}_r(\tau_2-s_k)-\mathcal{Q}_r(\tau_1-s_k)\Big]\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\right.\\
&\left.\times\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\right\|_{\mathcal{X}}\\
&+\left\|\int_0^{\tau_1}(\tau_1-s)^{r-1}[\mathcal{P}_r(\tau_2-s)-(\tau_1-s)]\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\right\|_{\mathcal{X}}\\
&+\left\|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{r-1}\mathcal{P}_r(\tau_2-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\right\|_{\mathcal{X}}\\
\le&\|\mathcal{Q}_r(\tau_2-s_k)-\mathcal{Q}_r(\tau_1-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n\Big]+\bar{\mathcal{C}}_i\\
&+\frac{\mathscr{M}_Ar}{\Gamma(r+1)}\|\mathcal{Q}_r(\tau_2-s_k)\mathcal{Q}_r(\tau_1-s_k)\|_{\mathcal{L}(\mathcal{X})}\\
&\times\int_0^{s_k}(s_k-s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds\\
&+\int_0^{\tau_1}(\tau_1-s)^{r-1}\|\mathcal{P}_r(\tau_2-s)\mathcal{P}_r(\tau_1-s)\|_{\mathcal{L}(\mathcal{X})}\\
&\times m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds\\
&\times\frac{\mathscr{M}_Ar}{\Gamma(r+1)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_s+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds.\end{aligned}$$

At the point when $\tau_2\to\tau_1$, the right hand side of the overhead inequality has a tendency to zero, afterwards by H(A6)-H(A8), $\mathcal{P}_r(t),\mathcal{Q}_r(t)$ are uniformly continuous, this demonstrates the equicontinuity.

We end of the step by proving that $\tilde{\Upsilon}_3$ is a $\hbar$-contraction. For any $\tilde{\mathcal{W}}\subset\mathscr{B}_h''$, $\tilde{\mathcal{W}}$ is piecewise equicontinuous since $\mathcal{P}_r(t)$ is equicontinuous. Here $\hbar_{PC}=\sup\{\hbar(\tilde{\mathcal{W}}(t)),$ $t\in[s_k,t_{k+1}]\}$, $k=0,1,2,\dots,m$. Then, for each bounded set $\tilde{\mathcal{W}}\in PC$, from the following H(A4)-H(A6), we have for $t\in[0,t_1]$,

$$\begin{aligned}\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t)=&\hbar\Big(\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big)\\
\le&\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t(t-s)^{r-1}\hbar\Big(\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)\Big)ds\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))\\
&\quad + \int_0^s \zeta(s,\tau)\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))d\tau\Big]ds\\
&\le \frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))+\zeta^*\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds\\
&\le \frac{\mathscr{M}_A(1+\zeta^*)}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)ds\sup_{0\le s\le T}\hbar(\tilde{\mathcal{W}}(s))\\
&\le \frac{\mathscr{M}_A(1+\zeta^*)t_1^r}{\Gamma(r+1)}\hbar(\tilde{\mathcal{W}})\int_0^t \tilde{\eta}(s)ds.
\end{aligned}
$$

For any $t\in(t_k,s_k]$, $k=1,2,\ldots,m$, we have

$$
\begin{aligned}
\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t) &= \hbar\Big(\mathfrak{I}_k(\tilde{\mathcal{W}}_t+\tilde{y}_t)+\mathscr{G}_k(t,\tilde{\mathcal{W}}_t+\tilde{y}_t)\Big)\\
&\le \mathcal{L}_k\sup_{-\infty\le\theta\le 0}\hbar\Big(\tilde{\mathcal{W}}(t+\theta)+\tilde{y}(t+\theta)\Big)+\tilde{\nu}_i\sup_{-\infty<\theta\le 0}\hbar\Big(\tilde{\mathcal{W}}(t+\theta)+\tilde{y}(t+\theta)\Big)\\
&\le (\mathcal{L}_k+\tilde{\nu}_i)\sup_{0\le\tau\le T}\hbar(\tilde{\mathcal{W}}(\tau))\\
&\le (\mathcal{L}_k+\tilde{\nu}_i)\hbar_{PC}(\tilde{\mathcal{W}}).
\end{aligned}
$$

Similarly, for any $t\in(s_k,t_{k+1}]$, $k=1,2,\ldots,m$, we have

$$
\begin{aligned}
\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t) &\le \hbar\Big(\mathcal{Q}_r(t-s_k)\mathfrak{I}_k(\tilde{\mathcal{W}}_t+\tilde{y}_t)\Big)+\mathscr{G}_k(t,\tilde{\mathcal{W}}_t+\tilde{y}_t)\\
&\quad+\hbar\Big(\mathcal{Q}_r(t-s_k)\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\
&\qquad\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big)\\
&\quad+\hbar\Big(\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big)\\
&\le \mathscr{M}_A\hbar\Big(\mathfrak{I}_k(\tilde{\mathcal{W}}_{s_k}+\tilde{y}_{s_k})+\mathscr{G}_k(s_k,\tilde{\mathcal{W}}_{s_k}+\tilde{y}_{s_k})\Big)\\
&\quad+\frac{\mathscr{M}_A^2}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\hbar\Big(\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)\Big)ds\\
&\quad+\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t(t-s)^{r-1}\hbar\Big(\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)\Big)ds\\
&\le \mathscr{M}_A\Big(\mathcal{L}_k\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s_k+\theta)+\tilde{y}(s_k+\theta))\\
&\quad+\tilde{\nu}_i\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s_k+\theta)+\tilde{y}(s_k+\theta))\Big)\\
&\quad+\frac{\mathscr{M}_A^2}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^s \zeta(s,\tau)\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))d\tau\Big]ds\\
&+\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))\\
&+\int_0^s \zeta(s,\tau)\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))d\tau\Big]ds\\
\le&\mathscr{M}_A(\mathcal{L}_k+\tilde{\nu}_i)\sup_{0\le\tau\le T}\hbar(\tilde{\mathcal{W}}(\tau))\\
&+\frac{\mathscr{M}_A^2}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))+\zeta^*\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds\\
&+\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))+\zeta^*\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds\\
\le&\mathscr{M}_A(\mathcal{L}_k+\tilde{\nu}_i)\hbar(\tilde{\mathcal{W}})+\frac{\mathscr{M}_A^2(1+\zeta^*)}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\tilde{\eta}(s)ds\sup_{0\le s\le T}\hbar(\tilde{\mathcal{W}}(s))\\
&+\frac{\mathscr{M}_A(1+\zeta^*)}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)ds\sup_{0\le s\le T}\hbar(\tilde{\mathcal{W}}(s))\\
\le&\mathscr{M}_A(\mathcal{L}_k+\tilde{\nu}_i)\hbar_{PC}(\tilde{\mathcal{W}})+\frac{\mathscr{M}_A^2(1+\zeta^*)(t_{k+1})^r}{\Gamma(r+1)}\hbar_{PC}(\tilde{\mathcal{W}})\int_0^{s_k}\tilde{\eta}(s)ds\\
&+\frac{\mathscr{M}_A(1+\zeta^*)(t_{k+1})^r}{\Gamma(r+1)}\hbar_{PC}(\tilde{\mathcal{W}})\int_0^t\tilde{\eta}(s)ds.
\end{aligned}
$$

Therefore, for all $t\in\mathcal{J}$,

$$
\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t)\le(\mathscr{M}_A+1)\Big((\mathcal{L}_k+\tilde{\nu}_i)+\frac{\mathcal{M}_A(1+\zeta^*)T^r}{\Gamma(r+1)}\int_0^T\tilde{\eta}(s)ds\Big)\hbar_{PC}(\tilde{\mathcal{W}})
$$

and

$$
\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})\le\iota\hbar_{PC}(\tilde{\mathcal{W}})<\hbar_{PC}(\tilde{\mathcal{W}}),
$$

where $\iota=\max_{1\le k\le m}(\mathscr{M}_A+1)\Big((\mathcal{L}_k+\tilde{\nu}_i)+\frac{\mathscr{M}_A(1+\zeta^*)T^r}{\Gamma(r+1)}\int_0^T\tilde{\eta}(s)ds\Big)<1$.

Therefore, $\tilde{\Upsilon}$ is $\hbar$-contraction. By Lemma 2.4, we conclude that $\tilde{\Upsilon}$ has at least one fixed point in $\tilde{y}\in\tilde{\mathcal{W}}\subset\mathscr{B}_h''$. Let $u(t)=\tilde{y}(t)+\tilde{z}(t)$ on $t\in(-\infty,\mathcal{J}]$. Then, u is a fixed point of the operator $\tilde{\Upsilon}$ which is the mild solution of the system (1.1)–(1.3) and the proof of theorem is complete. □

Theorem 3.2. *Assume that the hypotheses* H(A1)-H(A9) *are satisfied, then the system* (1.1)–(1.3) *has atleast one mild solution on* $\mathcal{J}$, *for some*

$$
\mathcal{Z}^*=\Big[\mathscr{M}_A\Big(\mathcal{L}_k+\tilde{\nu}_i\Big)+\frac{2\mathscr{M}_A^2\mathscr{M}_6 r\mathcal{J}^r}{\Gamma(1+r)}\int_0^t\eta(s)ds+\frac{2\mathscr{M}_A\mathscr{M}_6 r\mathcal{J}^r}{\Gamma(1+r)}\int_0^t\eta(s)ds\Big]<1.
$$

Proof. Define the operator $\Upsilon : \mathscr{B}_h^{'} \to \mathscr{B}_h^{'}$ by

$$(\Upsilon u)(t) = \begin{cases} \Phi(t), \quad t \in (-\infty, 0], \\ \mathcal{Q}_r(t)\Big[\Phi(0) + \mathcal{G}(0, \Phi(0))\Big] - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big)ds, \quad t \in [0, t_k], \\ k = 0, 1, 2, \dots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, 3, \dots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big)ds, \quad t \in (s_k, t_{k+1}], \\ k = 1, 2, \dots, m. \end{cases}$$

Here Υ is well defined and shows that the operator Υ satisfied the hypotheses of Lemma 2.4. By applying same techniques as in Theorem 3.1. The proof consists of following steps.

Step 1. We show that there exists some $q > 0$ such that $\tilde{\Upsilon}(\mathscr{B}_q) \subseteq \mathscr{B}_q$. If it is not true, then for each positive number q, there exist a function $\tilde{z}^q(.) \in \mathscr{B}_q$ and some $t \in \mathfrak{I}$ such that $\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| > q$.

Step 2. $\tilde{\Upsilon} : \mathscr{B}_h^{''} \to \mathscr{B}_h^{''}$ is continuous.

Step 3. $(\tilde{\Upsilon}_3 u)$ maps bounded sets into equicontinuous sets of $\mathscr{B}_h^{''}$.

Step 4. Mönch's condition holds.

Suppose that $\tilde{\mathcal{W}} \subseteq \mathscr{B}_h^{''}$ is countable and $\tilde{\mathcal{W}} \subseteq \mathrm{conv}(\{0\} \cup \tilde{\Upsilon}_3(\tilde{\mathcal{W}}))$. We show that $\hbar(\tilde{\mathcal{W}}) = 0$, where $\hbar$ is the Hausdorff measure of noncompactness. Without loss of generality, we may suppose that $\tilde{\mathcal{W}} = \{u_n\}_{n=1}^{+\infty}$. We can easily verify that $\tilde{\mathcal{W}}$ is bounded and equicontinuous.

Now we need to show that $\tilde{\Upsilon}_3(\tilde{\mathcal{W}}(t))$ is relatively compact in $\mathcal{X}$ for each $t \in \mathfrak{I}$.

Case 1. For each $t \in [0, t_1]$, we get

$$\begin{aligned} &\hbar\Big(\{\tilde{\Upsilon}_3\tilde{\mathcal{W}}(t)\}_{n=1}^{+\infty}\Big) \\ \leq &\hbar\bigg(\bigg\{\int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s) \\ &\times \mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)ds\bigg\}_{n=1}^{+\infty}\bigg) \\ \leq &\frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\hbar\bigg(\bigg\{\mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)\bigg\}_{n=1}^{+\infty}\bigg)ds \\ \leq &\frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\eta(s)\bigg[\sup_{-\infty<\theta\leq 0}\hbar\bigg(\Big\{\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta)\Big\}_{n=1}^{+\infty}\bigg) \\ &+ \hbar\bigg(\bigg\{\int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\bigg\}_{n=1}^{+\infty}\bigg)\bigg]\, ds \\ \leq &\bigg[\frac{2\mathscr{M}_A r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\bigg]\sup_{-\infty<\theta\leq 0}\hbar(\tilde{\mathcal{W}}(\tau))ds \end{aligned}$$

$$\leq \frac{\mathscr{M}_A t_1^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\ _{\hbar_{PC}}(\tilde{\mathcal{W}}).$$

Case 2. For any $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}\hbar\Big(\{\tilde{\Upsilon}_3\tilde{\mathcal{W}}(t)\}_{n=1}^{+\infty}\Big) =&\hbar\Big(\mathfrak{I}_k(\tilde{\mathcal{W}}_t + \tilde{y}_t) + \mathscr{G}_k(t, \tilde{\mathcal{W}}_t + \tilde{y}_t)\Big)\\ \leq&\mathcal{L}_k \sup_{-\infty\leq\theta\leq 0} \hbar(\tilde{\mathcal{W}}(t+\theta) + \tilde{y}(t+\theta))\\ &+ \tilde{\nu}_i \sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(t+\theta) + \tilde{y}(t+\theta))\\ \leq&(\mathcal{L}_k + \tilde{\nu}_i) \sup_{0\leq\tau\leq T} \hbar(\tilde{\mathcal{W}}(\tau)) \leq (\mathcal{L}_k + \tilde{\nu}_i)\hbar_{PC}(\tilde{\mathcal{W}}).\end{aligned}$$

Case 3. Now, for every $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}\hbar\Big(\{\tilde{\Upsilon}_3\tilde{\mathcal{W}}(t)\}_{n=1}^{+\infty}\Big) \leq&\hbar(\mathcal{Q}_r(t-s_k)\mathfrak{I}_k(u_{s_k}) + \mathscr{G}_k(s_k, \tilde{\mathcal{W}}_{s_k} + \tilde{y}_{s_k}))\\ &+ \hbar\Big(\mathcal{Q}_r(t-s_k)\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\ &\times \mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)ds\Big)\\ &+ \hbar\Big(\int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\\ &\quad \mathscr{F}(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)ds\Big)\\ \leq&\mathscr{M}_A\hbar(\mathfrak{I}_k(u_{s_k}) + \mathscr{G}_k(s_k, \tilde{\mathcal{W}}_{s_k} + \tilde{y}_{s_k})) + \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)}\int_0^{s_k}(s_k-s)^{r-1}\\ &\times \hbar\Big(\mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)\Big)ds\\ &+ \frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\\ &\times \hbar\Big(\mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)\Big)ds\\ \leq&\mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i \sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s_k+\theta) + \tilde{y}(s_k+\theta))\Big)\\ &+ \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)}\int_0^{s_k}(s_k-s)^{r-1}\eta(s)\Big[\sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))\\ &+ \int_0^s \zeta(s,\tau) \sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))d\tau\Big]ds\\ &+ \frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\eta(s)\Big[\sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))\\ &+ \int_0^s \zeta(s,\tau) \sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))d\tau\Big]ds\end{aligned}$$

$$
\begin{aligned}
&\le \mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i \sup_{-\infty \le t \le \mathfrak{T}} \hbar(\tilde{\mathcal{W}}(\tau))\Big) + \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)} \int_0^{s_k} (s_k - s)^{r-1} \\
&\quad \times \eta(s)\Big[\sup_{0\le\tau\le s} \hbar(\tilde{\mathcal{W}}(\tau)) + \zeta^* \sup_{0\le\tau\le s} \hbar(\tilde{\mathcal{W}}(\tau))\Big] ds \\
&\quad + \frac{2\mathscr{M}_A r}{\Gamma(1+r)} \int_0^{t} (t-s)^{r-1} \\
&\quad \times \eta(s)\Big[\sup_{0\le\tau\le s} \hbar(\tilde{\mathcal{W}}(\tau)) + \zeta^* \sup_{0\le\tau\le s} \hbar(\tilde{\mathcal{W}}(\tau))\Big] ds \\
&\le \mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i \hbar_{PC}(\tilde{\mathcal{W}})\Big) + \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)} (1+2\zeta^*)\mathscr{M}_6 \\
&\quad \times \int_0^{s_k} (s_k - s)^{r-1} \eta(s) ds \sup_{0\le s\le \mathfrak{T}} \hbar(\tilde{\mathcal{W}}(s)) \\
&\quad + \frac{2\mathscr{M}_A r}{\Gamma(1+r)} (1+2\zeta^*)\mathscr{M}_6 \int_0^t (t-s)^{r-1}\eta(s) ds \sup_{0\le s\le \mathfrak{T}} \hbar(\tilde{\mathcal{W}}(s)) \\
&\le \mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i \hbar_{PC}(\tilde{\mathcal{W}})\Big) \\
&\quad + \frac{2\mathscr{M}_A^2 r (t_{k+1})^r}{\Gamma(1+r)} (1+2\zeta^*)\mathscr{M}_6 \hbar_{PC}(\tilde{\mathcal{W}}) \int_0^{s_k} \eta(s) ds \\
&\quad + \frac{2\mathscr{M}_A r (t_{k+1})^r}{\Gamma(1+r)} (1+2\zeta^*)\mathscr{M}_6 \hbar_{PC}(\tilde{\mathcal{W}}) \int_0^{t} \eta(s) ds. \\
&\le \mathscr{M}_A\Big[\Big(\mathcal{L}_k + \tilde{\nu}_i\Big) + \Big(\frac{2\mathscr{M}_A r \mathfrak{T}^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6 \\
&\quad + \frac{2 r \mathfrak{T}^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\Big) \int_0^{\mathfrak{T}} \eta(s) ds\Big] \hbar_{PC}(\tilde{\mathcal{W}}).
\end{aligned}
$$

Along these lines, for all $t \in \mathfrak{I}$, we get

$$
\begin{aligned}
\hbar_{PC}(\tilde{\Upsilon}\tilde{\mathcal{W}})(t) \le \Big[&\mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i\Big) + \frac{2\mathscr{M}_A^2 \mathscr{M}_6 r \mathfrak{T}^r}{\Gamma(1+r)} \int_0^t \eta(s) ds \\
&+ \frac{2\mathscr{M}_A \mathscr{M}_6 r \mathfrak{T}^r}{\Gamma(1+r)} \int_0^t \eta(s) ds\Big] \hbar_{PC}(\tilde{\mathcal{W}}(\tau)),
\end{aligned}
$$

which implies, by Lemma 2.2, $\hbar_{PC}(\tilde{\Upsilon}(\tilde{\mathcal{W}})) \le \mathcal{Z}^* \hbar(\tilde{\mathcal{W}})$, where $\mathcal{Z}^*$ is defined in condition (3.2). Thus, from Mönch's condition, we get

$$
\hbar_{PC}(\tilde{\mathcal{W}}) \le \hbar_{PC}\Big(\mathrm{conv}(\{0\} \cup (\tilde{\Upsilon}(\tilde{\mathcal{W}})))\Big) = \hbar_{PC}(\tilde{\Upsilon}(\mathcal{W})) \le \mathcal{Z}^* \hbar_{PC}(\tilde{\mathcal{W}}),
$$

which implies that $\hbar_{PC}(\tilde{\mathcal{W}}) = 0$.

Hence, using Lemma 2.4, $\tilde{\Upsilon}$ has a fixed point $\tilde{y}$ in $\mathscr{B}_q$. Then, $u = \tilde{y} + \tilde{z}$ is a mild solution of system (1.1)–(1.3). This completes the proof. $\square$

4. Controllability Results

We consider the Controllability of Fractional Neutral Integro-Differential Equation and Non-Instantaneous impulses with infinite delay of the form

$$
\begin{aligned}
&(4.1) \quad {}^C D_t^r[u(t) - \mathcal{G}(t,u_t)] = \mathfrak{A}u(t) + \mathscr{F}\Big(t, u_t, \int_0^t \mathcal{H}(t,s,u_s)ds\Big) + \mathfrak{B}u(t),\\
&\qquad t \in (s_k, t_{k+1}], \quad k = 0,1,2,\ldots,m,\\
&(4.2) \quad u(t) = \mathfrak{I}(u(t_k)) + \mathscr{G}_k(t,u_t), \quad t \in (s_k, t_k], \quad k = 1,2,\ldots,m\\
&(4.3) \quad u(t) = \Phi(t), \quad t \in (-\infty, 0],
\end{aligned}
$$

where ${}^C D_t^r$ denotes the Caputo derivative with $r \in (0,1)$. The control function $u(.)$ is given by $L^2(\mathcal{J}, U)$, a Banach space of admissible control function, with U as a Banach space. $\mathfrak{B}$ is a bounded linear operator from U into $\mathcal{X}$. $u_t : (-\infty, 0] \to \mathcal{X}$, defined by $u_t(s) = u(t+s)$, belongs to some abstract phase space $\mathscr{B}_h$. $\mathcal{G}, \mathscr{F}$, $k = 0,1,2,\ldots,m$, $\mathscr{G}_k$, $k = 1,2,\ldots,m$ are appropriate function $0 = s_0 < t_1 < t_2 < \cdots < t_m < b$ are fixed number and $\Delta u(t_k) = \mathfrak{I}(u(t_k)) = u(t_k^+) - u(t_k^-)$. Let $u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of u at $t = t_k$.

Definition 4.1. A function $u : (-\infty, b] \to \mathcal{X}$ is called a mild solution of the control system (4.1)–(4.3) if $u_0 = \Phi \in \mathscr{B}_h$ on $(-\infty, 0]$ and the integral equation

$$
u(t) = \begin{cases}
\mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0,\Phi(0))] - \mathcal{G}(t,u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,u_s)ds\\
+ \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u(s)\Big]ds,\\
t \in [0,t_1], \quad k = 0,1,\ldots,m,\\
\mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t,u_t), \quad t \in (t_k, s_k], \quad k = 1,2,\ldots,m,\\
\mathcal{Q}_r(t-s_k)\mathscr{D}_k - \mathcal{G}(t,u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,u_s)ds\\
+ \int_0^t (t-s)^{r-1}\mathcal{P}_t(t-s)\Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u(s)\Big]ds,\\
t \in (s_k, t_{k+1}], k = 1,2,\ldots,m,
\end{cases}
$$

where

$$
\begin{aligned}
(4.4) \quad \mathscr{D}_k =& (\mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, u_{s_k})) - \int_0^{s_k} (s_k - s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k - s)\mathcal{G}(s, u(s_k))ds\\
&- \int_0^{s_k} (s_k - s)^{r-1}\mathcal{P}_r(s_k - s)\Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u(s)\Big]ds,\\
&k = 1,2,\ldots,m.
\end{aligned}
$$

Definition 4.2. System (4.1)–(4.3) is said to be controllable on $\mathcal{J}$ if for every continuous inital function $\Phi \in \mathscr{B}_h$, $u_1 \in \mathcal{X}$, there exists a control $u \in L^2(\mathcal{J}, U)$ such that the mild solution $u(t)$ of satisfies $u(b) = u_1$.

For the study of the system (4.1)–(4.3), we introduce the following assumption.

H(A10): (i) The linear operator $\mathfrak{B} : L^2(\mathcal{J}, U) \to L^1(\mathcal{J}, U)$ is bounded, $\Xi : L^2(\mathcal{J}, U) \to \mathcal{X}$ is defined by

$$\Xi\ u = \int_0^b (b-s)^{r-1} \mathcal{P}_r(t-s) \mathfrak{B} u(s) ds,$$

has an inverse operator Ξ^{-1} which takes values in $L^2(\mathcal{J}, U)/Ker\ \Xi$ and there exist two constants $\mathscr{M}_2, \mathscr{M}_3 > 0$ such that $\|\mathfrak{B}\| \le \mathscr{M}_2$ and $\|\Xi^{-1}\| \le \mathscr{M}_3$.

(ii) There exist a constant $q_0 \in (0, q)$ and $\mathcal{K}_\Xi \in L^{\frac{1}{q_0}}(\mathcal{J}, \mathbb{R}^+)$ such that for any bounded subset $\mathfrak{Q} \subset \mathcal{X}$, $\hbar((\Xi^{-1}\mathfrak{Q})(t)) \le \mathcal{K}_\Xi(t) \hbar(\mathfrak{Q})$.

The result is based on Mönch's fixed point theorem.

Theorem 4.1. *Assume that hypotheses* H(A1)-H(A8), H(A10) *are satisfied. Then, the system* (4.1)–(4.3) *is controllable on* $\mathcal{J}$, *provided by*

$$\left(\left[1 + \frac{2\mathscr{M}_A \mathscr{M}_2 \mathscr{M}_5 r}{\Gamma(1+r)} \right] (1+\zeta^*) \frac{2\mathscr{M}_A \mathscr{M}_6 r}{\Gamma(1+r)} + (\mathcal{L}_k + \tilde{\nu}) \right) < 1.$$

Proof. Using hypotheses H(A10), for an arbitrary function $u(\cdot) \in C$, we define the control $\widehat{X}_u(t)$ by

$$u_x(t) = \begin{cases} \Xi^{-1}\Bigg[\Bigg[u_b - \mathcal{Q}_r(b)\left(\Phi(0) + \mathcal{G}(0, \Phi(0))\right) - \mathcal{G}(b, u_b) \\ + \int_0^b (b-s)^{r-1} \mathfrak{A} \mathcal{P}_r(b-s) \mathcal{G}(s, u_s) ds \\ - \int_0^b (b-s)^{r-1} \mathcal{P}_r(b-s) \Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\Big) ds\Big]\Bigg]\Bigg](t), \\ t \in [0, t_1], \quad k = 0, 1, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ u_b - \mathcal{Q}_r(b - s_k)\mathscr{D}_k - \mathcal{G}(b, u_b) + \int_0^b (b-s)^{r-1} \mathfrak{A} \mathcal{P}_r(b-s) \mathcal{G}(s, u_s) ds \\ + \int_{s_k}^b (b-s)^{r-1} \mathcal{P}_r(b-s) \Big[\mathscr{F}\Big(s, x_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\Big) + \mathfrak{B} u_x(s)\Big] ds\Big], \\ t \in (s_k, t_{k+1}], \quad k = 1, 2, \ldots, m. \end{cases}$$

We show that, using this control, the operator $\Upsilon : \mathscr{B}_h' \to \mathscr{B}_h'$ defined by

$$\Upsilon u(t) = \begin{cases} \Phi(t), \quad t \in -(\infty, 0], \\ \mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0, \Phi(0))] - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1} \mathfrak{A} \mathcal{P}_r(t-s) \mathcal{G}(s, u_s) ds \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\Big) + \mathfrak{B} u_x(s)\Big] ds, \\ t \in [0, t_1], \quad k = 0, 1, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1} \mathfrak{A} \mathcal{P}_r(t-s) \mathcal{G}(s, u_s) ds \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\Big) + \mathfrak{B} u_x(s)\Big] ds, \\ t \in (s_k, t_{k+1}], \quad k = 1, 2, \ldots, m, \end{cases}$$

has a fixed points. The fixed point is then a solution of the given system. Clearly, $\Phi u(b) = u_1$, which implies the fractional system is controllable on $\mathcal{J}$.

Let $\Phi \in \mathscr{B}_h$, we define $\tilde{y}$ by

$$\tilde{y}(t) = \begin{cases} \Phi(t), & t \in (-\infty, 0], \\ \mathcal{Q}_r(t)\Phi(0), & t \in \mathcal{J}, \end{cases}$$

then $u(t) = \tilde{y}(t) + \tilde{z}(t)$, $t \in \mathcal{J}$. It is easy to see that u satisfies (4.4) if and only if $\tilde{z}$ satisfies $\tilde{z}_0 = 0$ and

$$\tilde{z}(t) = \begin{cases} 0, \quad t \leq 0, \\ \mathcal{Q}_r(t)\mathcal{G}(0,\Phi) - \mathcal{G}(t, \tilde{y}_t + \tilde{z}_t) - \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{y}_s + \tilde{z}_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s, \tilde{y}_s + \tilde{z}_s, \mathcal{H}(s,\tau,\tilde{y}_\tau + \tilde{z}_\tau)d\tau\Big) + \mathfrak{B}u_y(s)\Big]ds, \\ t \in [0, t_1], \quad k = 0, 1, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, \tilde{y}_t + \tilde{z}_t), t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, \tilde{y}_t + \tilde{z}_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{y}_s + \tilde{z}_s)ds \\ + \int_{s_k}^t (t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s, x_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u_y(s)\Big]ds, \\ t \in (s_k, t_{k+1}], \quad k = 1, 2, \ldots, m, \end{cases}$$

where

$$\begin{aligned} u_y(s) =& \Xi^{-1}\Big[u_1 - \mathcal{Q}_r(b)[\Phi(0) - \mathcal{G}(0, \Phi(0))] - \mathcal{G}(s, \tilde{y}_s + \tilde{z}_s) + \mathfrak{I}(u(t_k)) + \mathscr{G}_k(s, \tilde{y}_s + \tilde{z}_s) \\ &+ \int_0^b (b-s)^{r-1}\mathfrak{A}\mathcal{P}_r(b-s)\mathcal{G}(s, \tilde{y}_s + \tilde{z}_s)ds \\ &+ \int_0^b (b-s)^{r-1}\mathcal{P}_r(b-s)\mathscr{F}\Big(s, \tilde{y}_s + \tilde{z}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{y}_\tau + \tilde{z}_\tau)d\tau\Big)ds\Big](t). \end{aligned}$$

Let $\mathscr{B}_h^{''} = \{\tilde{z} \in \mathscr{B}_h^{'} : \tilde{z}_0 = 0 \in \mathscr{B}_h\}$. For any $\tilde{z} \in \mathscr{B}_h^{''}$,

$$\|\tilde{z}\|_b = \|\tilde{z}_0\|_{\mathscr{B}_h} + \sup\{\|\tilde{z}(s)\| : 0 \leq s \leq b\} = \sup\{\|\tilde{z}(s)\| : 0 \leq s \leq b\},$$

thus $(\mathscr{B}_h^{''}, \|\cdot\|_b)$ is a Banach space. Set $\mathscr{B}_q = \{\tilde{z} \in \mathscr{B}_h^{''} : \|\tilde{z}\|_b \leq q\}$ for some $q > 0$, $\mathscr{B}_q \subseteq \mathscr{B}_h^{''}$ is uniformly bounded, and for every $\tilde{z} \in \mathscr{B}_q$.

Define the operator $\tilde{\Upsilon} : \mathscr{B}_h'' \to \mathscr{B}_h''$ by

$$\tilde{\Upsilon}\tilde{z}(t) = \begin{cases} \Phi(t), \quad t \in -(\infty, 0], \\ \mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0, \Phi(0))] - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1} \mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u_y(s)\Big]ds, \\ t \in [0, t_1], \quad k = 0, 1, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m \\ \mathcal{Q}_r(t-s_k)\mathscr{D}_k - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u_y(s)\Big]ds, \\ t \in (s_k, t_{k+1}], \quad k = 1, 2, \ldots, m. \end{cases}$$

Thus, the operator Υ has a fixed point is equivalent to $\tilde{\Upsilon}$ has one. So our goal is to show that $\tilde{\Upsilon}$ has a fixed point and the proof is given in the following steps.

Step 1. There exists $q > 0$ such that $\tilde{\Upsilon}(\mathscr{B}_q) \subseteq \mathscr{B}_q$. If it is not true, then for each positive number q, there exists a function $\tilde{z}^q(\cdot) \in \mathscr{B}_q$ and some $t \in \mathfrak{J}$ such that $\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| > q$ for some $t \in \mathfrak{J}$.

Then by hypotheses H(A4) (iii), H(A5) (iii), H(A6) (ii), H(A10) (ii) and Lemma 2.1 (1), we have

$$\begin{aligned} q <& \|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \le \|\mathcal{Q}_r(t)\|_{\mathcal{L}(\mathcal{X})}\|\mathcal{G}(0,\Phi)\|_{\mathcal{X}} + \|\mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)\|_{\mathcal{X}} \\ &+ \Big\| \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)ds \Big\|_{\mathcal{X}} \\ &+ \Big\| \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_{y^q}(s)\Big]ds \Big\| \\ \le& \mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2] + \mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{M}_0\mathscr{C}_2 \\ &+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2)\frac{t_1^{r\beta}}{\beta} \\ &+ \frac{\mathscr{M}_A t_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathfrak{J}} m(s) \\ &+ \frac{\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\|u_{y^q}\|_{L^2} \\ =& \mathcal{J}_1, \end{aligned}$$

where

$$\begin{aligned} \|u_{y^q}\|_{L^2} = \mathscr{M}_3\Big[&\|u_1\| + \mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2] + \mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{M}_0\mathscr{C}_2 \\ &+ \mathcal{L}_k + \mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n] + \bar{\mathcal{C}}_i \\ &+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2)\frac{t_1^{r\beta}}{\beta} \end{aligned}$$

$$+\frac{\mathscr{M}_A\mathscr{M}_4 t_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{I}} m(s)\Big].$$

For any $t\in(t_k,s_k]$, $k=1,2,\dots,m$, we have

$$\begin{aligned}q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| &\leq \|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}\\ &\leq \mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i\\ &=\mathcal{J}_2,\end{aligned}$$

and for $t\in(s_k,t_{k+1}]$, $k=1,2,\dots,m$,

$$\begin{aligned}q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq&\|\mathcal{Q}_r(t-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|+\|\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\ &+\|\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\ &+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|\\ &+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\ &\times\Big[\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)+\mathfrak{B}_{u_y^q}(s)\Big]ds\Big\|_{\mathcal{X}}\Big]\\ &+\|\mathscr{G}(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}+\Big\|\int_0^t(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}\\ &+\Big\|\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)\\ &+\mathfrak{B}_{u_y^q}(s)\Big]ds\Big\|_{\mathcal{X}}\\ \leq&\mathscr{M}_A\Bigg[\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n]+\bar{\mathcal{C}}_i+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\ &+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{C}_2)\frac{(s_k)^{r\beta}}{\beta}\\ &+\frac{\mathscr{M}_A\mathscr{M}_4(s_k)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{I}} m(s)\\ &+\frac{\mathscr{M}_A\mathscr{M}_2(s_k^r)}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\mathscr{M}_3\Big[\|u_1\|+\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]\\ &+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2+\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i\\ &+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{t_1^{r\beta}}{\beta}\\ &+\frac{\mathscr{M}_A\mathscr{M}_4 t_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{I}} m(s)\Big]\Bigg]\\ &+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{(t_{k+1})^{r\beta}}{\beta}\\
&+ \frac{\mathscr{M}_A\mathscr{M}_4(t_{k+1})^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s)\\
&+ \frac{\mathscr{M}_A\mathscr{M}_2(s_k^r)}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\mathscr{M}_3\Big[\|u_1\|+\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]\\
&+ \mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2+\mathcal{L}_k+\mathfrak{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathfrak{C}}_i\\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{t_1^{r\beta}}{\beta}\\
&+ \frac{\mathscr{M}_A\mathscr{M}_4t_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s)\Big].
\end{aligned}
$$

Then, for all $t\in\mathfrak{I}$, we find that

$$
\begin{aligned}
\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \le& \mathscr{E}^* + \left(1+\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\right)\\
&\times\left[\left[(\mathcal{L}_k+\mathfrak{C}_i+\mathscr{M}_0\mathscr{C}_1)+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\right](\mathscr{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\right.\\
&\left.+\frac{\mathscr{M}_A\mathscr{M}_4(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega\Big(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\right]\\
&\times\sup_{t\in\mathfrak{I}} m(s)\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{E}^* =& \max_{1\le k\le m}\left\{\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]+\left(\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{2r-1}{2r-1}}+1\right)\right.\\
&\left.\times\left(\mathscr{M}_0\mathscr{C}_2+\tilde{C}_i+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_2\frac{T^{r\beta}}{\beta}\right)\right\}\\
&+\frac{\mathscr{M}_A T^r}{\Gamma(r+1)}\Omega\Big(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\Big]\sup_{t\in\mathfrak{I}} m(s).
\end{aligned}
$$

Combining the above equation

$$
\begin{aligned}
q <&\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\\
\le&\mathscr{E}^*+\left(1+\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\right)\left[\left[(\mathcal{L}_k+\mathfrak{C}_i+\mathscr{M}_0\mathscr{C}_1)+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\right]\right.\\
&\times(\mathscr{D}_1\|\tilde{z}\|_t+\tilde{c}_n)
\end{aligned}
$$

$$+\frac{\mathscr{M}_A\mathscr{M}_4(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega\Big(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\Bigg]$$
$$\times\sup_{t\in\mathfrak{I}}m(s)\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}.$$

Now dividing on both sides by q and taking the limit as $q\to\infty$, we get

$$\begin{aligned}
1\leq&\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\\
\leq&\mathscr{E}^*+\left(1+\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\right)\\
&\times\left[\left[(\mathcal{L}_k+\mathfrak{C}_i+\mathscr{M}_0\mathscr{C}_1)+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\right](\mathscr{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\right.\\
&\left.+\frac{\mathscr{M}_A\mathscr{M}_4(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega\Big(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\right]\sup_{t\in\mathfrak{I}}m(s)\\
&\times\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}.
\end{aligned}$$

We get $1\leq 0$. This is contradiction. Hence, for some integer $\tilde{\Upsilon}(\mathscr{B}_q)\subseteq\mathscr{B}_q$.

Step 2. $\tilde{\Upsilon}:\mathscr{B}_h''\to\mathscr{B}_h''$ is continuous,

$$\begin{aligned}
F_n(s)&=\mathscr{F}\left(s,\tilde{z}_s^{(n)}+\tilde{c}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau^{(n)}+\tilde{y}_\tau)d\tau\right),\\
F(s)&=\mathscr{F}\left(s,\tilde{z}_s+\tilde{c}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\right).
\end{aligned}$$

For this purpose let $\{\tilde{z}^{(n)}\}_{n=0}^{+\infty}\subseteq\mathscr{B}_h''$ with $\tilde{z}^{(n)}\to\tilde{z}$ in $\mathscr{B}_h''$. Then there is a number $c'>0$ such that $\|\tilde{z}^{(n)}(t)\|\leq c'$ for all n and a.e. $t\in\mathfrak{I}$, so $\tilde{z}^{(n)}\in\mathscr{B}_{c'}=\{\tilde{z}\in\mathscr{B}_h'':\|\tilde{z}\|_{\mathscr{B}_h''}\leq c'\}\subseteq\mathscr{B}_h''$ and $\tilde{z}\in B_{c'}$. From remark, we have $\|\tilde{z}_t+\tilde{y}_t\|_{\mathscr{B}_h}\leq c''$, $t\in\mathfrak{I}$.

By H(A4), H(A5), Remark $P_{12},P_{13},P_{14},P_{15}$, and Lebesgue's dominated convergence theroem, we obtain, for $t\in[0,t_1]$,

$$\begin{aligned}
&\|(\tilde{\Upsilon}\tilde{z}^n)(t)-(\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}}\\
\leq&\mathscr{M}_0\left\|(\mathfrak{A})^\beta\Big[\mathscr{G}(t,\tilde{z}_t^n+\tilde{y}_t)-\mathscr{G}(t,\tilde{z}_t+\tilde{y}_t)\Big]\right\|_{\mathcal{X}}+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{(t_1)^{r\beta}}{\beta}\\
&\times\int_0^t(t-s)^{r-1}\left\|(\mathfrak{A})^\beta\Big[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\Big]\right\|_{\mathcal{X}}ds\\
&+\frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)}\int_0^t(t-s)^{r-1}\left[\left\|\mathscr{F}\left(s,\tilde{z}_s^n+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau^n+\tilde{y}_\tau)d\tau\right)+\mathfrak{B}u_y^n\right.\right.\\
&\left.\left.-\mathscr{F}\left(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\right)+\mathfrak{B}u_y\right]\right\|_{\mathcal{X}}ds\to0\quad\text{as }n\to+\infty.
\end{aligned}$$

For all $t \in (t_k, s_k]$, $k = 1, 2, \dots, m$, we obtain

$$\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathfrak{X}} = 0.$$

In the same way, for all $t \in (s_k, t_{k+1}]$, $k = 1, 2, \dots, m$, we have

$$\begin{aligned}
&\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathfrak{X}} \\
\leq &\|\mathcal{Q}_r(t - s_k)\|_{\mathcal{L}(\mathfrak{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|_{\mathfrak{X}} + \|\mathscr{G}_k(s_k, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}) - \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\|_{\mathfrak{X}} \\
&+ \Big\|(\mathfrak{A})^{-\beta}\Big\|\Big\|(\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}) - (\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big\|_{\mathfrak{X}} \\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(s_k)^{r\beta}}{\beta}\int_0^{s_k}(s_k - s)^{r-1} \\
&\times \Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s_k, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}) - \mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big]\Big\|_{\mathfrak{X}} ds \\
&+ \frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)}\int_0^{s_k}(s_k - s)^{r-1}\Big\|\mathscr{F}\Big(s, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s, \tau, \tilde{z}^n_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u^n_y \\
&- \mathscr{F}\Big(s, \tilde{z}_{s_k} + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big\|_{\mathfrak{X}} ds\Big] \\
&+ \Big\|(\mathfrak{A})^{-\beta}\Big\|\Big\|(\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}^n_t + \tilde{y}_t) - (\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}_t + \tilde{c}_t)\Big\|_{\mathfrak{X}} \\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(t_{k+1})^{r\beta}}{\beta}\int_0^t \Big[\Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s, \tilde{z}^n_s + \tilde{y}_s) - (\mathfrak{A})^{\beta}\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)\Big]\Big\|_{\mathfrak{X}} \\
&+ \frac{\mathscr{M}_A(t_{k+1})^r}{\Gamma(r+1)}\int_0^t (s_k - s)^{r-1}\Big\|\mathscr{F}\Big(s, \tilde{z}^n_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}^n_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u^n_y \\
&- \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big\|_{\mathfrak{X}}\Big] ds \to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

It is simple to see that

$$\lim_{n\to+\infty}\Big\|(\tilde{\Upsilon}\tilde{z}^n) - (\tilde{\Upsilon}\tilde{z})\Big\|_{\mathscr{B}''_h} = 0.$$

Thus, $\tilde{\Upsilon}$ is continuous.

Step 3. $(\tilde{\Upsilon}_2\tilde{z})$ maps bounded into equicontinuous set of $\mathscr{B}''_h$.

Let $0 < \tau_1 < \tau_2 \leq t_1$. For each $\tilde{z} \in \mathscr{B}''_h$, we have

$$\begin{aligned}
&\Big\|(\tilde{\Upsilon}_2\tilde{z})(\tau_2) - (\tilde{\Upsilon}_2\tilde{z})(\tau_1)\Big\|_{\mathscr{B}''_h} \\
\leq &\Big\|\int_0^{\tau_2}(\tau_2 - s)^{r-1}\mathcal{P}_r(\tau_2 - s)\Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds\Big\| \\
&- \Big\|\int_0^{\tau_1}(\tau_1 - s)^{r-1}\mathcal{P}_r(\tau_1 - s)\Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds\Big\| \\
\leq &\Big\|\int_0^{\tau_1}(\tau_2 - s)^{r-1}\Big[\mathcal{P}_r(\tau_2 - s) - \mathcal{P}_r(\tau_1 - s)\Big]
\end{aligned}$$

$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds$$
$$+ \int_{\tau_1}^{\tau_2} \Big\|(\tau_2 - s)^{r-1}\mathcal{P}_r(\tau_2 - s) - (\tau_1 - s)^{r-1}\mathcal{P}_r(\tau_1 - s)\Big\|$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{c}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds.$$

For all $\tau_1, \tau_2 \in (t_k, s_k]$, $\tau_1 < \tau_2$, $k = 1, 2, \dots, m$, we have

$$\Big\|(\tilde{\Upsilon}_2\tilde{z})(\tau_2) - (\tilde{\Upsilon}_2\tilde{z})(\tau_1)\Big\|_{\mathscr{B}_h''} = \Big\|\mathfrak{I}_k(u(\tau_2) - u(\tau_1))\Big\|$$
$$+ \Big\|\mathscr{G}_k(\tau_2, \tilde{z}_{\tau_2} + \tilde{y}_{\tau_2}) - \mathscr{G}_k(\tau_1, \tilde{z}_{\tau_1} + \tilde{y}_{\tau_1})\Big\| = 0,$$

and for $\tau_2, \tau_1 \in (s_k, t_{k+1}]$, $\tau_1 < \tau_2$, $k = 1, 2, \dots, m$, we get

$$\Big\|(\tilde{\Upsilon}_2\tilde{z})(\tau_2) - (\tilde{\Upsilon}_2\tilde{z})(\tau_1)\Big\|_{\mathscr{B}_h''}$$
$$\leq \Big\|\Big[\mathcal{Q}_r(\tau_2 - s_k) - \mathcal{Q}_r(\tau_1 - s_k)\Big]\mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big\|$$
$$+ \Big\|\Big[\mathcal{Q}_r(\tau_2 - s_k) - \mathcal{Q}_r(\tau_1 - s_k)\Big]\int_0^{s_k}(s_k - s)^{r-1}\mathcal{P}_r(s_k - s)$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)ds + \mathfrak{B}u_y\Big]\Big\|ds$$
$$+ \Big\|\int_0^{\tau_1}(\tau_1 - s)^{r-1}[\mathcal{P}_r(\tau_2 - s) - (\tau_1 - s)]$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)ds + \mathfrak{B}u_y\Big]\Big\|ds$$
$$+ \Big\|\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{r-1}\mathcal{P}_r(\tau_2 - s)$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)ds + \mathfrak{B}u_y\Big]\Big\|ds$$
$$\leq \Big\|\mathcal{Q}_r(\tau_2 - s_k) - \mathcal{Q}_r(\tau_1 - s_k)\Big\|_{\mathcal{L}(\mathcal{X})}\Big[\mathcal{L}_k + \mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n]\Big] + \bar{\mathcal{C}}_i$$
$$+ \frac{\mathscr{M}_A r}{\Gamma(r+1)}\|\mathcal{Q}_r(\tau_2 - s_k)\mathcal{Q}_r(\tau_1 - s_k)\|_{\mathcal{L}(\mathcal{X})}$$
$$\times \int_0^{s_k}(s_k - s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{y}_s + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{y}_n))ds$$
$$+ \frac{\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2} + \int_0^{\tau_1}(\tau_1 - s)^{r-1}\|\mathcal{P}_r(\tau_2 - s)\mathcal{P}_r(\tau_1 - s)\|_{\mathcal{L}(\mathcal{X})}$$
$$\times m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_s + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))ds + \frac{\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2}$$
$$+ \int_0^{\tau_1}(\tau_1 - s)^{r-1}\|\mathcal{P}_r(\tau_2 - s)\mathcal{P}_r(\tau_1 - s)\|_{\mathcal{L}(\mathcal{X})}$$

$$\times m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_s + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))ds + \frac{\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2}$$
$$+ \frac{\mathscr{M}_A r}{\Gamma(r+1)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_s + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))ds$$
$$+ \frac{\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2}.$$

At the point when $\tau_2 \to \tau_1$, the right hand side of the above inequality has a tendency to zero. Therefore, $(\tilde{\Upsilon}_2\tilde{z})$ is equicontinuous on $\mathcal{J}$.

