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# APPROXIMATION BY CHLODOWSKY-TYPE OF SZÁSZ OPERATORS INCLUDING THE APPELL POLYNOMIALS OF CLASS $\mathbb{A}^{(2)}$

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Abstract. A Chlodowsky variation of generalized Szász type operators and a novel sequence of operators, containing the Appell polynomials of class $\mathbb{A}^{(2)}$, are the subjects of this study. Approximation properties and convergence results are given by using different types of modulus of continuity with the help of Steklov function. A weighted space of functions constructed on $[0,+\infty)$ is used to study the convergence features of these operators. Theoretical conclusions are demonstrated by using the Gould-Hopper and Hermite polynomials.

## 1. Introduction

A subfield of mathematical analysis is called approximation theory. It is the study of how to approximate mathematical functions using simpler or more computationally compliant approximations. Weierstrass used uniform approximation by polynomials to identify the set of continuous functions on a closed and bounded interval in 1885. The first illustration of these polynomials was provided by Bernstein. The Szász operators [5]

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{+\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{1.1}$$

is a well-known example of a linear positive operator where $f\in C[0,+\infty)$, $x\geq 0$, and $n\in\mathbb{N}$.

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Szász Chlodowsky operators defined as:

$$S_n(f;x) = e^{-\frac{nx}{b_n}} \sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) f\left(\frac{k}{n}b_n\right),$$

where $p_k(x) = \frac{x^k}{k!}$ and $b_n$ is a positive increasing sequence such that

$$\lim_{n\to+\infty} b_n = +\infty, \quad \lim_{n\to+\infty} \frac{b_n}{n} = 0.$$

Jakimovski and Leviatan [14] presented Szász-type operators in 1969 utilizing Appell polynomials, as shown in the following:

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{+\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad \text{for } n \in \mathbb{N}, \tag{1.2}$$

where $p_k(x)$, $k > 0$ are the Appell polynomials denoted by $g(u)e^{ux} = \sum_{k=0}^{+\infty} p_k(x)u^k$. Here $g(1) \neq 0$ and $g(u) = \sum_{k=0}^{+\infty} a_k u^k$ is an analytic function in the disk $|u| < R$, $R > 1$ In the case of $g(u) = 1$, then $p_k(x) = \frac{x^k}{k!}$ and from (1.2) we encounter again the Szász operators presented by (1.1).

The elaborative approximation features of Szász-type operators were lately explored in [1, 12, 15, 19, 21]. Atakut and Büyükyazıcı[3] presented the Stancu-type generalization of operators (1.2). Next, Ismail [13] obtained a new generalization of the Jakimovski and Leviatan operators (1.2) and the Szász operators (1.1) utilizing Sheffer polynomials. Let $H(u) = \sum_{k=1}^{+\infty} h_k u^k$, $h_1 \neq 0$, and $\mathbb{A}(u) = \sum_{k=0}^{+\infty} a_k u^k$, $a_0 \neq 0$, be analytic functions in the disc $|u| < R$, $R > 1$, where $h_k$ and $a_k$ are real. The Sheffer polynomials $p_k(x)$ have generating functions of the kind

$$\mathbb{A}(t)e^{xH(t)} = \sum_{k=0}^{+\infty} p_k(x)t^k, \quad |t| < R,$$

with the aid of adhering to limitations

(i) $p_k(x) \geq 0$ and for $x \in [0, +\infty)$;
(ii) $H'(1) = 1$ and $\mathbb{A}(1) \neq 0$.

Mursaleen et al. [18] described the following as the Chlodowsky variation of Szász-type operators containing Appell polynomials:

$$B_n^*(f;x) := \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}}} \sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) f\left(\frac{k}{n}b_n\right),$$

where $b_n$, $n \in \mathbb{N}$, is a positive increasing sequence such that

$$\lim_{n\to+\infty} b_n = +\infty, \quad \lim_{n\to+\infty} \frac{b_n}{n} = 0.$$

Additionally, Mursaleen et al. [18] presented Gould-Hopper polynomials and examples of Hermite polynomials. Kazmin [16] defined the Appell polynomials of class $\mathbb{A}^{(2)}$ and presented the generating function of this polynomial as follows:

$$\mathbb{A}(t)e^{xt} + D(t)e^{-xt} = \sum_{k=0}^{+\infty} p_k(x)t^k, \tag{1.3}$$

where

$$\mathbb{A}(t) = \sum_{k=0}^{+\infty} \frac{a_k}{k!}t^k \quad \text{and} \quad D(t) = \sum_{k=0}^{+\infty} \frac{d_k}{k!}t^k,$$

are formal power series identified at the disc $|u| < R$, $R > 1$, with $a_0^2 - d_0^2 \neq 0$.