Step 4. Mönch's condition holds.

Suppose that $\Xi \subseteq \mathscr{B}_h$ is countable and $\Xi \subseteq \operatorname{conv}(\{0\} \cup \tilde{\Upsilon}_2(\Xi))$. We show that $\hbar(\Xi) = 0$, where $\hbar$ is the Hausdorff MNC. Without loss of generality, we may suppose that $\Xi = \{\tilde{z}_n\}_{n=1}^{\infty}$ we can easily verify that Ξ is bounded and equicontinuous. Now we need to show that $\tilde{\Upsilon}_2(\Xi(t))$ is relatively compact in $\mathcal{X}$ for each $t \in \mathcal{J}$.

Case 1. For each $t \in [0, t_1]$, by Theorem 2.1 and we get

$$\begin{aligned}
\hbar\Big(\{\tilde{\Upsilon}_2\tilde{z}_{y_n}(t)\}_{n=1}^{+\infty}\Big) \leq& \hbar\Big(\int_0^b (b-s)^{r-1}\mathcal{P}_r(b-s)F_n(s)ds\Big)\\
\leq& \mathcal{K}_\Xi(s)\frac{2\mathscr{M}_A r}{\Gamma(r+1)}\int_0^b (b-s)^{r-1}\\
&\times \eta(s)\Big[\sup_{\infty<\theta\leq 0}\hbar\Big(\Big\{\tilde{z}^n(s+\theta) + \tilde{y}(s+\theta)\Big\}_{n=1}^{+\infty}\Big)\\
&+ \hbar\Big(\Big\{\int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau)d\tau\Big\}_{n=1}^{+\infty}\Big)\Big]ds\\
\leq& \mathcal{K}_\Xi(s)\frac{2\mathscr{M}_A r}{\Gamma(r+1)}\int_0^b (b-s)^{r-1}\eta(s)(1+2\zeta^*)\sup_{0\leq\tau\leq s}\hbar(\Xi(\tau))ds.
\end{aligned}$$

This implies that

$$\begin{aligned}
\hbar\Big(\{\Upsilon\tilde{z}_{y_n}(t)\}_{n=1}^{+\infty}\Big) \leq& \hbar\Big(\Big\{\int_0^b (b-s)^{r-1}\mathcal{P}_r(b-s)F_n(s)ds\Big\}_{n=1}^{+\infty}\Big)\\
&+ \hbar\Big(\Big\{\int_0^b (b-s)^{r-1}\mathcal{P}_r(b-s)\mathfrak{B}u_{y^n}(s)ds\Big\}_{n=1}^{+\infty}\Big)\\
\leq& \frac{2\mathscr{M}_A r}{\Gamma(r+1)}\int_0^b (b-s)^{r-1}\eta(s)ds(1+2\zeta^*)\sup_{0\leq\tau\leq s}\hbar(\Xi(\tau))\\
&+ (1+2\zeta^*)\frac{2\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(r+1)}\Big(\int_0^b (b-s)^{r-1}\mathcal{K}_\Xi(s)ds\Big)\\
&\times\Big[\frac{2\mathscr{M}_A r}{\Gamma(r+1)}\int_0^b (b-s)^{r-1}\eta(s)ds\Big]\sup_{0\leq\tau\leq s}\hbar(\Xi(\tau))ds\\
\leq& \Big[\frac{2\mathscr{M}_A t_1^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6
\end{aligned}$$

$$+\frac{2\mathscr{M}_A\mathscr{M}_2t_1^r}{\Gamma(1+r)}\mathscr{M}_5\frac{2\mathscr{M}_At_1^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\Big]\sup_{0\leq\tau\leq s}\hbar(\Xi(\tau))ds.$$

Case 2. For each $t\in(t_k,s_k]$, $k=1,2,\dots,m$, we have

$$\begin{aligned}\hbar\Big(\{\tilde{\Upsilon}_2\tilde{z}_{y_n}(t)\}_{n=1}^{\infty}\Big)=&\hbar\Big(\mathfrak{I}(u(t_k))+\mathscr{G}_k(t,\Xi_t+\tilde{y}_t)\Big)\\ \leq&\mathcal{L}_k\sup_{-\infty<\theta\leq0}\hbar\Big(\Big\{\Xi(t+\theta)+\tilde{y}(t+\theta)\Big\}_{n=1}^{+\infty}\\ &+\tilde{\nu}_i\sup_{-\infty<\theta\leq0}\hbar\Big(\Big\{\Xi(t+\theta)+\tilde{y}(t+\theta)\Big\}_{n=1}^{+\infty}\Big)\\ \leq&\mathcal{L}_k\sup_{0\leq\tau\leq\mathfrak{T}}\hbar(\Xi(\tau))+\tilde{\nu}_i\sup_{0\leq\tau\leq\mathfrak{T}}\hbar(\Xi(\tau))\leq(\mathcal{L}_k+\tilde{\nu}_i)\hbar_{PC}(\Xi).\end{aligned}$$

Case 3. Now, for any $t\in(s_k,t_{k+1}]$, $k=1,2,\dots,m$, we have

$$\begin{aligned}\hbar\Big(\{\tilde{\Upsilon}_2\tilde{z}_{y_n}(t)\}_{n=1}^{\infty}\Big)\leq&\hbar\Big(\mathcal{Q}_r(t-s_k)\mathfrak{I}(u(t_k))+\mathscr{G}_k(s_k,u_t)\Big)\\ &+\hbar\Big[\Big(\mathcal{Q}_r(t-s_k)\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\ &\times\mathscr{F}\Big(s,\Xi_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\Xi_\tau+\tilde{y}_\tau)d\tau+\mathfrak{B}u_y(s)\Big)ds\Big)\\ &+\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\\ &\times\mathscr{F}\Big(s,\Xi_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\Xi_\tau+\tilde{y}_\tau)d\tau+\mathfrak{B}u_y(s)\Big)ds\Big]\\ \leq&\mathcal{L}_k+\tilde{\nu}_i\sup_{0\leq\tau\leq\mathfrak{T}}\hbar(\Xi(\tau))+\Big[\frac{2\mathscr{M}_As_k^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6+\frac{2\mathscr{M}_A\mathscr{M}_2s_k^r}{\Gamma(r+1)}\mathscr{M}_5\\ &\times\frac{2\mathscr{M}_As_k^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6\Big]\sup_{0\leq\tau\leq s}\hbar(\Xi(\tau))\\ \leq&\Big[\mathcal{L}_k+\tilde{\nu}_i\hbar_{PC}(\Xi)+\frac{2\mathscr{M}_As_k^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6+\frac{2\mathscr{M}_A\mathscr{M}_2s_k^r}{\Gamma(r+1)}\mathscr{M}_5\\ &\times\frac{2\mathscr{M}_As_k^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6\Big]\sup_{0\leq\tau\leq s}\hbar(\Xi(\tau)).\end{aligned}$$

Therefore,

$$\hbar\Big(\tilde{\Upsilon}(\Xi)\Big)(t)\leq\Big(\Big[1+\frac{2\mathscr{M}_A\mathscr{M}_2\mathscr{M}_5r}{\Gamma(r+1)}\Big](1+2\zeta^*)\frac{2\mathscr{M}_A\mathscr{M}_6r}{\Gamma(r+1)}+\mathcal{L}_k+\tilde{\nu}_i\Big)\sup_{0\leq\tau\leq s}\hbar(\Xi(\tau)),$$

which implies that Lemma 2.4, $\hbar(\tilde{\Upsilon}(\Xi))\leq\check{\mathfrak{S}}^*\hbar(\Xi)$. Thus, from Mönch's condition, we get

$$\hbar(\Xi)\leq\hbar\Big(\operatorname{conv}\{0\}\cup(\tilde{\Upsilon}(\Xi))\Big)=\hbar(\tilde{\Upsilon}(\Xi))\leq\check{\mathfrak{S}}^*\hbar(\Xi),$$

which implies that $\hbar(\Xi)=0$.

Hence, using Lemma 2.4, $\tilde{\Upsilon}$ has a fixed point $\tilde{y}$ in $\mathscr{B}_h''$. Then $u = \tilde{y} + \tilde{z}$ is a mild solutions of system (4.1)-(4.3) satisfying $u(b) = u_1$. Therefore, system (4.1)–(4.3) is controllable on $\mathfrak{I}$. This completes the proof. ☐

5. Examples

Example 5.1. Now, we consider the space $\mathfrak{X} = \mathcal{L}^2([0,\pi],\mathbb{R})$ and the following fractional neutral partial diffferential equation with infinite delay:

$$(5.1)\quad {}^{C}D_t^r\Big[u(t,x) + \int_{-\infty}^{t} a(\theta,x)u(t,\theta)d\theta\Big]$$
$$=\frac{\partial^2}{\partial x^2}u(t,x) + \mu\Big(t, x_t(\cdot,u), \int_0^t \mu_i(t,s,x_t(\cdot,u))ds\Big), \quad t\in[s_k,t_{k+1}], \quad u\in[0,\pi],$$
$$(5.2)\quad u(t,x) = \mathfrak{I}(u(t,x)) + \mathscr{G}_k(t,u(t,x)), \quad t\in(t_k,s_k], \quad u\in[0,\pi],$$
$$(5.3)\quad u(t,0) = u(t,\pi) = 0, \quad t\in[0,\mathfrak{T}],$$
$$(5.4)\quad u(t,x) = \Phi(t,x), \quad -\infty \le t \le 0, \quad 0\le x\le \pi,$$

where $s_k \in (t_k, t_{k+1}]$, $k = 1,2,\dots,m$, in the partition $0 = t_0 < t_1 < \cdots < t_{m+1} = \mathfrak{T}$ of the interval $[0,\mathfrak{T}]$ with $s_0 = 0$ and u_t indicates the portion of the solution $u(\cdot,\cdot) : (-\infty,\mathfrak{T}]\times[0,\pi] \to \mathfrak{X}$, that is for any $t \ge 0$, $u_t(\cdot,\cdot) : (-\infty,0]\times[0,\pi]\to\mathfrak{X}$ is given by

$$u_t(\theta,x) = u(t+\theta,x), \quad \text{for } \theta\in(-\infty,0].$$

Let $\mathfrak{X} = L^2[0,\pi]$ and define $\mathfrak{A} : \mathfrak{D}(\mathfrak{A}) \subset \mathfrak{X} \to \mathfrak{X}$ by $\mathfrak{A}u = u''$, on

$$\mathfrak{D}(\mathfrak{A}) = \Big\{u\in\mathfrak{X} : \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in \mathfrak{X} \text{ and } u(0) = u(\pi) = 0\Big\}.$$

Then $\mathfrak{A}$ generates a infinitesimal generator of a analytic semigroup $\mathcal{Q}(t)_{t\ge 0}$ on $\mathfrak{X}$ and $\mathcal{Q}(t)$ is not a compact semigroup on $\mathfrak{X}$, with $\hbar(\mathcal{Q}(t)\mathfrak{D}) \le \hbar(\mathfrak{D})$, where $\hbar$ is the Hausdorff measure of noncompactness and there exists a constant an $\mathscr{M}_A \ge 1$ such that $\sup_{t\in\mathscr{I}}\|\mathcal{Q}(t)\| \le \mathscr{M}_A$. Define $f, g : [0,\pi]\times\mathfrak{X}\to\mathfrak{X}$ by

$$\begin{aligned} u(t)x &= u(t,x),\\ g(u)x &= \int_0^\pi a(\theta,x)u(\theta)d\theta,\\ u(t,x) &= \mathfrak{I}(u(t,x)) + \mathscr{G}_k(t,u(t,x)), \quad x\in[0,\pi],\\ f(t,\phi,\int_0^t \mathcal{H}(t,s,\phi)x &= \mu\left(t,\phi(\theta,x),\int_0^t \mu_1(t,s,\phi(\theta,x)ds\right), \quad \theta\in(-\infty,0],\\ \Phi(\theta)(x) &= \Phi(\theta,x), \quad \theta\in(-\infty,0], \quad x\in[0,\pi], \end{aligned}$$

with the following assumptions.

(i) For each $k = 0,1,2,\dots,m$, the function $\mathscr{F}$ is defined above by is continuous and we impose a suitable condition on F to satisfy the hypotheses H(A4)-H(A5).

(ii) For each $k = 1,2,\dots,m$, the function $\mathscr{G}_k$ is defined above by is continuous and we impose a suitable condition on G to satisfy the hypothesis H(A6).

With the above setting the system of equations (5.1)–(5.4) reduces to the system of equations (1.1)–(1.3) satisfying the hypotheses of Theorem 3.1 and hence, ensuring a mild solution on $(-\infty, \mathcal{T}]$.

Example 5.2. We consider the following fractional control impulsive system:

$$
\begin{aligned}
&(5.5) \qquad {}^C D_{0,t}^{\frac{3}{4}}[u(t,y) + \int_{-\infty}^{t} \mu_1(t,y,s)\Phi(s)(y)ds] \\
&\qquad = \frac{\partial^2}{\partial u^2} u(t,y) + \Xi\mu_2(t,y) \\
&(5.6) \qquad + \mu_2\Big(t, \int_{-\infty}^{t} \mu_3(s-t)u(s,y)ds, \int_0^t \int_{-\infty}^{0} \mu_4(s,y,\tau_1 - s)u(\tau_1,y)d\tau_1 ds\Big), \\
&(5.7) \qquad u(t,y) = \Phi(t,y), \quad t \in (-\infty, 0], \quad y \in [0,1], \\
&(5.8) \qquad u(t,y) = \mathfrak{I}(u(t_{\frac{1}{2}}, y)) + g(t, u(t,y)), \quad t \in \left(\frac{1}{2}, \frac{2}{3}\right],
\end{aligned}
$$

where ${}^C D_{0,t}^{\frac{3}{4}}$ is a Caputo fractional derivative of order $\Phi \in \mathscr{B}_h$, $\mu_2 : \mathcal{J} \times [0,1] \times [0,1]$ is continuous in t and Φ is continuous and satisfies certain smoothness conditions.

Let $U = Y = L^2(0,1)$ be endowed with the usual norm $\| \cdot \|_{L^2}$, and Let $\mathfrak{A} : \mathfrak{D}(\mathfrak{A}) \subset \mathcal{X} \to \mathcal{X}$ be defined by $\mathfrak{A}\mathfrak{W} = \mathfrak{W}''; \mathfrak{W} \in \mathfrak{D}(\mathfrak{A})$, where $\mathfrak{D}(\mathfrak{A}) = \{\mathfrak{W} \in \mathcal{X} : \mathcal{W}'' \in \mathcal{X}, \mathfrak{W}(0) = \mathfrak{W}(1) = 0\}$. It is well know that $\mathfrak{A}$ is an infinitesimal generator of a semigroup that $\{\mathcal{Q}_r(t) : t \geq 0\}$ in $\mathcal{X}$ and is given by $\mathcal{Q}_r(t)\mathfrak{W}(s) = \mathfrak{W}(t+s)$ for $\mathfrak{W} \in \mathcal{X}$. $\mathcal{Q}_r(t)$ is not a compact semigroup on $\mathcal{X}$ with $\hbar(\mathcal{Q}_r(t)\mathfrak{D}) \leq \hbar(\mathfrak{D})$, where $\hbar$ is the Hausdorff measure of noncompactness, and there exists $\mathscr{M}_A \leq 1$ such that $\sup_{t \in \mathcal{J}} \|\mathcal{Q}_r(t)\| \leq \mathscr{M}_A$. For the phase space, we choose $h = e^{2s}$, $s < 0$, then $l = \int_{\infty}^{0} h(s)ds = \frac{1}{2} < +\infty$ for $t \leq 0$, and we determine

$$\|\phi\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|\phi(\theta)\| ds.$$

Hence, for $(t, \phi) \in [0, \mathcal{T}] \times \mathscr{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \phi]$. Moreover, $t \to \mathfrak{W}(t^{\frac{3}{4}}\theta + s)x$ is equicontuinuous for $t \geq 0$ and $\theta \in (0, +\infty)$.

Define

$$
\begin{aligned}
{}^C D_{0,t}^{\frac{3}{4}} u(t)(y) &= \frac{\partial^{\frac{3}{4}}}{\partial t^{\frac{3}{4}}} u(t,y), \\
u(t)(y) &= u(t,y), \\
\mathcal{G}(t,\Phi)(y) &= \int_{-\infty}^{t} \mu_4(t,y,s)\Phi(s)(y)ds, \\
\mathscr{F}\Big(t, \Phi, \int_0^t \mathcal{H}(s,\Phi)ds\Big)(y) &= \mu_2\Big(t, \int_{-\infty}^{0} \mu_3(s)\Phi(s)(y)ds, \int_0^t \mathcal{H}(s,\Phi)(y)ds\Big).
\end{aligned}
$$

Let $\mathfrak{B} : \mathcal{X} \to \mathcal{X}$ be defined by

$$(\mathfrak{B}u)(t)(y) = \Xi\mu_2(t,y), \quad 0 < y < 1,$$

with this choice of $\mathfrak{A}, \mathfrak{B}$ and $\mathscr{F}$, system (5.6) can be rewritten as

$$\begin{aligned}
{}^{C}D_t^r[u(t)+\mathcal{G}(t,u_t)] =& \mathfrak{A}u(t)+\mathscr{F}\Big(t,u_t,\int_0^t \mathcal{H}(t,s,u_s)ds\Big)+\mathfrak{B}u(t), \quad t\in(s_k,t_{k+1}],\\
k =& 0,1,2,\dots,m,\\
u(t) =& \mathfrak{I}(u(t_k))+\mathscr{G}(t,u_t), \quad t\in(t_k,s_k],\\
u_0 =& \Phi\in\mathscr{B}_h, \quad (-\infty,0], \quad k=1,2,\dots,m.
\end{aligned}$$

For $y\in(0,1)$, the linear operator Ξ is given by

$$(\Xi u)(y)=\int_0^1(1-s)^{\frac{-1}{4}}\mathcal{P}_r(1-s)\mathfrak{W}\mu_2(s,y)ds,$$

where

$$\begin{aligned}
\mathcal{P}_r(t)\mathfrak{W}(s) &= \frac{3}{4}\int_0^\theta \theta\eta_{\frac{3}{4}}(\theta)\mathfrak{W}(t^{\frac{3}{4}}\theta+s)d\theta,\\
\wp_{\frac{3}{4}}(\theta) &= \frac{4}{3}\theta^{\frac{-7}{3}}\bar{\mathfrak{W}}_{\frac{3}{4}}(\theta^{\frac{-4}{3}}),\\
\bar{\mathfrak{W}}_{\frac{3}{4}}(\theta) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{\frac{-3n+4}{4}}\frac{\Gamma(\frac{3n+4}{4})}{n!}\sin\Big(\frac{3n\pi}{4}\Big), \quad \theta\in(0,+\infty).
\end{aligned}$$

Thus, under appropriate conditions on the functions $\mathscr{F}, \mathcal{G}, \mathscr{G}_k$ and $\mathfrak{I}_k$ as those in H(A1)-H(A9). We assume that Ξ satisfies H(A10), then all the conditions of Theorem 4.1 are satisfied. Hence, the system (5.5)–(5.8) is controllable on $\mathfrak{J}$.

6. Conclusion

In this paper, we have studied the existence, uniqueness and controllability results for fractional neutral integro-differential equation and non-instantaneous impulses with delay involving the Caputo derivatives in a Banach space. More precisely, some appropriate assumption, by utilizing the ideas and techniques of sectorial operator, the theory of fractional calculus, Darbo-sadovskii and Mönch's fixed point theorem via Hausdorff measure of noncompactness. Finally, an example is presented in the end to show the applications of the obtained abstract results.

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[1]Department Of Mathematics,
Erode Arts and Science College,
Erode-638009, Tamil Nadu, India
Email address: malarganesaneac@gmail.com
ORCID iD: https://orcid.org/0000-0002-5143-7048
Email address: ilavarasi99ravi@gmail.com
ORCID iD: https://orcid.org/0000-0001-8756-7162

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APPROXIMATION BY CHLODOWSKY-TYPE OF SZÁSZ OPERATORS INCLUDING THE APPELL POLYNOMIALS OF CLASS $\mathbb{A}^{(2)}$

KADIR KANAT[1*], MELEK SOFYALIOGLU AKSOY [1], AND HALIME ALTUNTAŞ [1]

Abstract. A Chlodowsky variation of generalized Szász type operators and a novel sequence of operators, containing the Appell polynomials of class $\mathbb{A}^{(2)}$, are the subjects of this study. Approximation properties and convergence results are given by using different types of modulus of continuity with the help of Steklov function. A weighted space of functions constructed on $[0, +\infty)$ is used to study the convergence features of these operators. Theoretical conclusions are demonstrated by using the Gould-Hopper and Hermite polynomials.

1. Introduction

A subfield of mathematical analysis is called approximation theory. It is the study of how to approximate mathematical functions using simpler or more computationally compliant approximations. Weierstrass used uniform approximation by polynomials to identify the set of continuous functions on a closed and bounded interval in 1885. The first illustration of these polynomials was provided by Bernstein. The Szász operators [5]

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{+\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1.1}$$

is a well-known example of a linear positive operator where $f \in C[0, +\infty)$, $x \geq 0$, and $n \in \mathbb{N}$.

Key words and phrases. Appell polynomials, weighted space, rate of convergence, Voronovskaya-type theorem.

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Szász Chlodowsky operators defined as:

$$S_n(f;x) = e^{-\frac{nx}{b_n}} \sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) f\left(\frac{k}{n}b_n\right),$$

where $p_k(x) = \frac{x^k}{k!}$ and b_n is a positive increasing sequence such that

$$\lim_{n\to+\infty} b_n = +\infty, \quad \lim_{n\to+\infty} \frac{b_n}{n} = 0.$$

Jakimovski and Leviatan [14] presented Szász-type operators in 1969 utilizing Appell polynomials, as shown in the following:

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{+\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad \text{for } n \in \mathbb{N}, \tag{1.2}$$

where $p_k(x)$, $k > 0$ are the Appell polynomials denoted by $g(u)e^{ux} = \sum_{k=0}^{+\infty} p_k(x)u^k$. Here $g(1) \neq 0$ and $g(u) = \sum_{k=0}^{+\infty} a_k u^k$ is an analytic function in the disk $|u| < R$, $R > 1$ In the case of $g(u) = 1$, then $p_k(x) = \frac{x^k}{k!}$ and from (1.2) we encounter again the Szász operators presented by (1.1).

The elaborative approximation features of Szász-type operators were lately explored in [1, 12, 15, 19, 21]. Atakut and Büyükyazıcı[3] presented the Stancu-type generalization of operators (1.2). Next, Ismail [13] obtained a new generalization of the Jakimovski and Leviatan operators (1.2) and the Szász operators (1.1) utilizing Sheffer polynomials. Let $H(u) = \sum_{k=1}^{+\infty} h_k u^k$, $h_1 \neq 0$, and $\mathbb{A}(u) = \sum_{k=0}^{+\infty} a_k u^k$, $a_0 \neq 0$, be analytic functions in the disc $|u| < R$, $R > 1$, where h_k and a_k are real. The Sheffer polynomials $p_k(x)$ have generating functions of the kind

$$\mathbb{A}(t)e^{xH(t)} = \sum_{k=0}^{+\infty} p_k(x)t^k, \quad |t| < R,$$

with the aid of adhering to limitations

(i) $p_k(x) \geq 0$ and for $x \in [0, +\infty)$;
(ii) $H'(1) = 1$ and $\mathbb{A}(1) \neq 0$.

Mursaleen et al. [18] described the following as the Chlodowsky variation of Szász-type operators containing Appell polynomials:

$$B_n^*(f;x) := \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}}} \sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) f\left(\frac{k}{n}b_n\right),$$

where b_n, $n \in \mathbb{N}$, is a positive increasing sequence such that

$$\lim_{n\to+\infty} b_n = +\infty, \quad \lim_{n\to+\infty} \frac{b_n}{n} = 0.$$

Additionally, Mursaleen et al. [18] presented Gould-Hopper polynomials and examples of Hermite polynomials. Kazmin [16] defined the Appell polynomials of class $\mathbb{A}^{(2)}$ and presented the generating function of this polynomial as follows:

$$\mathbb{A}(t)e^{xt} + D(t)e^{-xt} = \sum_{k=0}^{+\infty} p_k(x)t^k, \tag{1.3}$$

where

$$\mathbb{A}(t) = \sum_{k=0}^{+\infty} \frac{a_k}{k!}t^k \quad \text{and} \quad D(t) = \sum_{k=0}^{+\infty} \frac{d_k}{k!}t^k,$$

are formal power series identified at the disc $|u| < R$, $R > 1$, with $a_0^2 - d_0^2 \neq 0$.

By utilizing Appell polynomials of class $\mathbb{A}^{(2)}$ given by (1.3), Sucu and Varma [22] identify the sequence of operators for $x \in [0, +\infty)$

$$T_n(f; x) = \frac{1}{\mathbb{A}(1)e^{nx} + D(1)e^{-nx}} \sum_{k=0}^{+\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.4}$$

with the constraints $p_k(x) > 0$ for $k = 0, 1, 2 \ldots$, $\mathbb{A}(1) > 0$ and $D(1) > 0$. These limitations guarantee that the operators in (1.4) are positive. Keep in mind that the well-known Szász operators are produced once more for the specific choices $\mathbb{A}(t) = 1$ and $D(t) = 0$.

The structure of this work is as follows. We acquire test functions and central moments in the Section 2. In Section 3, we show how to use the first and second moduli of continuity to approximate solutions. Then, in Section 4, we examine the convergence features of newly constructed operators in weighted spaces with weighted norms on the interval $[0, +\infty)$. We get the rate of convergence utilizing the weighted modulus of continuity. Finally, we provide numerical examples that use orthogonal polynomials, such as Gould-Hopper and Hermite polynomials.

2. Approximation Properties of $\mathcal{B}_n^*$ Operators

Utilizing Appell polynomials of class $\mathbb{A}^{(2)}$, we examine the Chlodowsky variation of Szász-type operators [4] given by (1.3):

$$\mathcal{B}_n^*(f; x) = \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}} \sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) f\left(\frac{k}{n}b_n\right), \tag{2.1}$$

where b_n is a positive increasing sequence such that

$$\lim_{n\to+\infty} b_n = +\infty, \quad \lim_{n\to+\infty} \frac{b_n}{n} = 0.$$

We will employ the following test functions and suppose that the operators $\mathcal{B}_n^*$ are positive throughout the study:

$$e_i(t) = t^i, \quad i \in \{0, 1, 2, 3, 4\}.$$

In addition, assume that

$$\lim_{y\to+\infty} \frac{D^{(k)}(y)}{D(y)} = 1, \quad k \in \{0, 1, 2, 3, 4\}. \tag{2.2}$$

Lemma 2.1. *For all $x \in [0,+\infty)$, we have*

$$\mathcal{B}_n^*(e_0;x) = 1,$$

$$\mathcal{B}_n^*(e_1;x) = \frac{\mathbb{A}(1)e^{\frac{nx}{b_n}} - D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x + \frac{b_n}{n}\cdot\frac{\mathbb{A}'(1)e^{\frac{nx}{b_n}} - D'(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}},$$

$$\begin{aligned}\mathcal{B}_n^*(e_2;x) ={}& x^2 + \frac{b_n}{n}\cdot\frac{e^{\frac{nx}{b_n}}(\mathbb{A}(1)+2\mathbb{A}'(1)) - e^{-\frac{nx}{b_n}}(D(1)+2D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x\\ &+ \frac{b_n^2}{n^2}\cdot\frac{e^{\frac{nx}{b_n}}(\mathbb{A}'(1)+\mathbb{A}''(1)) - e^{-\frac{nx}{b_n}}(D'(1)+D''(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}},\end{aligned}$$

$$\begin{aligned}\mathcal{B}_n^*(e_3;x) ={}& \frac{\mathbb{A}(1)e^{\frac{nx}{b_n}} - D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^3\\ &+ \frac{b_n}{n}\cdot\frac{e^{\frac{nx}{b_n}}(3\mathbb{A}'(1)+3\mathbb{A}(1)) + e^{-\frac{nx}{b_n}}(3D'(1)+3D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^2\\ &+ \frac{b_n^2}{n^2}\cdot\frac{e^{\frac{nx}{b_n}}(3\mathbb{A}''(1)+6\mathbb{A}'(1)+\mathbb{A}(1)) + e^{-\frac{nx}{b_n}}(-3D''(1)-6D'(1)-D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x\\ &+ \frac{b_n^3}{n^3}\cdot\frac{e^{\frac{nx}{b_n}}(\mathbb{A}'''(1)+3\mathbb{A}''(1)+\mathbb{A}'(1)) + e^{-\frac{nx}{b_n}}(D'''(1)+3D''(1)+D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}},\end{aligned}$$

$$\begin{aligned}\mathcal{B}_n^*(e_4;x) ={}& x^4 + \frac{b_n}{n}\cdot\frac{e^{\frac{nx}{b_n}}(4\mathbb{A}'(1)+6\mathbb{A}(1)) - e^{-\frac{nx}{b_n}}(4D'(1)+6D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^3\\ &+ \frac{b_n^2}{n^2}\cdot\frac{e^{\frac{nx}{b_n}}(6\mathbb{A}''(1)+18\mathbb{A}'(1)+7\mathbb{A}(1)) + e^{-\frac{nx}{b_n}}(6D''(1)+18D'(1)+7D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^2\\ &+ \frac{b_n^3}{n^3}\left(\frac{e^{\frac{nx}{b_n}}(4\mathbb{A}'''(1)+18\mathbb{A}''(1)+14\mathbb{A}'(1)+\mathbb{A}(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right.\\ &\left.-\frac{e^{-\frac{nx}{b_n}}(4D'''(1)+18D''(1)+14D'(1)+D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right)x\\ &+ \frac{b_n^4}{n^4}\left(\frac{e^{\frac{nx}{b_n}}(\mathbb{A}^{iv}(1)+6\mathbb{A}'''(1)+7\mathbb{A}''(1)+\mathbb{A}'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right.\\ &\left.+\frac{e^{-\frac{nx}{b_n}}(D^{iv}(1)+6D'''(1)+7D''(1)+D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right).\end{aligned}$$

Proof. From the generating functions of the Appell polynomials of class $\mathbb{A}^{(2)}$ presented by (1.3), we obtain

$$
\begin{aligned}
\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) =& \mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}},\\
\sum_{k=0}^{+\infty} kp_k\left(\frac{nx}{b_n}\right) =& \frac{n}{b_n}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} - D(1)e^{-\frac{nx}{b_n}}\right)x + \mathbb{A}'(1)e^{\frac{nx}{b_n}} + D'(1)e^{-\frac{nx}{b_n}},\\
\sum_{k=0}^{+\infty} k^2p_k\left(\frac{nx}{b_n}\right) =& \frac{n^2}{b_n^2}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)x^2\\
&+ \frac{n}{b_n}\left((\mathbb{A}(1)+2\mathbb{A}'(1))e^{\frac{nx}{b_n}} - (D(1)+2D'(1))e^{-\frac{nx}{b_n}}\right)x\\
&+ \mathbb{A}'(1)e^{\frac{nx}{b_n}} + D'(1)e^{-\frac{nx}{b_n}} + \mathbb{A}''(1)e^{\frac{nx}{b_n}} + D''(1)e^{-\frac{nx}{b_n}},\\
\sum_{k=0}^{+\infty} k^3p_k\left(\frac{nx}{b_n}\right) =& \frac{n^3}{b_n^3}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} - D(1)e^{-\frac{nx}{b_n}}\right)x^3\\
&+ \frac{n^2}{b_n^2}\left((3\mathbb{A}'(1)+3\mathbb{A}(1))e^{\frac{nx}{b_n}} + (3D'(1)+3D(1))e^{-\frac{nx}{b_n}}\right)x^2\\
&+ \frac{n}{b_n}\Big((3\mathbb{A}''(1)+6\mathbb{A}'(1)+\mathbb{A}(1))e^{\frac{nx}{b_n}}\\
&-(3D''(1)+6D'(1)+D(1))e^{-\frac{nx}{b_n}}\Big)x\\
&+ e^{\frac{nx}{b_n}}(\mathbb{A}'''(1)+3\mathbb{A}''(1)+\mathbb{A}'(1)) + e^{-\frac{nx}{b_n}}(D'''(1)+3D''(1)+D'(1)),\\
\sum_{k=0}^{+\infty} k^4p_k\left(\frac{nx}{b_n}\right) =& \frac{n^4}{b_n^4}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)x^4\\
&+ \frac{n^3}{b_n^3}\left((4\mathbb{A}'(1)+6\mathbb{A}(1))e^{\frac{nx}{b_n}} - (4D'(1)+6D(1))e^{-\frac{nx}{b_n}}\right)x^3\\
&+ \frac{n^2}{b_n^2}\Big((6\mathbb{A}''(1)+18\mathbb{A}'(1)+7\mathbb{A}(1))e^{\frac{nx}{b_n}}\\
&+(6D''(1)+18D'(1)+7D(1))e^{-\frac{nx}{b_n}}\Big)x^2\\
&+ \frac{n}{b_n}\Big((4\mathbb{A}'''(1)+18\mathbb{A}''(1)+14\mathbb{A}'(1)+\mathbb{A}(1))e^{\frac{nx}{b_n}}\\
&-(4D'''(1)+18D''(1)+14D'(1)+D(1))e^{-\frac{nx}{b_n}}\Big)x\\
&+ e^{\frac{nx}{b_n}}\left((\mathbb{A}^{iv}(1)+6\mathbb{A}'''(1)+7\mathbb{A}''(1)+\mathbb{A}'(1))\right)\\
&+ e^{-\frac{nx}{b_n}}\left((D^{iv}(1)+6D'''(1)+7D''(1)+D'(1))\right).
\end{aligned}
$$

Given these equalities, we get the intended outcomes. □

Lemma 2.2. *The operators* (2.1) *confirm:*

$$\begin{aligned}
\mathcal{B}_n^*((e_1-x);x) =& \frac{-2D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}x+\frac{b_n}{n}\left(\frac{\mathbb{A}'(1)e^{\frac{nx}{b_n}}+D'(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right),\\
\mathcal{B}_n^*((e_1-x)^2;x) =& \frac{4D'(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}x^2+\frac{b_n}{n}\left(\frac{\mathbb{A}(1)e^{\frac{nx}{b_n}}-(4D'(1)+D(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x\\
&+\frac{b_n^2}{n^2}\left(\frac{(\mathbb{A}'(1)+\mathbb{A}''(1))e^{\frac{nx}{b_n}}+(D'(1)+D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right),\\
\mathcal{B}_n^*((e_1-x)^4;x) =& \left(\frac{-8\mathbb{A}(1)e^{\frac{nx}{b_n}}+8D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}+8\right)x^4\\
&+\frac{b_n}{n}\left(\frac{(-24D(1)-32D'(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x^3\\
&+\frac{b_n^2}{n^2}\left(\frac{3\mathbb{A}(1)e^{\frac{nx}{b_n}}+(11D(1)+48D'(1)+24D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x^2\\
&+\frac{b_n^3}{n^3}\left(\frac{(\mathbb{A}(1)+10\mathbb{A}'(1)+6\mathbb{A}''(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right.\\
&\left.-\frac{(D(1)+18D'(1)+30D''(1)+8D'''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x\\
&+\frac{b_n^4}{n^4}\left(\frac{(\mathbb{A}'(1)+7\mathbb{A}''(1)+6\mathbb{A}'''(1)+\mathbb{A}^{iv}(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right.\\
&\left.+\frac{(D'(1)+7D''(1)+6D'''(1)+D^{iv}(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right).
\end{aligned}$$

Proof. By using the linearity of the $\mathcal{B}_n^*$,

$$\begin{aligned}
\mathcal{B}_n^*((e_1-x);x) =& \mathcal{B}_n^*(e_1;x)-x\mathcal{B}_n^*(e_0;x),\\
\mathcal{B}_n^*((e_1-x)^2;x) =& \mathcal{B}_n^*(e_2;x)-2x\mathcal{B}_n^*(e_1;x)+x^2\mathcal{B}_n^*(e_0;x),\\
\mathcal{B}_n^*((e_1-x)^4;x) =& \mathcal{B}_n^*(e_4;x)-4x\mathcal{B}_n^*(e_3;x)+6x^2\mathcal{B}_n^*(e_2;x)-4x^3\mathcal{B}_n^*(e_1;x)\\
&+x^4\mathcal{B}_n^*(e_0;x),
\end{aligned}$$

we obtain the desired outcome of the lemma. □

Theorem 2.1. *Let*

$$E=\left\{f:\frac{f(x)}{1+x^2}\ \textit{is convergent as}\ x\to+\infty\right\}$$

and $\mathcal{B}_n^*$ be the operators given by (2.1)*. Then for any $f\in C[0,+\infty)\cap E$, the following relation holds*

$$\lim_{n\to+\infty}\mathcal{B}_n^*(f;x)=f(x)$$

uniformly on each compact subset of $[0,+\infty)$.

Proof. According to Lemma 2.1, we get

$$\lim_{n\to+\infty} \mathcal{B}_n^*(e_i;x) = e_i(x), \quad i\in\{0,1,2\}.$$

In every compact subset of $[0,+\infty)$, the stated convergence is uniformly confirmed. The desired outcome is obtained by applying the Korovkin theorem [2]. $\square$

3. Rate of Convergence

Definition 3.1. For any function $f\in\widetilde{C}[0,+\infty)$ and $\delta>0$, the modulus of continuity $\omega(f,\delta)$ of the function f [6] is identified by

$$\omega(f,\delta) = \sup_{\substack{x,y\in[0,+\infty)\\ |x-y|\le\delta}} |f(x)-f(y)|,$$

where the space of uniformly continuous functions is given by $\widetilde{C}[0,+\infty)$. Keep in mind that one can write

$$|f(x)-f(y)| \le \omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right), \tag{3.1}$$

for each $x\in[0,+\infty)$ and any $\delta>0$.

Theorem 3.1. *If* $f\in\widetilde{C}[0,+\infty)\cap E$, $\mathcal{B}_n^*$ *operators affirm the following inequality:*

$$|\mathcal{B}_n^*(f;x)-f(x)| \le 2\omega\left(f,\sqrt{\vartheta_n(x)}\right),$$

where

$$\vartheta := \vartheta_n(x) = (\mathcal{B}_n^*(t-x)^2;x). \tag{3.2}$$

Proof. By implementing the triangle inequality and the widely recognized feature of $\omega(f,\delta)$, we obtain

$$\begin{aligned}|\mathcal{B}_n^*(f;x)-f(x)| &= \left|\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left(f\left(\frac{k}{n}b_n\right)-f(x)\right)\right|\\ &\le \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|f\left(\frac{k}{n}b_n\right)-f(x)\right|.\end{aligned}$$

By using the equation (3.1) we have

$$\begin{aligned}|\mathcal{B}_n^*(f;x)-f(x)| \le& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left(1+\frac{1}{\delta}\left|\frac{k}{n}b_n-x\right|\right)\omega(f,\delta)\\ =&\omega(f,\delta)\left(\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\right.\\ &\left.+\frac{1}{\delta}\cdot\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n-x\right|\right).\end{aligned} \tag{3.3}$$

By applying Lemma 2.2 and considering the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n - x\right| &= \sum_{k=0}^{+\infty} \sqrt{p_k\left(\frac{nx}{b_n}\right)}\sqrt{p_k\left(\frac{nx}{b_n}\right)}\left|\frac{k}{n}b_n - x\right| \\
&\le \left(\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\right)^{\frac{1}{2}} \left(\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n - x\right|^2\right)^{\frac{1}{2}} \\
&= \left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)\left(\mathcal{B}_n^*((t-x)^2;x)\right)^{\frac{1}{2}}. \qquad (3.4)
\end{aligned}
$$

From the inequalities (3.3) and (3.4), we find that

$$
|\mathcal{B}_n^*(f;x) - f(x)| \le \left(1 + \frac{1}{\delta}\left(\mathcal{B}_n^*((t-x)^2;x)\right)^{\frac{1}{2}}\right)\omega(f,\delta), \tag{3.5}
$$

where $\vartheta_n(x)$ is given by (3.2). We get the desired result in inequality (3.5) by selecting $\delta = \sqrt{\vartheta_n(x)}$. □

Lemma 3.1. *For $0 < \alpha \le 1$ and $t_1, t_2 \in [0, +\infty)$, let us present the following function class:*

$$
Lip_M^{(\alpha)} = \{f : |f(t_1) - f(t_2)| \le M|t_1 - t_2|^\alpha\}.
$$

Theorem 3.2. *Suppose that $f \in Lip_M^{(\alpha)}$. Then, we attain*

$$
|\mathcal{B}_n^*(f;x) - f(x)| \quad \le \quad M[\mathcal{B}_n^*((t-x)^2;x)]^{\frac{\alpha}{2}}.
$$

Proof. Since $f \in Lip_M^{(\alpha)}$, we obtain

$$
\begin{aligned}
|\mathcal{B}_n^*(f;x) - f(x)| &= |\mathcal{B}_n^*(f(t) - f(x);x)| \\
&\le \mathcal{B}_n^*(|f(t) - f(x)|;x) \le M\mathcal{B}_n^*(|t-x|^\alpha;x).
\end{aligned}
$$

Finally, from Hölder inequality we deduce the following expression

$$
\begin{aligned}
\mathcal{B}_n^*(|t-x|^\alpha;x) =& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n - x\right|^\alpha \\
=& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty}\left[p_k\left(\frac{nx}{b_n}\right)^{\frac{2-\alpha}{2}}\right]\left[p_k\left(\frac{nx}{b_n}\right)^{\frac{\alpha}{2}}\right]\left|\frac{k}{n}b_n - x\right|^\alpha \\
\le& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)^{\frac{2-\alpha}{2}} \\
&\times\left(\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\right)^{\frac{2-\alpha}{2}} \\
&\times\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)^{\frac{\alpha}{2}} \\
&\times\left(\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left(\frac{k}{n}b_n - x\right)^2\right)^{\frac{\alpha}{2}}
\end{aligned}
$$

$$= (\mathcal{B}_n^*(1;x))^{\frac{2-\alpha}{2}} (\mathcal{B}_n^*((t-x)^2;x))^{\frac{\alpha}{2}}.$$

Thus, we achieve the desired results. $\square$

Definition 3.2. The second modulus of continuity of the function $f \in C[a,b]$ is identified by

$$\omega_2(f,\delta) := \sup_{0<t\leq\delta} ||f(\cdot+2t) - 2f(\cdot+t) + f(\cdot)||,$$

where $||f|| = \max_{x\in[a,b]} |f(x)|$.

Lemma 3.2 (Gavrea and Rasa [10])**.** *Suppose that we have the sequence of positive linear operators $g \in C^2[0,a]$ and $(\mathcal{B}_n^*)_{n\geq 0}$ with $\mathcal{B}_n^*(e_0;x) = 1$. Then,*

$$|\mathcal{B}_n^*(g;x) - g(x)| \leq ||g'||\sqrt{\mathcal{B}_n^*\left((t-x)^2;x\right)} + \frac{1}{2}||g''||\mathcal{B}_n^*\left((t-x)^2;x\right).$$

For $f \in C[a,b]$, the second-order Steklov function of f [24] is identified by

$$f_h(x) := \frac{1}{h}\int_{-h}^{h}\left(1 - \frac{|t|}{h}\right) f(h;x+t)dt, \quad x \in [a,b],$$

where $f(h;\cdot) : [a-h, b+h] \to \mathbb{R}$, $h > 0$, by

$$f(h;x) = \begin{cases} P_-(x), & a-h \leq x \leq a, \\ f(x), & a \leq x \leq b, \\ P_+(x), & b < x \leq b+h, \end{cases}$$

and P_-, P_+ are the linear best approximations to f given piecewisely.

Lemma 3.3 (Zhuk [24])**.** *Let f_h where $f \in [c,d]$ and $h \in \left(0, \frac{c-d}{2}\right)$ be the second-order Steklov function attached to f. Then, the inequalities*

(i) $||f_h - f|| \leq \frac{3}{4}\omega_2(f,h)$;

(ii) $||f_h''|| \leq \frac{3}{2h^2}\omega_2(f,h)$;

hold.

Theorem 3.3. *Assume that f is a continuous function on $[0,+\infty)$. Then, we attain*

$$|\mathcal{B}_n^*(f;x) - f(x)| \leq \frac{3}{4}(2+a+h^2)\omega_2(f,h) + \frac{2}{a}h^2||f||,$$

where

$$h := h_n(x) = (\mathcal{B}_n^*((t-x)^2;x))^{\frac{1}{4}}$$

and $\omega_2(f;h)$ is the second order modulus of continuity.

Proof. Let f_h be the second-order Steklov function of the function f. So, in view of $\mathcal{B}_n^*(1;x) = 1$, we attain

$$\begin{aligned} |\mathcal{B}_n^*(f;x) - f(x)| &\leq \mathcal{B}_n^*(|f-f_h|;x) + |\mathcal{B}_n^*(f_h;x) - f_h(x)| + |f_h(x) - f(x)| \\ &\leq 2||f-f_h|| + |\mathcal{B}_n^*(f_h;x) - f_h(x)|. \end{aligned} \tag{3.6}$$

Given $f_h \in C^2[0,a]$ and Lemma 3.2, it is evident that

$$(3.7) \quad |\mathcal{B}_n^*(f_h;x) - f_h(x)| \leq ||f_h'(x)||\sqrt{\mathcal{B}_n^*((t-x)^2;x)} + \frac{1}{2}||f_h''(x)||\mathcal{B}_n^*((t-x)^2;x).$$

The Landau inequality is defined from [8] as follows

$$||f'|| \leq 2||f||^{\frac{1}{2}}||f''||^{\frac{1}{2}}.$$

Additionally, by combining Lemma 3.3 with the Landau inequality,

$$||f_h^{'}|| \leq \frac{2}{a}||f_h|| + \frac{a}{2}||f_h^{''}|| \leq \frac{2}{a}||f|| + \frac{3a}{4h^2}\omega_2(f,h),$$

where $h = (\mathcal{B}_n^*((t-x)^2;x))^{\frac{1}{4}}$,

$$|\mathcal{B}_n^*(f_h;x) - f_h(x)| \leq \frac{2}{a}||f||h^2 + \frac{3a}{4}\omega_2(f,h) + \frac{3}{4}h^2\omega_2(f,h).$$

Now we use the inequality (3.7) in (3.6). Then with the help of Lemma 3.2

$$|\mathcal{B}_n^*(f;x) - f(x)| \leq \frac{2}{a}||f||h^2 + \frac{3}{4}(a+2+h^2)\omega_2(f,h)$$

is obtained and the proof is done. □

4. Weighted Approximation

To calculate the rate of convergence of the unbounded function described on $[0,+\infty)$, we require weighted spaces. Here, we work on the features of approximations of newly generated operators $\mathcal{B}_n^*$ on weighted spaces of exponentially growing functions on $[0,+\infty)$. At the beginning, we review the weighted spaces' notations. Let $\rho(x) = (1+x^2)$ be the weighted function and R_f be a positive constant.

$$B_\rho([0,+\infty)) = \left\{f : [0,+\infty) \to \mathbb{R} \mid |f(x)| \leq R_f\rho(x)\right\},$$

is a linear normed space equipped with $||f|| = \sup_{x\in[0,+\infty)} \frac{|f(x)|}{\rho(x)}$,

$$C_\rho([0,+\infty)) = \left\{f \in B_\rho([0,+\infty)) \mid f \text{ is continuous}\right\},$$

$$C_\rho^*([0,+\infty)) = \left\{f \in C_\rho([0,+\infty)) \mid \lim_{x\to+\infty}\frac{f(x)}{\rho(x)} < +\infty\right\}.$$

The relationship between these spaces can be expressed as follows: $C_\rho^*([0,+\infty)) \subset C_\rho([0,+\infty)]) \subset B_\rho([0,+\infty))$.

If f is not uniformly continuous on $[0,+\infty)$, then $\omega(f,\delta)$ does not tendency 0 as $\delta \to 0$. Therefore, Gadjieva and Dogru [9] defined the weighted modulus of continuity as follows in 1998:

$$\Omega(f;\delta) = \sup_{x\geq 0,|h|\leq\delta}\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}.$$

Yuksel and Ispir [23] identified the weighted modulus of continuity in 2006:

$$\Omega(f;\delta) = \sup_{x\geq 0}\sup_{0<|h|\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2},$$

where $f \in C^*_\rho[0,+\infty)$. We will give the properties of $\Omega(\cdot,\cdot)$ in the following lemma.

Lemma 4.1 ([23]). *If $f \in C^*_\rho[0,+\infty)$, then*

(i) *$\Omega(f,x)$ is monotone increasing function of δ;*
(ii) *$\lim_{\delta\to 0^+} \Omega(f,x) = 0$;*
(iii) *for any $\lambda \in [0,+\infty)$, $\Omega(f,\lambda x) \le (1+\lambda)\Omega(f,x)$.*

Using the weighted modulus of continuity, we will now find the rate of convergence for $f \in C^*_\rho[0,+\infty)$.

Theorem 4.1. *If $f \in C^*_\rho[0,+\infty)$, then*

$$\sup_{x\in[0,+\infty)} \frac{|\mathcal{B}^*_n(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2+M^*_0(n)+\sqrt{M^*_1(n)}\right)\Omega\left(f,\sqrt{M^*_0(n)}\right),$$

where

$$\begin{aligned} M^*_0(n) =& \frac{4D'(1)e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}} + \frac{1}{2}\cdot\frac{b_n}{n}\left(\frac{\mathbb{A}(1)e^{\frac{nx}{b_n}}-(4D'(1)+D(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right) \\ &+\frac{b_n^2}{n^2}\left(\frac{(\mathbb{A}'(1)+\mathbb{A}''(1))e^{\frac{nx}{b_n}}+(D'(1)+D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right),\end{aligned} \tag{4.1}$$

$$\begin{aligned} M^*_1(n) =& \left(\frac{16D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right) + \frac{3\sqrt{3}}{16}\cdot\frac{b_n}{n}\left(\frac{(-24D(1)-32D'(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right) \\ &+\frac{1}{4}\cdot\frac{b_n^2}{n^2}\left(\frac{3\mathbb{A}(1)e^{\frac{nx}{b_n}}+(11D(1)+48D'(1)+24D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right) \\ &+\frac{3\sqrt{3}}{16}\cdot\frac{b_n^3}{n^3}\left(\frac{(\mathbb{A}(1)+10\mathbb{A}'(1)+6\mathbb{A}''(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right. \\ &\left.-\frac{(D(1)+18D'(1)+30D''(1)+8D'''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right) \\ &+\frac{b_n^4}{n^4}\left(\frac{(\mathbb{A}'(1)+7\mathbb{A}''(1)+6\mathbb{A}'''(1)+\mathbb{A}^{(iv)}(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right. \\ &\left.+\frac{(D'(1)+7D''(1)+6D'''(1)+D^{(iv)}(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right).\end{aligned} \tag{4.2}$$

Proof. Based on Lemma 4.1 and the description of the weighted modulus of continuity, we get

$$\begin{aligned}|f(t)-f(x)| &\le \left(1+(x+|t-x|^2)\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f,\delta) \\ &\le 2(1+x^2)\left(1+(t-x)^2)\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f,\delta).\end{aligned}$$

Furthermore, applying $\mathcal{B}_n^*$ for both sides, we obtain

$$\begin{aligned}|\mathcal{B}_n^*(f;x)-f(x)| \le& 2(1+x^2)\\ &\times\left(1+\mathcal{B}_n^*\left((t-x)^2;x\right)+\mathcal{B}_n^*\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\right)\Omega(f,\delta).\end{aligned}$$

By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\mathcal{B}_n^*(f;x)-f(x)| \le& 2(1+x^2)\left(1+\mathcal{B}_n^*\left((t-x)^2;x\right)\right.\\ &+\frac{1}{\delta}\sqrt{\mathcal{B}_n^*\left((t-x)^2;x\right)}\\ &\left.+\frac{1}{\delta}\sqrt{\mathcal{B}_n^*\left((t-x)^4;x\right)}\sqrt{\mathcal{B}_n^*\left((t-x)^2;x\right)}\right)\Omega(f,\delta).\end{aligned}$$

From Lemma 2.2, we can write $\mathcal{B}_n^*\left((t-x)^2;x\right) \le M_0^*(n)(1+x^2)$ and $\mathcal{B}_n^*\left((t-x)^4;x\right) \le M_1^*(n)(1+x^2)^2$. Choosing $\delta = M_0^*(n)$, we have

$$\begin{aligned}|\mathcal{B}_n^*(f;x)-f(x)| \le& 2(1+x^2)\left(1+M_0^*(n)(1+x^2)+(1+x^2)^{\frac{1}{2}}\right.\\ &\left.+\sqrt{M_1^*(n)}(1+x^2)^{\frac{3}{2}}\Omega(f,\delta)\right)\\ \le& 2(1+x^2)^{\frac{5}{2}}\left(2+M_0^*(n)+\sqrt{M_1^*(n)}\right)\Omega(f,\delta).\end{aligned}$$

Finally, we achieve the desired result

$$\sup_{x\in[0,+\infty)}\frac{|\mathcal{B}_n^*(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2+M_0^*(n)+\sqrt{M_1^*(n)}\right)\Omega\left(f,\sqrt{M_0^*(n)}\right).$$

Here $M_0^*(n)$ and $M_1^*(n)$ are given by (4.1) and (4.2), respectively. ☐

5. Numerical Example

5.1. **Gould-Hopper polynomials.** Gould-Hopper polynomials [11] have the generating form

$$e^{ht^{d+1}}\exp(xt)=\sum_{k=0}^{+\infty} g_k^{d+1}(x,h)\frac{t^k}{k!},$$

and detailed representations can be obtained by

$$g_k^{d+1}(x,h)=\sum_{s=0}^{[\frac{k}{d+1}]}\frac{k!}{s!(k-(d+1)s)!}h^s x^{k-(d+1)s}. \tag{5.1}$$

The Gould-Hopper polynomials $g_k^{d+1}(x,h)$ are a set of Hermite type d-orthogonal polynomials [7]. d-orthogonality is introduced by Maroni [17] and Van [20]. By selecting the Appell polynomials of class $\mathbb{A}^{(2)}$ as follows

$$\mathbb{A}(t)=e^{ht^{d+1}},\quad D(t)=0.$$

Gould-Hopper polynomials can be obtained. Assuming $h \ge 0$, all of the restraint $D(1) \ge 0$, $\mathbb{A}(1) > 0$ and $p_k(x) > 0$ are satisfied for all values of $k = 0, 1 \dots.$ Using generating functions in (5.1), we obtain the series of operators in their obvious form, which includes Gould-Hopper polynomials $\mathcal{B}_n^G$

$$\mathcal{B}_n^G(f;x) = e^{-\frac{nx}{b_n}-h}\sum_{k=0}^{+\infty}\frac{g_k^{d+1}(\frac{nx}{b_n},h)}{k!}f\left(\frac{k}{n}b_n\right).$$

Lemma 5.1. *For the operators $\mathcal{B}_n^G$ one has*

$$\begin{aligned}
\mathcal{B}_n^G(e_0;x) =&1,\\
\mathcal{B}_n^G(e_1;x) =&x+\frac{b_n}{n}h(d+1),\\
\mathcal{B}_n^G(e_2;x) =&x^2+\frac{b_n}{n}(1+2h(d+1))x+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h),\\
\mathcal{B}_n^G(e_3;x) =&x^3+\frac{b_n}{n}3(h(d+1)+1)+\frac{b_n^2}{n^2}(3d(d+1)h+3(d+1)^2h^2+6h(d+1)+1))\\
&+\frac{b_n^3}{n^3}((d+1)^3h^3+3d(d+1)^2h^2+(d+1)d(d-1)h+3d(d+1)h\\
&+3(d+1)^2h^2+h(d+1)),\\
\mathcal{B}_n^G(e_4;x) =&x^4+\frac{b_n}{n}(4h(d+1)+6)x^3\\
&+\frac{b_n^2}{n^2}(6d(d+1)h+6(d+1)^2h^2+18h(d+1)+7)x^2\\
&+\frac{b_n^3}{n^3}(4(d+1)^3h^3+12d(d+1)^2h^2+4(d+1)d(d-1)h+18d(d+1)h\\
&+18(d+1)^2h^2+14h(d+1)+1)x+\frac{b_n^4}{n^4}((d+1)^4h^4+6(d+1)^3dh^3\\
&+3(d+1)^2d(2d-1)h^2+(d+1)^2d(d-1)h^2+(d-2)(d+1)d(d-1)h\\
&+6(d+1)^3h^3+18d(d+1)^2h^2+6(d+1)d(d-1)h+7(d+1)dh\\
&+7(d+1)^2h^2+h(d+1)).
\end{aligned}$$

Lemma 5.2. *For every $x\in[0,+\infty)$, the operators $\mathcal{B}_n^G$ confirm:*

$$\begin{aligned}
\mathcal{B}_n^G((t-x);x) =&\frac{b_n}{n}h(d+1),\\
\mathcal{B}_n^G((t-x)^2;x) =&\frac{b_n}{n}x+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h),\\
\mathcal{B}_n^G((t-x)^4;x) =&\frac{b_n^4}{n^4}\Big[(d+1)^4h^4+6(d+1)^3dh^3+3(d+1)^2d(2d-1)h^2\\
&+(d+1)^2d(d-1)h^2+(d-2)(d+1)d(d-1)h+6(d+1)^3h^3\\
&+18d(d+1)^2h^2+6(d+1)d(d-1)h+7d(d+1)h\\
&+7(d+1)^2h^2+h(d+1)\Big]+\frac{b_n^3}{n^3}\Big[4(d+1)^3h^3+6(d+1)dh
\end{aligned}$$

$$+6(d+1)^2h^2+10(d+1)h-4(d+1)^3h^3+1\Big]\,x+\frac{b_n^2}{n^2}3x^2.$$

Theorem 5.1 (Voronovskaja-type theorem)**.** *Let $f\in C^2[0,a]$. Then, one has*

$$\lim_{n\to+\infty}\frac{n}{b_n}[\mathcal{B}_n^G(f;x)-f(x)]=h(d+1)f'(x)+\frac{xf''(x)}{2}.$$

Proof. In light of the function f's Taylor formula, we determine

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+(t-x)^2\eta(t;x), \tag{5.2}$$

where $\eta(t;x)\in C[0,a]$ and $\lim_{t\to x}\eta(t;x)=0$. Implementing $\mathcal{B}_n^G$ to the both sides of (5.2), we attain

$$\mathcal{B}_n^G(f;x)=f(x)+f'(x)\mathcal{B}_n^G(t-x;x)+\frac{f''(x)}{2}\mathcal{B}_n^G((t-x)^2;x)+\mathcal{B}_n^G((t-x)^2\eta(t;x)). \tag{5.3}$$

According to Lemmas 5.1–5.2, (5.3) becomes

$$\mathcal{B}_n^G(f;x)=f(x)+f'(x)\frac{b_n}{n}h(d+1)+\frac{f''(x)}{2}\left[\frac{b_n}{n}x+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h\right]+I, \tag{5.4}$$

where

$$I:=e^{-\frac{nx}{b_n}-h}\sum_{k=0}^{+\infty}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right).$$

Let's now take the sum I as follows:

$$\begin{aligned} I=&e^{-\frac{nx}{b_n}-h}\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|\le\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right)\\ &+e^{-\frac{nx}{b_n}-h}\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right). \end{aligned} \tag{5.5}$$

From the continuity of function η, it results that for all $\epsilon>0$, there exists a positive δ such that if $\left|\left(\frac{k}{n}b_n\right)-x\right|\le\delta$, then $\left|\eta\left(\frac{k}{n}b_n;x\right)\right|<\epsilon$. Moreover, we can type $\left|\eta\left(\frac{k}{n}b_n;x\right)\right|<M$ for $\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta$ because the function η is limited. Given these facts, (5.5) implies

$$I\le\epsilon\mathcal{B}_n^G((t-x)^2;x)+Me^{-\frac{nx}{b_n}-h}\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right).$$

Taking into account the fact

$$\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\le\frac{1}{\delta^2}\mathcal{B}_n^G((t-x)^4;x)$$

in the last inequality, we attain

$$I \leq \epsilon \mathcal{B}_n^G((t-x)^2;x) + \frac{M}{\delta^2}\mathcal{B}_n^G((t-x)^4;x). \tag{5.6}$$

Substituting the inequality (5.6) in the equality (5.4), then from Lemma 5.2, we get

$$\begin{aligned}
\mathcal{B}_n^G(f;x) - f(x) \leq & f'(x)\frac{b_n}{n}h(d+1) \\
& + \left(\epsilon + \frac{f''(x)}{2}\right)\left[\frac{b_n}{n}x + \frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h\right] \\
& + \frac{M}{\delta^2}\left(\frac{b_n^4}{n^4}\left[(d+1)^4h^4 + 6(d+1)^3dh^3 + 3(d+1)^2d(2d-1)h^2\right.\right. \\
& +(d+1)^2d(d-1)h^2 + (d+4)(d+1)d(d-1)h + 6(d+1)^3h^3 \\
& \left. +18d(d+1)^2h^2 + 7d(d+1)h + 7(d+1)^2h^2 + h(d+1)\right] \\
& +\frac{b_n^3}{n^3}\left[4(d+1)^3h^3 + 6(d+1)dh + 6(d+1)^2h^2 + 10h(d+1)\right. \\
& \left.\left. -4(d+1)^3h^3 + 1\right]x + \frac{b_n^2}{n^2}3x^2\right).
\end{aligned}$$

Equivalently, we can type

(5.7)

$$\begin{aligned}
\mathcal{B}_n^G(f;x) - f(x) = & \mathcal{O}\left(\frac{b_n}{n}\right)\left(f'(x)h(d+1)\right. \\
& + \left(\epsilon + \frac{f''(x)}{2}\right)\left[x + \frac{b_n}{n}h(d+1)(1+d+(d+1)h\right] \\
& +\frac{M}{\delta^2}\left[\frac{b_n^3}{n^3}[(1+d)^4h^4 + 6(d+1)^3dh^3 + 3(d+1)^2d(2d-1)h^2\right. \\
& +(d+1)^2d(d-1)h^2 + (d-2)(d+1)d(d-1)h + 6(d+1)^3h^3 \\
& +18d(d+1)^2h^2 + 6(d+1)d(d-1)h + 7d(d+1)h + 7(d+1)^2h^2 \\
& +h(d+1)] + \frac{b_n^2}{n^2}\left[4(d+1)^3h^3 + 6(d+1)dh + 6(d+1)^2h^2\right. \\
& \left.\left.\left. +10h(d+1) - 4(d+1)^3h^3 + 1\right]x + \frac{b_n}{n}3x^2\right]\right).
\end{aligned}$$

Rewriting (5.7) as

$$\lim_{n\to+\infty}\frac{n}{b_n}[\mathcal{B}_n^G(f;x) - f(x)] = h(d+1)f'(x) + \frac{xf''(x)}{2},$$

after applying limits for $n \to +\infty$ completes the proof. $\square$

Theorem 5.2. *If $f \in C_\rho^*([0,+\infty))$, then*

$$\sup_{x\in[0,+\infty)} \frac{|\mathcal{B}_n^G(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2+M_0^G(n)+\sqrt{M_1^G(n)}\right)\Omega\left(f,\sqrt{M_0^G(n)}\right),$$

where

$$\begin{aligned}
M_0^G(n) =& \frac{b_n}{2n}+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h),\\
M_1^G(n) =& \frac{b_n^4}{n^4}\Big[(1+d)^4h^4+6(d+1)^3dh^3+3(d+1)^2d(2d-1)h^2+(d+1)^2d(d-1)h^2\\
&+(d-2)(d+1)d(d-1)h+6(d+1)^3h^3+18d(d+1)^2h^2+6(d+1)d(d-1)h\\
&+7d(d+1)h+7(d+1)^2h^2+h(d+1)\Big]+\frac{3\sqrt{3}}{16}\cdot\frac{b_n^3}{n^3}\Big[4(d+1)^3h^3+6(d+1)dh\\
&+6(d+1)^2h^2+10h(d+1)-4(d+1)^3h^3+1\Big]+\frac{3}{4}\cdot\frac{b_n^2}{n^2}.
\end{aligned}$$

Example 5.1. By taking $f(x)=\frac{x^2}{1+x^3}$, $b_n=n^{\frac{1}{3}}$ and $d=0.5$, we obtain the error approximation of the Chlodowsky variation of Szász-type operators including Gould-Hopper polynomials by using weighted modulus of continuity as we see in 5.1.

n	$h_1=1.5$	$h_2=2$	$h_3=3$
10	1.6880	2.0285	2.7854
10^2	0.4665	0.5228	0.6487
10^3	0.1791	0.1865	0.2049
10^4	0.0792	0.0800	0.0822
10^5	0.0365	0.0365	0.0368
10^6	0.0169	0.0169	0.0169
10^7	0.0078	0.0078	0.0078
10^8	0.0036	0.0036	0.0036

Table 1. Error of $\mathcal{B}_n^G(f;x)$ by using weighted modulus of continuity for $d=0.5$.

5.2. **Hermite polynomials of variance ε.** If $\mathbb{A}(t)=e^{\frac{-\varepsilon t^2}{2}}$, then $R_n(x)=H_n^{(\varepsilon)}(x)$ is the Hermite polynomials of variance ε which have the obvious presentation

$$H_n^{(\varepsilon)}(x)=\sum_{k=0}^{[\frac{n}{2}]}\left(-\frac{\varepsilon}{2}\right)^k\frac{1}{k!(n-2)!}x^{n-2k},$$

where $[\cdot]$ denotes the integer part. Denote by $\mathcal{B}_n^H$ the Chlodowsky variant of Szász-type operators containing the Hermite polynomials. Then,

$$\mathcal{B}_n^H(f;x) = e^{\frac{\varepsilon}{2} - \frac{n}{b_n}x} \sum_{k=0}^{+\infty} H_n^{(\varepsilon)}(x) f\left(\frac{k}{n} b_n\right).$$

For the operators $\mathcal{B}_n^H$, $(i), (ii)$ and assumptions (2.2) are confirmed with the assumption $\varepsilon \leq 0$.

Lemma 5.3. *We have the following results*

$$\begin{aligned}
\mathcal{B}_n^H(e_0;x) =& 1,\\
\mathcal{B}_n^H(e_1;x) =& x - \frac{b_n}{n}\varepsilon,\\
\mathcal{B}_n^H(e_2;x) =& x^2 + \frac{b_n}{n}(1-2\varepsilon)x + \frac{b_n^2}{n^2}(-2\varepsilon+\varepsilon^2),\\
\mathcal{B}_n^H(e_3;x) =& x^3 + \frac{b_n}{n}(-3\varepsilon+3)x^2 + \frac{b_n^2}{n^2}(-9\varepsilon+3\varepsilon^2+1)x + \frac{b_n^3}{n^3}(6\varepsilon^2-\varepsilon^3-4\varepsilon),\\
\mathcal{B}_n^H(e_4;x) =& x^4 + \frac{b_n}{n}(-4\varepsilon+6)x^3 + \frac{b_n^2}{n^2}(-24\varepsilon+6\varepsilon^2+7)x^2\\
&+ \frac{b_n^3}{n^3}(-4\varepsilon^3+18\varepsilon^2-20\varepsilon+1)x + \frac{b_n^4}{n^4}(\varepsilon^4-12\varepsilon^3+28\varepsilon^2-7\varepsilon-1).
\end{aligned}$$

Lemma 5.4. *For every $x \in [0,+\infty)$, the operators $\mathcal{B}_n^H$ satisfy*

$$\begin{aligned}
\mathcal{B}_n^H((t-x);x) &= -\frac{b_n}{n}\varepsilon,\\
\mathcal{B}_n^H((t-x)^2;x) &= \frac{b_n^2}{n^2}(-2\varepsilon+\varepsilon^2) + \frac{b_n}{n}x,\\
\mathcal{B}_n^H((t-x)^4;x) &= \frac{b_n^4}{n^4}(-8\varepsilon+28\varepsilon^2-12\varepsilon^3+\varepsilon^4) + \frac{b_n^3}{n^3}(1-16\varepsilon+6\varepsilon^2)x + \frac{b_n^2}{n^2}3x^2.
\end{aligned}$$

Lemma 5.5. *We have the following results*

$$\begin{aligned}
\lim_{n\to+\infty} \frac{n}{b_n}\mathcal{B}_n^H(t-x:x) &= -\varepsilon,\\
\lim_{n\to+\infty} \frac{n}{b_n}\mathcal{B}_n^H((t-x)^2:x) &= x,\\
\lim_{n\to+\infty} \frac{n^2}{b_n^2}\mathcal{B}_n^H((t-x)^4:x) &= 3x^2.
\end{aligned}$$

Theorem 5.3. *Let $f \in C^2[0,a]$ and $x \in [0,+\infty)$, we obtain*

$$\lim_{n\to+\infty} \frac{n}{b_n}[\mathcal{B}_n^H(f;x) - f(x)] = -\varepsilon f'(x) + \frac{xf''(x)}{2},$$

uniformly in each compact subset of $[0,+\infty)$.

Theorem 5.4. *If $f \in C_\rho^*([0,+\infty))$, then*

$$\sup_{x\in[0,+\infty)} \frac{|\mathcal{B}_n^H(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2+M_0^H(n)+\sqrt{M_1^H(n)}\right)\Omega\left(f,\sqrt{M_0^H(n)}\right),$$

where

$$M_0^H(n) = \frac{b_n^2}{n^2}(-2\varepsilon+\varepsilon^2)+\frac{b_n}{2n},$$
$$M_1^H(n) = \frac{b_n^4}{n^4}(-8\varepsilon+28\varepsilon^2-12\varepsilon^3+\varepsilon^4)+\frac{3\sqrt{3}}{16}\cdot\frac{b_n^3}{n^3}(1-16\varepsilon+6\varepsilon^2)+\frac{3}{4}\cdot\frac{b_n^2}{n^2}.$$

Example 5.2. By taking $f(x)=\frac{x^2}{1+x^3}$ and $b_n = n^{\frac{1}{4}}$, we can see the error estimation of the Chlodowsky variation of Szász-type operators including Hermite polynomials by the help of weighted modulus of continuity in Table 5.2.

n	$\varepsilon_1=-0.002$	$\varepsilon_2=1.5$	$\varepsilon_3=2.5$
10	0.6874	0.5950	0.8060
10^2	0.2916	0.2846	0.3022
10^3	0.1250	0.1245	0.1259
10^4	0.0532	0.0532	0.0533
10^5	0.0225	0.0225	0.0225
10^6	0.0095	0.0095	0.0095
10^7	0.0040	0.0040	0.0040
10^8	0.0016	0.0016	0.0016

TABLE 2. Error estimation of $\mathcal{B}_n^H(f;x)$ by using weighted modulus of continuity.

6. Conclusion

In our current research, we introduce the Chlodowsky variant of generalized Szász-type operators, a novel sequence of operators containing the Appell polynomials of class $\mathbb{A}^{(2)}$. The test functions and central moments of the operators were attained. Moreover, the rate of convergence is obtained through the use of the modulus of continuity by means of Steklov function. Then a Voronovskaya-type theorem for the quantitative asymtotic approximation is given. In the last section, it is shown that the Chlodowsky variant of generalized Szász-type operators Appell polynomials of class $\mathbb{A}^{(2)}$ reduce Gould-Hopper polynomials and Hermite polynomials under special choices.

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[1]Department of Mathematics,
Polatli Faculty of Science and Letters,
Haci Bayram Veli University

*Corresponding Author
Email address: kadir.kanat@hbv.edu.tr
ORCID iD: https://orcid.org/0000-0002-7738-903X
Email address: melek.sofyalioglu@hbv.edu.tr
ORCID iD: https://orcid.org/0000-0001-7837-2785
Email address: halime.altuntas@hbv.edu.tr
ORCID iD: https://orcid.org/0000-0002-4891-7964

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NOVEL WAVELET APPROACH FOR SOLVING FRACTIONAL BAGLEY-TORVIK PROBLEMS

MINE AYLIN BAYRAK[1], SERTAÇ ERMAN[2], ALI DEMIR[1], AND AHMET BÜYÜK[1]

Abstract. The primary purpose of this study is to construct truncated solutions for fractional Bagley-Torvik problems (FBT) by developing a novel method including newly defined Clique wavelets and collocation points. Clique wavelets are defined by utilizing Clique polynomials on $[0, 1]$. The convergence of this method is investigated and supported by illustrative examples through tables and figures. As a result, the efficiency and effectiveness of the method is proved by theorems and examples.

1. Introduction

In the mathematical modelling of various processes in physics,geophysics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, visco-elasticity, etc., fractional differential equations (FDE's) are utilized in common since they reflects the behaviour of the processes much more better than ordinary differential equations [1–13]. The properties of FDE's such as hereditary properties make them to play a significant role to analyze the present and future development of the processes [14, 15]. Therefore, the complicated processes can be modelled by FDE's with less difficulty compared with nonlinear differential equations [16]. As a result, many researchers use FDE's as an excellent tool to model and analyze a number of processes in science. However, to acquire analytical solutions of FDE's is much more difficult compared with ordinary differential equations. Consequently, a number of numerical methods such as Adomian decomposition method [17], homotopy perturbation method [18], the generalized

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Taylor collocation [19], a fractional linear multi-step method and a predictor-corrector (PC) method of the Adams type method [20], the Haar wavelet method [21, 22], an iterative reproducing kernel algorithm [23], etc. have been established to construct numerical or approximate solutions for them.

Wavelets can be defined as oscillatory functions with compact support. Diverse polynomials such as Chebyshev, Legendre, Hermite, Bernstein and Lucas polynomials [24–32] have been used in wavelet theory to develop some wavelet schemes to construct truncated or analytical solutions of differential equations. Generally orthogonality properties of these polynomials leads to a bases for important spaces in which a wide range of problems in system analysis, optimal control, numerical analysis, signal analysis and time-frequency analysis, etc., are solved analytically or approximately [33, 34].

FBT, proposed by Bagley and Torvik, is used to model visco-elastically damped structures where visco-elasticity is defined by a fractional derivative. FBT allows us to analyze the qualitative behavior of real material. Therefore it plays a prominent role in engineering and applied sciences. The order of the fractional derivative of damping term in FBT leads to the model of various processes but in FBT it is equal to $\frac{3}{2}$ which implies that damping depends on frequency. Moreover, in the modeling of the behavior of rigid plate or a gas embedded in a viscous fluid in a fluid [9,23]

In this study, we take a different approach by constructing Clique Wavelets to solve fractional Bagley-Torvik equation which have been used commonly in visco-elasticity theory. The novel contribution of this research is that the Clique wavelets are defined and used together with collocation points first time in this study. Even though Clique polynomials introduced by Fisher and Solow [35] are not orthogonal polynomials, they form a bases for $L_2[0, 1]$ which allows us to define Clique wavelets.

The following fractional FBT in Caputo sense is taken into consideration:

$$D^2y(t) + \lambda_1 D^\alpha y(t) + \lambda_2 y(t) = f(t), \tag{1.1}$$

$$y(0) = \mu_1, \quad \kappa_1 y'(0) + \kappa_2 y(1) = \mu_2, \tag{1.2}$$

where $0 < \alpha \leq 2$, the operator D^α denotes the Caputo fractional derivative of order α, λ_1 and λ_2 are constants and the function f is continuous on the interval $[0, 1]$.

The present paper is outline as follows. The fundamental definitions are given in Preliminaries. In Section 3, the establishment of wavelets which is used in suggested method is illustrated. In Section 4, the implementation and algorithm of proposed method is presented. The convergence analysis is given in Section 5. In the final section, four different examples are presented to illustrate the implementation and the effectiveness of the proposed method.

2. Preliminaries

Definition 2.1. The Riemann-Liouville integral for α is [3,36]:

$$J^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2. The α^{th} order fractional derivative in Caputo sense is given by [3,36]

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & m-1 < \alpha < m, m \in \mathbb{N}, \\ \frac{d^m f}{dt^m}(t), & \alpha = m. \end{cases} \tag{2.2}$$

Definition 2.3. The Clique polynomials on a bounded interval of the real line $[a, b]$, $b > a \geq 0$, are defined as follows:

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} t^k. \tag{2.3}$$

In general, the clique polynomial is given by,

$$P_n(t) = (1+t)^n = \binom{n}{0} + \binom{n}{1} t + \binom{n}{2} t^2 + \cdots + \binom{n}{n} t^n. \tag{2.4}$$

For $n = 0, 1$ in (2.4), we get $P_0(t) = 1$ and $P_1(t) = 1 + t$. This set of polynomials have the following recursive formulation

$$P_{n+1}(t) = (1+t) P_n(t), \quad n = 0, 1, \ldots \tag{2.5}$$

3. Construction of Clique Wavelets

In this section, Clique polynomials on the interval $[0, 1]$ are utilized to establish Clique wavelets $W_{j,i}(t)$ for $j = 0, 1, \ldots, 2^k - 1$ and $i = 0, 1, \ldots, n$ on $[0, 1)$ as in the following form:

$$\Psi_{j,i}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_i}} 2^{\frac{k}{2}} P_i(2^k t - j), & \text{if } t \in \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right), \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where k and n are non-negative integers and $\sigma_i = \frac{2^{2n}-1}{2n+1}$.

The sets of Clique wavelets are bases for $L_2[0, 1]$. The followings present the related Clique wavelets for $k = 1$ and $n = 2$

$$\Psi_{0,0}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_0}} \sqrt{2} P_0(2t), & \text{if } t \in \left[0, \frac{1}{2}\right), \\ 0, & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.2}$$

$$\Psi_{0,1}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_1}} \sqrt{2} P_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right), \\ 0, & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.3}$$

$$\Psi_{0,2}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_2}} \sqrt{2} P_2(2t), & \text{if } t \in \left[0, \frac{1}{2}\right), \\ 0, & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.4}$$

$$\Psi_{1,0}(t) = \begin{cases} 0, & \text{if } t \in \left[0, \frac{1}{2}\right), \\ \frac{1}{\sqrt{\sigma_0}}\sqrt{2}P_0(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.5}$$

$$\Psi_{1,1}(t) = \begin{cases} 0, & \text{if } t \in \left[0, \frac{1}{2}\right), \\ \frac{1}{\sqrt{\sigma_1}}\sqrt{2}P_1(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.6}$$

$$\Psi_{1,2}(t) = \begin{cases} 0, & \text{if } t \in \left[0, \frac{1}{2}\right), \\ \frac{1}{\sqrt{\sigma_2}}\sqrt{2}P_2(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right). \end{cases} \tag{3.7}$$

Therefore, an approximation of a function y in $L_2[0,1)$ can be constructed in terms of Clique wavelets as:

$$y_n(t) \approx \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t), \tag{3.8}$$

where the coefficients $c_{j,i}$ are obtained by the following inner product

$$c_{j,i} = \langle \Psi_{j,i}(t), y_n(t) \rangle = \int_0^1 \Psi_{j,i}(t) y_n(t) dt. \tag{3.9}$$

4. Implementation of the Proposed Method

In the establishment of the approximate solution y for the problem (1.1)–(1.2) in terms of special polynomials as

$$\sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t) \tag{4.1}$$

the following steps below are taken.

Step 1. Substituting the n^{th} degree approximation of (4.1) into the (1.1) yields the following equation:

$$\sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi''_{j,i}(t) + \lambda_1 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}D^\alpha\Psi_{j,i}(t) + \lambda_2 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t) = f(t), \tag{4.2}$$

where $m-1 < \alpha \leqslant m$.

Step 2. Collocating (4.2) at the nodes $t_r = \frac{r}{n}$, $r = 0, 1, 2, \ldots, n$, leads to the following system of fractional ordinary differential equations:

$$\sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi''_{j,i}(t_r) + \lambda_1 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}D^\alpha\Psi_{j,i}(t_r) + \lambda_2 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t_r) = f(t_r), \tag{4.3}$$

where $m-1 < \alpha \leqslant m$.

Step 3. Plugging the n^{th} degree approximation of (4.2) into in the initial and boundary conditions (1.2) yields the following system of algebraic equations, we can obtain $([\alpha]+1)$ equations as follows:

$$\sum_{j=0}^{2^k-1}\sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(0) = \mu_1, \tag{4.4}$$

$$\kappa_1\sum_{j=0}^{2^k-1}\sum_{i=0}^{n} c_{j,i}\Psi'_{j,i}(0) + \kappa_2\sum_{j=0}^{2^k-1}\sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(1) = \mu_2. \tag{4.5}$$

Step 4. Finally, a system of fractional algebraic equations are obtained. The unknown coefficients $c_{j,i}$, $j = 0, 1, \ldots, 2^k - 1$, $i = 0, 1, \ldots, n$, are determined by RPSM which allows us to construct the approximate solution $y_n(x, t)$.

5. Convergence and Error Analysis

The primary purpose of this section is to investigate the convergence analysis of the Clique polynomials in L_2 norm for fractional BTP. In other words, the series for an approximation of a function in terms of Clique polynomials converges to the exact solution as it is expanded by increasing the number of base elements. The definition of $L_2([0, 1])$ is given as [37]

$$L_2([0,1]) = \{\ell : [0,1] \to \mathbb{R} \mid \ell \text{ is a measurable and } \|\ell\| < +\infty\}, \tag{5.1}$$

where

$$\|\ell\|^2 = \int_0^1 |\ell(t)|^2 dt \tag{5.2}$$

indicates the induced norm related to the following inner product of the space $L_2([0, 1])$:

$$\langle \ell(t), r(t)\rangle = \int_0^1 \ell(t) r(t) dt. \tag{5.3}$$

We first consider a finite-dimensional subspace of $L_2([0, 1])$ of the form

$$\begin{aligned}\mathcal{C}_N = \operatorname{Span}\langle &\Psi_{0,0}(t), \Psi_{0,1}(t), \ldots, \Psi_{0,N}(t), \Psi_{1,0}(t), \Psi_{1,1}(t), \ldots, \Psi_{1,N}(t), , \ldots, \\ &\Psi_{2^k-1,0}(t), \Psi_{2^k-1,1}(t), \ldots, \Psi_{2^k-1,N}(t)\rangle.\end{aligned} \tag{5.4}$$

Clearly, $\dim(\mathcal{C}_N) = (2^k - 1) \times N$ and $\mathcal{C}_N$ is a complete subspace of $L_2([0, 1])$ since it is closed and finite-dimensional. Every function $u \in L_2([0, 1])$ has a unique best approximation $u^* \in \mathcal{C}_N$ in the following sense

$$\|u(t) - u^*(t)\| \leqslant \|u(t) - v(t)\|, \quad \text{for all } v \in \mathcal{C}_N. \tag{5.5}$$

Theorem 5.1. *Let ℓ_N denote the interpolating function of $u \in C^N([0, 1])$ at N Chebyshev nodes in the interval $[0, 1]$. Then, for every $t \in [0, 1]$, we have*

$$\|u(t) - \ell_N(t)\| \leqslant \frac{\|u\|_\infty}{2^{2N-1}N!}, \tag{5.6}$$

where $\|u\|_\infty := \max_{t\in[0,1]} |u^{(N)}(t)|$.

Since, the sets of Clique wavelets form a bases for $L_2([0,1])$, *every function* $u \in L_2([0,1])$ *can be represented by the series form in terms of Clique wavelet polynomials as*

$$u(t) = \sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty} c_{j,i}\Psi_{j,i}(t). \tag{5.7}$$

The restriction of it to the finite dimensional subspace $\mathcal{C}_N$ *of* $L_2([0,1])$, *a truncated series of* u *can be written as*

$$u(t) \approx u_N(t) = \sum_{j=0}^{2^k-1}\sum_{i=0}^{N} c_{j,i}\Psi_{j,i}(t). \tag{5.8}$$

Proof. See [38, 39]. □

Theorem 5.2. *For the best approximation* u_N *to* $u \in C^N([0,1]) \cap L_2([0,1])$ *in the space* $\mathcal{C}_N$, *the following inequality holds*

$$\|E_N(t)\|^2 = \|u(t) - u_N(t)\|^2 \le \|u(t) - v(t)\|^2, \quad \text{for all } v \in \mathcal{C}_N. \tag{5.9}$$

The above inequality holds for $v = \ell \in \mathcal{C}_N$. *As a result, we deduce that*

$$\|E_N(t)\|^2 \le \int_0^1 \left|\frac{\|u\|_\infty}{2^{2N+1}(N+1)!}\right|^2 dt \le \left[\frac{\|u\|_\infty}{2^{2N+1}(N+1)!}\right]^2. \tag{5.10}$$

As N *tends to infinity, the desired result is obtained.*

6. Illustrative Examples

The primary aim of this section is to present the implementation of the method by illustrative examples and check their accuracy.

Example 6.1. Consider the following fractional BTP as [3, 41]:

$$D^{\frac{3}{2}}y(t) + D^2y(t) + y(t) = 7t + t^3 + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}} + 1, \tag{6.1}$$

$$y(0) = y'(0) = 1, \tag{6.2}$$

for which $y(t) = t^3 + t + 1$ denotes the analytic solution.

The Clique wavelet solution of the problem (6.1)–(6.2) is the excellent truncated solution with higher accuracy for the exact solution compare to the other the existing methods. It is seen that the Figure 1 that as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer to the exact solution. As it is presented in Table 1 that the accuracy of the presented method is higher than the other existing methods.

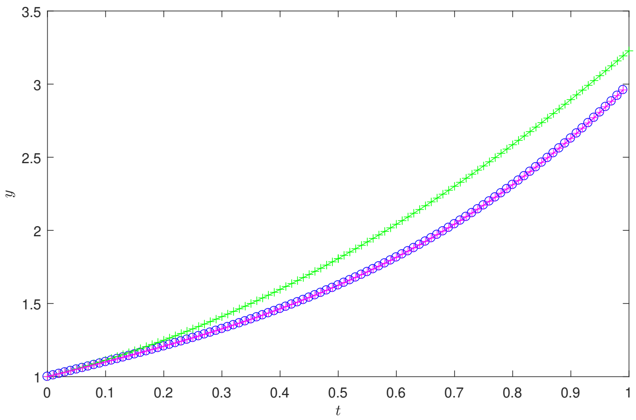

FIGURE 1. A Graphical presentation of truncated solutions (in green and pink) with regard to $n = 2, 3$ and exact solution (in blue) for Example 6.1.

TABLE 1. Comparison of results for $n = 3$.

t	VIM [44]	FIM [44]	SLC[43]	LWS[45]	CWM
0.01	0.08214	2.76e-3	2.57e-02	1.99e-15	2.22e-16
0.25	0.17315	3.42e-3	1.51e-01	1.22e-14	4.44e-16
0.5	0.10515	1.01e-3	5.44e-01	4.90e-14	0
0.75	1.34104	4.98e-3	1.083448	1.10e-13	0
1	4.11359	5.01e-3	1.676130	1.96e-13	0

Example 6.2. Consider the following fractional BTP as [40]:

$$D^2y(t) - \frac{2}{5}D^{\frac{3}{2}}y(t) - \frac{1}{2}y(t) = -\frac{1}{2}t^3 + \frac{3}{4}t^2 + \frac{183}{32}t - 3 - \frac{4}{5}\cdot\frac{\sqrt{t}(-3+4t)}{\sqrt{\pi}}, \tag{6.3}$$

$$y(0) = 0, \quad y'(0) = \frac{9}{16}, \tag{6.4}$$

for which $y(t) = t^3 - \frac{3}{2}t^2 + \frac{9}{16}t$ denotes to the analytic solution.

The Clique wavelet solution of the problem (6.3)–(6.4) is the excellent truncated solution with higher accuracy for the exact solution compare to the other the existing methods. It is seen that the Figure 2 that as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer to the exact solution.

As it is presented in Table 2 that the accuracy of the presented method is higher than the other existing methods.

Example 6.3. Consider the following fractional BTP as [41]:

$$D^2y(t) + D^{\frac{3}{2}}y(t) + y(t) = t^3 + 5t + 8\frac{t^{\frac{3}{2}}}{\sqrt{\pi}}, \tag{6.5}$$

$$y(0) = y'(0) = 0, \tag{6.6}$$

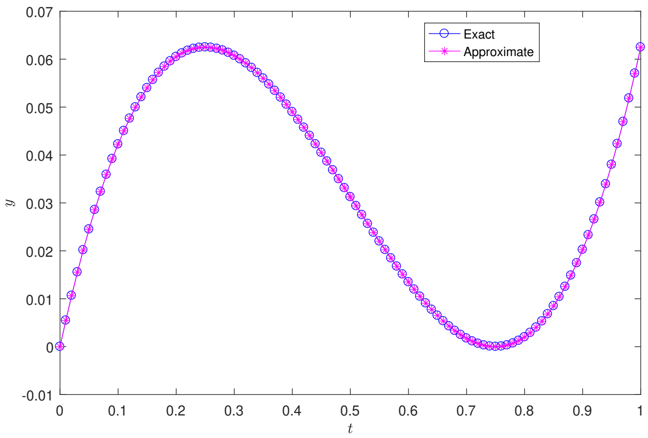

FIGURE 2. A Graphical presentation of truncated solutions (in pink) with regard to $n = 4$ and exact solution (in blue) for Example 6.2.

TABLE 2. Comparison of results for various n.

x	$N = 2$	$N = 3$	$N = 4$
0.1	1.71e-02	7.13e-04	6.25e-16
0.25	9.74e-02	3.70e-03	5.00e-16
0.50	3.27e-01	9.77e-03	1.11e-16
0.75	5.95e-01	1.07e-02	1.67e-16
1	8.09e-01	1.11e-13	2.22e-16

for which $y(t) = t(t^2 - 1)$ denotes the analytic solution.

The Clique wavelet solution of the problem (6.5)–(6.6) is the excellent truncated solution with higher accuracy for the exact solution compare to the other the existing methods. It is seen from the Figure 1, as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer to the exact solution. As it is presented in Table 3 that the accuracy of the presented method is higher than the other existing methods.

Example 6.4. Consider the following fractional BTP as [42]:

$$D^2y(t) - D^\alpha y(t) = -1 - E_\alpha((t-1)^\alpha), \quad 0 < \alpha \leqslant 1, \tag{6.7}$$

$$y(0) = y(1) = 0, \tag{6.8}$$

for which $y(t) = t(1 - E_\alpha((t-1)^\alpha))$ denotes to the analytic solution.

The Clique wavelet solution of the problem (6.7)–(6.8) is the excellent truncated solution with higher accuracy for the exact solution compare to the other existing methods. It is seen from the Figure 4, as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer the exact solution. As it is presented in Table 4 that the accuracy of the presented method is higher than the other existing methods.

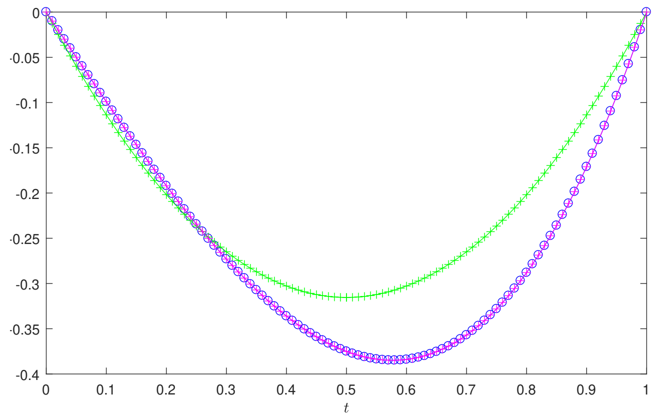

FIGURE 3. A Graphical presentation of truncated solutions (in green and pink) with regard to $n = 2, 3$ and exact solution (in blue) for Example 6.3

TABLE 3. Comparison of results for various n.

x	$N = 3$	[45]
0.2	5.551e-17	9.575e-15
0.4	1.665e-16	3.758e-14
0.6	1.110e-16	8.426e-14
0.8	0	1.497e-13
1	0	0

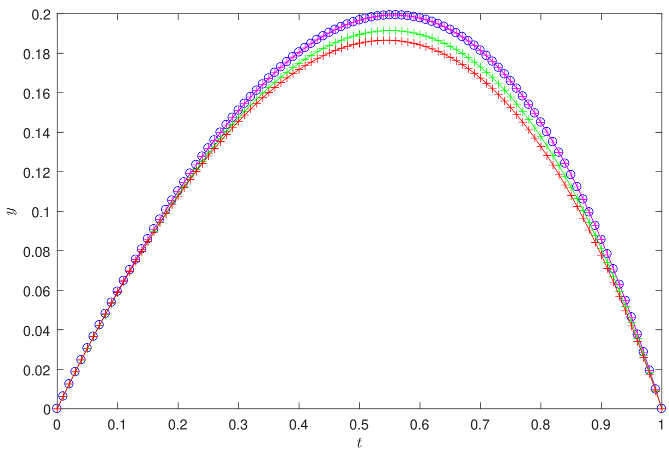

FIGURE 4. A Graphical presentation of truncated solutions (in pink, green and red) with regard to $\alpha = 1, 0.9, 9.8$ and exact solution (in blue) for Example 6.4.

7. CONCLUSION

In this research, a novel method is developed by defining Clique wavelets and using them with collocation points to construct truncated solutions of the Bagley-Torvik

TABLE 4. Comparison of results for various n.

x	$N=2$	$N=3$	$N=4$	$N=5$
0.1	1.29e-02	2.71e-03	2.47e-04	9.35e-05
0.25	1.86e-02	3.63e-03	5.47e-04	8.75e-05
0.50	4.07e-03	1.20e-01	2.76e-05	4.05e-05
0.75	1.53e-02	9.60e-02	1.05e-04	2.51e-05
1	0	0	4.16e-17	2.22e-16

initial value problems. Its convergence analysis is also presented and supported by illustrative examples. It is shown that obtained truncated solutions have higher accuracy compare to other methods. As a result, the effectiveness and accuracy of this Clique wavelets method is investigated and illustrated. This effective approach can be also utilized to obtain other fractional problems in applied sciences. In the future work, this method will be taken into consideration to construct truncated solutions of time-fractional initial value problems and inverse problems.

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[1]Department of Mathematics,
Faculty of Arts and Science,
University of Kocaeli
Email address: aylin@kocaeli.edu.tr
ORCID iD: https://orcid.org/0000-0001-7716-3455

[2]Department of Engineering Fundamental Sciences,
Sakarya University of Applied Sciences
Email address: sertacerman@subu.edu.tr
ORCID iD: https://orcid.org/0000-0002-3156-5173
Email address: ademir@kocaeli.edu.tr
ORCID iD: https://orcid.org/0000-0003-3425-1812
Email address: ahmetbuyuk89@gmail.com
ORCID iD: https://orcid.org/0000-0003-1812-7889

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RANDIĆ INDEX OF A GRAPH WITH SELF-LOOPS

HARSHITHA A[1], SABITHA D'SOUZA[1*], AND PRADEEP G. BHAT[1]

ABSTRACT. Let $G(n,m)$ be a simple graph with vertex set V and $S \subseteq V$ with $|S| = \sigma$. The graph G_S is obtained by adding a self-loop to each vertex of the graph G in the set S. The Randić index of a graph is one of the important topological indices which has its application in chemistry. In this manuscript, the Randić index of a graph with self-loops is defined and are obtained some bounds for the same.

1. INTRODUCTION

Let $G_S(n, m+\sigma)$ be a graph obtained by attaching a self-loop to each vertices in the set $S \subseteq V(G)$ of a simple graph $G(n,m)$, where $|S| = \sigma$. Degree of a vertex in a graph G is the number of edges incident on a vertex. The notation $\deg_G(v)$ represents the degree of a vertex v in the graph G. A self-loop contributes 2 to the number of edges incident on a vertex. The Randić index is one of the most studied degree-based topological index in the literature which has various applications in chemistry and pharmacology. Randić index was introduced by M. Randić [1] in 1976 and it is defined as

$$R(G) = \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{\deg_G(v_i)\deg_G(v_j)}}.$$

For more studies on Randić index, one can refer the papers [2–6]. All the results with regards to Randić index are obtained for a simple graphs. In this paper, the authors define Randić index of a graph with self-loops. Let G_S be a graph obtained by attaching a self-loop to each vertices in the set $S \subseteq V$ of vertices of the graph

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$G(V,E)$, where $|S| = \sigma$. The Randić index of G_S is defined as

$$\begin{aligned} R(G_S) &= \sum_{v_iv_j \in E(G_S)} \frac{1}{\sqrt{\deg_{G_S}(v_i)\deg_{G_S}(v_j)}} \\ &= \sum_{\substack{v_iv_j \in E(G) \\ \wedge\ v_i,v_j \in V-S}} \frac{1}{\sqrt{\deg_G(v_i)\deg_G(v_j)}} + \sum_{\substack{v_iv_j \in E(G) \\ \wedge\ v_i \in S, v_j \in V-S}} \frac{1}{\sqrt{(\deg_G(v_i)+2)\deg_{G_S}(v_j)}} \\ &\quad + \sum_{\substack{v_iv_j \in E(G) \\ \wedge\ v_i,v_j \in S}} \frac{1}{\sqrt{(\deg_G(v_i)+2)(\deg_G(v_j)+2)}} + \sum_{v_i \in S} \frac{1}{\deg_G(v_i)+2}. \end{aligned}$$

A graph is a tree if it is connected and acyclic. In a tree, the vertex with degree 1 is called a pendant vertex and the vertex with degree 2 or more is called an internal vertex. The notation $\langle S \rangle$ represents the graph induced by the vertices of the set S.

For all notations and terminology, the reader is directed to the references [7, 8].

2. Main Results

The Randić index of a graph G_S may increase, decrease or equal to the Randić index of the graph G. For instance, consider a path graph P_4 with path $v_1v_2v_3v_4$. Let $S = \{v_1\}$. Then, $R((P_4)_S) = 1.9486$, which is more than $R(P_4) = 1.9142$. For the same graph P_4, if $S = \{v_2\}$, then $R((P_4)_S) = 1.8106$ which is less than Randić index of P_4. For the path graph $P_2 = \{v_1, v_2\}$ with $S = \{v_1, v_2\}$, $R(P_2) = R((P_2)_S) = 1$.

Theorem 2.1. *Let $G(V,E)$ be a r-regular graph and $S \subseteq V$ with $|S| = \sigma$. For the graph G_S, obtained by attaching a self-loops to each vertices of S,*

$$R(G_S) = \frac{m_S + \sigma}{r+2} + \frac{m_{V-S}}{r} + \frac{m - m_S - m_{V-S}}{\sqrt{r(r+2)}},$$

where $m = |E(G)|$, $m_S = |E(\langle S \rangle)|$ and $m_{V-S} = |E(\langle V - S \rangle)|$.

Proof. Let G_S be a graph obtained by attaching a self-loop to each vertex in the set $S \subseteq V$ of a r-regular graph $G(V,E)$. Consider,

$$\begin{aligned} R(G_S) = &\sum_{\substack{v_iv_j \in E(G) \\ \wedge\ v_i,v_j \in S}} \frac{1}{\sqrt{(r+2)^2}} + \sum_{\substack{v_iv_j \in E(G) \\ \wedge\ v_i \in S, v_j \notin S}} \frac{1}{\sqrt{r(r+2)}} + \sum_{\substack{v_iv_j \in E(G) \\ \wedge\ v_i,v_j \notin S}} \frac{1}{\sqrt{r^2}} \\ &+ \sum_{v_i \in S} \frac{1}{\sqrt{(r+2)^2}}. \end{aligned}$$

Let $|E| = m$, m_S be the number of edges of $\langle S \rangle$, and m_{V-S} be the number of edges of $\langle V - S \rangle$. Therefore,

$$R(G_S) = \frac{m_S + \sigma}{r+2} + \frac{m_{V-S}}{r} + \frac{m - m_S - m_{V-S}}{\sqrt{r(r+2)}}. \qquad \square$$

Theorem 2.2. *Let G be a r-regular graph of order n and size m and G_S be a graph obtained by attaching a self-loop to all the vertices of G. Then,*

$$R(G_S) = R(G) = \frac{n}{2}.$$

Proof. Let G_S be a graph obtained by attaching a self-loop to all the vertices of the graph G. Then,

$$R(G_S) = \frac{m+\sigma}{r+2}.$$

But $\sigma = n$ and $m = \frac{nr}{2}$ for r-regular graph. Therefore,

$$R(G_S) = \frac{nr+2n}{2(r+2)} = \frac{n}{2}.$$

Also,

$$R(G) = \frac{m}{r} = \frac{nr}{2r} = \frac{n}{2}.$$

Therefore, if $\sigma = n$,

$$R(G_S) = R(G) = \frac{n}{2}. \qquad \square$$

Theorem 2.3. *Let G_S be a graph obtained by attaching a self-loop to each vertices in the set $S \subseteq V$ of a graph $G(n, m)$. If $|S| = \sigma = n$, then*

$$\frac{m+n}{\Delta+2} \leq R(G_S) \leq \frac{m+n}{\delta+2}.$$

Upper and lower bound sharpness occur for the regular graph.

Proof. Let G be a graph and G_S be a graph obtained by attaching a self-loop to all the vertices of G. Then,

$$R(G_S) = \sum_{v_iv_j \in E(G)} \frac{1}{\sqrt{(\deg_G(v_i)+2)(\deg_G(v_j)+2)}} + \sum_{i=1}^{n} \frac{1}{\deg_G(v_i)+2}.$$

But,

$$\begin{aligned}\sqrt{(\deg_G(v_i)+2)(\deg_G(v_j)+2)} &= \sqrt{\deg_G(v_i)\deg_G(v_j) + 2(\deg_G(v_i)+\deg_G(v_j)) + 4}\\ &\leq \sqrt{\Delta^2+4\Delta+4}\\ &= \Delta+2.\end{aligned}$$

This implies,

$$\frac{1}{\sqrt{(\deg_G(v_i)+2)(\deg_G(v_j)+2)}} \geq \frac{1}{\Delta+2}.$$

Also,

$$\frac{1}{\deg_G(v_i)+2} \geq \frac{1}{\Delta+2}.$$

Therefore,

$$R(G_S) \geq \frac{m}{\Delta+2} + \frac{n}{\Delta+2} \geq \frac{m+n}{\Delta+2}.$$

Similarly,

$$\frac{1}{\sqrt{(\deg_G(v_i)+2)(\deg_G(v_j)+2)}} \leq \frac{1}{\delta+2}$$

and

$$\frac{1}{\deg_G(v_i)+2} \leq \frac{1}{\delta+2}.$$

Therefore,

$$R(G_S) \leq \frac{m}{\delta+2} + \frac{n}{\delta+2} \leq \frac{m+n}{\delta+2}.$$

If G is a regular graph, then $\deg_G(v_i) = \Delta = \delta = r$, for each $i = 1, 2, \ldots, n$ and therefore,

$$R(G_S) = \frac{m+n}{r+2}. \qquad \square$$

Theorem 2.4. *Let T be a tree of order n having k-pendant vertices and T_S be a graph obtained by adding a self-loop to each pendant vertex. Then,*

$$\frac{n-1}{3} + \frac{n-1}{\sqrt{3(n-1)}} \leq R(T_S) \leq \frac{k}{3} + \frac{k}{\sqrt{6}} + \frac{n+k-1}{2}.$$

Lower bound sharpness occurs for a star graph and upper bound sharpness occurs for a path graph.

Proof. Let T_S be a graph obtained by adding a self-loop to all k-pendant vertices of a tree of order n. Now,

$$\begin{aligned} R(T_S) =& \sum_{\substack{v_iv_j \in E(T) \\ v_i, v_j \notin S}} \frac{1}{\sqrt{\deg_T(v_i)\deg_T(v_j)}} + \sum_{i=1}^{k} \frac{1}{\sqrt{\deg_T(v_i)(\deg_T(v_k)+2)}} \\ &+ \sum_{i=1}^{k} \frac{1}{\sqrt{(\deg_T(v_k)+2)^2}} \\ \leq& \frac{n-k-1}{2} + \frac{k}{\sqrt{6}} + \frac{k}{3}. \end{aligned}$$

Equality holds for a path graph since each internal vertex of a path graph is of degree 2.

Now, for upper bound, $\deg_T(v_i) \leq n-1$ and therefore $\frac{1}{\sqrt{\deg_T(v_i)}} \geq \frac{1}{n-1}$. Consider,

$$\begin{aligned}
R(T_S) = &\sum_{\substack{v_iv_j \in E(T)\\ v_i,v_j \notin S}} \frac{1}{\sqrt{\deg_T(v_i)\deg_T(v_j)}} + \sum_{i=1}^{k} \frac{1}{\sqrt{\deg_T(v_i)(\deg_T(v_k)+2)}}\\
&+ \sum_{i=1}^{k} \frac{1}{\sqrt{(\deg_T(v_k)+2)^2}}\\
\geq &\frac{n-1}{\sqrt{3(n-1)}} + \frac{n-1}{3}.
\end{aligned}$$

Equality holds for a star graph since maximum degree of an internal vertex of a star graph is $n-1$. □

Theorem 2.5. *Let T be a tree of order n having k-pendant vertices and T_S be a graph obtained by adding a self-loop to each internal vertex. Then,*

$$\frac{\sqrt{n+1}+(n-1)}{n+1} \leq R(T_S) \leq \frac{2n-1}{4}.$$

Lower bound sharpness occurs for a star graph and upper bound sharpness occurs for a path graph.

Proof. Let T be a tree of order n having k-pendant vertices and T_S be a graph obtained by adding a self-loop to each internal vertex. Let v_k and v_i represent pendant vertex and internal vertex, respectively. Now, consider

$$\begin{aligned}
R(T_S) = &\sum_{v_iv_j \in E(T)} \frac{1}{\sqrt{(\deg_T(v_i)+2)(\deg_T(v_j)+2)}} + \sum_{i=1}^{n-k} \frac{1}{\sqrt{(\deg_T(v_i)+2)\deg_T(v_k)}}\\
&+ \sum_{i=1}^{n-k} \frac{1}{\sqrt{(\deg_T(v_i)+2)^2}}\\
\leq &\frac{n-k-1}{4} + \frac{k}{2} + \frac{n-k}{4}\\
= &\frac{2n-1}{4}.
\end{aligned}$$

Equality holds for a path graph since each internal vertex of a path graph is of degree 2.

For upper bound, $\deg_T(v_i) \leq n-1$ and therefore $\frac{1}{\sqrt{\deg_T(v_i)+2}} \geq \frac{1}{n+1}$. Consider

$$\begin{aligned}R(T_S) =& \sum_{v_iv_j \in E(T)} \frac{1}{\sqrt{(\deg_T(v_i)+2)(\deg_T(v_j)+2)}} + \sum_{i=1}^{n-k} \frac{1}{\sqrt{(\deg_T(v_i)+2)\deg_T(v_k)}}\\ &+ \sum_{i=1}^{n-k} \frac{1}{\sqrt{(\deg_T(v_i)+2)^2}}\\ \geq& \frac{1}{n+1} + \frac{n-1}{\sqrt{n+1}}.\end{aligned}$$

Equality holds for a star graph since maximum degree of an internal vertex of a star graph is $n-1$. ☐

Theorem 2.6. *Let G be a bipartite graph with partition $V = \{V_1, V_2\}$ and G_S be a graph obtained by adding a self-loop to each vertex of S in G. Let $S_1, S_2 \subseteq S$ with $S_1 \cup S_2 = S$, $S_1 \cap V_2 = \emptyset$, $S_2 \cap V_1 = \emptyset$, $|S_1| = \sigma_1$ and $|S_2| = \sigma_2$. Then,*

$$\begin{aligned}R(G_S) \geq& \frac{m\langle S_1 \cup S_2\rangle}{\sqrt{(m+2)(n+2)}} + \frac{m\langle V - (S_1 \cup S_2)\rangle}{\sqrt{mn}}\\ &+ \frac{m\langle S_1 \cup (V_2 - S_2)\rangle}{\sqrt{m(n+2)}} + \frac{m\langle S_2 \cup (V_1 - S_1)\rangle}{\sqrt{n(m+2)}} + \frac{\sigma_1}{n+2} + \frac{\sigma_2}{m+2}.\end{aligned}$$

The bound sharpness occurs for the complete bipartite graph.

Proof. Let G_S be a graph obtained by adding a self-loop to each vertex in the set $S \subseteq V$ of a bipartite graph G with partition $V = \{V_1, V_2\}$. Also, let $S = S_1 \cup S_2$, with $S_1 \cap V_2 = S_2 \cap V_1 = \emptyset$. Then,

$$\begin{aligned}R(G_S) =& \sum_{\substack{v_iv_j \in E(G)\ \wedge\\ v_i, v_j \notin S}} \frac{1}{\sqrt{\deg_G(v_i)\deg_G(v_j)}}\\ &+ \sum_{\substack{v_iv_j \in E(G)\ \wedge\\ v_i, v_j \in S}} \frac{1}{\sqrt{(\deg_G(v_i)+2)(\deg_G(v_j)+2)}}\\ &+ \sum_{\substack{v_iv_j \in E(G)\ \wedge\\ v_i \in S, v_j \notin S}} \frac{1}{\sqrt{(\deg_G(v_i)+2)\deg_G(v_j)}}\\ &+ \sum_{\substack{v_iv_j \in E(G)\ \wedge\\ v_i \notin S, v_j \in S}} \frac{1}{\sqrt{\deg_G(v_i)(\deg_G(v_j)+2)}}\\ &+ \sum_{v_i \in S_1} \frac{1}{\sqrt{(\deg_G(v_i)+2)^2}} + \sum_{v_j \in S_2} \frac{1}{\sqrt{(\deg_G(v_j)+2)^2}}.\end{aligned}$$

But, $\deg_G(v_i)\deg_G(v_j) \le mn$, $\deg_G(v_i) \le n$ if $v_i \in S_1$ and $\deg_G(v_j) \le m$ if $v_j \in S_2$. Therefore,

$$\begin{aligned} R(G_S) \geq & \frac{m\langle S_1 \cup S_2\rangle}{\sqrt{(m+2)(n+2)}} + \frac{m\langle V - (S_1 \cup S_2)\rangle}{\sqrt{mn}} \\ & + \frac{m\langle S_1 \cup (V_2 - S_2)\rangle}{\sqrt{m(n+2)}} + \frac{m\langle S_2 \cup (V_1 - S_1)\rangle}{\sqrt{n(m+2)}} + \frac{\sigma_1}{n+2} + \frac{\sigma_2}{m+2}. \end{aligned}$$

Equality holds for a complete bipartite graph since $\deg_G(v_i)\deg_G(v_j) = mn$, $\deg_G(v_i) = n$ if $v_i \in S_1$ and $\deg_G(v_j) = m$ if $v_j \in S_2$. □

Theorem 2.7. *Let $K_{m,n}$, $m \le n$, be a complete bipartite graph with partition $V = \{V_1, V_2\}$, $(K_{m,n})_S{}'$ be a graph obtained by attaching a self-loop to each vertex of V_1 and $(K_{m,n})_S{}''$ be a graph obtained by attaching a self-loop to each vertex of V_2. Then,*

$$R(K_{m,n})_S{}' \le R((K_{m,n})_S{}'').$$

Equality holds if and only if $m = n$.

Proof. Let $(K_{m,n})_S{}'$ be a graph obtained by attaching a self-loop to each vertex of V_1 and $(K_{m,n})_S{}''$ be a graph obtained by attaching a self-loop to each vertex of V_2 of a complete bipartite graph $K_{m,n}$, $m \le n$, with partition $V = \{V_1, V_2\}$. Now, $R((K_{m,n})_S{}') = \frac{mn}{\sqrt{m(n+2)}} + \frac{m}{n+2}$ and $R((K_{m,n})_S{}'') = \frac{mn}{\sqrt{n(m+2)}} + \frac{n}{m+2}$. From this, one can easily observe that

$$R(K_{m,n})_S{}' \le R((K_{m,n})_S{}'').$$

If $m = n$, then $m(n+2) = n(m+2)$, $m+2 = n+2$ and therefore $R((K_{m,n})_S{}'') = R((K_{m,n})_S{}')$. Conversely, if $m \ne n$, then $R((K_{m,n})_S{}'') \ne R((K_{m,n})_S{}')$ since $m(n+2) \ne n(m+2)$ and $m+2 \ne n+2$. □

3. Future Scope

(a) Characterize the class of graphs for which Randić index of a graph with self-loops is more than the Randić index of a simple graph and vice versa.
(b) Obtain the minimum and maximum Randić index for the class of graphs of given order.

4. Conclusion

The Randić index of a graph with self-loop is defined and bounds for Randić index of regular graph, tree and complete bipartite graph with self-loops are obtained.

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[1]Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal-576104, India
Email address: harshuarao@gmail.com
ORCID iD: https://orcid.org/0000-0002-1028-3265
Email address: sabitha.dsouza@manipal.edu
ORCID iD: https://orcid.org/0000-0002-1028-3265
Email address: pg.bhat@manipal.edu
ORCID iD: https://orcid.org/0000-0003-2179-6207

*Corresponding author

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ESSENTIAL NORM OF GENERALIZED INTEGRATION OPERATOR BETWEEN ZYGMUND TYPE SPACES

MOSTAFA HASSANLOU[1*] AND FARIBA ALIGHADR[2]

Abstract. Considering the generalized integration operator

$$\left(C^n_{\varphi,g} f\right)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)\; d\xi,$$

between two Zygmund type spaces, the essential norm of the operator will be estimated. Here φ is an analytic self-map on $\mathbb{D}$, $n \in \mathbb{N}$ and $g \in H(\mathbb{D})$. As a result, a criteria for the compactness of the above operator is given in the paper.

1. Introduction

First we bring some notation and information will be used in the paper. A main problem concerning composition operator or other classical operators is to relate the function-theoretic properties of the inducing function of the operator and operator-theoretic properties of it's own operator while acts on various Banach spaces of analytic functions. For example, it is well known that the composition operator C_φ is bounded on the classical Hardy, Bergman and Bloch spaces. See the excellent books [2, 13] for complete study on the composition operators and classical spaces of analytic functions.

Throughout the paper $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Every analytic self-map φ of the unit disk $\mathbb{D}$ induces through composition a linear composition operator C_φ from $H(\mathbb{D})$ to itself which is defined as

$$C_\varphi(f)(z) = f(\varphi(z)).$$

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The above operator is generalized by Li and Stević in [7] as follows:

$$\left(C_{\varphi}^{g} f\right)(z)=\int_{0}^{z} f'(\varphi(\xi)) g(\xi)\, d\xi, \quad f \in H(\mathbb{D}), z \in \mathbb{D},$$

where $g \in H(\mathbb{D})$. The following operator is a generalization of C_{φ}^{g} on the unit disk

$$\left(C_{\varphi,g}^{n} f\right)(z)=\int_{0}^{z} f^{(n)}(\varphi(\xi)) g(\xi)\, d\xi, \quad f \in H(\mathbb{D}), z \in \mathbb{D},$$

where $n \in \mathbb{N}$. If $n=1$, then we write $C_{\varphi,g}^{n}=C_{\varphi}^{g}$. This integral type operator has been investigated by many authors, see, e.g., [1,5–8,19,20].

Some characterizations of the boundedness, compactness and estimating the essential norm of generalized integration operators or other classical operators on or into the Zygmund type spaces can be found in [1,4–7,9–12,19–22].

The class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\|=\sup \frac{\left|f\left(e^{i(\theta+h)}\right)+f\left(e^{i(\theta-h)}\right)-2 f\left(e^{i\theta}\right)\right|}{h}<+\infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h>0$, is denoted by $\mathcal{Z}$ and called Zygmund space. The Closed Graph Theorem together [3, Theorem 5.3] implies that $f \in \mathcal{Z}$ if and only if

$$\sup_{z \in \mathbb{D}}\left(1-|z|^{2}\right)|f''(z)|<+\infty. \tag{1.1}$$

The space $\mathcal{Z}$ is a Banach space with the following norm

$$\|f\|_{\mathcal{Z}}=|f(0)|+|f'(0)|+\sup_{z \in \mathbb{D}}\left(1-|z|^{2}\right)|f''(z)|. \tag{1.2}$$

Motivated by the above definition, the Zygmund type space $\mathcal{Z}_{\alpha}$, $\alpha>0$, is defined as the space of all analytic functions f on $\mathbb{D}$ for which $\sup_{z\in\mathbb{D}}\left(1-|z|^{2}\right)^{\alpha}|f''(z)|<+\infty$ and the norm on this spaces is

$$\|f\|_{\mathcal{Z}_{\alpha}}=|f(0)|+|f'(0)|+\sup_{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha}|f''(z)|.$$

The little Zygmund type space of $\mathbb{D}$, denoted by $\mathcal{Z}_{\alpha,0}$, is the closed subspace of $\mathcal{Z}_{\alpha}$ consisting of functions f with

$$\lim_{|z| \rightarrow 1}\left(1-|z|^{2}\right)^{\alpha}|f''(z)|=0.$$

Zygmund type spaces can be generalized on the set the functions defined on or the weight function instead of $(1-|z|^{2})^{\alpha}$. We refer, for some researches considered Zygmund type spaces, to [5–8,11,12,15,19–22].

Also, for $\alpha>0$ the Bloch type space $\mathcal{B}^{\alpha}$ is the space of the functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^{\alpha}}=|f(0)|+\sup_{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha}|f'(z)|<+\infty.$$

The essential of the operator $T : X \to Y$ between Banach spaces is the distance of T from the space of compact operators from X to Y. In other words

$$\|T\|_{e,X\to Y} = \inf \|T - S\|,$$

where the infimum is taken all over the compact operators $S : X \to Y$ and $\|.\|$ is the operator norm. The operator is compact if and only if $\|T\|_{e,X\to Y} = 0$.

The aim of the paper is to approximate the essential norm of the operator $C^n_{\varphi,g}$ between Zygmund type space $\mathcal{Z}_\alpha$. Before that, we characterize bounded generalized integration operators on these spaces. It should be noted that the operator norm can be computed from boundedness conditions.

In this paper, for real scalars A and B, the notation $\preceq$ means $A \leq cB$ for some positive constant c. Also, the notation $A \approx B$ means $A \preceq B$ and $B \preceq A$.

2. Bounded Operator

In this section, we first state some facts about functions in Zygmund type spaces and then characterize the bounded operators. If $f \in \mathcal{Z}_\alpha$, then

$$|f''(z)| \leq \frac{\|f\|_{\mathcal{Z}_\alpha}}{(1-|z|^2)^\alpha}. \tag{2.1}$$

It is known that for $f \in \mathcal{B}^\alpha$

$$|f^{(k)}(z)| \leq \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+k-1}},$$

where $k \geq 2$, see for example [17]. So, for $f \in \mathcal{Z}_\alpha$ we get

$$|f^{(k+1)}(z)| \leq \frac{\|f\|_{\mathcal{Z}_\alpha}}{(1-|z|^2)^{\alpha+k-1}}. \tag{2.2}$$

For estimating $|f(z)|$ and $|f'(z)|$ the following lemma is applied, see [4].

Lemma 2.1. *For every $f \in \mathcal{Z}_\alpha$ where $\alpha > 0$ we have*

(i) $|f'(z)| \leq \frac{2}{1-\alpha}||f||_{\mathcal{Z}_\alpha}$ *and* $|f(z)| \leq \frac{2}{1-\alpha}||f||_{\mathcal{Z}_\alpha}$ *for every* $0 < \alpha < 1$;

(ii) $|f'(z)| \leq 2||f||_{\mathcal{Z}} \log \frac{1}{1-|z|}$ *and* $|f(z)| \leq ||f||_{\mathcal{Z}}$ *for* $\alpha = 1$;

(iii) $|f'(z)| \leq \frac{2}{\alpha-1} \cdot \frac{||f||_{\mathcal{Z}_\alpha}}{(1-|z|)^{\alpha-1}}$, *for every* $\alpha > 1$;

(iv) $|f(z)| \leq \frac{2}{(\alpha-1)(2-\alpha)}||f||_{\mathcal{Z}_\alpha}$, *for every* $1 < \alpha < 2$;

(v) $|f(z)| \leq 2||f||_{\mathcal{Z}_2} \log \frac{2}{1-|z|}$, *for* $\alpha = 2$;

(vi) $|f(z)| \leq \frac{2}{(\alpha-1)(\alpha-2)} \cdot \frac{||f||_{\mathcal{Z}_\alpha}}{(1-|z|)^{\alpha-2}}$, *for every* $\alpha > 2$.

Because of the above estimations, we divide the results for $n \geq 2$ and $n = 1$.

Theorem 2.1. *Let $n \geq 2$ and $0 < \alpha, \beta < +\infty$. Then, $C^n_{\varphi,g} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded if and only if*

$$M_1 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}} < +\infty, \quad M_2 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}} < +\infty.$$

Proof. Suppose that $C^n_{\varphi,g} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded. So there exists a positive constant C such that for every $f \in \mathcal{Z}_\alpha$

$$\|C^n_{\varphi,g} f\|_{\mathcal{Z}_\beta} \leq C\|f\|_{\mathcal{Z}_\alpha}.$$

We prove that $M_1 < +\infty$. The proof of the other one is similar. Since any polynomial belongs to $\mathcal{Z}_\alpha$, by taking the function $f_1(z) = z^n$ in the above inequality, we obtain

$$R_1 = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta |g'(z)| < +\infty. \tag{2.3}$$

Similarly by using $f_2(z) = z^{n+1}$, we get

$$\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta |\varphi'(z)g(z) + \varphi(z)g'(z)| < +\infty. \tag{2.4}$$

According to (2.3), (2.4) and boundedness of φ, we obtain that

$$R_2 = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta |\varphi'(z)| \cdot |g(z)| < +\infty. \tag{2.5}$$

Fix $a \in \mathbb{D}$ and define the functions F_a as follows

$$F_a(z) = C_F \frac{(1 - |a|^2)^3}{(1 - \overline{a}z)^{\alpha+1}} - D_F \frac{(1 - |a|^2)^2}{(1 - \overline{a}z)^{\alpha}}, \tag{2.6}$$

where $C_F = \frac{\alpha}{\overline{a}^{n+1}}$ and $D_F = \frac{\alpha(\alpha+n)}{\overline{a}^{n+1}}$. It can be checked that $F_a \in \mathcal{Z}_\alpha$, $F_a^{(n)}(a) = 0$ and

$$F_a^{(n+1)}(a) = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{(1 - |a|^2)^{\alpha+n-1}}.$$

Also, $\sup_{\frac{1}{2}<|a|<1} \|F_a\|_{\mathcal{Z}_\alpha} < +\infty$. Then, using the boundedness of the operator, we have

$$\begin{aligned}
\|C^n_{\varphi,g} F_{\varphi(a)}\|_{\mathcal{Z}_\beta} &\geq \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta \left|\left(C^n_{\varphi,g} F_{\varphi(a)}\right)''(z)\right| \\
&\geq \left(1 - |a|^2\right)^\beta \left|\varphi'(a)g(a)F^{(n+1)}_{\varphi(a)}(\varphi(a))\right| - \left(1 - |a|^2\right)^\beta \left|g'(a)F^{(n)}_{\varphi(a)}(\varphi(a))\right| \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+n)\,(1 - |a|^2)^\beta\, |\varphi'(a)| \cdot |g(a)|}{(1 - |\varphi(a)|^2)^{\alpha+n-1}}.
\end{aligned}$$

Hence,

$$\sup_{1/2<|\varphi(a)|<1} \frac{(1 - |a|^2)^\beta |\varphi'(a)| \cdot |g(a)|}{(1 - |\varphi(a)|^2)^{\alpha+n-1}} \preceq \sup_{1/2<|\varphi(a)|<1} \|C^n_{\varphi,g} F_{\varphi(a)}\|_{\mathcal{Z}_\beta} < +\infty. \tag{2.7}$$

On the other hand

$$\sup_{|\varphi(a)|\leq 1/2} \frac{(1 - |a|^2)^\beta |\varphi'(a)| \cdot |g(a)|}{(1 - |\varphi(a)|^2)^{\alpha+n-1}} \preceq \sup_{|\varphi(a)|\leq 1/2} \left(1 - |a|^2\right)^\beta |\varphi'(a)| \cdot |g(a)| \leq R_2 < +\infty. \tag{2.8}$$

From (2.7) and (2.8), $M_1 < +\infty$. In order to prove $M_2 < +\infty$ we use the functions

$$G_a(z) = C_G \frac{(1-|a|^2)^3}{(1-\overline{a}z)^{\alpha+1}} - D_G \frac{(1-|a|^2)^2}{(1-\overline{a}z)^{\alpha}}, \tag{2.9}$$

where $C_G = \frac{\alpha}{\overline{a}^n}$ and $D_G = \frac{\alpha+n+1}{\overline{a}^n}$. Then, $G_a^{(n+1)}(a) = 0$ and

$$G_a^{(n)}(a) = -\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{(1-|a|^2)^{\alpha+n-2}}.$$

In the rest of the proof is used the same method and do not bring here.

Conversely, suppose that M_1 and M_2 are finite. Let $f \in \mathcal{Z}_\alpha$. The equation (2.2) implies that

$$\begin{aligned}
\|C^n_{\varphi,g} f\|_{\mathcal{Z}_\beta} =& |f^{(n)}(\varphi(0))| \cdot |g(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \Big|\varphi'(z) g(z) f^{(n+1)}(\varphi(z)) \\
&+ g'(z) f^{(n)}(\varphi(z))\Big| \\
\le& \frac{|g(0)|}{(1-|\varphi(0)|^2)^{\alpha+n-2}} \|f\|_{\mathcal{Z}_\alpha} + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}} \|f\|_{\mathcal{Z}_\alpha} \\
&+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}} \|f\|_{\mathcal{Z}_\alpha}.
\end{aligned}$$

So, $C^n_{\varphi,g} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded. □

Theorem 2.2. *Let $n=1$. Then, $C^g_\varphi = C^1_{\varphi,g} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded if and only if*

$$M_3 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^\alpha} < +\infty$$

and

(i) *for $0<\alpha<1$, $g \in \mathcal{B}^\beta$;*
(ii) *for $\alpha = 1$, $\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(z)| \log (1-|\varphi(z)|^2)^{-1} < +\infty$;*
(iii) *for $\alpha > 1$, $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} < +\infty$.*

Proof. To prove that $M_3 < +\infty$, we use the same method as in the proof of Theorem 2.1. Also the proof of part (iii) is similar, take $n=1$ in the proof of Theorem 2.1. Moreover the part (ii) comes from [7] with a simple modification for $\beta > 0$.

Now suppose that $M_3 < +\infty$ and $g \in \mathcal{B}^\beta$. Then,

$$\begin{aligned}
\|C^g_\varphi f\|_{\mathcal{Z}_\beta} \le& |g(0) f'(\varphi(0))| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |\varphi'(z) g(z) f''(\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |g'(z) f'(\varphi(z))| \\
\le& \frac{2}{1-\alpha} |g(0)| \cdot \|f\|_{\mathcal{Z}_\alpha} + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^\alpha} \|f\|_{\mathcal{Z}_\alpha}
\end{aligned}$$

$$+\frac{2}{1-\alpha}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}|g'(z)|\cdot\|f\|_{\mathcal{Z}_\alpha}.$$

So, $C_\varphi^g:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ is bounded. Now we assume that the operator is bounded. We just prove that $g\in\mathcal{B}^\beta$ and this comes from applying C_φ^g to the function $f_0(z)=z$. $\square$

3. Essential Norm

In the following theorem we bring an approximation for the essential norm, as a tool for investigate the compactness of $C^n_{\varphi,g}:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$.

Theorem 3.1. *Let $n\geq 2$ and $\alpha,\beta>0$. If $C^n_{\varphi,g}:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ is bounded, then*

$$\|C^n_{\varphi,g}\|_e\approx\max\left\{\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}},\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}}\right\}. \tag{3.1}$$

As a result of this theorem, we can find a characterization for compactness of the operator: $C^n_{\varphi,g}:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ is compact if and only if

$$\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}=0,\quad \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}}=0.$$

Proof of Theorem 3.1. Define the operator $K_r:\mathcal{Z}_\alpha\to\mathcal{Z}_\alpha$ by $K_rf=f_r$, where $0<r<1$ and $f_r(z)=f(rz)$. Then, K_r is a compact operator. Let $\{r_j\}$ be a sequence in $(0,1)$ and $r_j\to1$ as $j\to+\infty$. So $C^n_{\varphi,g}K_{r_j}$ is compact $\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$. Let $f\in\mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha}\leq1$. Then,

$$\begin{aligned}
&\|(C^n_{\varphi,g}-C^n_{\varphi,g}K_{r_j})f\|_{\mathcal{Z}_\beta}\\
=&|g(0)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(0))\right|\\
&+\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\beta\left|\varphi'(z)g(z)\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))+g'(z)\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\
\leq&|g(0)|\cdot\left|(f-f_{r_j})^{(n)}(\varphi(0))\right|\\
&+\sup_{|\varphi(z)|\leq r_N}\left(1-|z|^2\right)^\beta|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\
&+\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^\beta|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\
&+\sup_{|\varphi(z)|\leq r_N}\left(1-|z|^2\right)^\beta|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\
&+\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^\beta|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|,
\end{aligned}$$

where $N\in\mathbb{N}$ such that $r_j\geq2/3$ for $j\geq N$. Since $f_{r_j}\to f$ uniformly on compact subsets of $\mathbb{D}$, then

$$\limsup_{j\to+\infty}|g(0)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(0))\right|=0$$

and also

$$\begin{aligned}&\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left(1-|z|^2\right)^{\beta}|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\ \le& R_2\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|=0\end{aligned}$$

and

$$\begin{aligned}&\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left(1-|z|^2\right)^{\beta}|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\ \le& R_1\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|=0,\end{aligned}$$

where R_1 and R_2 come from equations (2.3) and (2.5). So, we have

$$\begin{aligned}&\limsup_{j\to+\infty}\|(C^n_{\varphi,g}-C^n_{\varphi,g}K_{r_j})f\|_{\mathcal{Z}_\beta}\\ \le&\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^{\beta}|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\ &+\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^{\beta}|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\ \preceq& 2\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\frac{(1-|z|^2)^{\beta}|\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}\\ &+2\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\frac{(1-|z|^2)^{\beta}|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}}.\end{aligned}$$

Definition of essential norm implies that

$$\begin{aligned}\|C^n_{\varphi,g}\|_e\le&\limsup_{j\to+\infty}\|C^n_{\varphi,g}-C^n_{\varphi,g}K_{r_j}\|\\ =&\limsup_{j\to+\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\|(C^n_{\varphi,g}-C^n_{\varphi,g}K_{r_j})f\|_{\mathcal{Z}_\beta}\\ \preceq&\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^{\beta}|\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}+\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^{\beta}|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}},\end{aligned}$$

and the upper bound for the essential norm finds. In order to prove the lower bound, let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to1$, $k\to+\infty$. We use the sequence of functions $f_k(z)=F_{\varphi(z_k)}(z)$ as in (2.6). Then, $\{f_k\}$ is bounded and converges to zero uniformly on compact subsets of $\mathbb{D}$. Moreover, $\sup_k\|f_k\|_{\mathcal{Z}_\alpha}<+\infty$. So for any compact operator $K:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ we have

$$\begin{aligned}\|C^n_{\varphi,g}-K\|\succeq&\limsup_{k\to+\infty}\|(C^n_{\varphi,g}-K)f_k\|_{\mathcal{Z}_\beta}\\ \ge&\limsup_{k\to+\infty}\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\beta}-\limsup_{k\to+\infty}\|Kf_k\|_{\mathcal{Z}_\beta}\\ =&\limsup_{k\to+\infty}\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\beta}\end{aligned}$$

$$\geq \limsup_{k\to+\infty} \left(1-|z_k|^2\right)^\beta |\varphi'(z_k)| \cdot |g(z_k)| \cdot \left|f_k^{(n+1)}(\varphi(z_k))\right|$$
$$\approx \limsup_{k\to+\infty} \frac{(1-|z_k|^2)^\beta |\varphi'(z_k)| \cdot |g(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n-1}}.$$

Hence,

$$\|C^n_{\varphi,g}\|_e = \inf_K \|C^n_{\varphi,g} - K\| \succeq \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}.$$

If we use the sequence $g_k(z) = G_{\varphi(z_k)}(z)$ as in (2.9), then it can be obtained

$$\|C^n_{\varphi,g}\|_e \succeq \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}}.$$

Now the proof of theorem is completed. □

In the case $n = 1$, the following theorem can be applied.

Theorem 3.2. *Let $\alpha, \beta > 0$. If $C^g_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded, then the following hold.*

(i) *If $0 < \alpha < 1$, then*

$$\|C^g_\varphi\|_e \approx \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}.$$

(ii) *If $\alpha = 1$, then*

$$\|C^g_\varphi\|_e \approx \max \left\{ \begin{array}{l} \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |g'(z)| \log (1-|\varphi(z)|^2)^{-1}, \\ \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|\cdot|g(z)|}{1-|\varphi(z)|^2} \end{array} \right\}.$$

(iii) *If $\alpha > 1$, then*

$$\|C^g_\varphi\|_e \approx \max \left\{ \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}, \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \right\}.$$

Proof. (i) The upper bound can be obtained from the proof of Theorem 3.1 using Lemma 2.1 and the lower bound can be obtained from the proof of Theorem 3.1.

(ii) The upper bound can be obtained from the proof of Theorem 3.1 using Lemma 2.1 and the lower bound can be obtained from the proof of Theorem 3.1 using the functions $f_k(z) = F_{\varphi(z_k)}(z)$ and

$$g_k(z) = \frac{\overline{\varphi(z_k)}z - 1}{\overline{\varphi(z_k)}} \left[\left(1 + \log \frac{1}{1-\overline{\varphi(z_k)}z}\right)^2 + 1 \right] \left(\log \frac{1}{1-|\varphi(z_k)|^2}\right)^{-1}.$$

(iii) The proof is similar to the proof of Theorem 3.1. □

As a result, we can find a characterization for compactness. The operator $C^g_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is compact if and only if

(i) $0 < \alpha < 1$:

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0;$$

(ii) $\alpha = 1$:

$$\limsup_{|\varphi(z)|\to 1} \left(1-|z|^2\right)^\beta |g'(z)| \log\left(1-|\varphi(z)|^2\right)^{-1} = 0,$$

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{1-|\varphi(z)|^2} = 0,$$

which is Theorem 6 of [7], $\beta = 1$;

(iii) $\alpha > 1$:

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0, \quad \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0.$$

Remark 3.1. Let $g(z) = \varphi'(z)$. Then, $\left(C_\varphi^g f\right)(z) = \int_0^z f'(\varphi(\xi))\varphi'(\xi)\, d\xi = C_\varphi f(z)$, the composition operator. So, the essential norm and also characterization for compactness of composition operator $C_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ can be obtained which is known in the literature.

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[1]Engineering Faculty of Khoy,
Urmia University of Technology,
Urmia, Iran
Email address: m.hassanlou@uut.ac.ir
ORCID iD: https://orcid.org/0000-0002-9213-2574

[2]Faculty of Mathematics, Statistics and Computer Science,
University of Tabriz,
Tabriz, Iran
Email address: faribaalighadr@gmail.com
ORCID iD: https://orcid.org/0000-0001-5682-9334

*Corresponding author

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ON THE ASYMPTOTIC BEHAVIORS ASSOCIATED WITH THE DAVISON FUNCTIONAL EQUATION

MOHAMMAD AMIN TAREEGHEE[1], ABBAS NAJATI[1*], AND JAE-HYEONG BAE[2]

Abstract. We prove the Hyers-Ulam stability of the Davison functional equation

$$f(x+xy)+f(y)=f(x+y)+f(xy),$$

for a class of mappings from a normed algebra $\mathcal{A}$ (with a unit element 1) into a Banach space $\mathcal{B}$, on the restricted domain $\{(x,y)\in\mathcal{A}\times\mathcal{A}:\ \min\{\|x\|,\|y\|\}\geqslant d\}$, where $d>0$ is a constant. As a result, we obtain some asymptotic behaviors of Davison mappings. In addition, we obtain the corollary that for every mapping g from a normed algebra $\mathcal{A}$ into a normed space $\mathcal{B}$, and for all positive real numbers r,s, one of the following two conditions must be valid:

$$\sup_{x,y\in\mathcal{A}}\|g(x+y)+g(xy)-g(x+xy)-g(y)\|\cdot\|x\|^r\cdot\|y\|^s=+\infty$$

or

$$g(x+y)+g(xy)=g(x+xy)+g(y).$$

1. Introduction and preliminaries

The functional equation

$$f(x+xy)+f(y)=f(x+y)+f(xy), \tag{1.1}$$

was proposed by Davison [2] at the 17th International Symposium on Functional Equations. He inquired about its general solution for mappings from a commutative field $\mathbb{F}$ to another commutative field $\mathbb{K}$. At the same symposium, Benz [1] provided the general continuous solution to the functional equation (1.1) when f is an unknown mapping from the real numbers to the real numbers. In 2000, Girgensohn and Lajkó [4] characterized the general solution of (1.1) without requiring any regular condition.

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They showed that if $f : \mathbb{R} \to \mathbb{R}$ is a solution of (1.1), then f can be expressed as $f(x) = A(x) + b$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive mapping and $b \in \mathbb{R}$ is any constant. Furthermore, they derived the general solution of the pexiderized version of (1.1). In a separate work, Davison [3] determined the solution of (1.1) when the domain of the unknown mapping f is the ring of integers $\mathbb{Z}$ or the set of natural numbers $\mathbb{N}$. Najati and Sahoo [9] introduced two pexiderized functional equations of Davison type and obtained their general solutions.

Jung and Sahoo [6] were the first to study the Hyers-Ulam stability of the Davison functional equation (1.1). The pexiderized functional equation

$$f(xy) + f(x + y) = g(xy + x) + g(y)$$

was investigated for the Hyers-Ulam stability in [5]. Studying the Hyers-Ulam stability of Davison functional equation (1.1) and its pexiderized version on restricted domains would be interesting topics. Let $\mathcal{A}$ be a normed algebra and consider

$$\begin{aligned}
D_1 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \min\{\|x\|, \|y\|\} \geqslant d\}, \\
D_2 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \|x\| \geqslant d\}, \\
D_3 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \|y\| \geqslant d\}, \\
D_4 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \|x\| + \|y\| \geqslant d\}, \\
D_5 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \max\{\|x\|, \|y\|\} \geqslant d\},
\end{aligned}$$

where $d > 0$ is a real constant. It is clear that $D_1 \subseteq D_j$ for $2 \leqslant j \leqslant 5$. The primary objective of this current paper is to investigate the Hyers-Ulam stability of (1.1) on the unbounded restricted domain D_1. As a consequence, we obtain a hyperstability result for the Davison functional equation (1.1). This leads us to deduce the slightly surprising result that for any mapping f, from a normed algebra $\mathcal{A}$ into a normed space $\mathcal{B}$, and for all positive real numbers $r, s > 0$ one of the following two conditions must hold true:

(i) $\sup_{x,y\in\mathcal{A}} \|f(x + xy) + f(y) - f(x + y) - f(xy)\| \cdot \|x\|^r \cdot \|y\|^s = +\infty$,
(ii) $f(x + xy) + f(y) = f(x + y) + f(xy), \ x, y \in \mathcal{A}$.

Also (ii) is equivalent to

$$\sup_{x,y\in\mathcal{A}} \|f(x + xy) + f(y) - f(x + y) - f(xy)\|(\|x\|^r + \|y\|^s) = +\infty.$$

2. Stability and Hyperstability

The following lemma plays a key role in proving the main theorem.

Lemma 2.1. *Let $\varepsilon \geqslant 0$ and $d > 0$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a mapping from a normed algebra $\mathcal{A}$ (with unit element 1) to a normed space $\mathcal{B}$ satisfying*

$$\|f(x + y) + f(xy) - f(x + xy) - f(y)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.1}$$

Then,

$$\|f(x + 4y) + f(x + 4y + 1) - f(4y) - f(4y + 1) - f(2x + 2y) + f(2y)\| \leqslant 3\varepsilon, \tag{2.2}$$

for all $x, y \in \mathcal{A}$*, with* $\min\{\|x\|, \|y\|\} \geqslant d+1$*. Moreover,*

$$\Big\|f(-2x)+f(2x)-f(x)-f(-x)\Big\| \leqslant 39\varepsilon, \quad \|x\| \geqslant 4d+4, \tag{2.3}$$

$$\|-f(-4x+1)-f(2x)+f(-2x)+f(1)\| \leqslant 12\varepsilon, \quad \|x\| \geqslant 2d+2, \tag{2.4}$$

$$\|f(2x)-2f(x)+f(0)\| \leqslant 213\varepsilon, \quad \|x\| \geqslant 12d+12. \tag{2.5}$$

Proof. Replace y by $y+1$ in (2.1) to obtain

$$\|f(x+y+1)+f(xy+x)-f(2x+xy)-f(y+1)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d+1. \tag{2.6}$$

Adding (2.6) and (2.1), one obtains

$$\|f(x+y+1)+f(x+y)+f(xy)-f(2x+xy)-f(y)-f(y+1)\| \leqslant 2\varepsilon, \tag{2.7}$$

for all $x, y \in \mathcal{A}$, with $\min\{\|x\|, \|y\|\} \geqslant d+1$. By substituting $4y$ for y in (2.7), we obtain

$$\|f(x+4y+1)+f(x+4y)+f(4xy)-f(2x+4xy)-f(4y)-f(4y+1)\| \leqslant 2\varepsilon, \tag{2.8}$$

for all $x, y \in \mathcal{A}$, with $\min\{\|x\|, \|y\|\} \geqslant d+1$. Replacing x by $2x$ and y by $2y$ in (2.1), one obtains

$$\|f(2x+2y)+f(4xy)-f(2x+4xy)-f(2y)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.9}$$

Using (2.8) and (2.9), we get (2.2).

By substituting $-2x$ for x and x for y in (2.2), we obtain

$$\|2f(2x)+f(2x+1)-f(4x)-f(4x+1)-f(-2x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.10}$$

Also, replacing x by $2x$ and y by $\frac{x}{2}$ in (2.2), we get

$$\|f(4x)+f(4x+1)-f(2x)-f(2x+1)-f(5x)+f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.11}$$

Adding (2.10) and (2.11), we obtain

$$\|f(2x)-f(-2x)-f(5x)+f(x)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.12}$$

By substituting $-3x$ for x and $\frac{x}{2}$ for y in (2.2), we have

$$\|f(-x)+f(-x+1)-f(2x)-f(2x+1)-f(-5x)+f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.13}$$

Add (2.10) and (2.13), to get

$$\begin{aligned} \|f(2x)-f(4x)-f(4x+1)-f(-2x)+f(-x) \\ +f(-x+1)-f(-5x)+f(x)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant 2d+2. \end{aligned} \tag{2.14}$$

Replacing x by $3x$ and y by $-x$ in (2.2), we have

$$\|f(-x)+f(-x+1)-f(-4x)-f(-4x+1)-f(4x)+f(-2x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.15}$$

By (2.14) and (2.15), we conclude

$$\begin{aligned} \|f(-4x)+f(-4x+1)-2f(-2x)+f(2x) \\ -f(4x+1)-f(-5x)+f(x)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant 2d+2. \end{aligned} \tag{2.16}$$

By substituting $-x$ for x in equation (2.10) and then combining the result with inequalities (2.10) and (2.16), we arrive at

$$\begin{aligned}\|f(-2x) - 2f(2x) - f(-5x) + f(x) + f(-2x+1)\\ - f(2x+1) + f(4x)\| \leqslant 15\varepsilon, \quad \|x\| \geqslant 2d+2.\end{aligned} \tag{2.17}$$

If we substitute $-4x$ for x and $\frac{x}{2}$ for y in (2.2), we can obtain

$$\|f(-2x) + f(-2x+1) - f(2x) - f(2x+1) - f(-7x) + f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.18}$$

It can be inferred from equations (2.17) and (2.18) that

$$\| - f(2x) - f(-5x) + f(4x) + f(-7x)\| \leqslant 18\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.19}$$

Replacing x by $-x$ in (2.19) and adding the resultant to (2.12), we obtain

$$\|f(7x) + f(-4x) - f(2x) - f(x)\| \leqslant 24\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.20}$$

Replacing x by $2x$ and y by $\frac{3x}{2}$ in (2.2), we have

$$\|f(8x) + f(8x+1) - f(6x) - f(6x+1) - f(7x) + f(3x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.21}$$

Also, replacing x by $-2x$ and y by $2x$ in (2.2), we get

$$\|f(6x) + f(6x+1) - f(8x) - f(8x+1) - f(0) + f(4x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.22}$$

Add (2.21) and (2.22), to obtain

$$\| - f(7x) + f(3x) + f(4x) - f(0)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.23}$$

Also, adding (2.20) and (2.23), we arrive at

$$\|f(-4x) + f(4x) + f(3x) - f(2x) - f(x) - f(0)\| \leqslant 30\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.24}$$

Putting $y = \frac{x}{2}$ in (2.2), we have

$$\|f(3x+1) - f(2x) - f(2x+1) + f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.25}$$

Now, replacing x by $\frac{x}{2}$ in (2.22) and combining the resultant to (2.25), we conclude

$$\|f(3x) - f(4x) - f(4x+1) + 2f(2x) + f(2x+1) - f(x) - f(0)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.26}$$

It follows from (2.10) and (2.26) that

$$\|f(3x) + f(-2x) - f(x) - f(0)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.27}$$

So, by combining (2.24) and (2.27), we get (2.3).

By substituting $3x - y$ for x in (2.2), we get the following inequality:

$$\|f(3x+3y) + f(3x+3y+1) - f(6x) - f(4y) - f(4y+1) + f(2y)\| \leqslant 3\varepsilon,$$

for all $x, y \in \mathcal{A}$, with $\min\{\|3x - y\|, \|y\|\} \geqslant d+1$. If we put $y = -x$ in the above inequality, we can rewrite it as:

$$\|f(0) + f(1) - f(6x) - f(-4x) - f(-4x+1) + f(-2x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.28}$$

Finally, by substituting $2x$ for x in (2.27) and adding the result to (2.28), we arrive at inequality (2.4).

Replacing x by $-x$ in (2.10) and then combining the resultant inequality with (2.4), one obtains

$$\|f(-4x) - f(-2x+1) - f(-2x) + f(1)\| \leqslant 15\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.29}$$

Also, replacing x by $\frac{x}{2}$ in (2.4) and then combining the resultant inequality with (2.29), we conclude

$$\|f(-4x) - f(-2x) + f(x) - f(-x)\| \leqslant 27\varepsilon, \quad \|x\| \geqslant 4d+4. \tag{2.30}$$

Substitute $2x$ for x in (2.27) and then combining the obtained inequality with (2.30), we obtain the following inequality:

$$\|f(6x) + f(-2x) - f(x) + f(-x) - f(2x) - f(0)\| \leqslant 36\varepsilon, \quad \|x\| \geqslant 4d+4. \tag{2.31}$$

Inequality (2.3) gives us

$$\Big\|2f(-4x) + 2f(4x) - 2f(2x) - 2f(-2x)\Big\| \leqslant 78\varepsilon, \quad \|x\| \geqslant 4d+4.$$

By (2.3), (2.31) and the above inequality, we conclude

$$\|f(6x) - 2f(2x) - 2f(x) + 2f(4x) + 2f(-4x) - f(0)\| \leqslant 153\varepsilon, \quad \|x\| \geqslant 4d+4. \tag{2.32}$$

By multiplying (2.24) by 2 and adding the result to (2.32), we get

$$\|f(6x) - 2f(3x) + f(0)\| \leqslant 213\varepsilon, \quad \|x\| \geqslant 4d+4.$$

This can be rewritten as inequality (2.5), which is the desired result. □

Now we are ready to prove the main theorem.

Theorem 2.1. *Take $\varepsilon \geqslant 0$, $d > 0$. Let $\mathcal{A}$ be a normed algebra* (*with unit element* 1) *and $\mathcal{B}$ a Banach space. If a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies* (2.1)*, then there is a unique additive mapping $\varphi : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - \varphi(x) - f(0)\| \leqslant 640\varepsilon, \quad x \in \mathcal{A}. \tag{2.33}$$

Proof. By Lemma 2.1, f fulfills (2.5). Then, for all integers n, m with $n \geqslant m \geqslant 0$, we have

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} + \sum_{i=m}^{n} \frac{f(0)}{2^{i+1}}\right\| \leqslant \sum_{i=m}^{n} \frac{213\varepsilon}{2^{i+1}}, \quad \|x\| \geqslant 12d+12. \tag{2.34}$$

Therefore, $\{\frac{f(2^n x)}{2^n}\}_n$ is a Cauchy sequence for all $x \in \mathcal{A}$. Define $\varphi : \mathcal{A} \to \mathcal{B}$ by $\varphi(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{2^n}$ for all $x \in \mathcal{A}$. Obviously, $\varphi(2x) = 2\varphi(x)$ for all $x \in \mathcal{A}$. Therefore, by (2.3) we infer that φ is odd. So, by (2.4), we conclude

$$\varphi(x) = \lim_{n\to+\infty} \frac{f(2^n x + 1)}{2^n}, \quad x \in \mathcal{A}.$$

Hence, it follows from (2.2) that

$$2\varphi(x+4y) + \varphi(2y) = 2\varphi(4y) + \varphi(2x+2y), \quad x, y \in \mathcal{A}. \tag{2.35}$$

Since $\varphi(2x) = 2\varphi(x)$, (2.35) can be written as

$$\varphi(2x+8y) = 3\varphi(2y) + \varphi(2x+2y), \quad x, y \in \mathcal{A}. \tag{2.36}$$

Putting $x = -y$ in (2.36) and using $\varphi(0) = 0$, we conclude

$$\varphi(3y) = 3\varphi(y), \quad y \in \mathcal{A}. \tag{2.37}$$

Hence, (2.36) and (2.37) yield

$$\varphi(2x+8y) = \varphi(6y) + \varphi(2x+2y), \quad x, y \in \mathcal{A}. \tag{2.38}$$

Replacing y by $\frac{y}{6}$ and x by $\frac{x}{2} - \frac{y}{6}$ in (2.38), we deduce that φ is an additive mapping.

By setting $m = 0$ and letting n approach infinity in (2.34), we arrive at

$$\|f(x) - \varphi(x) - f(0)\| \leqslant 213\varepsilon, \quad \|x\| \geqslant 12d + 12. \tag{2.39}$$

For $y \in \mathcal{A} \setminus \{0\}$ we can choose $x \in \mathcal{A}$ such that

$$\min\{\|x\|, \|xy\|, \|x+y\|, \|x+xy\|\} \geqslant 12d + 12.$$

By (2.39), we have the following inequalities

$$\begin{aligned} \|-f(x+y) + \varphi(x+y) + f(0)\| &\leqslant 213\varepsilon, \\ \|-f(xy) + \varphi(xy) + f(0)\| &\leqslant 213\varepsilon, \\ \|f(x+xy) - \varphi(x+xy) - f(0)\| &\leqslant 213\varepsilon. \end{aligned}$$

Combining the previous inequalities and (2.1), we get

$$\|f(y) - \varphi(y) - f(0)\| \leqslant 640\varepsilon.$$

Since this inequality holds for $y = 0$, we deduce (2.33) which is what we wanted to prove. ☐

As a result, we conclude that if a mapping f satisfies (1.1) on a certain subset $D \subseteq \mathcal{A}$, then f fulfills (1.1) on the entire $\mathcal{A}$.

In the subsequent results, $\mathcal{A}$ denotes a normed algebra with unit element and $\mathcal{B}$ is a normed space.

Corollary 2.1. *Suppose that a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies one of the following assertions:*

(i) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0$, $\min\{\|x\|, \|y\|\} \geqslant d$,
(ii) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0$, $\max\{\|x\|, \|y\|\} \geqslant d$,
(iii) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0$, $\|x\| + \|y\| \geqslant d$,
(iv) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0$, $\|x\| \geqslant d$,
(v) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0$, $\|y\| \geqslant d$,

for some constant $d > 0$. Then, $f - f(0)$ is additive on $\mathcal{A}$.

Proof. Since $(ii) - (v)$ imply (i), we only need to deal with (i). Applying Lemma 2.1 for $\varepsilon = 0$ we deduce

$$f(2x) = 2f(x) - f(0), \quad \|x\| \geqslant 12d + 12.$$

By induction on n, one obtains

$$f(2^n x) = 2^n f(x) - (2^n - 1)f(0), \quad \|x\| \geqslant 12d + 12.$$

This yields the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is convergent for all $x \in \mathcal{A}$. We define

$$\varphi(x) := \lim_{n\to+\infty} \frac{f(2^n x)}{2^n}, \quad x \in \mathcal{A}.$$

By applying some parts of the proof of Theorem 2.1, we deduce that φ is additive and $\varphi(x) = f(x) - f(0)$ for all $x \in \mathcal{A}$. This ends the proof. $\square$

In the following, we investigate a result that concerns some asymptotic properties related to Davison mappings.

Corollary 2.2. *Suppose that a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies one of the following conditions:*

(i) $\lim_{\min\{\|x\|,\|y\|\}\to+\infty} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(ii) $\lim_{\max\{\|x\|,\|y\|\}\to+\infty} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(iii) $\lim_{\|x\|+\|y\|\to+\infty} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(iv) $\lim_{\|x\|\to+\infty} \sup_{y\in\mathcal{A}} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(v) $\lim_{\|y\|\to+\infty} \sup_{x\in\mathcal{A}} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0.$

Then, $f - f(0)$ is additive on $\mathcal{A}$.

Proof. It is clear that (i) is a consequence of $(ii) - (v)$. Therefore, we only consider (i). Let $\varepsilon > 0$ be any given real number and $\widetilde{\mathcal{B}}$ be the completion of $\mathcal{B}$. From (i), we can find $d_\varepsilon > 0$ such that

$$\|f(x+y) + f(xy) - f(x+xy) - f(y)\| < \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d_\varepsilon.$$

By applying Theorem 2.1 we obtain a constant $K > 0$ and an additive mapping $\varphi_\varepsilon : \mathcal{A} \to \widetilde{\mathcal{B}}$ that satisfy

$$\|\varphi_\varepsilon(x) - f(x) + f(0)\| \leqslant K\varepsilon, \quad x \in \mathcal{A}.$$

So,

$$\begin{aligned}\|f(x+y) - f(x) - f(y) + f(0)\| &\leqslant \|f(x+y) - \varphi_\varepsilon(x+y) - f(0)\| \\ &\quad + \|\varphi_\varepsilon(x) - f(x) + f(0)\| \\ &\quad + \|\varphi_\varepsilon(y) - f(y) + f(0)\| \leqslant 3K\varepsilon, \quad x, y \in \mathcal{A}.\end{aligned}$$

Because ε was chosen arbitrarily, we conclude that $f(x+y) = f(x) + f(y) - f(0)$ for every $x, y \in \mathcal{A}$. This yields that $f - f(0)$ is additive on $\mathcal{A}$. $\square$

Corollary 2.3. *Take $\delta, \varepsilon \geqslant 0$ and suppose that $p, q < 0$ are real numbers and a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies*

$$\|f(x+y)+f(xy)-f(x+xy)-f(y)\| \leqslant \varepsilon\|x\|^p\|y\|^q+\delta(\|x\|^p+\|y\|^q),$$

for all $x, y \in \mathcal{A}$ with $\min\{\|x\|, \|y\|\} \geqslant d$, where $d > 0$ is a constant. Then, $f - f(0)$ is additive on $\mathcal{A}$.

As a result, we can deduce the slightly surprising result that for any mapping f, from a normed algebra $\mathcal{A}$ into a normed space $\mathcal{B}$, and for all positive real numbers $r, s > 0$ one of the following two conditions must hold true:

(i) $\sup_{x,y\in\mathcal{A}} \|f(x+xy)+f(y)-f(x+y)-f(xy)\| \cdot \|x\|^r \cdot \|y\|^s = +\infty$,
(ii) $f(x+xy)+f(y) = f(x+y)+f(xy),\ x, y \in \mathcal{A}$.

Also (ii) is equivalent to

$$\sup_{x,y\in\mathcal{A}} \|f(x+xy)+f(y)-f(x+y)-f(xy)\|(\|x\|^r+\|y\|^s) = +\infty.$$

Corollary 2.4. *Take $\delta, \varepsilon > 0$ and $d > 0$. Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a mapping such that $F(x_0, y_0) \neq 0$ for some $x_0, y_0 \in \mathcal{A}$ with $\min\{\|x_0\|, \|y_0\|\} \geqslant d$ and there are real numbers $p, q < 0$ such that*

$$\|F(x,y)\| \leqslant \varepsilon\|x\|^p\|y\|^q+\delta(\|x\|^p+\|y\|^q), \quad \min\{\|x\|, \|y\|\} \geqslant d.$$

Then, there does not exist any mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies

$$f(x+y)+f(xy) = f(x+xy)+f(y)+F(x,y). \tag{2.40}$$

Proof. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a solution of (2.40). So,

$$\|f(x+y)+f(xy)-f(x+xy)-f(y)\| \leqslant \varepsilon\|x\|^p\|y\|^q+\delta(\|x\|^p+\|y\|^q),$$

where $\min\{\|x\|, \|y\|\} \geqslant d$. Consequently, based on the previous lemma, it can be concluded that $f - f(0)$ is additive on $\mathcal{A}$, which implies that $F(x_0, y_0) = 0$. This contradicts our initial assumption. $\square$

3. Conclusions

The Hyers-Ulam stability of the Davison functional equation has been investigated in previous studies [5–8]. In all of them, a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies the inequality

$$\|f(x+y)+f(xy)-f(x+xy)-f(y)\| \leqslant \varepsilon,$$

on the whole space $\mathcal{A}$. Studying the stability problems of the Davison functional equation on a restricted domain will also be an intriguing area of research. In more specific terms, we investigated whether a true additive mapping exists close to a mapping $f : \mathcal{A} \to \mathcal{B}$ that fulfills the aforementioned inequality only in the restricted domain $D_1 = \{(x, y) \in \mathcal{A} \times \mathcal{A} :\ \min\{\|x\|, \|y\|\} \geqslant d\}$. Consequently, we will be able to derive certain asymptotic behaviors of Davison mappings. Of course, it should be noted that this issue has been investigated on the domain $D_2 = \{(x, y) \in \mathcal{A} \times \mathcal{A} :\ \|x\| \geqslant d\}$, which contains D_1. The value derived from the estimate (2.33) is relatively large. It is anticipated that smaller values may be attainable through an alternative proof method. Therefore, an unresolved question arises: does the constant in inequality (2.33) represent the optimal estimate?

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[1]Department of Mathematics,
Faculty of Sciences,
University of Mohaghegh Ardabili,
Ardabil, Iran.
Email address: mohammadamint@uma.ac.ir
ORCID iD: https://orcid.org/0009-0000-8028-1495
Email address: a.nejati@yahoo.com
ORCID iD: https://orcid.org/0000-0001-7563-0507

*Corresponding Author

[2]Humanitas College,
Kyung Hee University,
Yongin 17104, Korea
Email address: jhbae@khu.ac.kr
ORCID iD: https://orcid.org/0000-0001-9513-3743

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ON GENERALIZED COMMUTATIVE JACOBSTHAL QUATERNIONS

DOROTA BRÓD[1], ANETTA SZYNAL-LIANA[1], AND IWONA WŁOCH[1]

Abstract. In this paper, we introduce and study generalized commutative Jacobsthal quaternions and their one-parameter generalization. We present some fundamental properties of them, among others the Binet formula, Catalan, Cassini, d'Ocagne and Vajda identities. Moreover, we give the generating functions and summation formulas for these numbers.

1. Introduction

Quaternions were introduced in 1843 by W. Hamilton for representing vectors in the space as follows. A quaternion q is a hyper-complex number represented by an equation $q = a + bi + cj + dk$, where a, b, c, d are real numbers and i, j, k are standard orthonormal basis in $\mathbb{R}^3$, which satisfy the quaternion multiplication rules

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j \quad \text{and} \quad ij = -ji = k.$$

From the above rules, it immediately follows that multiplication of quaternions is not commutative. For these reasons, it is not easy to study the problems of quaternion algebra, however Hamiltonian quaternions were modified or generalized in this direction so that commutative property in multiplication is possible, see [14].

Let $\mathbb{H}^c_{\gamma\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1e_1 + x_2e_2 + x_3e_3, \tag{1.1}$$

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where quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta, \tag{1.2}$$

$$e_1 e_2 = e_2 e_1 = e_3, \quad e_2 e_3 = e_3 e_2 = \beta e_1, \quad e_3 e_1 = e_1 e_3 = \alpha e_2, \tag{1.3}$$

and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$. The generalized commutative quaternions were introduced in [16]. They generalize elliptic quaternions ($\alpha < 0$, $\beta = 1$), parabolic quaternions ($\alpha = 0$, $\beta = 1$), hyperbolic quaternions ($\alpha > 0$, $\beta = 1$), bicomplex numbers ($\alpha = -1$, $\beta = -1$), complex hyperbolic numbers ($\alpha = -1$, $\beta = 1$) and hyperbolic complex numbers ($\alpha = 1$, $\beta = -1$).

Similarly to classical quaternions, we can define the conjugate of the generalized commutative quaternion $\mathbf{x}$ as $\overline{\mathbf{x}} = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3$ and norm $N(\mathbf{x})$ of $\mathbf{x}$ of the form $N(\mathbf{x}) = \mathbf{x} \cdot \overline{\mathbf{x}} = \overline{\mathbf{x}} \cdot \mathbf{x}$. By simple calculations we obtain $N(\mathbf{x}) = x_0^2 - x_1^2\alpha - x_2^2\beta - x_3^2\alpha\beta - 2x_2x_3\beta e_1 - 2x_1x_3\alpha e_2 - 2x_1x_2e_3$.

Note that associative quaternions can be divided into two families, the first family of noncommutative quaternions, for example Hamiltonian, hyperbolic, split quaternions and the second family of commutative quaternions including in particular generalized Serge quaternions, dual quaternions, see [14].

The theory of quaternions is complemented by Fibonacci type sequences, i.e., sequences defined by a second-order linear homogeneous recurrence relations with real coefficients, see [8, 10, 15, 17]. In this paper, we use the Jacobsthal sequence for studying commutative quaternions. Let us recall necessary definitions.

The Jacobsthal sequence $\{J_n\}$ was introduced by Horadam [9]. The first ten terms of the sequence are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171$. This sequence is defined by the following recurrence relation

$$J_n = J_{n-1} + 2J_{n-2}, \quad \text{for } n \geq 2, \tag{1.4}$$

with initial conditions $J_0 = 0$, $J_1 = 1$. The Binet formula of this sequence has the form

$$J_n = \frac{2^n - (-1)^n}{3}, \quad \text{for } n \geq 0.$$

The Jacobsthal numbers are a special case of Horadam numbers W_n defined by the recurrence

$$W_n = pW_{n-1} - qW_{n-2}, \quad \text{for } n \geq 2,$$

with fixed real numbers W_0 and W_1 and $p, q \in \mathbb{Z}$.

Fibonacci type numbers and their generalizations have applications in the theory of hypercomplex numbers, see for example [1, 7, 12, 18].

Many authors have studied some generalizations of the recurrence of the Jacobsthal sequence, see [5, 6, 11] and generalizations of hypercomplex Jacobsthal numbers, see [3,13]. In [2], a one-parameter generalization of the Jacobsthal numbers was introduced. We recall this generalization.

Let $n \geq 0$, $r \geq 0$ be integers. The nth r-Jacobsthal number $J_{r,n}$ is defined as follows

$$J_{r,n} = 2^r J_{r,n-1} + (2^r + 4^r) J_{r,n-2}, \quad \text{for } n \geq 2, \tag{1.5}$$

with $J_{r,0} = 1,\ J_{r,1} = 1 + 2^{r+1}$.

For $r = 0$, we have $J_{0,n} = J_{n+2}$. By (1.5) we obtain

$$
\begin{aligned}
J_{r,0} &= 1,\\
J_{r,1} &= 2 \cdot 2^r + 1,\\
J_{r,2} &= 3 \cdot 4^r + 2 \cdot 2^r,\\
J_{r,3} &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r,\\
J_{r,4} &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r,\\
J_{r,5} &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r.
\end{aligned}
\tag{1.6}
$$

The r-Jacobsthal numbers have a graph interpretation, they can be used for the counting of independent sets of special classes of graphs, see [2]. We will recall some properties of the r-Jacobsthal numbers.

Theorem 1.1 (Binet formula [2]). *Let $n \geq 0$, $r \geq 0$ be integers. Then, the nth r-Jacobsthal number is given by*

$$J_{r,n} = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}\gamma^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}\delta^n,$$

where

$$\gamma = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \delta = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Theorem 1.2 ([2]). *Let $n \geq 1$, $r \geq 0$ be integers. Then,*

$$\sum_{l=0}^{n-1} J_{r,l} = \frac{J_{r,n} + (2^r + 4^r)J_{r,n-1} - 2 - 2^r}{4^r + 2^{r+1} - 1}.$$

Theorem 1.3 (Convolution identity [2]). *Let n, m, r be integers such that $m \geq 2, n \geq 1$, $r \geq 0$. Then,*

$$J_{r,m+n} = 2^r J_{r,m-1} J_{r,n} + (4^r + 8^r) J_{r,m-2} J_{r,n-1}.$$

Theorem 1.4 ([2]). *The generating function of the sequence of r-Jacobsthal numbers has the following form*

$$f(t) = \frac{1 + (1 + 2^r)t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

2. Generalized Commutative Jacobsthal Quaternions

For $n \geq 0$, the nth generalized commutative Jacobsthal quaternion is defined by the following relation

$$gc\mathfrak{J}_n = J_n + J_{n+1}\, e_1 + J_{n+2}\, e_2 + J_{n+3}\, e_3, \tag{2.1}$$

where J_n is the nth Jacobsthal number and e_1, e_2, e_3 are quaternionic units which satisfy the rules (1.2) and (1.3).

Analogously, for $n \geq 0$ and $r \geq 0$, we define the nth generalized commutative r-Jacobsthal quaternion

$$gc\mathcal{J}_{r,n} = J_{r,n} + J_{r,n+1}\, e_1 + J_{r,n+2}\, e_2 + J_{r,n+3}\, e_3, \tag{2.2}$$

where $J_{r,n}$ is the nth r-Jacobsthal number.

By (1.6) and (2.2), we obtain

$$\begin{aligned} gc\mathcal{J}_{r,0} =& 1 + (2^{r+1} + 1)e_1 + (3 \cdot 4^r + 2^{r+1})e_2 + (5 \cdot 8^r + 5 \cdot 4^r + 2^r)e_3, \\ gc\mathcal{J}_{r,1} =& 2^{r+1} + 1 + (3 \cdot 4^r + 2^{r+1})e_1 + (5 \cdot 8^r + 5 \cdot 4^r + 2^r)e_2 \\ & + (8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r)e_3, \\ gc\mathcal{J}_{r,2} =& 3 \cdot 4^r + 2^{r+1} + (5 \cdot 8^r + 5 \cdot 4^r + 2^r)e_1 \\ & + (8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r)e_2 \\ & + (13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r)e_3. \end{aligned} \tag{2.3}$$

Using the fact that $J_{0,n} = J_{n+2}$, we get $gc\mathcal{J}_{0,n} = gc\mathcal{J}_{n+2}$.

By the definition of the generalized commutative r-Jacobsthal quaternion, we obtain the following recurrence relation.

Proposition 2.1. *Let $n \geq 2$, $r \geq 0$ be integers. Then,*

$$gc\mathcal{J}_{r,n} = 2^r gc\mathcal{J}_{r,n-1} + (2^r + 4^r) gc\mathcal{J}_{r,n-2},$$

where $gc\mathcal{J}_{r,0}$, $gc\mathcal{J}_{r,1}$ are given by (2.3)*.*

In [16], the authors introduced generalized commutative Horadam quaternions

$$gc\mathcal{H}_n = W_n + W_{n+1}\, e_1 + W_{n+2}\, e_2 + W_{n+3}\, e_3,$$

where W_n is the nth Horadam number and e_1, e_2, e_3 are quaternionic units which satisfy the rules (1.2) and (1.3). It was proved the following result.

Theorem 2.1 (Binet formula for generalized commutative Horadam quaternions, [16])**.** *Let $n \geq 0$ be an integer. Then,*

$$gc\mathcal{H}_n = At_1^n \hat{t_1} + Bt_2^n \hat{t_2},$$

where

$$\begin{aligned} t_1 =& \frac{1}{2}\left(p - \sqrt{p^2 - 4q}\right), \quad t_2 = \frac{1}{2}\left(p + \sqrt{p^2 - 4q}\right), \\ \hat{t_1} =& 1 + t_1 e_1 + t_1^2 e_2 + t_1^3 e_3, \quad \hat{t_2} = 1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3, \\ A =& \frac{W_1 - W_0 t_2}{t_1 - t_2}, \quad B = \frac{W_0 t_1 - W_1}{t_1 - t_2}. \end{aligned}$$

By Theorem 2.1, we get the Binet formula for the generalized commutative r-Jacobsthal quaternions and for the generalized commutative Jacobsthal quaternions.

Corollary 2.1 (Binet formula for generalized commutative r-Jacobsthal quaternions)**.** *Let $n \geq 0$, $r \geq 0$ be integers. Then,*

$$gc\mathfrak{J}_{r,n} = C_1\underline{\gamma}\gamma^n + C_2\underline{\delta}\delta^n, \tag{2.4}$$

where

$$\begin{aligned} \gamma =& 2^{r-1} + \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \quad \delta = 2^{r-1} - \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \\ \underline{\gamma} =& 1 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3, \quad \underline{\delta} = 1 + \delta e_1 + \delta^2 e_2 + \delta^3 e_3, \\ C_1 =& \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} + 3\cdot 2^r + 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}, \quad C_2 = \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} - 3\cdot 2^r - 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}. \end{aligned}$$

Corollary 2.2 (Binet formula for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$ be an integer. Then,*

$$gc\mathfrak{J}_n = \frac{1}{3}\Big[2^n(1 + 2e_1 + 4e_2 + 8e_3) - (-1)^n(1 - e_1 + e_2 - e_3)\Big].$$

Proof. By Corollary 2.1, for $r = 0$, we have $C_1 = \frac{4}{3}$, $C_2 = -\frac{1}{3}$, $\gamma = 2$, $\delta = -1$, and

$$\begin{aligned} gc\mathfrak{J}_{0,n} &= \frac{4}{3}\cdot 2^n(1 + 2e_1 + 4e_2 + 8e_3) - \frac{1}{3}(-1)^n(1 - e_1 + e_2 - e_3) \\ &= \frac{1}{3}\cdot 2^{n+2}(1 + 2e_1 + 4e_2 + 8e_3) - \frac{1}{3}(-1)^{n+2}(1 - e_1 + e_2 - e_3) \\ &= gc\mathfrak{J}_{n+2}. \end{aligned}$$

□

3. Properties of the Generalized Commutative r-Jacobsthal Quaternions

In this section, we give some identities such as Catalan, Cassini, d'Ocagne, and Vajda identities for the generalized commutative r-Jacobsthal quaternions. Moreover, we present convolution identity, a summation formula, and generating function for the generalized commutative r-Jacobsthal quaternions. In particular, we get analogous results for the generalized commutative Jacobsthal quaternions.

Theorem 3.1 (General bilinear index-reduction formula for the generalized commutative r-Jacobsthal quaternions)**.** *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then,*

$$gc\mathfrak{J}_{r,a}\cdot gc\mathfrak{J}_{r,b} - gc\mathfrak{J}_{r,c}\cdot gc\mathfrak{J}_{r,d} = -\frac{(1+2^r)^2}{4\cdot 2^r + 5\cdot 4^r}\underline{\gamma}\underline{\delta}\left(\gamma^a\delta^b + \gamma^b\delta^a - \gamma^c\delta^d - \gamma^d\delta^c\right).$$

Proof. By formula (2.4), we get

$$\begin{aligned} & gc\mathfrak{J}_{r,a}\cdot gc\mathfrak{J}_{r,b} - gc\mathfrak{J}_{r,c}\cdot gc\mathfrak{J}_{r,d} \\ =& (C_1\underline{\gamma}\gamma^a + C_2\underline{\delta}\delta^a)(C_1\underline{\gamma}\gamma^b + C_2\underline{\delta}\delta^b) - (C_1\underline{\gamma}\gamma^c + C_2\underline{\delta}\delta^c)(C_1\underline{\gamma}\gamma^d + C_2\underline{\delta}\delta^d) \\ =& C_1^2\underline{\gamma}^2(\gamma^{a+b} - \gamma^{c+d}) + C_2^2\underline{\delta}^2(\delta^{a+b} - \delta^{c+d}) + C_1C_2\underline{\gamma}\underline{\delta}(\gamma^a\delta^b - \gamma^c\delta^d + \gamma^b\delta^a - \gamma^d\delta^c). \end{aligned}$$

Using the fact that $a + b = c + d$ and $C_1C_2 = -\frac{(1+2^r)^2}{4\cdot 2^r + 5\cdot 4^r}$, we get the result. □

Moreover, by simple calculations, we get

$$\begin{aligned}\underline{\gamma\delta} = \underline{\delta\gamma} =& 1 + \gamma\delta\alpha + \gamma^2\delta^2\beta + \gamma^3\delta^3\alpha\beta + (\gamma+\delta)(1+\gamma^2\delta^2\beta)e_1 \\ &+ (\gamma^2+\delta^2)(1+\gamma\delta\alpha)e_2 + (\gamma^3+\delta^3+\gamma\delta(\gamma+\delta))e_3.\end{aligned}$$

Using the equalities

$$\begin{aligned}\gamma+\delta &= 2^r,\\ \gamma-\delta &= \sqrt{4\cdot 2^r + 5\cdot 4^r},\\ \gamma\delta &= -(4^r+2^r),\\ \gamma^2+\delta^2 &= (\gamma+\delta)^2 - 2\gamma\delta = 3\cdot 4^r + 2^{r+1},\\ \gamma^3+\delta^3 &= (\gamma+\delta)^3 - 3\gamma\delta(\gamma+\delta) = 4\cdot 8^r + 3\cdot 4^r,\end{aligned}$$

we have

$$\begin{aligned}\underline{\gamma\delta} = \underline{\delta\gamma} =& 1 - (4^r+2^r)\alpha + (4^r+2^r)^2\beta - (4^r+2^r)^3\alpha\beta \\ &+ 2^r(1+(4^r+2^r)^2\beta)e_1 + (2^{r+1}+3\cdot 4^r)(1-(4^r+2^r)\alpha)e_2 \\ &+ (2\cdot 4^r + 3\cdot 8^r)e_3.\end{aligned} \tag{3.1}$$

It is easily seen that for special values of a, b, c, d, by Theorem 3.1, we get new identities for generalized commutative r-Jacobsthal quaternions:

- for $a = b = n$, $c = n - m$ and $d = n + m$ Catalan identity,
- for $a = b = n$, $c = n - 1$ and $d = n + 1$ Cassini identity,
- for $a = n$, $b = m + 1$, $c = n + 1$ and $d = m$ d'Ocagne identity,
- for $a = m + k$, $b = n - k$, $c = m$ and $d = n$ Vajda identity.

Corollary 3.1 (Catalan identity for generalized commutative r-Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq m$. Then,*

$$gc\mathcal{J}_{r,n}^2 - gc\mathcal{J}_{r,n-m}\cdot gc\mathcal{J}_{r,n+m} = -\frac{(-4^r-2^r)^n(1+2^r)^2}{4\cdot 2^r + 5\cdot 4^r}\underline{\gamma\delta}\left(2 - \left(\frac{\gamma}{\delta}\right)^m - \left(\frac{\delta}{\gamma}\right)^m\right),$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

Corollary 3.2. (*Cassini identity for generalized commutative r-Jacobsthal quaternions*) *Let $n \geq 1$, $r \geq 0$ be integers. Then,*

$$gc\mathcal{J}_{r,n}^2 - gc\mathcal{J}_{r,n-1}\cdot gc\mathcal{J}_{r,n+1} = (-4^r-2^r)^{n-1}(1+2^r)^2\underline{\gamma\delta},$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

Corollary 3.3. (*d'Ocagne identity for the generalized commutative r-Jacobsthal quaternions*) *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq m$. Then,*

$$\begin{aligned}&gc\mathcal{J}_{r,n}\cdot gc\mathcal{J}_{r,m+1} - gc\mathcal{J}_{r,n+1}\cdot gc\mathcal{J}_{r,m}\\ =&\frac{(1+2^r)^2\sqrt{4\cdot 2^r + 5\cdot 4^r}}{4\cdot 2^r + 5\cdot 4^r}(-4^r-2^r)^m\underline{\gamma\delta}\left(\gamma^{n-m} - \delta^{n-m}\right),\end{aligned}$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

Corollary 3.4 (Vajda identity for the generalized commutative r-Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, $k \geq 0$, $r \geq 0$ be integers such that $n \geq k$ and $n \geq m$. Then,*

$$\begin{aligned} & gc\mathfrak{J}_{r,m+k} \cdot gc\mathfrak{J}_{r,n-k} - gc\mathfrak{J}_{r,m} \cdot gc\mathfrak{J}_{r,n} \\ &= -\frac{(1+2^r)^2(-4^r-2^r)^m}{4 \cdot 2^r + 5 \cdot 4^r} \underline{\gamma\delta} \left(\delta^{n-m} \left[\left(\frac{\gamma}{\delta}\right)^k - 1 \right] + \gamma^{n-m} \left[\left(\frac{\delta}{\gamma}\right)^k - 1 \right] \right), \end{aligned}$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

In particular, by Theorem 3.1, we obtain the following identities for generalized commutative Jacobsthal quaternions.

Corollary 3.5 (Catalan identity for generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, be integers such that $n \geq m$. Then,*

$$\begin{aligned} & gc\mathfrak{J}_n^2 - gc\mathfrak{J}_{n-m} \cdot gc\mathfrak{J}_{n+m} \\ =& \frac{1}{9}(-2)^{n-m} \left((-2)^m - 1\right)^2 (1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3). \end{aligned}$$

Corollary 3.6 (Cassini identity for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 1$ be an integer. Then,*

$$gc\mathfrak{J}_n^2 - gc\mathfrak{J}_{n-1} \cdot gc\mathfrak{J}_{n+1} = (-2)^{n-1}(1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3).$$

Corollary 3.7 (d'Ocagne identity for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$ be integers. Then,*

$$\begin{aligned} & gc\mathfrak{J}_n \cdot gc\mathfrak{J}_{m+1} - gc\mathfrak{J}_{n+1} \cdot gc\mathfrak{J}_m \\ =& \frac{1}{3}(-2)^m \left(2^{n-m} - (-1)^{n-m}\right) (1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3). \end{aligned}$$

Corollary 3.8 (Vajda identity for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, $k \geq 0$ be integers such that $n \geq k$. Then,*

$$\begin{aligned} & gc\mathfrak{J}_{m+k} \cdot gc\mathfrak{J}_{n-k} - gc\mathfrak{J}_m \cdot gc\mathfrak{J}_n \\ &= -\frac{1}{9}(-2)^m \left((-1)^{n-m} \left[(-2)^k - 1\right] + 2^{n-m} \left[\left(-\frac{1}{2}\right)^k - 1 \right] \right) \\ &\quad \times (1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3). \end{aligned}$$

Now, we give the convolution identity for the generalized commutative r-Jacobsthal quaternions.

Theorem 3.2. *Let $m \geq 2, n \geq 1$, $r \geq 0$ be integers. Then,*

$$\begin{aligned} 2gc\mathfrak{J}_{r,m+n} =& 2^r gc\mathfrak{J}_{r,m-1} \cdot gc\mathfrak{J}_{r,n} + (4^r + 8^r) gc\mathfrak{J}_{r,m-2} \cdot gc\mathfrak{J}_{r,n-1} \\ & + J_{r,m+n} - \alpha J_{r,m+n+2} - \beta J_{r,m+n+4} - 2gc\mathfrak{J}_{r,m+n+3}\, e_3 + \alpha\beta J_{r,m+n+6}. \end{aligned} \tag{3.2}$$

Proof. By simple calculations we get

$$\begin{aligned}
& gc\mathfrak{J}_{r,m-1} \cdot gc\mathfrak{J}_{r,n} \\
=& J_{r,m-1}J_{r,n} + J_{r,m-1}J_{r,n+1}e_1 + J_{r,m-1}J_{r,n+2}e_2 + J_{r,m-1}J_{r,n+3}e_3 \\
& + J_{r,m}J_{r,n}e_1 + \alpha J_{r,m}J_{r,n+1} + J_{r,m}J_{r,n+2}e_3 + \alpha J_{r,m}J_{r,n+3}e_2 \\
& + J_{r,m+1}J_{r,n}e_2 + J_{r,m+1}J_{r,n+1}e_3 + \beta J_{r,m+1}J_{r,n+2} + \beta J_{r,m+1}J_{r,n+3}e_1 \\
& + J_{r,m+2}J_{r,n}e_3 + \alpha J_{r,m+2}J_{r,n+1}e_2 + \beta J_{r,m+2}J_{r,n+2}e_1 + \alpha\beta J_{r,m+2}J_{r,n+3}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& gc\mathfrak{J}_{r,m-2} \cdot gc\mathfrak{J}_{r,n-1} \\
=& J_{r,m-2}J_{r,n-1} + J_{r,m-2}J_{r,n}e_1 + J_{r,m-2}J_{r,n+1}e_2 + J_{r,m-2}J_{r,n+2}e_3 \\
& + J_{r,m-1}J_{r,n-1}e_1 + \alpha J_{r,m-1}J_{r,n} + J_{r,m-1}J_{r,n+1}e_3 + \alpha J_{r,m-1}J_{r,n+2}e_2 \\
& + J_{r,m}J_{r,n-1}e_2 + J_{r,m}J_{r,n}e_3 + \beta J_{r,m}J_{r,n+1} + \beta J_{r,m}J_{r,n+2}e_1 \\
& + J_{r,m+1}J_{r,n-1}e_3 + \alpha J_{r,m+1}J_{r,n}e_2 + \beta J_{r,m+1}J_{r,n+1}e_1 + \alpha\beta J_{r,m+1}J_{r,n+2}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
& 2^r\, gc\mathfrak{J}_{r,m-1} \cdot\, gc\mathfrak{J}_{r,n} + (4^r + 8^r) gc\mathfrak{J}_{r,m-2} \cdot\, gc\mathfrak{J}_{r,n-1} \\
=& 2^r J_{r,m-1}J_{r,n} + (4^r + 8^r)\, J_{r,m-2}J_{r,n-1} \\
& + [2^r\, J_{r,m-1}J_{r,n+1} + (4^r + 8^r)\, J_{r,m-2}J_{r,n} + 2^r\, J_{r,m}J_{r,n} + (4^r + 8^r)\, J_{r,m-1}J_{r,n-1}]e_1 \\
& + [2^r\, J_{r,m-1}J_{r,n+2} + (4^r + 8^r)\, J_{r,m-2}J_{r,n+1} + 2^r\, J_{r,m+1}J_{r,n} + (4^r + 8^r)\, J_{r,m}J_{r,n-1}]e_2 \\
& + [2^r\, J_{r,m-1}J_{r,n+3} + (4^r + 8^r)\, J_{r,m-2}J_{r,n+2} + 2^r\, J_{r,m+2}J_{r,n} + (4^r + 8^r)\, J_{r,m+1}J_{r,n-1}]e_3 \\
& + \alpha(2^r\, J_{r,m}J_{r,n+1} + (4^r + 8^r)\, J_{r,m-1}J_{r,n}) + \beta(2^r\, J_{r,m+1}J_{r,n+2} + (4^r + 8^r)\, J_{r,m}J_{r,n+1}) \\
& + \alpha[2^r\, J_{r,m}J_{r,n+3} + (4^r + 8^r)\, J_{r,m-1}J_{r,n+2} + 2^r\, J_{r,m+2}J_{r,n+1} + (4^r + 8^r)\, J_{r,m+1}J_{r,n}]e_2 \\
& + [2^r\, J_{r,m+1}J_{r,n+1} + (4^r + 8^r)\, J_{r,m}J_{r,n} + 2^r\, J_{r,m}J_{r,n+2} + (4^r + 8^r)\, J_{r,m-1}J_{r,n+1}]e_3 \\
& + \beta[2^r\, J_{r,m+2}J_{r,n+2} + (4^r + 8^r)\, J_{r,m+1}J_{r,n+1} + 2^r J_{r,m+1}J_{r,n+3} + (4^r + 8^r) J_{r,m}J_{r,n+2}]e_1 \\
& + \alpha\beta(2^r J_{r,m+2}J_{r,n+3} + (4^r + 8^r) J_{r,m+1}J_{r,n+2}).
\end{aligned}$$

Using Theorem 1.3, we get

$$\begin{aligned}
& 2^r\, gc\mathfrak{J}_{r,m-1} \cdot\, gc\mathfrak{J}_{r,n} + (4^r + 8^r)\, gc\mathfrak{J}_{r,m-2} \cdot\, gc\mathfrak{J}_{r,n-1} \\
=& J_{r,m+n} + 2[J_{r,m+n+1}e_1 + J_{r,m+n+2}e_2 + J_{r,m+n+3}e_3] \\
& + \alpha J_{r,m+n+2} + \beta J_{r,m+n+4} + 2J_{r,m+n+3}e_3 \\
& + 2\alpha J_{r,m+n+4}e_2 + 2\beta J_{r,m+n+5}e_1 + \alpha\beta J_{r,m+n+6} \\
=& 2\, gc\mathfrak{J}_{r,m+n} - J_{r,m+n} + 2(J_{r,m+n+3} + J_{r,m+n+4}e_1 + J_{r,m+n+5}e_2 \\
& + J_{r,m+n+6}e_3)e_3 + \alpha J_{r,m+n+2} + \beta J_{r,m+n+4} - \alpha\beta J_{r,m+n+6} \\
=& 2\, gc\mathfrak{J}_{r,m+n} + 2gc\mathfrak{J}_{r,m+n+3}\, e_3 - J_{r,m+n} + \alpha J_{r,m+n+2} \\
& + \beta J_{r,m+n+4} - \alpha\beta J_{r,m+n+6}.
\end{aligned}$$

Hence, we get the result. □

Remark 3.1. The formula (3.2) can be written in the following way

$$\begin{aligned} gc\mathfrak{J}_{r,m+n} =& 2^r gc\mathfrak{J}_{r,m-1} \cdot gc\mathfrak{J}_{r,n} + (4^r + 8^r) gc\mathfrak{J}_{r,m-2} \cdot gc\mathfrak{J}_{r,n-1} \\ & - e_1 gc\mathfrak{J}_{r,m+n+1} - e_2 gc\mathfrak{J}_{r,m+n+2} - e_3 gc\mathfrak{J}_{r,m+n+3}. \end{aligned}$$

Corollary 3.9. *Let $m \geq 1$, $n \geq 1$ be integers. Then,*

$$\begin{aligned} 2gc\mathfrak{J}_{m+n} =& gc\mathfrak{J}_{m-1} \cdot gc\mathfrak{J}_{n} + 2gc\mathfrak{J}_{m} \cdot gc\mathfrak{J}_{n-1} \\ & + J_{m+n} - \alpha J_{m+n+2} - \beta J_{m+n+4} - 2gc\mathfrak{J}_{m+n+3}\, e_3 + \alpha\beta J_{m+n+6}. \end{aligned}$$

The next theorem presents a summation formula for the generalized commutative r-Jacobsthal quaternions.

Theorem 3.3. *Let $n \geq 1$, $r \geq 0$ be integers. Then,*

$$\begin{aligned} \sum_{l=1}^{n} gc\mathfrak{J}_{r,l} =& \frac{gc\mathfrak{J}_{r,n+1} + (2^r + 4^r)\, gc\mathfrak{J}_{r,n} - (1 + e_1 + e_2 + e_3)(1 + 3 \cdot 2^r + 4^r)}{4^r + 2^{r+1} - 1} \\ & - (2^{r+1} + 1)e_1 - (2^{r+2} + 3 \cdot 4^r + 1)e_2 - (1 + 5 \cdot 2^r + 8 \cdot 4^r + 5 \cdot 8^r)e_3. \end{aligned}$$

Proof. By Theorem 1.2, we get $\sum\limits_{l=1}^{n} J_{r,l} = \frac{J_{r,n+1} + (2^r + 4^r)J_{r,n} - 1 - 3 \cdot 2^r - 4^r}{4^r + 2^{r+1} - 1}$. Hence, we have

$$\begin{aligned} \sum_{l=1}^{n} gc\mathfrak{J}_{r,l} =& \sum_{l=1}^{n} (J_{r,l} + J_{r,l+1}e_1 + J_{r,l+2}e_2 + J_{r,l+3}e_3) \\ =& \sum_{l=1}^{n} J_{r,l} + \sum_{l=1}^{n} J_{r,l+1}e_1 + \sum_{l=1}^{n} J_{r,l+2}e_2 + \sum_{l=1}^{n} J_{r,l+3}e_3 \\ =& \frac{1}{4^r + 2^{r+1} - 1} \Big[J_{r,n+1} + (2^r + 4^r)J_{r,n} - 1 - 3 \cdot 2^r - 4^r \\ & + (J_{r,n+2} + (2^r + 4^r)J_{r,n+1} - 1 - 3 \cdot 2^r - 4^r)e_1 \\ & + (J_{r,n+3} + (2^r + 4^r)J_{r,n+2} - 1 - 3 \cdot 2^r - 4^r)e_2 \\ & + (J_{r,n+4} + (2^r + 4^r)J_{r,n+3} - 1 - 3 \cdot 2^r - 4^r)e_3 \Big] \\ & - J_{r,1}e_1 - (J_{r,1} + J_{r,2})e_2 - (J_{r,1} + J_{r,2} + J_{r,3})e_3. \end{aligned}$$

By simple calculations, we obtain

$$\begin{aligned} \sum_{l=1}^{n} gc\mathfrak{J}_{r,l} =& \frac{1}{4^r + 2^{r+1} - 1} \Big[J_{r,n+1} + J_{r,n+2}e_1 + J_{r,n+3}e_2 + J_{r,n+4}e_3 \\ & + (2^r + 4^r)(J_{r,n} + J_{r,n+1}e_1 + J_{r,n+2}e_2 + J_{r,n+3}e_3) \\ & - (1 + 3 \cdot 2^r + 4^r)(1 + e_1 + e_2 + e_3) \Big] - (2^{r+1} + 1)e_1 \\ & - (2^{r+2} + 3 \cdot 4^r + 1)e_2 - (1 + 5 \cdot 2^r + 8 \cdot 4^r + 5 \cdot 8^r)e_3 \\ =& \frac{gc\mathfrak{J}_{r,n+1} + (2^r + 4^r)\, gc\mathfrak{J}_{r,n} - (1 + e_1 + e_2 + e_3)(1 + 3 \cdot 2^r + 4^r)}{4^r + 2^{r+1} - 1} \\ & - (2^{r+1} + 1)e_1 - (2^{r+2} + 3 \cdot 4^r + 1)e_2 - (1 + 5 \cdot 2^r + 8 \cdot 4^r + 5 \cdot 8^r)e_3. \quad \square \end{aligned}$$

Corollary 3.10. *Let $n \geq 1$ be an integer. Then,*

$$\sum_{l=1}^{n} gc\mathcal{J}_l = \frac{1}{2}(gc\mathcal{J}_{n+2} - gc\mathcal{J}_2).$$

Using the following identities ([4])

$$\sum_{l=0}^{n} J_{2l} = \frac{1}{3}(2J_{2n+1} - n - 2), \tag{3.3}$$

$$\sum_{l=0}^{n} J_{2l+1} = \frac{1}{3}(2J_{2n+2} + n + 1), \tag{3.4}$$

we get the next results for the generalized commutative Jacobsthal quaternions.

Theorem 3.4. *Let $n \geq 1$ be an integer. Then,*

(i) $\sum\limits_{l=1}^{n} gc\mathcal{J}_{2l} = \frac{1}{3}\left(2gc\mathcal{J}_{2n+1} - n(2gc\mathcal{J}_2 - gc\mathcal{J}_3) - 2gc\mathcal{J}_1\right);$

(ii) $\sum\limits_{l=1}^{n} gc\mathcal{J}_{2l-1} = \frac{1}{3}\left(2gc\mathcal{J}_{2n} + n(2gc\mathcal{J}_2 - gc\mathcal{J}_3) - 2gc\mathcal{J}_0\right);$

(iii) $\sum\limits_{l=1}^{k} gc\mathcal{J}_{n+l} = \frac{1}{2}\left(gc\mathcal{J}_{n+k+2} - gc\mathcal{J}_{n+1}\right).$

Proof. (i)

$$\begin{aligned}\sum_{l=1}^{n} gc\mathcal{J}_{2l} =& J_2 + J_4 + \cdots + J_{2n} + (J_3 + J_5 + \cdots + J_{2n+1})e_1 \\ &+ (J_4 + J_6 + \cdots + J_{2n+2})e_2 + (J_5 + J_7 + \cdots + J_{2n+3})e_3.\end{aligned}$$

Using twice (3.3) and (3.4), we have

$$\begin{aligned}\sum_{l=1}^{n} gc\mathcal{J}_{2l} =& \frac{1}{3}\Big(2J_{2n+1} - n - 2) + (2J_{2n+2} + n - 2)e_1 \\ &+ (2J_{2n+3} - n - 6)e_2 + (2J_{2n+4} + n - 10)e_3\Big) \\ =& \frac{1}{3}(2(J_{2n+1} + J_{2n+2}e_1 + J_{2n+3}e_2 + J_{2n+4}e_3) - n(1 - e_1 + e_2 - e_3) \\ &- 2(1 + e_1 + 3e_2 + 5e_3)) = \frac{1}{3}(2gc\mathcal{J}_{2n+1} - n(2gc\mathcal{J}_2 - gc\mathcal{J}_3) - 2gc\mathcal{J}_1).\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{l=1}^{n} gc\mathcal{J}_{2l-1} =& J_1 + J_3 + \cdots + J_{2n-1} + (J_2 + J_4 + \cdots + J_{2n})e_1 \\ &+ (J_3 + J_5 + \cdots + J_{2n+1})e_2 + (J_4 + J_6 + \cdots + J_{2n+2})e_3.\end{aligned}$$

By (3.3) and (3.4) we get

$$\sum_{l=1}^{n} gc\mathcal{J}_{2l-1} = \frac{1}{3}\Big(2J_{2n} + n + (2J_{2n+1} - n - 2)e_1$$

$$
\begin{aligned}
&+ (2J_{2n+2} + n - 2)e_2 + (2J_{2n+3} - n - 6)e_3\Big) \\
=&\frac{1}{3}(2(J_{2n} + J_{2n+1}e_1 + J_{2n+2}e_2 + J_{2n+3}e_3) + n(1 - e_1 + e_2 - e_3) \\
&- 2(e_1 + e_2 + 3e_3)) = \frac{1}{3}(2gc\mathfrak{J}_{2n} + n(2gc\mathfrak{J}_2 - gc\mathfrak{J}_3) - 2gc\mathfrak{J}_0).
\end{aligned}
$$

(iii) By Corollary 3.10, we get

$$
\begin{aligned}
\sum_{l=1}^{k} gc\mathfrak{J}_{n+l} =& \sum_{l=1}^{n+k} gc\mathfrak{J}_l - \sum_{l=1}^{n-1} gc\mathfrak{J}_l \\
=&\frac{1}{2}(gc\mathfrak{J}_{n+k+2} - gc\mathfrak{J}_2) - \frac{1}{2}(gc\mathfrak{J}_{n+1} - gc\mathfrak{J}_2) \\
=&\frac{1}{2}\left(gc\mathfrak{J}_{n+k+2} - gc\mathfrak{J}_{n+1}\right).
\end{aligned}
$$

□

At the end, we give the generating function for the generalized commutative r-Jacobsthal quaternions and the generalized commutative Jacobsthal quaternions.

Theorem 3.5. *The generating function for the generalized commutative r-Jacobsthal quaternions has the following form*

$$
f(t) = \frac{gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t}{1 - 2^r t - (2^r + 4^r)t^2}.
$$

Proof. Let

$$
f(t) = gc\mathfrak{J}_{r,0} + gc\mathfrak{J}_{r,1}t + gc\mathfrak{J}_{r,2}t^2 + \cdots + gc\mathfrak{J}_{r,n}t^n + \cdots
$$

be the generating function of the generalized commutative r-Jacobsthal quaternions. Then,

$$
\begin{aligned}
2^r t f(t) =& 2^r gc\mathfrak{J}_{r,0}t + 2^r gc\mathfrak{J}_{r,1}t^2 + 2^r gc\mathfrak{J}_{r,2}t^3 \\
&+ \cdots + 2^r gc\mathfrak{J}_{r,n-1}t^n + \cdots, \\
(2^r + 4^r)t^2 f(t) =& (2^r + 4^r) gc\mathfrak{J}_{r,0}t^2 + (2^r + 4^r) gc\mathfrak{J}_{r,1}t^3 \\
&+ (2^r + 4^r) gc\mathfrak{J}_{r,2}t^4 + \cdots \\
&+ (2^r + 4^r) gc\mathfrak{J}_{r,n-2}t^n + \cdots.
\end{aligned}
$$

By Proposition 2.1, we get

$$
\begin{aligned}
f(t) - 2^r t f(t) - (2^r + 4^r)t^2 f(t) =& gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t \\
&+ (gc\mathfrak{J}_{r,2} - 2^r gc\mathfrak{J}_{r,1} - (2^r + 4^r) gc\mathfrak{J}_{r,0})t^2 + \cdots \\
=& gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t.
\end{aligned}
$$

Thus,

$$
f(t) = \frac{gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t}{1 - 2^r t - (2^r + 4^r)t^2}.
$$

Using equality (2.3), we obtain

$$\begin{aligned} gc\mathfrak{J}_{r,0} =& 1 + (2^{r+1} + 1)e_1 + (3 \cdot 4^r + 2^{r+1})e_2 \\ & + (5 \cdot 8^r + 5 \cdot 4^r + 2^r)e_3, \\ gc\mathfrak{J}_{r,1} - 2^r\, gc\mathfrak{J}_{r,0} =& 2^r + 1 + (4^r + 2^r)e_1 + (2 \cdot 8^r + 3 \cdot 4^r + 2^r)e_2 \\ & + (3 \cdot 16^r + 5 \cdot 8^r + 2 \cdot 4^r)e_3. \end{aligned}$$

□

Corollary 3.11. *The generating function for the generalized commutative Jacobsthal quaternions has the following form*

$$f(t) = \frac{e_1 + e_2 + 3e_3 + (1 + 2e_2 + 2e_3)t}{1 - t - 2t^2}.$$

Concluding Remarks

In this paper, we introduced and studied generalized commutative Jacobsthal quaternions and their one-parameter generalization - generalized commutative r-Jacobsthal quaternions. The Jacobsthal sequence can also be generalized using Jacobsthal polynomials. It will be interesting to continue this research by defining r-Jacobsthal polynomials and then examining generalized r-Jacobsthal commutative quaternion polynomials.

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[1]The Faculty of Mathematics and Applied Physics,
Rzeszow University of Technology,
al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
Email address: dorotab@prz.edu.pl
ORCID iD: https://orcid.org/0000-0001-5181-1725
Email address: aszynal@prz.edu.pl
ORCID iD: https://orcid.org/0000-0001-5508-0640
Email address: iwloch@prz.edu.pl
ORCID iD: https://orcid.org/0000-0002-9969-0827

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APPROXIMATE CONSERVATION LAWS AND SYMMETRY OPERATORS FOR FRACTIONAL DIFFERENTIAL HARRY-DYM EQUATION WITH A SMALL PERTURBATION PARAMETER

HAMID ERFANIAN ORAEI DEHROKHI[1], S. REZA HEJAZI[1], AND ELHAM LASHKARIAN[1]

Abstract. The approximate Lie group analysis of differential equations is applied in order to find symmetry operators of time-fractional Harry-Dym equation. First the method of finding symmetries is extended to approximate fractional differential equations and the corresponding reduced form of the equation are derived. The Riemann-Liouville and Caputo definitions are used in this case. Then, the perturbed conservation laws are computed with the modified version of Noether's theorem based on the formal Lagrangian.

1. Introduction

Lie symmetries of differential equations are so powerful tools for study structure of a given system of differential equations specially the exact solutions. Several problems in physics, enginnering, economics, etc., are described by differential equations, thus, the importance of this field shows itself more here [4, 5, 7, 11, 13, 15–17, 20, 30, 37, 39–42, 47]. Nowadays this theory is extended on fractional differential equations (FDEs).

Fractional partial differential equations (FPDEs) are widely used to describe various physical effects and many complex phenomena and the other various field such as: electrochemistry, quantitative biology, engineering, mechanics and etc. [28, 50]. Also the use of fractional differentiation for the mathematical modeling of real world has been widespread at the recent years. For example the optical soliton perturbation with fractional temporal evolution is one of the viable means to address a growing problem in telecommunication industry, namely the Internet bottleneck. For another

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example the use of fractional differentiation for the mathematical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic material properties and etc. [31]. Consequently, several excellent and illustrative articles are written about this method [14, 32, 38, 43, 48]. Also we have some other valuable papers contain diverse methods to study fractional differential equations [12, 25, 27, 29, 44]

Conservation laws are the consequences of fundamental properties of nature. It is well known that they have a close connection with symmetries. For integer-order differential equations with Lagrangians, conservation laws can be found by corresponding variational symmetries using Noether's theorem; which makes a correspondence between a variational symmetry and a conservation law of a system (see, e.g., [21, 40]). As a limitation of Noether's theorem is what if the system has no any Lagrangian. So, for differential equations without Lagrangians their conservation laws can be constructed using several approaches, such as the concept of non-linear self-adjointness as the modification of Noether's theorem [1, 6, 18, 19, 22, 52], the direct construction method [2, 3], the method of A-operators [8].

However, conservation laws can be constructed not only for integer-order differential equations, but also for fractional differential equations (FDEs). Such equations contain differential and integral operators of non-integer order [26, 49], and they have received significant interest over the last few decades (see books [26, 51] and references therein). At present, FDEs are successfully used in science and engineering as mathematical models of systems and phenomena with memory and spatial non-locality [9, 45]. A problem of constructing conservation laws for FDEs is investigated by many researchers (see, e.g., [10] and references therein). In [35] the concept of non-linear self-adjointness was adopted for FDEs without Lagrangians, and in [36] the explicit algorithm for constructing conservation laws for a wide class of such equations using their Lie point symmetrie was developed.

The main purpose of the paper is to calculate the fractional symmetries of the perturbed equation

$$D_t^\alpha u = 2\left(\frac{1}{\sqrt{u}}\right)_{xxx} + \epsilon u_t, \quad \alpha \in (0,1). \tag{1.1}$$

The paper is outlined as follows. Some general and standard definitions and concepts of fractional derivatives including the basic results of the symmetry analysis of perturbed fractional differential equations are given in Section 2. Section 3 is devoted for the computation of symmetries and reduced equations constructed by the invariants of the operators. Finally, the method of finding conservation laws based on the modified Noether's theorem is extended to perturbed fractional differential equations in order to give the conservation laws of the Eq. (1.1).

2. Definitions for Approximating Fractional Differential Equations

For simplicity, we restrict our attention to the case of FDE with fractional derivatives with respect to only one independent variable x^1:

$$F(x, u, u_1, \ldots, u_k, {}_aD_{x^1}^{\alpha_0}u, \ldots, {}_aD_x^{\alpha_m}u, {}_{x^1}D_b^{\beta_0}u, \ldots, {}_{x^1}D_b^{\beta_l}u) = 0, \tag{2.1}$$

where $k, l, m \in \mathbb{N}$, $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_m$, $0 < \beta_0 < \beta_1 < \cdots < \beta_l$, ${}_aD_{x_1}^{\alpha_i}u$ and ${}_{x_1}D_b^{\beta_j}u$ are the left-sided and the right-sided fractional derivatives, respectively [26]. In Eq. (2.1) $u = u(x)$ is a function of independent variables $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, $x_1 \in (a, b)$, and

$$u_{(s)} \equiv \left\{u_{i_1 \cdots i_s}\right\} = \left\{\frac{\partial^s u(x)}{\partial x^{i_1} \cdots \partial x^{i_s}}\right\}, \quad i_1, \ldots, i_s = 1, \ldots, n, s = 1, \ldots, k, \tag{2.2}$$

are the so-called successive derivatives of differential variable u with respect to the independent variables ξ (see, e.g., [20]). If the orders of fractional derivatives in Eq. (2.1) are all nearly integers, then this equation can be approximated by a differential equation with small parameters which can be extracted from these orders. In a particular case when orders of all fractional derivatives in Eq. (2.1) have the same small deviation from the nearest integer number, this equation can be approximated by a differential equation with a single small parameter. In this case Eq. (2.1) can be written as

$$F\left(x, u, u_1, \ldots, u_k, {}_aD_{x^1}^{\alpha}u, \ldots, {}_aD_{x^1}^{\alpha+m}u, {}_{x^1}D_b^{\beta}u, \ldots, {}_{x^1}D_b^{\beta+l}u\right) = 0, \tag{2.3}$$

where $\alpha \in (0, 1)$, $k, m \in \mathbb{N}$.

We assume that the left- and right-sided Riemann-Liouville fractional derivatives

$$({}_\alpha D_{x^1}^{\alpha+j}u)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x^1}\right)^{j+1}\int_\alpha^{x^1}\frac{u(\xi, x^2, \ldots, x^n)}{(x^1-\xi)^\alpha}d\xi, \tag{2.4}$$

$$({}_{x^1}D_b^{\alpha+j}u)(x) = \frac{(-1)^{j+1}}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x^1}\right)^{j+1}\int_{x^1}^{b}\frac{u(\xi, x^2, \ldots, x^n)}{(x^1-\xi)^\alpha}d\xi, \tag{2.5}$$

or the Caputo fractional derivatives

$$({}_\alpha^c D_{x^1}^{\alpha+j}u)(x) = \frac{1}{\Gamma(1-\alpha)}\int_\alpha^{x^1}\frac{\partial^{j+1}u(\xi, x^2, \ldots, x^n)}{\partial \xi^{j+1}(x^1-\xi)^\alpha}d\xi, \tag{2.6}$$

$$({}_\alpha^c D_b^{\alpha+j}u)(x) = \frac{(-1)^{j+1}}{\Gamma(1-\alpha)}\int_{x^1}^{b}\frac{\partial^{j+1}u(\xi, x^2, \ldots, x^n)}{\partial \xi^{j+1}(x^1-\xi)^\alpha}d\xi, \tag{2.7}$$

can be used in Eq. (2.3) as ${}_\alpha D_{x^1}^{\alpha+j}$ and ${}_{x^1}D_b^{\alpha+j}$, $\alpha \in (0, 1)$, $j = 0, 1, \ldots, m$.

Let in Eq. (2.3) $\alpha = \epsilon$ or $\alpha = 1-\epsilon$, where ϵ is a small parameter: $0 \prec \epsilon \preceq 1$ function $u(x)$ is such that the left-sided Riemann-Liouville fractional derivatives ${}_\alpha D_{x^1}^{\alpha+\epsilon}u$, $j = 0, 1, \ldots$, and ${}_{x^1}D_b^{\alpha-\epsilon}u$, $j = 1, 2, \ldots$, exist and at each point $x^1 \in (a, b)$ they can be expanded into the series

$${}_\alpha D_{x_1}^{j\pm\epsilon} = \sum \binom{j \pm \epsilon}{s}\frac{(x^1-\alpha)^{s-j\pm\epsilon}}{\Gamma(1+s-j\pm\epsilon)} \cdot \frac{\partial^s u(\xi, x^2, \ldots, x^n)}{\partial \xi^s},$$

where $\binom{j\pm\epsilon}{s} = \frac{\Gamma(1+j\pm\epsilon)}{\Gamma(1+s-j\pm\epsilon)s!}$ is a binomial coefficient. Then the following expansion is valid

$$ {}_{\alpha}D^{j\pm\epsilon}_{x_1}\frac{\partial^j u}{\partial(x^1)^j} \pm \epsilon \left\{ \begin{array}{c} [\psi(j+1)-\ln(x^1-\alpha)]\dfrac{\partial^j u}{\partial(x^1)^j} \\ -\sum_{s=1,s\neq j}^{+\infty} \dfrac{(-1)^{s-j}}{(s-j)}\cdot\dfrac{j!}{s!}(x^1-\alpha)^{s-j}\dfrac{\partial^s u}{\partial(x^1)^s} \end{array} \right\} + \mathcal{O}(\epsilon). $$

Here $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function, and j is an integer number so that $j\pm\epsilon > 0$.

Similar expansion can be obtained for the right-sided Riemann-Liouville fractional derivative (2.4) and (2.5). In [37] it was proved that

$$ {}_{x^1}D^{j\pm\epsilon}_{b}\frac{\partial^j u}{\partial(x^1)^j} \pm \epsilon \left\{ \begin{array}{c} [\psi(j+1)-\ln(x^1-\alpha)]\dfrac{\partial^j u}{\partial(x^1)^j} \\ -\sum_{s=1,s\neq j}^{+\infty} \dfrac{(-1)^{s-j}}{(s-j)}\cdot\dfrac{j!}{s!}(x^1-\alpha)^{s-j}\dfrac{\partial^s u}{\partial(x^1)^s} \end{array} \right\} + \mathcal{O}(\epsilon). $$

The expansion of Caputo's fractional derivatives is obtained similarly [26].

Definition 2.1. Relations between Caputo and Riemann-Liouville fractional derivatives are established by [49],

$$ {}^{C}_{\alpha}D^{\alpha+j}_{x^1}u = {}_{\alpha}D^{\alpha+j}_{x^1}u - \sum_{s=0}^{j}\frac{(x^1-a)^{s-j-\alpha}}{\Gamma(1+s-j-\alpha)}\cdot\frac{\partial^s u}{\partial(x^1)^s}\bigg|_{x^1=a}, $$
$$ {}^{C}_{x^1}D^{\alpha+j}_{\alpha}u = {}_{\alpha}D^{\alpha+j}_{b}u - \sum_{s=0}^{j}\frac{(b-x^1)^{s-j-\alpha}}{\Gamma(1+s-j-\alpha)}\cdot\frac{\partial^s u}{\partial(x^1)^s}\bigg|_{x^1=b}. $$

For the left-sided Caputo fractional derivative (2.6) and (2.7) it can be proved that

$$ {}^{C}_{\alpha}D^{j\pm\epsilon}_{x^1}u = {}_{\alpha}D^{j\pm\epsilon}_{x^1}u \pm \sum_{s=0}^{j-1}(-1)^{s-j}(j-s-1)!(x^1-\alpha)^{s-j}\frac{\partial^s u}{\partial(x^1)^s}\bigg|_{x^1=\alpha} + g(x+\alpha), \tag{2.8} $$

where

$$ g(x+\alpha) = \begin{cases} -[1+\epsilon(\psi(j+1)-\ln(x^1-\alpha)]\dfrac{\partial^j u}{\partial(x^1)^j}\bigg|_{x^1=\alpha}, & \text{for } {}^{C}_{\alpha}D^{j+\epsilon}_{x^1}u, \\ 0, & \text{for } {}^{C}_{\alpha}D^{j+\epsilon}_{x^1}u. \end{cases} $$

Expansions for the right-sided Caputo derivatives (2.7) can be obtained from Eq. (2.8) by changing the point $x^1 = a$ to $x^1 = b$ and by changing the expression $(x^1 - a)$ to $(b - x^1)$.

If $\alpha = \epsilon$ or $\alpha = 1 - \epsilon$ then, Eq. (2.3) can be approximated by the perturbed integer-order differential equation

$$ F_{(0)}(x, u_1, \dots u_l) + \epsilon F_{(1)}(x, u_1, \dots, u_l, D^{l+1}_{x^1}u, D^{l+2}_{x^1}u, \dots) \simeq 0. \tag{2.9} $$

Here $l = \max\{k, m\}$ for $\alpha = 1 - \epsilon$ and $l = \max\{k, m - 1\}$ for $\alpha = \epsilon$.

Definition 2.2. The generator of an approximate Lie point transformation group [32, 34]

$$\begin{aligned}\tilde{x}^i \simeq & f^i(x,u,\alpha,\epsilon) \equiv f^i_{(0)}(x,u,\alpha) + \epsilon f^i_{(1)}(x,u,\alpha), \quad i = 1,2,\ldots,n,\\ \tilde{u} \simeq & g(x,u,\alpha,\epsilon) \equiv g_{(0)}(x,u,\alpha) + \epsilon g_{(1)}(x,u,\alpha),\end{aligned}$$

satisfying the conditions $\tilde{x}^i|_{\alpha=0} \simeq x^i$ and $\tilde{u}|_{\alpha=0} \simeq u$, is a first-order differential operator

$$V \simeq V_0 + \epsilon V_1 \equiv \left(\xi^i_0(x,u) + \epsilon \xi^i_1(x,u)\right)\frac{\partial}{\partial x^i} + (\eta^i_0(x,u) + \epsilon\eta^i_1(x,u))\frac{\partial}{\partial u}, \tag{2.10}$$

where

$$\begin{aligned}\xi^i_0(x,u) &= \frac{\partial f^i_0(x,u,\alpha)}{\partial \alpha}\Big|_{\alpha=0}, \quad \xi^i_1(x,u) = \frac{\partial f^i_1(x,u,\alpha)}{\partial \alpha}\Big|_{\alpha=0},\\ \eta_0(x,u) &= \frac{\partial g_0(x,u,\alpha)}{\partial \alpha}|_{\alpha=0}, \qquad \eta_1(x,u) = \frac{\partial g_1(x,u,\alpha)}{\partial \alpha}\Big|_{\alpha=0}, \quad i = 1,2,\ldots,n.\end{aligned}$$

Definition 2.3. For (2.9) the operator (2.10) satisfying the equation [33],

$$V(f_0 + \epsilon f_1)|_{(2,9)} \simeq 0,$$

is called an approximate symmetry of (2.9).

3. Symmetry Analysis on Perturbed Equation

Constructing approximate symmetries first, you need to apply the extension ${}_{\alpha}D^{j\pm\epsilon}_{x^1}$ for equation by setting $\alpha = 1 - \epsilon$. The approximation of Eq. (1.1) in the form

$$\begin{aligned}f_0 + \epsilon f_1 =& u_t + 2\left(\frac{1}{\sqrt{u}}\right)_{xxx}\\ &+ \epsilon\left[(\ln t + \gamma - 1)u_t + \frac{u}{t} + \sum_{n=1}^{+\infty}\frac{(-1)^n t^n}{n(n+1)!}D^{n+1}_t u\right] \approx 0,\end{aligned} \tag{3.1}$$

is achieved.

It follows from Eq. (3.1) that $\epsilon u_t \approx 0$. Therefore, $\epsilon D^{n+1}_t u \cong 0$ for $n \geq 0$ and the infinite series in (3.1) is also approximately equal to zero. Thus, (3.1) takes a very simple form

$$u_t + 2\left(\frac{1}{\sqrt{u}}\right)_{xxx} + \epsilon\frac{u}{t} \approx 0. \tag{3.2}$$

To begin the process we need to get symmetries of perturbed equation (3.1). These operators are found via a systematic computations. Thus, the Lie algebra G of the symmetries is spanned with the following geometric vector fields:

$$\begin{aligned}&V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad V_4 = t\frac{\partial}{\partial t} + \frac{2u}{3}\frac{\partial}{\partial u},\\ &V_5 = \frac{x^2}{2}\frac{\partial}{\partial x} - 2xu\frac{\partial}{\partial u}, \quad V_6 = \epsilon\frac{\partial}{\partial t}, \quad V_7 = \epsilon\frac{\partial}{\partial x}.\end{aligned}$$

4. Reduction

The process of reduction for perturbed FDE is as same as the others. In this section some reduced form of the Eq. (3.1) are given.

Case 1. For the symmetry V_1, the corresponding characteristic equation is

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}.$$

Integration the group trajectories provides the following similarity variable and function

$$u = g(r), \quad r = x,$$

for reduction. Thus, the reduced equation

$$\frac{-4g'''g^2 + 18g'g''g - 15g'^3}{4g^{7/2}} + \epsilon\left[(\ln t + \gamma - 1) + \frac{g}{t} + \sum_{n=1}^{+\infty} \frac{(-1)^n t^n}{n(n+1)!} D_t^{n+1} u\right] \approx 0,$$

is obtained.

Case 2. Symmetry V_2, yields the characteristic equation

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}.$$

Integration this system gives the following similarity variable and function

$$u = g(r), \quad r = t.$$

So, the reduced equation

$$g' + \epsilon\left[(\ln t + \gamma - 1)g' + \frac{g}{t} + \sum_{n=1}^{+\infty} \frac{(-1)^n t^n}{n(n+1)!} D_t^{n+1} u\right] \approx 0,$$

is constructed.

Case 3. The operator V_3, has the characteristic equation

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{-2u}.$$

The solution of this system gives the similarity variable and function

$$u = g(r)x^{-2}, \quad r = t.$$

So, the reduced equation is

$$g'x^{-2} + \epsilon\left[(\ln t + \gamma - 1)g'x^{-2} + \frac{g}{tx^2} + \sum_{n=1}^{+\infty} \frac{(-1)^n t^n}{n(n+1)!} D_t^{n+1} u\right] \approx 0.$$

Case 4. Finally for the symmetry V_4, we conclude

$$\frac{dt}{0} = \frac{dx}{\frac{1}{2}x^2} = \frac{du}{-2xu},$$

as the system of charachteristics. Integration provides the following similarity variable and function

$$u = g(r)x^{-4}, \quad r = x.$$

These variables reduced the Eq. (2.2) to

$$g'x^{-4}+\epsilon\left[(\ln t+\gamma-1)g'x^{-4}+\frac{gx^{-4}}{t}+\sum_{n=1}^{+\infty}\frac{(-1)^n t^n}{n(n+1)!}D_t^{n+1}u\right]\approx 0.$$

5. Approximate Conservation Laws

There are several methods in order to find conservation laws of a given system of differential equations such as Noether's theorem, Boyer's generalization of Noether's theorem, direct method, homotopy operator method, Ibragimov's method etc. All cases have some advantages together with limitations. An illustrative comparison between these methods with so many examples is coming in [46]. In this section approximate conservation laws will be constructed by using approximate symmetries in a systematic method. This method is based on the extension of concept of non-linear self-adjointness by Ibragimov [19, 23] which presents a formal Lagrangian of perturbed Eq. (2.9). After introducing the adjoint equation and examining non-linear selfadjointness condition, conservation laws will be obtained in definite relationships.

5.1. **Basic definitions for constructing conservation laws.**

Definition 5.1. The approximate formal Lagrangian is defined by [24],

$$\mathcal{L}\simeq\mathcal{L}_0+\epsilon\mathcal{L}_1\equiv vF_0+v\epsilon F_1, \tag{5.1}$$

where v is a new dependent variable.

Then, the approximately adjoint equation can be written as

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial u}\approx&F^*_{(0)}(x,u,v,u_1,v_1,\dots,v_l,u_l)\\&+\epsilon F^*_{(1)}(x,u,v,u_1,v_1,\dots,v_i,u_l,u_l,D_{x^1}^{l+1}u,D_{x^1}^{l+2}u,D_{x^1}^{l+2}v,\dots)\simeq 0,\end{aligned}$$

where

$$\frac{\partial}{\partial u}=\frac{\partial}{\partial u}+\sum_{s=1}^{+\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}},$$

is the variational derivative. Here D_i denotes the operator of total differentiation with respect to x^i defined by

$$D_i=\frac{\partial}{\partial x^i}+u_i\frac{\partial}{\partial u}+v_i\frac{\partial}{\partial v}+\sum_{s=1}^{+\infty}\left(u_{i_1\cdots i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}+v_{i_1\cdots i_s}\frac{\partial}{\partial v_{i_1\cdots i_s}}\right).$$

Any approximate symmetry (2.10) of Eq. (2.9) leads to a conservation law

$$D_i(C^i)=0,\quad C^i=C^i_0+\epsilon C^i_1,$$

where

$$C_0^i=Q_0\left[\frac{\partial\mathcal{L}_0}{\partial u_i}+\sum_{s=1}^{l-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial\mathcal{L}_0}{\partial u_{i_1\cdots i_s}}\right]+\sum_{r=1}^{l-1}D_{k_1}\cdots D_{k_r}(Q_0)$$

$$\times \left[\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r}} + \sum_{s=1}^{l-r-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r i_1\cdots i_s}}\right],$$

$$C_1^i = Q_1\left[\frac{\partial \mathcal{L}_0}{\partial u_i} + \sum_{s=1}^{l-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_0}{\partial u_{i_1\cdots i_s}}\right] + \sum_{r=1}^{l-1} D_{k_1}\cdots D_{k_r}(Q_1)$$
$$\times \left[\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r}} + \sum_{s=1}^{l-r-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r i_1\cdots i_s}}\right]$$
$$+ Q_0\left[\frac{\partial \mathcal{L}_1}{\partial u_i} + \sum_{s=1}^{+\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_1}{\partial u_{i_1\cdots i_s}}\right] + \sum_{r=1}^{+\infty} D_{k_1}\cdots D_{k_r}(Q_1)$$
$$\times \left[\frac{\partial \mathcal{L}_1}{\partial u_{ik_1\cdots k_r}} + \sum_{s=1}^{+\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_1}{\partial u_{ik_1\cdots k_r i_1\cdots i_s}}\right],$$

where

$$Q_0 = \eta_0 - \xi_0^i u_i, \quad Q_1 = \eta_1 - \xi_1^i u_i,$$

and $\mathcal{L}_0, \mathcal{L}_1$ are defined in (5.1) upon the substitution

$$v \approx \varphi_0(x,u) + \epsilon\varphi_1(x,u) \neq 0.$$

5.2. **Approximate conservation laws for Eq. (1.1).** Symmetries $V_1, \ldots, V_5$ are stable since they do not vanish at $\epsilon = 0$. These symmetries can be used for constructing stable approximate conservation laws of Eq. (3.2). Approximate symmetry V_6 is the essential symmetry and can gives the essential approximate conservation laws. Symmetry V_7 is inessential and should be omitted.

The equation approximately adjoint to Eq. (3.2) can be obtained from Eq. (3.1) using the formal Lagrangian (5.1) written for Eq. (3.2). Direct calculations give

$$F_0^* + \epsilon F_1^* \equiv v_t - \frac{15v_x^3}{4u^{7/2}} + \frac{9v_x v_{xx}}{2u^{5/2}} - \frac{v_{xxx}}{u^{3/2}}$$
$$+ \epsilon\left[(\ln t + \gamma - 1)v_t + \sum_{s=1}^{+\infty}\sum_{r=0}^{s}\binom{s-1}{r-1}\frac{t^r}{r(r+1)!}D_t^{r+1}v\right] \approx 0.$$

It can be proved that Eq. (3.2) is approximately non-linearly self-adjoint. Indeed, substituting Eqs. (3.2) and (5.1) into

$$F_0^*|_{v(x,t,u)\approx\varphi(0)(x,t,u)+\epsilon\varphi(1)(x,t,u)} + \epsilon F_1^*|_{v(x,t,u)\approx\varphi(0)(x,t,u)} \approx \lambda_0 F_0 + \epsilon(\lambda_0 F_1 + \lambda_1 F_0),$$

and taking into account the representation $v(x,t,u) \approx \varphi(0)(x,t,u) + \epsilon\varphi(1)(x,t,u)$, we obtain the approximate equation for $\varphi(0)$ and $\varphi(1)$. Splitting this equation by ϵ yields

$$v_t = \varphi_{(0)u}u_t + \epsilon\varphi_{(1)u}u_t + \varphi_{(0)t} + \epsilon\varphi_{(1)t},$$
$$v_x = \varphi_{(0)u}u_x + \epsilon\varphi_{(1)u}u_x + \varphi_{(0)x} + \epsilon\varphi_{(1)x},$$
$$v_{xx} = \varphi_{(0)ux}u_x + \varphi_{(0)u}u_{xx} + \varphi_{(0)xx} + \epsilon\varphi_{(1)ux}u_x + \epsilon\varphi_{(1)u}u_{xx} + \epsilon\varphi_{(1)xx},$$
$$v_{xxx} = \varphi_{(0)uxx}u_x + 2\varphi_{(0)ux}u_{xx} + \varphi_{(0)u}u_{xxx} + \varphi_{(0)xxx} + \epsilon\varphi_{(1)uxx}u_x + 2\epsilon\varphi_{(1)uxx}u_{xx},$$

$$+ \epsilon\varphi_{(1)u}u_{xxx} + \epsilon\varphi_{(1)xxx}.$$

Consequently, we have,

$$\begin{aligned}
&\varphi_{(0)u}u_x + \varphi_{(0)t} + \epsilon(\varphi_{(1)u}u_t + \varphi_{(1)t}) \\
&- \frac{15(\varphi_{(0)u}u_x + \varphi_{(0)x} + \epsilon\varphi_{(1)u}u_x + \epsilon\varphi_{(1)x})^3}{8u^{7/2}} \\
&+ \frac{9(\varphi_{(0)u}u_x + \varphi_{(0)x} + \epsilon\varphi_{(1)u}u_x + \epsilon\varphi_{(1)x})(\varphi_{(0)xx}u_x + \varphi_{(0)u}u_{xx} + \varphi_{(0)xx} + \epsilon\varphi_{(1)ux}u_x}{4u^{5/2}} \\
&+ \frac{\epsilon\varphi_{(1)u}u_{xx} + \epsilon\varphi_{(1)xx})}{4u^{5/2}} - \varphi_{(0)uxx}u_x + 2\varphi_{(0)ux}u_{xx} + \varphi_{(0)u}u_{xxx} + \varphi_{(0)xxx} \\
&+ \epsilon\varphi_{(1)uxx}u_x + 2\epsilon\varphi_{(1)ux}u_{xx} + \epsilon\varphi_{(1)u}u_{xxx} + \frac{\epsilon\varphi_{(1)xxx}}{2u^{3/2}} \\
=&\lambda_0\left(u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}}\right) + \epsilon\Bigg\{\lambda_1u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}} \\
&+ \lambda_0\left[(\ln t + \gamma - 1)u_t + \frac{u}{t} + \sum_{n=1}^{+\infty}\frac{(-1)^nt^n}{n(n+1)!}D_t^{n+1}u\right]\Bigg\}.
\end{aligned}$$

Then,

$$v(x,t,u) \equiv \frac{\lambda_0}{u^{3/2}} + \epsilon\left(\lambda_1u^{-3/2} + \lambda_0\frac{x}{t} + C_1\right)$$

and

$$\begin{aligned}
\mathcal{L} \approx \frac{\lambda_0}{u^{3/2}}&\left(u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}}\right) + \epsilon\Bigg\{\left(\lambda_1u^{-3/2} + \lambda_0\frac{x}{t} + C_1\right) \\
&\times\left(u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}}\right) + \frac{\lambda_0}{u^{3/2}}\Bigg[(\ln t + \gamma - 1)u_t \\
&+ \frac{u}{t} + \sum_{n=1}^{+\infty}\frac{(-1)^nt^n}{n(n+1)!}D_t^{n+1}u\Bigg]\Bigg\}.
\end{aligned}$$

Thus, the conserved vectors for Eq. (1.1) are constructed by the formula

$$\begin{aligned}
C_0^x =&Q_0\left(\frac{45\lambda_0u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right) + D_{xx}(Q_0)\left(\frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right), \\
C_1^x =&Q_1\left(\frac{45\lambda_0u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right) + D_{xx}(Q_1)\left(\frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right) \\
&+ Q_0\left[\frac{-45\lambda_1u_x^2}{64u^{21/2}} - \frac{9\lambda_1u_{xx}}{u^{5/2}} - \lambda_1\left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{u}t} + \frac{x}{t}\right)\right] \\
&+ D_{xx}(Q_0)\left(\lambda_1u^{-3/2} + \lambda_0\frac{x}{t} + 1\right),
\end{aligned}$$

$$
\begin{aligned}
C_0^t =& Q_0 \frac{\lambda_0}{u^{3/2}},\\
C_1^t =& \frac{\lambda_0}{u^{3/2}} \Bigg\{ Q_1 + Q_0 \left[\lambda_1 u^{-3/2} + \lambda_0 \frac{x}{t} + 1 + \ln t + \gamma + \sum_{i=1}^{+\infty} D_t^i(Q_0) \frac{t^i}{i(i+1)!} \right] \\
&+ \sum_{s=1}^{+\infty} (-1)^s D_t^s(Q0) \left[\sum_{k=s}^{+\infty} \frac{t^k}{k(k+1)!} \right] \Bigg\},
\end{aligned}
$$

where

$$
C^x = C_0^x + \epsilon C_1^x, \quad C^t = C_0^t + \epsilon C_1^t.
$$

Now, the conservation laws of the Eq. (1.1) with respect to the symmetry operators are listed below.

(I) For V_1 we have $Q_0 = -u_t$ and $Q_1 = 0$, so component of approximate conservation laws is

$$
\begin{aligned}
C^x =& -u_t \left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5} \right) \\
&- u_t \left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}} - \frac{9\lambda_1 u_{xx}}{u^{5/2}} - \lambda_1 \left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{u}t} + \frac{x}{t} \right) \right],\\
C^t =& -u_t \frac{\lambda_0}{u^{3/2}} + \frac{\lambda_0}{u^{3/2}} \Bigg\{ - u_t \Bigg[\lambda_1 u^{-3/2} + \lambda_0 \frac{x}{t} + 1 + \ln t + \gamma \\
&+ \sum_{i=1}^{+\infty} D_t^i(-u_t) \frac{t^i}{i(i+1)!} \Bigg] + \sum_{s=1}^{+\infty} (-1)^s D_t^s(-u_t) \sum_{k=s}^{+\infty} \frac{t^k}{k(k+1)!} \Bigg\}.
\end{aligned}
$$

(II) Symmetry V_2, $Q_0 = -u_x$ and $Q_1 = 0$ yield

$$
\begin{aligned}
C^x =& -u_x \left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5} \right) \\
&- u_x \left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}} - \frac{9\lambda_1 u_{xx}}{u^{5/2}} - \lambda_1 \left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{u}t} + \frac{x}{t} \right) \right],\\
C^t =& -u_x \frac{\lambda_0}{u^{3/2}} + \frac{\lambda_0}{u^{3/2}} \Bigg\{ - u_x \Bigg[\lambda_1 u^{-3/2} + \lambda_0 \frac{x}{t} + 1 + \ln t + \gamma \\
&+ \sum_{i=1}^{+\infty} D_t^i(-u_x) \frac{t^i}{i(i+1)!} \Bigg] + \sum_{s=1}^{+\infty} (-1)^s D_t^s(-u_x) \sum_{k=s}^{+\infty} \frac{t^k}{k(k+1)!} \Bigg\}.
\end{aligned}
$$

(III) For V_3, the charachterisctics $Q_0 = -2u - xu_x$ and $Q_1 = 0$ are derived. In this case we have

$$
C^x = (-2u - xu_x) \left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5} \right)
$$

$$
\begin{aligned}
&-(2u-xu_x)\left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}}-\frac{9\lambda_1 u_{xx}}{u^{5/2}}-\lambda_1\left(\frac{1}{u^{3/2}}+\frac{1}{\sqrt{ut}}+\frac{x}{t}\right)\right],\\
C^t &= (-2u-xu_x)\frac{\lambda_0}{u^{3/2}}+\frac{\lambda_0}{u^{3/2}}\Bigg\{\Bigg[(-2u-xu_x)(\lambda_1 u^{-3/2}+\lambda_0\frac{x}{t}+1\\
&+\ln t+\gamma+\sum_{i=1}^{+\infty} D_t^i(-2u-xu_x)\frac{t^i}{i(i+1)!}\Bigg]\\
&+\sum_{s=1}^{+\infty}(-1)^s D_t^s(-2u-xu_x)\sum_{k=s}^{+\infty}\frac{t^k}{k(k+1)!}\Bigg\}.
\end{aligned}
$$

(IV) The components of conservation laws for V_4 with respect to $Q_0=\dfrac{2}{3}u-tu_t$ and $Q_1=0$ are

$$
\begin{aligned}
C^x &= \left(\frac{2}{3}u-tu_t\right)\left(\frac{45\lambda_0 u_x^2}{64u^{12}}+\frac{9u_x^2\lambda_0}{2u^5}-\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)\\
&-\left(\frac{2}{3}u-tu_t\right)\left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}}-\frac{9\lambda_1 u_{xx}}{u^{5/2}}-\lambda_1\left(\frac{1}{u^{3/2}}+\frac{1}{\sqrt{ut}}+\frac{x}{t}\right)\right],\\
C^t &= \left(\frac{2}{3}u-tu_t\right)\frac{\lambda_0}{u^{3/2}}+\frac{\lambda_0}{u^{3/2}}\Bigg\{\left(\frac{2}{3}u-tu_t\right)\Bigg[\lambda_1 u^{-3/2}+\lambda_0\frac{x}{t}+1\\
&+\ln t+\gamma+\sum_{i=1}^{+\infty} D_t^i\left(\frac{2}{3}u-tu_t\right)\frac{t^i}{i(i+1)!}\Bigg]+\sum_{s=1}^{+\infty}(-1)^s D_t^s\left(\frac{2}{3}u-tu_t\right)\\
&\times\sum_{k=s}^{+\infty}\frac{t^k}{k(k+1)!}\Bigg\}.
\end{aligned}
$$

(V) Now consider V_5. This operator has the characteristics $Q_0=-2xu-\dfrac{x^2}{2}u_x$ and $Q_1=0$. So, the conserved vectors are

$$
\begin{aligned}
C^x &= \left(-2xu-\frac{x^2}{2}u_x\right)\left(\frac{45\lambda_0 u_x^2}{64u^{12}}+\frac{9u_x^2\lambda_0}{2u^5}-\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)\\
&-\left(-2xu-\frac{x^2}{2}u_x\right)\left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}}-\frac{9\lambda_1 u_{xx}}{u^{5/2}}-\lambda_1\left(\frac{1}{u^{3/2}}+\frac{1}{\sqrt{ut}}+\frac{x}{t}\right)\right],\\
C^t &= \left(-2xu-\frac{x^2}{2}u_x\right)\frac{\lambda_0}{u^{3/2}}+\frac{\lambda_0}{u^{3/2}}\Bigg\{\left(-2xu-\frac{x^2}{2}u_x\right)\Bigg[\lambda_1 u^{-3/2}+\lambda_0\frac{x}{t}+1\\
&+\ln t+\gamma+\sum_{i=1}^{+\infty} D_t^i\left(-2xu-\frac{x^2}{2}u_x\right)\frac{t^i}{i(i+1)!}\Bigg]
\end{aligned}
$$

$$+\sum_{s=1}^{+\infty}(-1)^s D_t^s\left(-2xu-\frac{x^2}{2}u_x\right)\sum_{k=s}^{+\infty}\frac{t^k}{k(k+1)!}\Bigg\}.$$

(VI) The approximate symmetry V_6 with $Q_0=0$ and $Q_1=-u_t$ yields

$$C^x=-\epsilon\left[u_t\left(\frac{45\lambda_0 u_x^2}{64u^{12}}+\frac{9u_x^2\lambda_0}{2u^5}-\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)+u_{xxt}\left(\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)\right],$$

$$C_0^t=-\epsilon u_t\frac{\lambda_0}{u^{3/2}}.$$

(VII) Finally for V_7 ,$Q_0=0$ and $Q_1=-u_x$ we have

$$C^x=-\epsilon\left[u_x\left(\frac{45\lambda_0 u_x^2}{64u^{12}}+\frac{9u_x^2\lambda_0}{2u^5}-\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)+u_{xxt}\left(\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)\right],$$

$$C_0^t=-\epsilon u_x\frac{\lambda_0}{u^{3/2}}.$$

6. Concluding Remark

Recently, trending in FDEs led to proliferation of studies that it became an increasing interest in mathematics. Most studies in the field of FDEs have only focused of approximate analysis with some standard analytic methods. Symmetry analysis of differential equations is fast becoming a key instrument in all equations arising from a natural phenomenon. A key aspect of Lie group theory of differential equations is that this study provides an exciting opportunity to advance our knowledge about the exact solutions and geometric structures of the given system. Over the past years there has been an increasing interest in this field because the method could be implemented in all types of system of differential equations. This method is based on finding some differential operators (vector fields) called symmetries in order to find the exact solutions of differential equations. These operators are the largest local group of transformations acting on the independent and dependent variables of the system with the property that they transform solutions of the system to other solutions. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system [33].

Also there are several methods for finding conservation laws of a system of differential equations; such as Noether's method, Boyer's generalization of Noether's method, direct method, homotopy operator method, Ibragimov's method (modified version of Noether's theorem), etc. All five of these methods have some limitations in their use.

As it is mentioned in the paper the modified Noether's theorem is used for finding the conservation laws of Eq. (1.1). We note that there are several limitations to Noether's theorem. It is restricted to variational systems. Consequently, to be applicable to a given system as written, the given system must be of even order, have the same number of dependent variables as the number of equations in the system and have no dissipation. There is also the difficulty of finding symmetries admitted by the action functional. Moreover, the use of Noether's theorem to find conservation laws is coordinate-dependent. On comparing Ibragimov's method with Noether's theorem, we find that this method is similar to Noether's theorem which requires a Lagrangian to exist, however, Lagrangians exists only for very special types of differential equations within the construct of Noether's theorem. Ibragimov has attempted to overcome this difficulty by defining an adjoint equation for non-linear differential equations and constructing a Lagrangian for an arbitrary (linear and non-linear) equation considered together with its adjoint equation. With this reason Ibragimov's method is applied in this paper.

As mentioned in the paper, Lukashchuk discussed the basis of extended Lie group theory on approximate FDEs in [34]. The main target of this paper was to give a comprehensive analysis of the Lie group theory of perturbed fractional Harry-Dym equation. The work is done due to the developed method by Lukashchuk. This method is implemented by replacing the small perturbed parameter with a fractional expression of a FDE. For the next step we found symmetries and conservation laws of the equation. This is followed with the help of approximated Lie theory of FDEs.

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[1]Department of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Semnan, Iran.
Email address: h-erfanian94@gmail.com
ORCID iD: https://orcid.org/0000-0002-9457-1917
Email address: r_hejazi@shahroodut.ac.ir
ORCID iD: https://orcid.org/0000-0002-5121-4899
Email address: lashkarianelham@yahoo.com
ORCID iD: https://orcid.org/0009-0008-3444-2647

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JORDAN HIGHER DERIVATIONS ON PRIME HILBERT C*-MODULES

SAYED KHALIL EKRAMI

Abstract. Let $\mathcal{M}$ be a Hilbert C*-module. A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$ with $\varphi_0 = I$, is said to be a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a, b\rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a),$$

for all $a, b \in \mathcal{M}$ and each non-negative integer n. In this paper, we show that, if $\mathcal{M}$ is prime, then every Hilbert C*-module Jordan higher derivation $\{\varphi_n\}_{n=0}^{+\infty}$ on $\mathcal{M}$, is a Hilbert C*-module higher derivation on $\mathcal{M}$. As a consequence, we show that every Hilbert C*-module Jordan derivation on $\mathcal{M}$, is a Hilbert C*-module derivation on $\mathcal{M}$.

1. Introduction

The notion of a Hilbert C*-module initiated as a generalization of a Hilbert space in which the inner product takes its values in a C*-algebra (see [13]). Let $\mathcal{A}$ be a C*-algebra. An inner product $\mathcal{A}$-module is a complex linear space $\mathcal{M}$ which is a left $\mathcal{A}$-module with compatible scalar multiplication $\lambda(ax) = (\lambda a)x = a(\lambda x)$ $(\lambda \in \mathbb{C}, x \in \mathcal{M}, a \in \mathcal{A})$, together with an $\mathcal{A}$-valued inner product $(x, y) \mapsto \langle x, y\rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that for each $x, y, z \in \mathcal{M}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$,

(i) $\langle x, x\rangle \geq 0$ and the equality holds if and only if $x = 0$;
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;
(iii) $\langle ax, y\rangle = a\langle x, y\rangle$;

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(iv) $\langle x, y\rangle^* = \langle y, x\rangle$.

It follows from the above conditions that $\langle x, x\rangle$ is a positive element in C*-algebra $\mathcal{A}$, the inner product is conjugate-linear in its second variable and $\langle x, ay\rangle = \langle x, y\rangle a^*$ for all $x, y \in \mathcal{M}$ and $a \in \mathcal{A}$. An inner product $\mathcal{A}$-module $\mathcal{M}$ which is complete with respect to the norm $\|x\|_{\mathcal{M}} = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module over the C*-algebra $\mathcal{A}$. For example, every C*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module under the $\mathcal{A}$-valued inner product $\langle a, b\rangle = ab^*$ $(a, b \in \mathcal{A})$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module. The notion of a right Hilbert $\mathcal{A}$-module can be defined similarly.

A Hilbert C*-module $\mathcal{M}$ is said to be *prime*, if for elements a, b of $\mathcal{M}$, $\langle a, \mathcal{M}\rangle b = 0$ implies that $a = 0$ or $b = 0$. Equivalently, $\mathcal{M}$ is called prime, if for elements a, b of $\mathcal{M}$, validity the equation $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$, implies that $a = 0$ or $b = 0$. $\mathcal{M}$ is said to be *semiprime*, if $\langle a, \mathcal{M}\rangle a = 0$ implies that $a = 0$. Trivially any prime Hilbert C*-module $\mathcal{M}$ is semiprime.

Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a C*-algebra $\mathcal{A}$. A mapping $T : \mathcal{M} \to \mathcal{N}$ is said to be adjointable, if there exists a mapping $S : \mathcal{N} \to \mathcal{M}$ such that $\langle T(x), y\rangle = \langle x, S(y)\rangle$ for all $x \in D_T \subseteq \mathcal{M}$, $y \in D_S \subseteq \mathcal{N}$. The unique mapping S is denoted by T^* and is called the adjoint of T. It is well known that any adjointable mapping $T : \mathcal{M} \to \mathcal{N}$ is $\mathcal{A}$-linear (that is $T(ax + \lambda y) = aT(x) + \lambda T(y)$ for all $x, y \in \mathcal{M}, a \in \mathcal{A}, \lambda \in \mathbb{C}$) and bounded.

A linear mapping $\psi : \mathcal{M} \to \mathcal{M}$ is called a *Hilbert C*-module derivation* on $\mathcal{M}$, if it satisfies the equation

$$\psi(\langle a, b\rangle c) = \langle \psi(a), b\rangle c + \langle a, \psi(b)\rangle c + \langle a, b\rangle \psi(c),$$

for all $a, b, c \in \mathcal{M}$. ψ is called a *Hilbert C*-module Jordan derivation* on $\mathcal{M}$, if it satisfies the equation

$$\psi(\langle a, b\rangle a) = \langle \psi(a), b\rangle a + \langle a, \psi(b)\rangle a + \langle a, b\rangle \psi(a),$$

for all $a, b \in \mathcal{M}$. Note that every Hilbert C*-module derivation is a Hilbert C*-module Jordan derivation. But the converse is not true in general.

Remark 1.1. Every adjointable mapping $\psi : \mathcal{M} \to \mathcal{M}$ satisfying $\psi^* = -\psi$ is a Hilbert C*-module derivation. Infact if $\psi^* = -\psi$, then $\langle \psi(a), b\rangle c + \langle a, \psi(b)\rangle c = 0$ for all $a, b, c \in \mathcal{M}$. Moreover

$$\begin{aligned}\langle \psi(\langle a, b\rangle c), x\rangle &= \Big\langle \langle a, b\rangle c, \psi^*(x)\Big\rangle = \langle a, b\rangle\langle c, \psi^*(x)\rangle = \langle a, b\rangle\langle \psi(c), x\rangle \\ &= \Big\langle \langle a, b\rangle \psi(c), x\Big\rangle,\end{aligned}$$

for all $a, b, c, x \in \mathcal{M}$ which implies that $\psi(\langle a, b\rangle c) = \langle a, b\rangle \psi(c)$ for all $a, b, c \in \mathcal{M}$.

Example 1.1. Let $M_2(\mathbb{C})$ be the C*-algebra of 2×2 complex matrices. The mapping $\psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$\psi(A) = \begin{bmatrix} a_{21} & a_{22} \\ -a_{11} & -a_{12} \end{bmatrix},$$

for all $A = [a_{ij}] \in M_2(\mathbb{C})$, is a Hilbert C*-module derivation on $M_2(\mathbb{C})$.

A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$, with $\varphi_0 = I$ (the identity mapping on $\mathcal{M}$) is called a Hilbert C*-module higher derivation on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a,b\rangle c) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c),$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n.

Example 1.2. Let ψ be a Hilbert C*-module derivation on $\mathcal{M}$. Then the sequence $\{\varphi_n\}_{n=0}^{+\infty}$ of linear mappings on $\mathcal{M}$ defined by $\varphi_0 = I$ and

$$\varphi_n(\langle a,b\rangle c) = \sum_{\substack{i+j+k=n\\ 0\le i,j,k\le n}} \frac{1}{i!j!k!}\langle \psi^i(a), \psi^j(b)\rangle \psi^k(c),$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n (in which $\psi^0 = I$), is a Hilbert C*-module higher derivation on $\mathcal{M}$. The four terms of this Hilbert C*-module higher derivation are

$$\begin{aligned}
\varphi_0(\langle a,b\rangle c) =& \langle a,b\rangle c,\\
\varphi_1(\langle a,b\rangle c) =& \langle \psi(a),b\rangle c + \langle a,\psi(b)\rangle c + \langle a,b\rangle\psi(c),\\
\varphi_2(\langle a,b\rangle c) =& \frac{1}{2}\langle \psi^2(a),b\rangle c + \frac{1}{2}\langle a,\psi^2(b)\rangle c + \frac{1}{2}\langle a,b\rangle\psi^2(c)\\
&+ \langle \psi(a),\psi(b)\rangle c + \langle \psi(a),b\rangle\psi(c) + \langle a,\psi(b)\rangle\psi(c),\\
\varphi_3(\langle a,b\rangle c) =& \frac{1}{6}\langle \psi^3(a),b\rangle c + \frac{1}{6}\langle a,\psi^3(b)\rangle c + \frac{1}{6}\langle a,b\rangle\psi^3(c)\\
&+ \frac{1}{2}\langle \psi^2(a),\psi(b)\rangle c + \frac{1}{2}\langle \psi^2(a),b\rangle\psi(c) + \frac{1}{2}\langle \psi(a),\psi^2(b)\rangle c\\
&+ \frac{1}{2}\langle a,\psi^2(b)\rangle\psi(c) + \frac{1}{2}\langle \psi(a),b\rangle\psi^2(c) + \frac{1}{2}\langle a,\psi(b)\rangle\psi^2(c)\\
&+ \langle \psi(a),\psi(b)\rangle\psi(c).
\end{aligned}$$

A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$, with $\varphi_0 = I$, is called a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$, if

$$\varphi_n(\langle a,b\rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a),$$

for all $a, b \in \mathcal{M}$ and each non-negative integer n.

When $\{\varphi_n\}_{n=0}^{+\infty}$ is a Hilbert C*-module higher derivation (Jordan higher derivation), φ_1 is a Hilbert C*-module derivation (Jordan derivation). Trivially every Hilbert C*-module higher derivation is a Hilbert C*-module Jordan higher derivation. But the converse is not true in general.

The classical result due to Herstein [11] was extended for higher derivations by Haetinger [9], who proved that every Jordan higher derivation on a prime ring of characteristic different from two is a higher derivation. Further, Ferrero and Haetinger

[8] established that on a 2-torsion free semiprime ring every Jordan triple higher derivation, is a higher derivation. In this paper we prove that if $\mathcal{M}$ is a prime Hilbert C*-module, then every Hilbert C*-module Jordan higher derivation on $\mathcal{M}$, is a Hilbert C*-module higher derivation on $\mathcal{M}$. As a consequence, we show that every Hilbert C*-module Jordan derivation on $\mathcal{M}$, is a Hilbert C*-module derivation on $\mathcal{M}$.

For more information about Hilbert C*-module derivations and Hilbert C*-module higher derivations the reader can see [6, 16]. Also for information about derivations and higher derivations on algebras, the reader refer to [1–5, 7, 10, 12, 14, 15, 17, 18].

2. The Result

Let $\mathcal{M}$ be a Hilbert C*-module and I be the identity mapping on $\mathcal{M}$. A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$, with $\varphi_0 = I$, is said to be a

(i) *Hilbert C*-module higher derivation* on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a, b\rangle c) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c), \tag{2.1}$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n;

(ii) *Hilbert C*-module Jordan higher derivation* on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a, b\rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a), \tag{2.2}$$

for all $a, b \in \mathcal{M}$ and each non-negative integer n.

Trivially every Hilbert C*-module higher derivation is a Hilbert C*-module Jordan higher derivation. But the converse is not true in general. In this section, we prove that on a prime Hilbert C*-module $\mathcal{M}$, every Hilbert C*-module Jordan higher derivation is a Hilbert C*-module higher derivation. Before proving the result, we need some lemmas.

Lemma 2.1. *Let $\mathcal{M}$ be a Hilbert C*-module and $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$ be a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$. Then,*

$$\varphi_n(\langle a, b\rangle c + \langle c, b\rangle a) = \sum_{i+j+k=n} \left(\langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c) + \langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(a)\right), \tag{2.3}$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n.

Proof. Replacing a by $a + c$ in (2.2), we get

$$\varphi_n(\langle a + c, b\rangle (a + c)) = \sum_{i+j+k=n} \langle \varphi_i(a + c), \varphi_j(b)\rangle \varphi_k(a + c),$$

which implies that

$$\begin{aligned}
&\varphi_n(\langle a, b\rangle a + \langle c, b\rangle a + \langle a, b\rangle c + \langle c, b\rangle c)\\
&= \sum_{i+j+k=n} \Big(\langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a) + \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c)\\
&\quad + \langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(a) + \langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(c)\Big),
\end{aligned}$$

for all $a, b, c \in \mathcal{M}$. Since $\varphi_n(\langle a, b\rangle a) = \sum_{i+j+k=n}\langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a)$ and $\varphi_n(\langle c, b\rangle c) = \sum_{i+j+k=n}\langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(c)$, canceling these terms from both sides of the above equation, we get the equation (2.3). □

Lemma 2.2. *Let $\mathcal{M}$ be a 2-torsion-free semiprime Hilbert C^*-module and $a, b \in \mathcal{M}$. If $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, then $\langle a, x\rangle b = \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. If $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$, then $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$.*

Proof. Let $a, b \in \mathcal{M}$. Suppose that $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Then, we have

$$\begin{aligned}\Big\langle \langle a, x\rangle b, y\Big\rangle \langle a, x\rangle b &= -\Big\langle \langle b, x\rangle a, y\Big\rangle \langle a, x\rangle b = -\langle b, x\rangle\langle a, y\rangle\langle a, x\rangle b = -\Big\langle b, \langle y, a\rangle x\Big\rangle \langle a, x\rangle b\\ &= -\Big\langle \langle b, \langle y, a\rangle x\rangle a, x\Big\rangle b = \Big\langle \langle a, \langle y, a\rangle x\rangle b, x\Big\rangle b = \Big\langle \langle a, x\rangle\langle a, y\rangle b, x\Big\rangle b\\ &= \langle a, x\rangle\langle a, y\rangle\langle b, x\rangle b = \langle a, x\rangle\Big\langle \langle a, y\rangle b, x\Big\rangle b = -\langle a, x\rangle\Big\langle \langle b, y\rangle a, x\Big\rangle b\\ &= -\langle a, x\rangle\langle b, y\rangle\langle a, x\rangle b = -\Big\langle \langle a, x\rangle b, y\Big\rangle \langle a, x\rangle b,\end{aligned}$$

for all $y \in \mathcal{M}$, which implies that $\Big\langle \langle a, x\rangle b, y\Big\rangle \langle a, x\rangle b = 0$ for all $y \in \mathcal{M}$. Since $\mathcal{M}$ is semiprime, we get $\langle a, x\rangle b = 0$ and so $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$.

Now suppose that $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$. Then, we have

$$\Big\langle \langle b, x\rangle a, y\Big\rangle \langle b, x\rangle a = \langle b, x\rangle\langle a, y\rangle\langle b, x\rangle a = \langle b, x\rangle\Big\langle \langle a, y\rangle b, x\Big\rangle a = 0,$$

for all $y \in \mathcal{M}$. Then semiprimeness of $\mathcal{M}$ implies that $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. □

Lemma 2.3. *Let $\mathcal{M}$ be a 2-torsion-free Hilbert C^*-module. Then the following conditions are equivalent.*

(i) *$\mathcal{M}$ is a prime Hilbert C^*-module.*
(ii) *For $a, b \in \mathcal{M}$, validity of $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, implies that $a = 0$ or $b = 0$.*
(iii) *For $a, b \in \mathcal{M}$, validity of $\langle a, x\rangle a = \langle b, x\rangle b$ for all $x \in \mathcal{M}$, implies that $a = b$ or $a = -b$.*

Proof. (i)⇒(ii) If $\mathcal{M}$ is a prime Hilbert C*-module and $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, then by Lemma 2.2, $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$ and then by primeness of $\mathcal{M}$, $a = 0$ or $b = 0$.

(ii)⇒(i) Suppose that $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$. Then $\Big\langle \langle b, x\rangle a, y\Big\rangle \langle b, x\rangle a = \langle b, x\rangle\langle a, y\rangle\langle b, x\rangle a = \langle b, x\rangle\Big\langle \langle a, y\rangle b, x\Big\rangle a = 0$ which implies that $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Hence $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$ and therefore $a = 0$ or $b = 0$. Thus, $\mathcal{M}$ is a prime.

(ii)⇒(iii) Let $\langle a, x\rangle a = \langle b, x\rangle b$ for all $x \in \mathcal{M}$. Then $\langle a-b, x\rangle(a+b)+\langle a+b, x\rangle(a-b) = 0$ for all $x \in \mathcal{M}$. Thus, $a - b = 0$ or $a + b = 0$.

(iii)⇒(ii) Let $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Then, $\langle a - b, x\rangle(a - b) = \langle a + b, x\rangle(a + b)$ for all $x \in \mathcal{M}$. Hence, $a - b = a + b$ or $a - b = -(a + b)$. That is $a = 0$ or $b = 0$. □

Lemma 2.4. *Let $\mathcal{M}$ be a 2-torsion-free semiprime Hilbert C^*-module and $\Delta, \Omega : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be mappings which are additive in each variable and $\Delta(a,b,a) = \Omega(a,b,a) = 0$ for all $a, b \in \mathcal{M}$. If*

$$\langle \Delta(a,b,c), x\rangle \Omega(a,b,c) = 0, \tag{2.4}$$

for all $a,b,c,x \in \mathcal{M}$, then $\langle \Delta(a,b,c), x\rangle \Omega(c,b,a) = 0$ for all $a,b,c,x \in \mathcal{M}$.

Proof. Suppose that $\langle \Delta(a,b,c), x\rangle \Omega(a,b,c) = 0$ for all $a,b,c,x \in \mathcal{M}$. Then, by Lemma 2.2, we get $\langle \Omega(a,b,c), x\rangle \Delta(a,b,c) = 0$ for all $a,b,c,x \in \mathcal{M}$.

Replacing a and c by $a+c$ in (2.4), we have

$$\langle \Delta(a+c,b,a+c), x\rangle \Omega(a+c,b,a+c) = 0,$$

which implies that

$$\langle \Delta(a,b,c), x\rangle \Omega(c,b,a) + \langle \Delta(c,b,a), x\rangle \Omega(a,b,c) = 0,$$

for all $a,b,c,x \in \mathcal{M}$. It follows from

$$\begin{aligned}
&\Big\langle \langle \Delta(a,b,c), x\rangle \Omega(c,b,a), y\Big\rangle \langle \Delta(a,b,c), x\rangle \Omega(c,b,a) \\
&= - \Big\langle \langle \Delta(a,b,c), x\rangle \Omega(c,b,a), y\Big\rangle \langle \Delta(c,b,a), x\rangle \Omega(a,b,c) \\
&= - \langle \Delta(a,b,c), x\rangle \langle \Omega(c,b,a), y\rangle \langle \Delta(c,b,a), x\rangle \Omega(a,b,c) \\
&= - \langle \Delta(a,b,c), x\rangle \Big\langle \langle \Omega(c,b,a), y\rangle \Delta(c,b,a), x\Big\rangle \Omega(a,b,c) = 0,
\end{aligned}$$

and semiprimeness of $\mathcal{M}$ that $\langle \Delta(a,b,c), x\rangle \Omega(c,b,a) = 0$ for all $a,b,c,x \in \mathcal{M}$. □

Lemma 2.5. *Let $\mathcal{M}$ be a Hilbert C^*-module. Then for all $a,b,c,x \in \mathcal{M}$ we have*

$$\Big\langle a, \Big\langle b, \langle c, x\rangle c\Big\rangle b\Big\rangle a = \Big\langle \langle a,b\rangle c, x\Big\rangle \langle c,b\rangle a.$$

Proof. Let $a,b,c,x \in \mathcal{M}$, then

$$\begin{aligned}
\Big\langle a, \Big\langle b, \langle c, x\rangle c\Big\rangle b\Big\rangle a &= \Big\langle a, \langle b,c\rangle \langle x,c\rangle b\Big\rangle a = \Big\langle a, \langle x,c\rangle b\Big\rangle \langle c,b\rangle a \\
&= \langle a,b\rangle \langle c,x\rangle \langle c,b\rangle a = \Big\langle \langle a,b\rangle c, x\Big\rangle \langle c,b\rangle a.
\end{aligned}$$

□

Theorem 2.1. *Let $\mathcal{M}$ be a 2-torsion-free prime Hilbert C^*-module. Then, every Hilbert C^*-module Jordan higher derivation on $\mathcal{M}$ is a Hilbert C^*-module higher derivation on $\mathcal{M}$.*

Proof. Let $\{\varphi_n\}_{n=0}^{+\infty}$ be a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$ and $a,b,c \in \mathcal{M}$. Define

$$\Delta_n(a,b,c) := \varphi_n(\langle a,b\rangle c) - \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c), \tag{2.5}$$

for each non-negative integer n and $\Omega(a,b,c) := \langle a,b\rangle c - \langle c,b\rangle a$. Trivially $\Delta_n(a,b,a) = \Omega(a,b,a) = 0$ for all $n \in \mathbb{N}$, $\Delta_n(a,b,c) + \Delta_n(c,b,a) = 0$ and $\Omega(a,b,c) + \Omega(c,b,a) = 0$.

We have

$$\begin{aligned}
S =&\varphi_n\Big(\Big\langle a, \Big\langle b, \langle c, x\rangle c\Big\rangle b\Big\rangle a + \Big\langle c, \Big\langle b, \langle a, x\rangle a\Big\rangle b\Big\rangle c\Big)\\
=&\sum_{i+j+k=n}\Big(\Big\langle \varphi_i(a), \varphi_j(\Big\langle b, \langle c, x\rangle c\Big\rangle b)\Big\rangle \varphi_k(a) + \Big\langle \varphi_i(c), \varphi_j(\Big\langle b, \langle a, x\rangle a\Big\rangle b)\Big\rangle \varphi_k(c)\Big)\\
=&\sum_{i+p+q+r+k=n}\Big(\Big\langle \varphi_i(a), \Big\langle \varphi_p(b), \varphi_q(\langle c, x\rangle c)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(a)\\
&+\Big\langle \varphi_i(c), \Big\langle \varphi_p(b), \varphi_q(\langle a, x\rangle a)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(c)\Big)\\
=&\sum_{i+p+s+t+u+r+k=n}\Big(\Big\langle \varphi_i(a), \Big\langle \varphi_p(b), \langle \varphi_s(c), \varphi_t(x)\rangle \varphi_u(c)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(a)\\
&+\Big\langle \varphi_i(c), \Big\langle \varphi_p(b), \langle \varphi_s(a), \varphi_t(x)\rangle \varphi_u(a)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(c)\Big)\\
=&\sum_{i+p+s+t+u+r+k=n}\Big(\Big\langle \langle \varphi_i(a), \varphi_p(b)\rangle \varphi_s(c), \varphi_t(x)\Big\rangle \langle \varphi_u(c), \varphi_r(b)\rangle \varphi_k(a)\\
&+\Big\langle \langle \varphi_i(c), \varphi_p(b)\rangle \varphi_s(a), \varphi_t(x)\Big\rangle \langle \varphi_u(a), \varphi_r(b)\rangle \varphi_k(c)\Big),
\end{aligned}$$

for all $x \in \mathcal{M}$. On the other hand, using Lemmas 2.5 and 2.1, we get

$$\begin{aligned}
S =& \varphi_n\Big(\Big\langle \langle a, b\rangle c, x\Big\rangle \langle c, b\rangle a + \Big\langle \langle c, b\rangle a, x\Big\rangle \langle a, b\rangle c\Big)\\
=&\sum_{i+j+k=n}\Big(\Big\langle \varphi_i(\langle a, b\rangle c), \varphi_j(x)\Big\rangle \varphi_k(\langle c, b\rangle a) + \Big\langle \varphi_i(\langle c, b\rangle a), \varphi_j(x)\Big\rangle \varphi_k(\langle a, b\rangle c)\Big),
\end{aligned}$$

for all $x \in \mathcal{M}$. It follows from above equations that

$$\begin{aligned}
(2.6)\qquad &\sum_{i+j+k=n}\Big(\Big\langle \varphi_i(\langle a, b\rangle c), \varphi_j(x)\Big\rangle \varphi_k(\langle c, b\rangle a) + \Big\langle \varphi_i(\langle c, b\rangle a), \varphi_j(x)\Big\rangle \varphi_k(\langle a, b\rangle c)\Big)\\
&=\sum_{i+p+s+t+u+r+k=n}\Big(\Big\langle \langle \varphi_i(a), \varphi_p(b)\rangle \varphi_s(c), \varphi_t(x)\Big\rangle \langle \varphi_u(c), \varphi_r(b)\rangle \varphi_k(a)\\
&\quad+\Big\langle \langle \varphi_i(c), \varphi_p(b)\rangle \varphi_s(a), \varphi_t(x)\Big\rangle \langle \varphi_u(a), \varphi_r(b)\rangle \varphi_k(c)\Big),
\end{aligned}$$

for all $x \in \mathcal{M}$.

Now we use induction on n. Putting $n = 1$ in the above equation and canceling the like terms from both sides of this equation and then arranging them, we get

$$\begin{aligned}
&\langle \Delta_1(a, b, c), x\rangle \langle c, b\rangle a + \langle \langle c, b\rangle a, x\rangle \Delta_1(a, b, c)\\
&+ \langle \Delta_1(c, b, a), x\rangle \langle a, b\rangle c + \langle \langle a, b\rangle c, x\rangle \Delta_1(c, b, a) = 0,
\end{aligned}$$

for all $x \in \mathcal{M}$. Since $\Delta_1(c, b, a) = -\Delta_1(a, b, c)$, we get

$$\begin{aligned}
&\langle \Delta_1(a, b, c), x\rangle \langle c, b\rangle a + \langle \langle c, b\rangle a, x\rangle \Delta_1(a, b, c)\\
&- \langle \Delta_1(a, b, c), x\rangle \langle a, b\rangle c - \langle \langle a, b\rangle c, x\rangle \Delta_1(a, b, c) = 0,
\end{aligned}$$

which implies that

$$\Big\langle \Delta_1(a,b,c),x\Big\rangle \Omega(c,b,a)+\Big\langle \Omega(c,b,a),x\Big\rangle \Delta_1(a,b,c)=0,$$

for all $x\in \mathcal{M}$ and since $\Omega(c,b,a)=-\Omega(a,b,c)$, then

$$\Big\langle \Delta_1(a,b,c),x\Big\rangle \Omega(a,b,c)+\Big\langle \Omega(a,b,c),x\Big\rangle \Delta_1(a,b,c)=0,$$

for all $x\in \mathcal{M}$. Since $\mathcal{M}$ is semiprime, it follows from Lemma 2.2, that

$$\Big\langle \Delta_1(a,b,c),x\Big\rangle \Omega(a,b,c)=\Big\langle \Omega(a,b,c),x\Big\rangle \Delta_1(a,b,c)=0,$$

for all $x\in \mathcal{M}$. Since $\mathcal{M}$ is prime, it follows from Lemma 2.3 that $\Delta_1(a,b,c)=0$ or $\Omega(a,b,c)=0$. If $\Delta_1(a,b,c)=0$, then $\varphi_1(\langle a,b\rangle c)=\langle \varphi_1(a),b\rangle c+\langle a,\varphi_1(b)\rangle c+\langle a,b\rangle \varphi_1(c)$, and so φ_1 is a Hilbert C*-module derivation. If $\Omega(a,b,c)=0$, then $\langle a,b\rangle c=\langle c,b\rangle a$. Thus it follows from Lemma 2.1 that φ_1 is a Hilbert C*-module derivation.

Now suppose that for all $1\le \ell\le n-1$, φ_ℓ satisfies the equation

$$\varphi_\ell(\langle a,b\rangle c)=\sum_{i+j+k=\ell}\langle \varphi_i(a),\varphi_j(b)\rangle \varphi_k(c). \tag{2.7}$$

We will show that the equation (2.7) is true for $\ell=n$.

Note that equation (2.6) can be written as

$$\begin{aligned}\sum_{j=0}^{n}\sum_{i+k=n-j}&\Big(\Big\langle \varphi_i(\langle a,b\rangle c),\varphi_j(x)\Big\rangle \varphi_k(\langle c,b\rangle a)+\Big\langle \varphi_i(\langle c,b\rangle a),\varphi_j(x)\Big\rangle \varphi_k(\langle a,b\rangle c)\Big)\\ =\sum_{t=0}^{n}&\sum_{i+p+s+u+r+k=n-t}\Big(\Big\langle \langle \varphi_i(a),\varphi_p(b)\rangle \varphi_s(c),\varphi_t(x)\Big\rangle \langle \varphi_u(c),\varphi_r(b)\rangle \varphi_k(a)\\ &+\Big\langle \langle \varphi_i(c),\varphi_p(b)\rangle \varphi_s(a),\varphi_t(x)\Big\rangle \langle \varphi_u(a),\varphi_r(b)\rangle \varphi_k(c)\Big),\end{aligned} \tag{2.8}$$

for all $x\in \mathcal{M}$. In (2.8), for $1\le j\le n$ we have $i+k=n-j<n$ and then $i,k<n$. This implies that φ_i,φ_k satisfy (2.7). Thus we can cancel the like terms of both sides of equation (2.8). In fact the equation (2.8) reduces to the following equation for the case that $j=0$:

$$\begin{aligned}&\sum_{i+k=n}\Big(\Big\langle \varphi_i(\langle a,b\rangle c),x\Big\rangle \varphi_k(\langle c,b\rangle a)+\Big\langle \varphi_i(\langle c,b\rangle a),x\Big\rangle \varphi_k(\langle a,b\rangle c)\Big)\\ =&\sum_{i+p+s+u+r+k=n}\Big(\Big\langle \langle \varphi_i(a),\varphi_p(b)\rangle \varphi_s(c),x\Big\rangle \langle \varphi_u(c),\varphi_r(b)\rangle \varphi_k(a)\\ &+\Big\langle \langle \varphi_i(c),\varphi_p(b)\rangle \varphi_s(a),x\Big\rangle \langle \varphi_u(a),\varphi_r(b)\rangle \varphi_k(c)\Big),\end{aligned}$$

which implies that

$$
\begin{aligned}
&\Big\langle \varphi_n(\langle a,b\rangle c),x\Big\rangle\langle c,b\rangle a+\Big\langle \varphi_n(\langle c,b\rangle a),x\Big\rangle\langle a,b\rangle c\\
&+\Big\langle \langle a,b\rangle c,x\Big\rangle\varphi_n(\langle c,b\rangle a)+\Big\langle \langle c,b\rangle a,x\Big\rangle\varphi_n(\langle a,b\rangle c)\\
&+\sum_{\substack{i+k=n\\1\le i,k\le n-1}}\Big(\Big\langle \varphi_i(\langle a,b\rangle c),x\Big\rangle\varphi_k(\langle c,b\rangle a)+\Big\langle \varphi_i(\langle c,b\rangle a),x\Big\rangle\varphi_k(\langle a,b\rangle c)\Big)\\
=&\sum_{i+p+s=n}\Big\langle \langle\varphi_i(a),\varphi_p(b)\rangle\varphi_s(c),x\Big\rangle\langle c,b\rangle a+\Big\langle \langle\varphi_i(c),\varphi_p(b)\rangle\varphi_s(a),x\Big\rangle\langle a,b\rangle c\\
&+\sum_{u+r+k=n}\Big\langle \langle a,b\rangle c,x\Big\rangle\langle\varphi_u(c),\varphi_r(b)\rangle\varphi_k(a)+\Big\langle \langle c,b\rangle a,x\Big\rangle\langle\varphi_u(a),\varphi_r(b)\rangle\varphi_k(c)\\
&+\sum_{\substack{i+p+s+u+r+k=n\\1\le i+p+s,u+r+k\le n-1}}\Big(\Big\langle \langle\varphi_i(a),\varphi_p(b)\rangle\varphi_s(c),x\Big\rangle\langle\varphi_u(c),\varphi_r(b)\rangle\varphi_k(a)\\
&+\Big\langle \langle\varphi_i(c),\varphi_p(b)\rangle\varphi_s(a),x\Big\rangle\langle\varphi_u(a),\varphi_r(b)\rangle\varphi_k(c)\Big).
\end{aligned}
$$

Canceling the like terms from both sides of the above equation and then arranging them, we get

$$
\begin{aligned}
&\langle\Delta_n(a,b,c),x\rangle\langle c,b\rangle a+\langle\langle c,b\rangle a,x\rangle\Delta_n(a,b,c)\\
&+\langle\Delta_n(c,b,a),x\rangle\langle a,b\rangle c+\langle\langle a,b\rangle c,x\rangle\Delta_n(c,b,a)=0,
\end{aligned}
$$

for all $x\in\mathcal{M}$. A similar argument as used for $n=1$, shows that

$$\Big\langle\Delta_n(a,b,c),x\Big\rangle\Omega(a,b,c)=\Big\langle\Omega(a,b,c),x\Big\rangle\Delta_n(a,b,c)=0,$$

for all $x\in\mathcal{M}$. It follows from primeness of $\mathcal{M}$ that $\Delta_n(a,b,c)=0$ or $\Omega(a,b,c)=0$. In each case, it follows that the equation (2.7) holds for $\ell=n$. This proves that $\{\varphi_n\}_{n=0}^{+\infty}$ is a Hilbert C*-module higher derivation on $\mathcal{M}$. $\square$

The next corollary follows from Theorem 2.1.

Corollary 2.1. *Let $\mathcal{M}$ be a 2-torsion-free prime Hilbert C*-module. Then every Hilbert C*-module Jordan derivation on $\mathcal{M}$ is a Hilbert C*-module derivation on $\mathcal{M}$.*

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Department of Mathematics,
Payame Noor University,
P.O. Box 19395-3697, Tehran, Iran.
Email address: ekrami@pnu.ac.ir, khalil.ekrami@gmail.com
ORCID iD: https://orcid.org/0000-0002-6233-5741

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PICTURE FUZZY ORDERING AND $\mathbb{D}^*$-BASED LATTICES

DJAZIA SAADI[1] AND ABDELAZIZ AMROUNE[1]

ABSTRACT. In this paper, some fundamental concepts related to fuzzy relations, fuzzy lattices, intuitionistic fuzzy relations, and intuitionistic fuzzy lattices are extended to the picture fuzzy setting. Also, the structure of the set $\mathbb{D}^*$ of membership values of the picture fuzzy set that plays the role of a prototype for the picture fuzzy set was studied, and some of its basic properties were discussed. Furthermore, we have introduced the concepts of picture fuzzy filters in a crisp lattice, crisp filters in a picture fuzzy lattice, and picture fuzzy filters in a picture fuzzy lattice, and some of their properties, subtle differences, and extensions in terms of picture fuzzy sets are proved. As well as giving many characterizations of picture fuzzy filters in a picture fuzzy lattice. Finally, we present the necessary and sufficient requirements for a picture fuzzy subset to be a picture fuzzy prime filter.

1. INTRODUCTION

Many problems in daily life contain various levels of uncertainty. Since existing standard mathematical tools may not model such uncertainties, new ones are needed. Fuzzy sets [35] and intuitionistic fuzzy sets [9], were introduced to deal with uncertainty, are some of the well-known mathematical tools for the aforesaid purpose.

Zadeh introduced the ideas of fuzzy sets and fuzzy relations initially, followed by Goguen [26, 35, 36]. Many authors have investigated various approaches to fuzzy lattices, fuzzy filters, and related concepts, see [2–5, 19, 33]. Birkhoff introduced filters in 1935 [12] and Cartan in 1937 [17, 18]. They are basic in algebra and play a major role in the study of fuzzy logic. From a logical perspective, filters correspond to

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collections of provable formulae. The filters theory, which has numerous applications in mathematics such as logic and topology, seems to have received more attention in recent years. In [32], Venugopalan presented the idea of a fuzzy ordered set (foset) and defined the upper bound $U(A)$ and lower bound $L(A)$ of a set A. He also defined the supremum and infimum of a set A. In 1990, Bo and Wangming proposed the ideas of fuzzy sublattices and fuzzy ideals of a lattice [14]. As a fuzzy algebra, a fuzzy lattice was defined by Ajmal and Thomas [2] and fuzzy sublattices were described by the same authors. Using their α-cuts, Chon defined and characterized fuzzy lattices in 2009 and established several fundamental properties of fuzzy lattices [19]. Mezzomo [28] used a fuzzy partial order relation to describe fuzzy lattices. Additionally, he defined filters and α-filters of fuzzy lattices and he used their α-cuts to describe them.

In 1983, Atanassov first introduced intuitionistic fuzzy sets (IFSs) to address the issue of non-membership [8]. This concept was followed by the introduction in 1984, of a generalized intuitionistic fuzzy set, known as ”intuitionistic L-fuzzy set“ [11]. It has been shown to be particularly useful for dealing with ambiguity. Based on the intuitionistic fuzzy set concept of Atanassov, Burillo and Bustince [15] introduced the notion of intuitionistic fuzzy relation. Specifically, they presented the intuitionistic fuzzy order relation as a logical extension of the fuzzy order relation introduced by Zadeh [36] before. Many authors have studied the concept of intuitionistic fuzzy order, intuitionistic fuzzy lattice, intuitionistic fuzzy filter and intuitionistic fuzzy ideal [2, 6, 31, 37].

Although these sets can model many problems, there are much more problems and uncertainties in real life that they fail to model. For instance, in voting for an election, the decisions of the electorate may be split into three types: yes, no, and abstain. To model this problem and the problems similar to it, Cuong and Kreinovich put forward in 2013 [21] a new concept called ”picture fuzzy sets“. This idea is an intriguing development of both ”fuzzy sets“ and ”intuitionistic fuzzy sets“. The idea of an element's positive, negative, and neutral membership degrees with a sum less than or equal to one is the main contribution of Cuong Bui Cong and Vladik Kreinovich. This gives an unusual but great idea of what a mathematician and a lot of logic are like.

Not only does the resulting notion have a beautiful mathematical structure with connections to various fields of mathematics, but it also has a broad range of applications outside mathematics, for example in decision-making [7, 25, 34], Medical Diagnosis [23], investment risk [13] and other applications [1, 30].

The picture fuzzy set is one of the most reliable techniques to handle the uncertainties in the data throughout the decision-making process, where an intuitionistic fuzzy set may not yield satisfactory outcomes. It is an effective mathematical tool for dealing with uncertain real-life issues. It can function extremely effectively in ambiguous situations that call for responses of the yes, no, abstain, and rejection types. Fetanat and Tayebi are doing research to try to combine a new decision support system (DSS) with a picture fuzzy set based combined compromise solution (PFS-CoCoSo) [24].

Since $\mathbb{D}^*$ is a complete lattice, it is possible to use membership degrees with more freedom by interpreting picture fuzzy sets as $\mathbb{D}^*$-fuzzy sets.

The rest of this paper is structured as follows. Section 2 presents some essential concepts relevant to fuzzy sets, intuitionistic fuzzy set theory, picture fuzzy sets, and the structure set $\mathbb{D}^*$ (the set of membership values of a picture fuzzy set). In Section 3, we define the picture fuzzy relation and study its main properties. Section 4 extends the notion of fuzzy lattices and intuitionistic fuzzy lattices studied in [19] to picture fuzzy cases. As a consequence, we extend some results of [19] to picture fuzzy cases. Section 5 introduces the concept of a picture fuzzy sub-lattice, a picture fuzzy filter in a lattice. We achieve this by generalizing some existing notions and results in Zadeh's fuzzy sets and Antanssov's intuitionistic fuzzy sets (see [2–5, 33]) to the picture fuzzy case. In Section 6, we extend the notion of a crisp filter and fuzzy filter in a fuzzy lattice [29] to a crisp filter and a picture fuzzy filter in a picture fuzzy lattice and we give more characterizations of them. Section 7 focuses on prime filters and picture fuzzy prime filters of a picture fuzzy lattice. Finally, we present some concluding remarks in Section 7.

2. Preliminaries

This section provides a brief introduction to fuzzy sets, intuitionistic fuzzy sets and picture fuzzy sets.

Definition 2.1 ([35]). Suppose that X is a non-empty set. A fuzzy set E in X is given by $E = \{(x, \mu_E(x)) \mid x \in X\}$, with $\mu_E : X \to [0,1]$ represents the degree of membership of x in E.

Definition 2.2 ([9]). Suppose that X is a non-empty set. An intuitionistic fuzzy set E on X is given by $E = \{\langle x, \mu_E(x), \nu_E(x)\rangle \mid x \in X\}$, with $\mu_E : X \to [0,1]$ and $\nu_E : X \to [0,1]$ denote respectively the degree of membership and the degree of non-membership of x in E. The functions μ_E and ν_E should satisfy the condition: $\mu_E(x) + \nu_E(x) \leq 1$, for any $x \in X$.

Many authors have discussed related concepts such as fuzzy relations, intuitionistic fuzzy relations, fuzzy lattices, intuitionistic fuzzy lattices, etc. (see [2–5, 9, 10, 16, 19, 26, 29, 33, 35, 36]).

Definition 2.3 ([21]). Suppose that X is a non-empty set. A picture fuzzy set E on X is given by $E = \{\langle x, \mu_E(x), \eta_E(x), \nu_E(x)\rangle \mid x \in X\}$, where $\mu_E(x), \eta_E(x), \nu_E(x) \in [0,1]$ denote respectively the degree of positive membership of x in E, degree of neutral membership of x in E and degree of negative membership of x in E. μ_E, η_E and ν_E satisfy the condition $\mu_E(x) + \eta_E(x) + \nu_E(x) \leq 1$, for any $x \in X$.

The quantity $\pi(x) = 1 - (\mu_E(x) + \eta_E(x) + \nu_E(x))$ is called the degree of refusal membership of x in E.

According to [20, 27], consider the set $\mathbb{D}^*$ defined by:

$$\mathbb{D}^* = \left\{a = (a_1, a_2, a_3) \in [0,1]^3 \mid a_1 + a_2 + a_3 \leq 1\right\}.$$

This set plays the role of a prototype of a picture fuzzy set, and the study of this set allows us to perform picture fuzzy sets operations using these of $\mathbb{D}^*$.

Note that for $a \in \mathbb{D}^*$, a_1, a_2 and a_3 refer to the first, second and third components of a, i.e., $a = (a_1, a_2, a_3)$.

Obviously, for each picture fuzzy subset $E = \{\langle x, \mu_E(x), \eta_E(x), \nu_E(x)\rangle \mid x \in X\}$, corresponds to a $\mathbb{D}^*$-fuzzy subset, i.e., a mapping $E : X \to \mathbb{D}^*$ given by $E(x) = (\mu_E(x), \eta_E(x), \nu_E(x)) \in \mathbb{D}^*$.

Definition 2.4 ([20, 27])**.** For all $a, b \in \mathbb{D}^*$, we define the order relation $\preceq$ on $\mathbb{D}^*$ by $a \preceq b$ if and only if ($a_1 < b_1$ and $a_3 \geq b_3$) or ($a_1 = b_1$ and $a_3 > b_3$) or ($a_1 = b_1$ and $a_3 = b_3$ and $a_2 \leq b_2$), for all $a, b \in \mathbb{D}^*$.

Note that $(\mathbb{D}^*, \preceq)$ is a bounded lattice with top element $1_{\mathbb{D}^*} = (1, 0, 0)$ and bottom element $0_{\mathbb{D}^*} = (0, 0, 1)$. And for each $a, b \in \mathbb{D}^*$, $a \curlywedge b$ and $a \curlyvee b$ are given by

$$a \curlywedge b = \begin{cases} a, & \text{if } a \preceq b, \\ b, & \text{if } b \preceq a, \\ (a_1 \wedge b_1, 1 - (a_1 \wedge b_1) - (a_3 \vee b_3), a_3 \vee b_3), & \text{otherwise,} \end{cases}$$

$$a \curlyvee b = \begin{cases} b, & \text{if } a \preceq b, \\ a, & \text{if } b \preceq a, \\ (a_1 \vee b_1, 0, a_3 \wedge b_3), & \text{otherwise.} \end{cases}$$

Concerning this definition it is worth pointing out the following aspect.

- If $a_1 \neq 0$, then $a_3 \neq 1$.
- $a = (a_1, a_2, a_3) \succ 0_{\mathbb{D}^*}$ equivalent ($a_1 > 0$) or ($a_2 > 0$) or ($a_1 = 0$ and $a_3 < 1$).
- The components of non-comparable elements $a, b \in \mathbb{D}^*$ verify that ($a_1 > b_1$ and $a_3 > b_3$) or ($a_1 < b_1$ and $a_3 < b_3$). We write $a \parallel b$.

Following that, we will discuss some fundamental properties for the order of $\mathbb{D}^*$ that will be useful in the sequel.

Proposition 2.1. *Let $a, b, c, d \in \mathbb{D}^*$. Then,*

(1) $a \curlywedge b \preceq a$, $a \curlywedge b \preceq b$;
(2) $a \preceq a \curlyvee b$, $b \preceq a \curlyvee b$;
(3) $a \curlywedge b \preceq a \curlyvee b$;
(4) $a \succ 0_{\mathbb{D}^*}$ *and* $b \succ 0_{\mathbb{D}^*}$ *if and only if* $a \curlywedge b \succ 0_{\mathbb{D}^*}$;
(5) $a \succeq c$ *and* $b \succeq c$ *if and only if* $a \curlywedge b \succeq c$;
(6) $a \succ 0_{\mathbb{D}^*}$ *or* $b \succ 0_{\mathbb{D}^*}$ *if and only if* $a \curlyvee b \succ 0_{\mathbb{D}^*}$;
(7) *if* $a \succeq c$ *or* $b \succeq c$, *then* $a \curlyvee b \succeq c$;
(8) $a \preceq c$ *and* $b \preceq c$ *if and only if* $a \curlyvee b \preceq c$;
(9) *if* $b \preceq c$, *then* $a \curlyvee b \preceq a \curlyvee c$ *and* $a \curlywedge b \preceq a \curlywedge c$;
(10) $(a \curlyvee b) \curlywedge (c \curlyvee d) \succeq (a \curlywedge c) \curlyvee (a \curlywedge d) \curlyvee (b \curlywedge c) \curlyvee (b \curlywedge d)$.

Proof. Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{D}^*$.

(4) Suppose that $a \succ 0_{\mathbb{D}^*}$, $b \succ 0_{\mathbb{D}^*}$. Then,

$$a \curlywedge b = \begin{cases} a, & \text{if } a \preceq b, \\ b, & \text{if } b \preceq a, \\ (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3), & \text{otherwise.} \end{cases}$$

The result is clear if $a \curlywedge b = a$ or $a \curlywedge b = b$, it remains to prove that the property is true in the case $a \curlywedge b = (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3)$.

Since $a \succ 0_{\mathbb{D}^*}$ and $b \succ 0_{\mathbb{D}^*}$, it follows that

$$\begin{cases} a_1 > 0 \text{ and } a_3 < 1 & (1) \\ \text{or} & \\ a_1 = 0 \quad \text{and} \quad a_3 < 1 & (2) \end{cases} \quad \text{and} \quad \begin{cases} b_1 > 0 \text{ and } b_3 < 1 & (3) \\ \text{or} & \\ b_1 = 0 \text{ and } b_3 < 1 & (4). \end{cases}$$

Then we distinguish four cases.

Case 01: If we have (1) and (3), i.e., ($a_1 > 0$ and $a_3 < 1$) and ($b_1 > 0$ and $b_3 < 1$), then $a_1 \wedge b_1 > 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 02: If we have (1) and (4), i.e., ($a_1 > 0$ and $a_3 < 1$) and ($b_1 = 0$ and $b_3 < 1$), then $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 03: If we have (2) and (3), i.e., ($a_1 = 0$ and $a_3 < 1$) and ($b_1 > 0$ and $b_3 < 1$), then $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 04: If we have (2) and (4), i.e., ($a_1 = 0$ and $a_3 < 1$) and ($b_1 = 0$ and $b_3 < 1$), then $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Conversely, suppose that $a \curlywedge b \succ 0_{\mathbb{D}^*}$ and $a = 0_{\mathbb{D}^*}$. Then $a \curlywedge b = 0_{\mathbb{D}^*} \curlywedge b = 0_{\mathbb{D}^*}$, for each $b \in \mathbb{D}^*$. This is a contradiction. Thus, $a \curlywedge b \succ 0_{\mathbb{D}^*}$ implies $a \succ 0_{\mathbb{D}^*}$ and $b \succ 0_{\mathbb{D}^*}$.

(6) Suppose that $a \succ 0_{\mathbb{D}^*}$. Since $a \curlyvee b \succeq a$, then $a \curlyvee b \succ 0_{\mathbb{D}^*}$.

Conversely, suppose that $a \curlyvee b \succ 0_{\mathbb{D}^*}$, $a = 0_{\mathbb{D}^*}$ and $b = 0_{\mathbb{D}^*}$. Then, $a \curlyvee b = 0_{\mathbb{D}^*} \curlyvee 0_{\mathbb{D}^*} = 0_{\mathbb{D}^*}$. This is a contradiction. Thus, $a \curlyvee b \succ 0_{\mathbb{D}^*}$ implies $a \succ 0_{\mathbb{D}^*}$ or $b \succ 0_{\mathbb{D}^*}$.

The fact that $\mathbb{D}^*$ is a lattice, the rest of the properties are clear. □

3. Picture Fuzzy Relations

Here, we recall the definition of a picture fuzzy relation and investigate its main properties. We denote by $\mathbb{D}_0^* = \mathbb{D}^* - \{0_{\mathbb{D}^*}\}$.

Definition 3.1 ([20])**.** Suppose that X is a non-empty set. A picture fuzzy relation $\mathcal{R} : X \times X \to \mathbb{D}^*$ is defined by $\mathcal{R}(x, y) = (\mu_{\mathcal{R}}(x, y), \eta_{\mathcal{R}}(x, y), \nu_{\mathcal{R}}(x, y))$, for all $x, y \in X$, where $\mu_{\mathcal{R}} : X \times X \to [0, 1]$, $\eta_{\mathcal{R}} : X \times X \to [0, 1]$ and $\nu_{\mathcal{R}} : X \times X \to [0, 1]$ satisfying the condition $0 \le \mu_{\mathcal{R}}(x, y) + \eta_{\mathcal{R}}(x, y) + \nu_{\mathcal{R}}(x, y) \le 1$, for every $(x, y) \in X \times X$.

In the sequel, $PFR(X)$ denotes the set of all the picture fuzzy relations on X.

Definition 3.2. Suppose that X is a non-empty set and $\mathcal{R}, \mathcal{P} \in PFR(X)$. Using Definition 2.4, we can define the following.

(i) The picture fuzzy inclusion by $\mathcal{R} \subseteq \mathcal{P}$ if and only if, for all $x, y \in X$,
$(\mu_{\mathcal{R}}(x,y), \eta_{\mathcal{R}}(x,y), \nu_{\mathcal{R}}(x,y)) \preceq (\mu_{\mathcal{P}}(x,y), \eta_{\mathcal{P}}(x,y), \nu_{\mathcal{P}}(x,y))$.

(ii) The picture fuzzy intersection $\mathcal{R} \cap \mathcal{P}$ by $(\mathcal{R} \cap \mathcal{P})(x,y) = \mathcal{R}(x,y) \curlywedge \mathcal{P}(x,y)$, for all $x, y \in X$.

(iii) The picture fuzzy union $\mathcal{R} \cup \mathcal{P}$ by $(\mathcal{R} \cup \mathcal{P})(x,y) = \mathcal{R}(x,y) \curlyvee \mathcal{P}(x,y)$, for all $x, y \in X$.

(iv) The support of $\mathcal{R}$ by $S(\mathcal{R}) = \{(x,y) \in X^2 \mid \mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}\}$.

(v) The kernel of $\mathcal{R}$ by $\ker(\mathcal{R}) = \{(x,y) \in X^2 \mid \mathcal{R}(x,y) = 1_{\mathbb{D}^*}\}$.

(vi) For all $\alpha \in \mathbb{D}_0^*$, we define the α-cut of $\mathcal{R}$ by $\mathcal{R}_\alpha = \{(x,y) \in X^2 \mid \mathcal{R}(x,y) \succeq \alpha\}$.

Definition 3.3. Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. We say that $\mathcal{R}$ is

(i) reflexive if and only if $\mathcal{R}(x,x) = (1,0,0)$ for all $x \in X$;

(ii) perfect antisymmetric, if for every $x, y \in X$ with $x \neq y$ and $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(y,x) = 0_{\mathbb{D}^*}$;

(iii) transitive if and only if for all $x, y, z \in X$, $\mathcal{R}(x,z) \succeq \mathcal{R}(x,y) \curlywedge \mathcal{R}(y,z)$.

Remark 3.1. The following statement is equivalent to the definition of perfect antisymmetry: for all $x, y \in X$, if $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$, then $x = y$.

Definition 3.4. Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. Then, $\mathcal{R}$ is called a picture fuzzy ordering, or a partial picture fuzzy ordering, if it is reflexive, perfect antisymmetric and transitive.

A picture fuzzy poset (PF-poset, for short) is a set with a picture fuzzy partial order relation.

Example 3.1. Let $X = \{x_1, x_2, x_3\}$ and let $\mathcal{R} \in PFR(X)$ be given by

$\mathcal{R}$	x_1	x_2	x_3
x_1	$(1.00, 0.00, 0.00)$	$(0.30, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
x_2	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
x_3	$(0.00, 0.30, 0.20)$	$(0.00, 0.00, 0.00)$	$(1.00, 0.00, 0.00)$

.

It is obvious that $(X, \mathcal{R})$ is a picture fuzzy ordering.

Definition 3.5. A picture fuzzy ordering $\mathcal{R}$ is linear (or total) on X if for any $x, y \in X$, either $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ or $\mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$.

A linearly PF-poset $(X, \mathcal{R})$ or a picture fuzzy chain is a PF-poset $(X, \mathcal{R})$ in which $\mathcal{R}$ is linear.

Lemma 3.1. *Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. If $\mathcal{R}$ is a picture fuzzy ordering relation on X, then $S(\mathcal{R})$ and* $\ker(\mathcal{R})$ *are order relations on X.*

Proof. Suppose that $(X, \mathcal{R})$ is a PF-poset. The reflexivity of $S(\mathcal{R})$ is direct. Since $\mathcal{R}(x,x) = (1,0,0) \succ 0_{\mathbb{D}^*}$ for all $x \in X$, then $(x,x) \in S(\mathcal{R})$.

For the antisymmetry, suppose that $(x,y), (y,x) \in S(\mathcal{R})$, i.e., $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$. Then, from the perfect antisymmetric of $\mathcal{R}$, we obtain $x = y$.

Concerning the transitivity, suppose that $(x, y), (y, z) \in S(\mathcal{R})$, that is, $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$. Since $\mathcal{R}(x, z) \succeq \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$, using Proposition 2.1 (4), we obtain $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ for all $x, z \in X$, thus $(x, z) \in S(\mathcal{R})$.

Therefore, $S(\mathcal{R})$ is a partial order relation on X.

Similarly, we obtain the same result for $\ker(\mathcal{R})$. □

Remark 3.2. The fact that $S(\mathcal{R})$ is a partial order relation on X does not imply that $\mathcal{R}$ is a picture fuzzy ordering relation on X.

Example 3.2. Let $X = a, b$. Consider the relation $\mathcal{R}$ defined on X by

$\mathcal{R}$	a	b
a	$(0.10, 0.30, 0.00)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.50, 0.03, 0.20)$

, its support is given by

$S(\mathcal{R})$	a	b
a	1	0
b	0	1

.

It is not difficult to see that $S(\mathcal{R})$ is a partial order relation on X but $\mathcal{R}$ is not a picture fuzzy ordering relation on X.

Proposition 3.1. *Suppose that X is a non-empty set and $\mathcal{R} \in PFR(X)$. $\mathcal{R}$ is a picture fuzzy ordering relation if and only if all cuts $\mathcal{R}_\alpha$ are order relations on X, for any $\alpha \in \mathbb{D}_0^*$.*

Proof. Let $\alpha \in \mathbb{D}_0^*$. Suppose $(X, \mathcal{R})$ is a PF-poset and let $x \in X$. Since $\mathcal{R}(x, x) = (1, 0, 0)$, then $\mathcal{R}(x, x) \succeq \alpha$, for all $\alpha \in \mathbb{D}_0^*$, so $(x, x) \in \mathcal{R}_\alpha$. Thus $\mathcal{R}_\alpha$ is reflexive.

Suppose that $(x, y), (y, x) \in \mathcal{R}_\alpha$, then $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, x) \succeq \alpha$. This implies that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$. From the perfect antisymmetric of $\mathcal{R}$, we obtain $x = y$. Thus, $\mathcal{R}_\alpha$ is antisymmetric.

Suppose that $(x, y) \in \mathcal{R}_\alpha$ and $(y, z) \in \mathcal{R}_\alpha$. Then, $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, z) \succeq \alpha$.

From the transitivity of $\mathcal{R}$, we obtain $\mathcal{R}(x, z) \succeq \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z) \succeq \alpha$, this implies that $(x, z) \in \mathcal{R}_\alpha$. Thus, $\mathcal{R}_\alpha$ is transitive.

Hence, if $\mathcal{R}$ is a picture fuzzy ordering relation, then all cuts $\mathcal{R}_\alpha$ are order relations on X.

Conversely, assume that for all $\alpha \in \mathbb{D}_0^*$, $\mathcal{R}_\alpha$ is a partial ordering relation on X.

If $\alpha = 1_{\mathbb{D}^*} \in \mathbb{D}_0^*$, then $(x, x) \in \mathcal{R}_{1_{\mathbb{D}^*}}$ for all $x \in X$, i.e., $\mathcal{R}(x, x) = 1_{\mathbb{D}^*}$. Thus, $\mathcal{R}$ is reflexive.

Suppose that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$. Then, there exist $\alpha, \beta \in \mathbb{D}_0^*$ such that $\mathcal{R}(x, y) = \alpha$ and $\mathcal{R}(y, x) = \beta$. Put $\gamma = \alpha \curlywedge \beta$.

It is obvious that $\mathcal{R}(x, y) \succeq \gamma$ and $\mathcal{R}(x, y) \succeq \gamma$, that is, $(x, y) \in \mathcal{R}_\gamma$ and $(y, x) \in \mathcal{R}_\gamma$. From the antisymmetry of $\mathcal{R}_\gamma$ we obtain $x = y$. Thus, $\mathcal{R}$ is perfect antisymmetric.

Let $x, y, z \in X$, and put $\alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$.

If $\alpha = 0_{\mathbb{D}^*}$, it is obvious that $\alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z) \preceq \mathcal{R}(x, z)$.

If $\alpha \succ 0_{\mathbb{D}^*}$, then we have $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, z) \succeq \alpha$, that is, $(x, y) \in \mathcal{R}_\alpha$ and $(y, z) \in \mathcal{R}_\alpha$. Using the transitivity of $\mathcal{R}_\alpha$ we obtain $(x, z) \in \mathcal{R}_\alpha$, i.e., $\mathcal{R}(x, z) \succeq \alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$. Thus, $\mathcal{R}$ is transitive. Hence, $\mathcal{R}$ is a picture fuzzy ordering. □

4. Picture Fuzzy Lattices

In the following, we first extend the notions of fuzzy lattice and intuitionistic fuzzy lattice studied in [2–5, 19, 22, 33], to picture fuzzy cases. As a consequence, we extend some results in this direction.

Definition 4.1. Suppose that $(X, \mathcal{R})$ is a PF-poset and E is a non-empty subset of X. An element $u \in X$ is an upper bound of E if for all $x \in E$, $\mathcal{R}(x, u) \succ 0_{\mathbb{D}^*}$. An upper bound u_0 of E is the least upper bound of E if for any upper bound u of $E, \mathcal{R}(u_0, u) \succ 0_{\mathbb{D}^*}$. An element $l \in X$ is a lower bound of E if for all $x \in E$, $\mathcal{R}(l, x) \succ 0_{\mathbb{D}^*}$. A lower bound l_0 of E is the greatest lower bound of E if for any lower bound l of E, $\mathcal{R}(l, l_0) \succ 0_{\mathbb{D}^*}$.

$x \sqcup y$ and $x \sqcap y$ denote respectively the least upper bound and the greatest lower bound of $\{x, y\}$.

Remark 4.1. Note that the least upper bound and the greatest lower bound of any picture fuzzy subset are unique when they exist. (The uniqueness comes from the perfect antisymmetry of $\mathcal{R}$).

Definition 4.2. A PF-poset $(X, \mathcal{R})$ is a picture fuzzy lattice (PFL, for short) if and only if for all $x, y \in X$, $x \sqcup y$ and $x \sqcap y$ exist.

Example 4.1. In Example 3.1, $(X, \mathcal{R})$ is a PFL. Indeed, $x_1 \sqcap x_2 = x_1$, $x_1 \sqcap x_3 = x_3$ and $x_2 \sqcap x_3 = x_3$. Also, $x_1 \sqcup x_2 = x_2$, $x_1 \sqcup x_3 = x_1$ and $x_2 \sqcup x_3 = x_2$.
The boundaries' remainders are obtained using commutativity and idempotence.

The proofs of the following two propositions are straightforward.

Proposition 4.1. *For a PFL $(X, \mathcal{R})$, let $x, y, z \in X$. Then,*

(1) $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(y, x \sqcup y) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(x \sqcap y, y) \succ 0_{\mathbb{D}^*}$*;*
(2) $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$ *implies* $\mathcal{R}(x \sqcup y, z) \succ 0_{\mathbb{D}^*}$*;*
(3) $\mathcal{R}(z, x) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(z, y) \succ 0_{\mathbb{D}^*}$ *implies* $\mathcal{R}(z, x \sqcap y) \succ 0_{\mathbb{D}^*}$*;*
(4) $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ *if and only if* $x \sqcup y = y$*;*
(5) $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ *if and only if* $x \sqcap y = x$*;*
(6) *if* $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$*, then* $\mathcal{R}(x \sqcap y, x \sqcap z) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(x \sqcup y, x \sqcup z) \succ 0_{\mathbb{D}^*}$.

Proposition 4.2. *For a PFL $(X, \mathcal{R})$, let $x, y, z \in X$. Then,*

(1) $x \sqcup x = x$, $x \sqcap x = x$*;*
(2) $x \sqcup y = y \sqcup x$, $x \sqcap y = y \sqcap x$*;*
(3) $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$, $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$*;*
(4) $(x \sqcup y) \sqcap x = x$, $(x \sqcap y) \sqcup x = x$.

We now turn to a characterization of the relationship between a PFL and its level sets.

Proposition 4.3. *For a PF-poset $(X, \mathcal{R})$. If $(X, \mathcal{R}_\alpha)$ are lattices for all $\alpha \in \mathbb{D}_0^*$, then $(X, \mathcal{R})$ is a PFL.*

Proof. For a PF-poset $(X, \mathcal{R})$, assume that $(X, \mathcal{R}_\alpha)$ are crisp lattices. Let $\alpha \in \mathbb{D}_0^*$. For all $x, y \in X$, there exists $u_0 \in X$, such that $(x, u_0) \in \mathcal{R}_\alpha$, $(y, u_0) \in \mathcal{R}_\alpha$, and $(u_0, u) \in \mathcal{R}_\alpha$, for every upper bound u of $\{x, y\}$. Then, there exists u_0 such that $\mathcal{R}(x, u_0) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(y, u_0) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(u_0, u) \succ 0_{\mathbb{D}^*}$ for all upper bound u of $\{x, y\}$. Hence there exists a least upper bound u_0 of $\{x, y\}$ on $(X, \mathcal{R})$. In a similar way, there exists a greatest lower bound l_0 of $\{x, y\}$ on $(X, \mathcal{R})$. Thus, $(X, \mathcal{R})$ is a PFL. ☐

Remark 4.2. If $(X, \mathcal{R})$ is a PFL, then $(X, \mathcal{R}_\alpha)$ may not be a crisp lattice. Indeed.

Example 4.2. Let $(X, \mathcal{R})$ be a PFL, where $X = \{a, b, c, d\}$ and $\mathcal{R}$ defined by the following table

$\mathcal{R}$	a	b	c	d
a	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
b	$(0.30, 0.00, 0.40)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
c	$(0.50, 0.20, 0.10)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
d	$(0.70, 0.00, 0.20)$	$(0.4., 0.10, 0.50)$	$(0.10, 0.00, 0.60)$	$(1.00, 0.00, 0.00)$

.

Consider the relation $\mathcal{R}_{(0.5,0.2,0.3)}$

$\mathcal{R}_{(0.5,0.2,0.3)}$	a	b	c	d
a	1	0	0	0
b	0	1	0	0
c	1	0	1	0
d	1	0	0	1

.

$(X, \mathcal{R}_{0.5,0.2,0.3})$ is a poset.

It is not difficult to see that $\{b, c\}$ has neither the least upper bound nor the greatest lower bound. So, $(X, \mathcal{R}_{0.5,0.2,0.3})$ is a poset but not a crisp lattice.

5. Picture Fuzzy Filters in a Lattice

This section introduces picture fuzzy sub-lattice and picture fuzzy filter in a crisp lattice by inspiring those of fuzzy and intuitionistic fuzzy case. We accomplish this by generalizing some existing results in the Zadeh's fuzzy sets and Antanssov's intuitionistic fuzzy sets (see [2–5, 33]) to the picture fuzzy case.

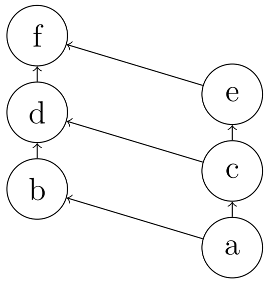
In the following, the symbol $\preceq$ indicates the picture fuzzy ordering defined on the set $\mathbb{D}^*$ as seen in Definition 2.4 and $\succeq$ is its dual.

Definition 5.1. For a crisp lattice $(X, \leqslant, \wedge, \vee)$, let E be a picture fuzzy subset on X. Then, E is called a picture fuzzy sublattice of $(X, \leqslant, \wedge, \vee)$, if for all $x, y \in X$

(i) $E(x \wedge y) \succeq E(x) \curlywedge E(y)$;
(ii) $E(x \vee y) \succeq E(x) \curlywedge E(y)$.

Example 5.1. Let $X = \{a, b, c, d, e, f\}$ and let $(X, \leqslant)$ be the lattice given by the following table and represented by the given Hasse diagram .

$\mathcal{R}$	a	b	c	d	e	f
a	1	1	1	1	1	1
b	0	1	0	1	0	1
c	0	0	1	1	1	1
d	0	0	0	1	0	1
e	0	0	0	0	1	1
f	0	0	0	0	0	1



The picture fuzzy subset E defined on X by

X	$E(x)$
a	$(0.10, 0.50, 0.40)$
b	$(0.20, 0.10, 0.40)$
c	$(0.10, 0.20, 0.30)$
d	$(0.20, 0.00, 0.40)$
e	$(0.10, 0.20, 0.30)$
f	$(0.40, 0.40, 0.10)$

is a picture fuzzy sublattice of $(X, \leqslant)$.

Definition 5.2. Let $(X, \leq, \wedge, \vee)$ be a crisp lattice. A picture fuzzy subset E of X is called a picture fuzzy filter (PFF, for short) of X, if for all $x, y \in X$

(i) $E(x \wedge y) \succeq E(x) \curlywedge E(y)$;
(ii) $E(x \vee y) \succeq E(x) \curlyvee E(y)$.

Example 5.2. Consider the lattice $(X, \leqslant)$ given in Example 5.1. The picture fuzzy subset E on X defined by

X	$E(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$
c	$(0.10, 0.30, 0.50)$
d	$(0.10, 0.20, 0.40)$
e	$(0.10, 0.30, 0.50)$
f	$(0.40, 0.10, 0.30)$

is a PFF of $(X, \leqslant)$.

Remark 5.1. Every PFF is a picture fuzzy sublattice. However, the opposite is not correct.

Example 5.3. The PFF E given in Example 5.2 is a picture fuzzy sublattice.

But the picture fuzzy sublattice E given in Example 5.1 is not a PFF, since $E(c \vee b) = (0.2, 0, 0.4)$, $E(c) \curlyvee E(b) = (0.2, 0, 0.3)$ and $(0.2, 0, 0.4) \not\succeq (0.2, 0, 0.3)$.

Proposition 5.1. *Let $(X, \leq)$ be a crisp lattice. If E, F are PFFs (resp. sublattices) of $(X, \leqslant)$, then $E \cap F$ is also a PFF (resp. sublattice) of $(X, \leqslant)$.*

Proof. Let E and F be two PFFs of $(X, \leqslant)$.

We have $(E \cap F)(x) = E(x) \curlywedge F(x)$. Then,

$$\begin{aligned}
(E \cap F)(x \wedge y) &= E(x \wedge y) \curlywedge F(x \wedge y) \\
&\succeq (E(x) \curlywedge E(y)) \curlywedge (F(x) \curlywedge F(y)) \\
&= (E(x) \curlywedge F(x)) \curlywedge (E(y) \curlywedge F(y)) \\
&= (E \cap F)(x) \curlywedge (E \cap F)(y), \\
(E \cap F)(x \vee y) &= E(x \vee y) \curlywedge F(x \vee y) \\
&\succeq (E(x) \curlyvee E(y)) \curlywedge (F(x) \curlyvee F(y)) \\
&\succeq (E(x) \curlywedge F(x)) \curlyvee (E(x) \curlywedge F(y)) \\
&\quad \curlyvee (E(y) \curlywedge F(x)) \curlyvee (E(y) \curlywedge F(y)) \\
&\succeq (E(x) \curlywedge F(x)) \curlyvee (E(y) \curlywedge F(y)) \\
&= (E \cap F)(x) \curlyvee (E \cap F)(y).
\end{aligned}$$

Hence, $E \cap F$ is a PFFs of $(X, \leqslant)$.

The same argument can be applied to the picture fuzzy sublattice case. □

Remark 5.2. The union of two PFFs (resp. sublattices) of $(X, \leqslant)$ need not be a PFF (resp. sublattice) of $(X, \leqslant)$.

Example 5.4. Consider the lattice $(X, \leqslant)$ given in Example 5.1. Let E be the PFF given in Example 5.2 and consider the PFF E^* on X by

X	$E^*(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.10, 0.40, 0.50)$
c	$(0.00, 0.00, 1.00)$
d	$(0.30, 0.10, 0.40)$
e	$(0.00, 0.00, 1.00)$
f	$(0.30, 0.20, 0.40)$

.

Then,

X	$(E \cup E^*)(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.10, 0.40, 0.50)$
c	$(0.10, 0.30, 0.50)$
d	$(0.30, 0.10, 0.40)$
e	$(0.10, 0.30, 0.50)$
f	$(0.40, 0.10, 0.30)$

.

Since $(E \cup E^*)(c \wedge b) = 0_{\mathbb{D}^*}, (E \cup E^*)(c) \curlywedge (E \cup E^*)(b) = (0.1, 0.3, 0.5)$ and $(0, 0, 1) \nsucceq (0.1, 0.3, 0.5)$, this implies that $E \cup E^*$ is not a PFF.

Proposition 5.2. *Let $(X, \leq)$ be a crisp lattice. If E is a PFF of $(X, \leqslant)$, then $S(E)$ and* $\ker(E)$ *are crisp filters on $(X, \leqslant)$.*

Recall that $S(E) = \{x \in X \mid E(x) \succ 0_{\mathbb{D}^*}\}$, $\ker(E) = \{x \in X \mid E(x) = 1_{\mathbb{D}^*}\}$.

Proof. Suppose that E is a PFF of $(X, \leqslant)$.

(i) Let $x \in S(E)$ and $y \in X$ such that $x \leqslant y$, it follows that $E(x) \succ 0_{\mathbb{D}^*}$ and $x \vee y = y$. Since $E(y) = E(x \vee y) \succeq E(x) \curlyvee E(y) \succ 0_{\mathbb{D}^*}$, then $y \in S(E)$.

(ii) Let $x, y \in S(E)$. We prove that $x \wedge y \in S(E)$. $x, y \in S(E)$ implies $E(x) \succ 0_{\mathbb{D}^*}$ and $E(y) \succ 0_{\mathbb{D}^*}$. Since $E(x \wedge y) \succeq E(x) \curlywedge E(y)$, then according to Proposition 2.1(4), $E(x \wedge y) \succ 0_{\mathbb{D}^*}$. Hence, $x \wedge y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Theorem 5.1. *Let $(X, \leq)$ be a crisp lattice. A picture fuzzy subset E of X is a PFF if and only if its α-cuts are filters of $(X, \leqslant)$ for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_0^*$.*

Proof. Suppose that E is a PFF on $(X, \leqslant)$ and prove that E_α are filters of $(X, \leqslant)$, for all $\alpha \in \mathbb{D}_0^*$.

(i) Let $x \in E_\alpha$ and $y \in X$ such that $x \leqslant y$. Then, $E(x) \succeq \alpha$ and $x \vee y = y$. It follows that $E(y) = E(x \vee y) \succeq E(x) \curlyvee E(y) \succeq \alpha$. Hence, $y \in E_\alpha$.

(ii) Let $x, y \in E_\alpha$. Then, it holds that $E(x) \succeq \alpha$ and $E(y) \succeq \alpha$. Since $E(x \wedge y) \succeq E(x) \curlywedge E(y) \succeq \alpha$, thus $x \wedge y \in E_\alpha$.

Conversely, suppose that E_α are filters of $(X, \leqslant)$, for all $\alpha \in \mathbb{D}_0^*$, and show that E is a PFF on $(X, \leqslant)$.

(i) Let $x, y \in X$ and let $\alpha \in \mathbb{D}_0^*$. Put $E(x) \curlywedge E(y) = \alpha$. We have $E(x) \succeq E(x) \curlywedge E(y) = \alpha$ and $E(y) \succeq E(x) \curlywedge E(y) = \alpha$, that is, $x, y \in E_\alpha$. Since E_α is a filter, then $x \wedge y \in E_\alpha$. This implies that $E(x \wedge y) \succeq \alpha = E(x) \curlywedge E(y)$.

(ii) Let $x, y \in X$ and let $\alpha, \beta \in \mathbb{D}_0^*$ such that $E(x) = \alpha$ and $E(y) = \beta$. Then $x \in E_\alpha$ and $y \in E_\beta$. Since E_α and E_β are filters, it follow that $x \vee y \in E_\alpha$ and $x \vee y \in E_\beta$, i.e., $E(x \vee y) \succeq \alpha$ and $E(x \vee y) \succeq \beta$.
Hence, $E(x \vee y) \succeq \alpha \curlyvee \beta = E(x) \curlyvee E(y)$.

□

6. Filters in a Picture Fuzzy Lattice

This section extends the notion of a crisp filter and fuzzy filter in a fuzzy lattice [29] to a crisp filter and a PFF in a PFL as well as providing more characterizations of them.

Definition 6.1. For a PFL $(X, \mathcal{R}, \sqcap, \sqcup)$, let E be a non-empty subset of X. E is a crisp filter on $(X, \mathcal{R}, \sqcap, \sqcup)$ if the following conditions are satisfied.

(F1) If $x \in X, y \in E$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$, then $x \in E$.

(F2) If $x, y \in E$, then $x \sqcap y \in E$.

Definition 6.2. For a PFL $(X, \mathcal{R}, \sqcap, \sqcup)$, let E be a picture fuzzy subset of X. E is a PFF on $(X, \mathcal{R}, \sqcap, \sqcup)$ if it satisfies the following conditions:

(PFF1) $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$, for all $x, y \in X$;

(PFF2) $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ implies $E(x) \preceq E(y)$, for all $x, y \in X$.

Proposition 6.1. *Let* $(X, \mathcal{R})$ *be a picture fuzzy lattice. If* E *and* F *are two PFFs of* $(X, \mathcal{R})$, *then* $E \cap F$ *is a PFF of* $(X, \mathcal{R})$.

Proof. Suppose that E and F are two PFFs of $(X, \mathcal{R})$. Then, for all $x, y \in X$,

$$\begin{aligned}(E \cap F)(x \sqcap y) &= E(x \sqcap y) \curlywedge F(x \sqcap y) \\ &\succeq (E(x) \curlywedge E(y)) \curlywedge (F(x) \curlywedge F(y)) \\ &= (E(x) \curlywedge F(x)) \curlywedge (E(y) \curlywedge F(y)) \\ &= (E \cap F)(x) \curlywedge (E \cap F)(y).\end{aligned}$$

On the other hand, if $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $E(x) \preceq E(y)$ and $F(x) \preceq F(y)$, this implies that $E(x) \curlywedge F(x) \preceq E(y) \curlywedge F(y)$. That is, $(E \cap F)(x) \preceq (E \cap F)(y)$. ☐

Remark 6.1. The union of two PFFs is not always a PFF, as demonstrated in the following example.

Example 6.1. Let $X = \{0, a, b, 1\}$ and assume that $(X, \mathcal{R})$ is the lattice given by the following table

$\mathcal{R}$	0	a	b	1
0	$(1.00, 0.00, 0.00)$	$(0.20, 0.30, 0.50)$	$(0.30, 0.10, 0.50)$	$(0.30, 0.00, 0.40)$
a	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.30, 0.00, 0.40)$
b	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.20, 0.30, 0.40)$
1	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$

.

We define the two PFFs E_1 and E_2 on $(X, \mathcal{R})$ by

X	$E_1(x)$	$E_2(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.20, 0.40, 0.30)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.30, 0.20, 0.20)$	$(0.30, 0.20, 0.10)$

.

Then,

X	$(E_1 \cap E_2)(x)$	$(E_1 \cup E_2)(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.00, 0.00, 1.00)$	$(0.20, 0.40, 0.30)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.30, 0.20, 0.20)$	$(0.30, 0.20, 0.10)$

.

It is obvious that $E_1 \cap E_2$ is a PFF of $(X, \mathcal{R})$, but $E_1 \cup E_2$ is not a PFF. Indeed, $(E_1 \cup E_2)(a \sqcap b) = 0_{\mathbb{D}^*}$, $(E_1 \cup E_2)(a) \curlywedge (E_1 \cup E_2)(b) = (0.1, 0.6, 0.3)$ and $(0, 0, 1) \not\succeq (0.1, 0.6, 0.3)$.

Proposition 6.2. *For a PFL* $(X, \mathcal{R})$, *let* E *be a picture fuzzy subset on* X. *If* E *is a PFF of* $(X, \mathcal{R})$, *then* $S(E)$ *and* $\ker(E)$ *are crisp filters of* $(X, \mathcal{R})$.

Proof. Suppose that E is a PFF of $(X, \mathcal{R})$ and let $x, y \in X$.

(F1) If $x \in S(E)$ and $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ implies $E(y) \succeq E(x) \succ 0_{\mathbb{D}^*}$. Thus, $y \in S(E)$.

(F2) If $x, y \in S(E)$, then $E(x) \succ 0_{\mathbb{D}^*}$ and $E(y) \succ 0_{\mathbb{D}^*}$. Since $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$, according to Proposition 2.1 (4), $E(x \sqcap y) \succ 0_{\mathbb{D}^*}$, i.e., $x \sqcap y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Proposition 6.3. *For a PFL $(X,\mathcal{R})$, let E be a picture fuzzy subset on X. E is a PFF of $(X,\mathcal{R})$ if and only if its α-cuts are crisp filters of $(X,\mathcal{R})$.*

Proof. Let $x,y\in X$ and let $\alpha\in\mathbb{D}_0^*$. Suppose that E is a PFF of $(X,\mathcal{R})$.

(F1) If $x\in E_\alpha$ and $\mathcal{R}(x,y)\succ 0_{\mathbb{D}^*}$, then $E(x)\succeq\alpha$ and $E(y)\succeq E(x)\succeq\alpha$. Hence, $y\in E_\alpha$.

(F2) If $x,y\in E_\alpha$, then $E(x)\succeq\alpha$ and $E(y)\succeq\alpha$, then $E(x\sqcap y)\succeq E(x)\curlywedge E(y)\succeq\alpha$, That is, $x\sqcap y\in E_\alpha$.

Conversely, suppose that E_α are crisp filters of $(X,\mathcal{R})$, for all $\alpha\in\mathbb{D}_0^*$.

(PFF1) Put $E(x)\curlywedge E(y)=\beta$. It is obvious that, if $E(x)\curlywedge E(y)=0_{\mathbb{D}^*}$, then $E(x\sqcap y)\succeq E(x)\curlywedge E(y)$. When β is greater than $0_{\mathbb{D}^*}$, we have $E(x)\succeq\beta$ and $E(y)\succeq\beta$ it follows that $x,y\in E_\beta$. Then $x\sqcap y\in E_\beta$. Hence, $E(x\sqcap y)\succeq\beta=E(x)\curlywedge E(y)$.

(PFF2) Suppose that $\mathcal{R}(x,y)\succ 0_{\mathbb{D}^*}$ and put $E(x)=\gamma$. The case $\gamma=0_{\mathbb{D}^*}$ is trivially. When γ is greater than $0_{\mathbb{D}^*}$, then, $x\in E_\gamma$ implies $y\in E_\gamma$. Hence, $E(y)\succeq\gamma=E(x)$. □

Now, we give some characterizations of PFFs of a PFL.

Theorem 6.1. *E is a PFF of a PFL $(X,\mathcal{R})$ if and only if it satisfies* (PFF1) *and*

(PFF3) *$E(x)\succeq E(x\sqcap y)\curlywedge E(y)$, for all $x,y\in X$.*

Proof. Suppose that E is a PFF of $(X,\mathcal{R})$. It suffices to prove (PFF3).

Since $\mathcal{R}(x\sqcap y,x)\succ 0_{\mathbb{D}^*}$, then according to (PFF1) and (PFF2), $E(x)\succeq E(x\sqcap y)\succeq E(x\sqcap y)\curlywedge E(y)$.

Conversely, suppose that (PFF1) and (PFF3) are satisfied. If $\mathcal{R}(x,y)\succ 0_{\mathbb{D}^*}$, then $x\sqcap y=x$ implies $E(y)\succeq E(x\sqcap y)\curlywedge E(x)=E(x)$. □

Theorem 6.2. *E is a PFF of a PFL $(X,\mathcal{R})$ if and only if it satisfies*

(PFF4) *$\mathcal{R}(x\sqcap y,z)\succ 0_{\mathbb{D}^*}$ implies $E(z)\succeq E(x)\curlywedge E(y)$, for all $x,y,z\in X$.*

Proof. Let $x,y,z\in X$. Suppose that E is a PFF of $(X,\mathcal{R})$.

If $\mathcal{R}(x\sqcap y,z)\succ 0_{\mathbb{D}^*}$, then according to (PFF1) and (PFF2), $E(z)\succeq E(x\sqcap y)\succeq E(x)\curlywedge E(y)$.

Conversely, suppose that (PFF4) is satisfied. Then the following hold.

(PFF1) Since $\mathcal{R}(x\sqcap y,x\sqcap y)\succ 0_{\mathbb{D}^*}$, then $E(x\sqcap y)\succeq E(x)\curlywedge E(y)$.

(PFF2) If $\mathcal{R}(x,y)\succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(x\sqcap x,y)\succ 0_{\mathbb{D}^*}$. It follows that $E(y)\succeq E(x)\curlywedge E(x)$, that is, $E(y)\succeq E(x)$. □

Theorem 6.3. *E is a PFF of a PFL $(X,\mathcal{R})$ if and only if it satisfies* (PFF1) *and*

(PFF5) *$E(x\sqcup y)\succeq E(x)$, for all $x,y\in X$.*

Proof. Let $x,y\in X$. Suppose that E is a PFF of $(X,\mathcal{R})$. It suffices to prove (PFF5). Since $\mathcal{R}(x,x\sqcup y)\succ 0_{\mathbb{D}^*}$, then $E(x\sqcup y)\succeq E(x)$.

Conversely, suppose that (PFF1) and (PFF5) are satisfied. If $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$, then $x \sqcup y = y$. Hence, $E(x) \preceq E(x \sqcup y) = E(y)$. ☐

Theorem 6.4. *E is a PFF of a PFL $(X,\mathcal{R})$ if and only if it satisfies*

(PFF6) *for all $x, y \in X, E(x \sqcap y) = E(x) \curlywedge E(y)$.*

Proof. Suppose that E is a PFF of $(X,\mathcal{R})$.

In view of the definition of a PFF, it suffices to show that $E(x \sqcap y) \preceq E(x) \curlywedge E(y)$.

Let $x, y \in X$. Since $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(x \sqcap y, y) \succ 0_{\mathbb{D}^*}$, then $E(x \sqcap y) \preceq E(x)$ and $E(x \sqcap y) \preceq E(y)$. Hence $E(x \sqcap y) \preceq E(x) \curlywedge E(y)$.

Conversely, suppose that $E(x \sqcap y) = E(x) \curlywedge E(y)$, for all $x, y \in X$.

- $E(x \sqcap y) = E(x) \curlywedge E(y)$ implies (PFF1).
- If $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$, then $x \sqcap y = x$. Thus $E(x) = E(x \sqcap y) = E(x) \curlywedge E(y)$, that is, $E(x) \preceq E(y)$. ☐

7. Picture Fuzzy Prime Filters in a Picture Fuzzy Lattice

Prime filters, as well as picture fuzzy prime filters of a PFL, are the topic of this section's discussion.

Definition 7.1. Suppose that $(X,\mathcal{R},\sqcap,\sqcup)$ is a PFL and E is a crisp filter of $(X,\mathcal{R})$. Then E is called a crisp prime filter if for all $x, y \in X$, $x \sqcup y \in E$ imply that $x \in E$ or $y \in E$.

Definition 7.2. Suppose that $(X,\mathcal{R},\sqcap,\sqcup)$ is a PFL and E is a PFF of $(X,\mathcal{R})$. E is called a picture fuzzy prime filters (PFPF, for short) if for any $x, y \in X$, $E(x \sqcup y) = E(x) \curlyvee E(y)$.

Remark 7.1. The intersection of two PFPFs of $(X,\mathcal{R})$ does not be necessarily a PFPF of $(X,\mathcal{R})$.

Example 7.1. Consider the lattice $(X,\mathcal{R})$ given in Example 6.1.

Let E_1, E_2 be two PFPFs on $(X,\mathcal{R})$ defined by

X	$E_1(x)$	$E_2(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.20, 0.40, 0.30)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.20, 0.40, 0.30)$	$(0.10, 0.50, 0.20)$

Then,

X	$(E_1 \cap E_2)(x)$
0	$(0.00, 0.00, 1.00)$
a	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$
1	$(0.10, 0.60, 0.30)$

It is easy to check that $E_1 \cap E_2$ is a PFF. But $(E_1 \cap E_2)(a \sqcup b) = (E_1 \cap E_2)(1) = (0.10, 0.60, 0.30)$. In the other hand, $(E_1 \cap E_2)(a) \curlyvee (E_1 \cap E_2)(b) = (0,0,1) \curlyvee (0,0,1) = (0,0,1) \neq (0.10, 0.60, 0.30)$. Hence, the PFF $E_1 \cap E_2$ is not prime.

Proposition 7.1. *Suppose that* $(X, \mathcal{R})$ *is a PFL. If* E *is a PFPF on* $(X, \mathcal{R})$*, then* $S(E)$ *and* $\ker(E)$ *are crisp prime filters on* $(X, \mathcal{R})$.

Proof. Suppose that E is a PFPF on $(X, \mathcal{R})$. From Proposition 6.2, it holds that $S(E)$ is a filter on $(X, \mathcal{R})$. We then demonstrate that $S(E)$ is prime.

Let $x, y \in X$. If $x \vee y \in S(E)$, then $E(x \vee y) \succ 0_{\mathbb{D}^*}$. Since $E(x \vee y) = E(x) \curlyvee E(y)$, this implies from Proposition 2.1 (6) that $E(x) \succ 0_{\mathbb{D}^*}$ or $E(y) \succ 0_{\mathbb{D}^*}$. Hence, either $x \in S(E)$ or $y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Theorem 7.1. *Suppose that* $(X, \mathcal{R})$ *is a PFL and let* E *be a picture fuzzy subset on* X*. If for all* $\alpha \in \mathbb{D}_0^*$*,* E_α *are prime filters on* $(X, \mathcal{R})$*, then* E *is a PFPF on* $(X, \mathcal{R})$*.*

Proof. Suppose that E_α are prime filters on $(X, \mathcal{R})$, for all $\alpha \in \mathbb{D}_0^*$. From Proposition 6.3, E is a PFF on $(X, \mathcal{R})$. It remains to show the primality of E, i.e., for all $x, y \in X$, $E(x \sqcup y) = E(x) \curlyvee E(y)$.

Put $E(x \sqcup y) = \alpha$, then $x \sqcup y \in E_\alpha$. Since E_α is a prime filter, this implies that $x \in E_\alpha$ or $y \in E_\alpha$. Hence, $E(x) \curlyvee E(y) \succeq \alpha = E(x \sqcup y)$.

In contrast, since $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x \sqcup y) \succ 0_{\mathbb{D}^*}$ these imply from (PFF1) that $E(x) \preceq E(x \sqcup y)$ and $E(y) \preceq E(x \sqcup y)$. Hence, $E(x) \curlyvee E(y) \preceq E(x \sqcup y)$. □

Remark 7.2. Unlike the fuzzy case, the converse implication in Proposition 7.1 is not true.

Example 7.2. Let $(X, \mathcal{R})$ be the lattice given as follows

$\mathcal{R}$	0	a	b	c	1
0	$1_{\mathbb{D}^*}$	$(0.20, 0.01, 0.70)$	$(0.40, 0.20, 0.30)$	$(0.50, 0.10, 0.30)$	$1_{\mathbb{D}^*}$
a	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$(0.40, 0.10, 0.50)$	$(0.10, 0.00, 0.60)$	$(0.70, 0.00, 0.20)$
b	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$(0.30, 0.00, 0.40)$
c	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$(0.50, 0.20, 0.10)$
1	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$

.

The following table represented two picture fuzzy subsets: E_1 and its support, and E_2 and its kernel

X	$E_1(x)$	$E_2(x)$	$S(E_1)(x)$	$\ker(E_2)(x)$
0	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	0	0
a	$(0.10, 0.60, 0.30)$	$0_{\mathbb{D}^*}$	1	0
b	$(0.10, 0.40, 0.10)$	$(0.10, 0.30, 0.20)$	1	0
c	$(0.20, 0.10, 0.30)$	$1_{\mathbb{D}^*}$	1	1
1	$(0.20, 0.60, 0.10)$	$1_{\mathbb{D}^*}$	1	1

.

It is easy to see that $S(E_1)$ and $\ker(E_2)$ are crisp prime filters on $(X, \mathcal{R})$, but E_1 and E_2 are not PFPFs on $(X, \mathcal{R})$.

8. Conclusion and Future Work

After refining the $(\mathbb{D}^*, \preceq)$ laws associated with $\preceq$ so that $\mathbb{D}^*$ is a complete lattice, and investigating the algebraic structure of $\mathbb{D}^*$ (A study made in a paper that will appear in TWS), it is customary to study picture fuzzy lattices, picture fuzzy sub-lattice and picture fuzzy filter in a crisp lattice by inspiring those of fuzzy and intuitionistic fuzzy case in this paper. Also, we have studied prime filters and picture fuzzy prime filters of a picture fuzzy lattice.

In future work, we plan to characterize principal picture fuzzy filters (resp. picture fuzzy ideals) on a picture fuzzy lattice. Also, we intend to introduce the notion for picture fuzzy homomorphism and picture fuzzy isomorphism and do some picture fuzzy isomorphism theorems of picture fuzzy lattices. In particular, we characterize some quotients of picture fuzzy lattice classes by their picture fuzzy ideals. Also, we study t-picture fuzzy lattices (picture fuzzy lattices w.r.t. a triangular norm) and picture fuzzy t-filters.

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[1]Department of Mathematics,
University of M'sila,
M'sila 28000, Algeria.
Email address: djazia.saadi@univ-msila.dz
ORCID iD: https://orcid.org/0000-0003-2149-2481
Email address: abdelaziz.amroune@univ-msila.dz
ORCID iD: https://orcid.org/0000-0002-0778-6566

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