By utilizing Appell polynomials of class $\mathbb{A}^{(2)}$ given by (1.3), Sucu and Varma [22] identify the sequence of operators for $x \in [0, +\infty)$

$$T_n(f;x) = \frac{1}{\mathbb{A}(1)e^{nx} + D(1)e^{-nx}} \sum_{k=0}^{+\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.4}$$

with the constraints $p_k(x) > 0$ for $k = 0, 1, 2\ldots$, $\mathbb{A}(1) > 0$ and $D(1) > 0$. These limitations guarantee that the operators in (1.4) are positive. Keep in mind that the well-known Szász operators are produced once more for the specific choices $\mathbb{A}(t) = 1$ and $D(t) = 0$.

The structure of this work is as follows. We acquire test functions and central moments in the Section 2. In Section 3, we show how to use the first and second moduli of continuity to approximate solutions. Then, in Section 4, we examine the convergence features of newly constructed operators in weighted spaces with weighted norms on the interval $[0, +\infty)$. We get the rate of convergence utilizing the weighted modulus of continuity. Finally, we provide numerical examples that use orthogonal polynomials, such as Gould-Hopper and Hermite polynomials.

## 2. Approximation Properties of $\mathcal{B}_n^*$ Operators

Utilizing Appell polynomials of class $\mathbb{A}^{(2)}$, we examine the Chlodowsky variation of Szász-type operators [4] given by (1.3):

$$\mathcal{B}_n^*(f;x) = \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}} \sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) f\left(\frac{k}{n}b_n\right), \tag{2.1}$$

where $b_n$ is a positive increasing sequence such that

$$\lim_{n\to+\infty} b_n = +\infty, \quad \lim_{n\to+\infty} \frac{b_n}{n} = 0.$$

We will employ the following test functions and suppose that the operators $\mathcal{B}_n^*$ are positive throughout the study:

$$e_i(t) = t^i, \quad i \in \{0, 1, 2, 3, 4\}.$$

In addition, assume that

$$\lim_{y\to+\infty} \frac{D^{(k)}(y)}{D(y)} = 1, \quad k \in \{0, 1, 2, 3, 4\}. \tag{2.2}$$

**Lemma 2.1.** *For all $x \in [0, +\infty)$, we have*

$$\mathcal{B}_n^*(e_0; x) = 1,$$

$$\mathcal{B}_n^*(e_1; x) = \frac{\mathbb{A}(1)e^{\frac{nx}{b_n}} - D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x + \frac{b_n}{n} \cdot \frac{\mathbb{A}'(1)e^{\frac{nx}{b_n}} - D'(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}},$$

$$\begin{aligned}\mathcal{B}_n^*(e_2; x) =& x^2 + \frac{b_n}{n} \cdot \frac{e^{\frac{nx}{b_n}}(\mathbb{A}(1) + 2\mathbb{A}'(1)) - e^{-\frac{nx}{b_n}}(D(1) + 2D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x \\ &+ \frac{b_n^2}{n^2} \cdot \frac{e^{\frac{nx}{b_n}}(\mathbb{A}'(1) + \mathbb{A}''(1)) - e^{-\frac{nx}{b_n}}(D'(1) + D''(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}},\end{aligned}$$

$$\begin{aligned}\mathcal{B}_n^*(e_3; x) =& \frac{\mathbb{A}(1)e^{\frac{nx}{b_n}} - D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^3 \\ &+ \frac{b_n}{n} \cdot \frac{e^{\frac{nx}{b_n}}(3\mathbb{A}'(1) + 3\mathbb{A}(1)) + e^{-\frac{nx}{b_n}}(3D'(1) + 3D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^2 \\ &+ \frac{b_n^2}{n^2} \cdot \frac{e^{\frac{nx}{b_n}}(3\mathbb{A}''(1) + 6\mathbb{A}'(1) + \mathbb{A}(1)) + e^{-\frac{nx}{b_n}}(-3D''(1) - 6D'(1) - D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x \\ &+ \frac{b_n^3}{n^3} \cdot \frac{e^{\frac{nx}{b_n}}(\mathbb{A}'''(1) + 3\mathbb{A}''(1) + \mathbb{A}'(1)) + e^{-\frac{nx}{b_n}}(D'''(1) + 3D''(1) + D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}},\end{aligned}$$

$$\begin{aligned}\mathcal{B}_n^*(e_4; x) =& x^4 + \frac{b_n}{n} \cdot \frac{e^{\frac{nx}{b_n}}(4\mathbb{A}'(1) + 6\mathbb{A}(1)) - e^{-\frac{nx}{b_n}}(4D'(1) + 6D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^3 \\ &+ \frac{b_n^2}{n^2} \cdot \frac{e^{\frac{nx}{b_n}}(6\mathbb{A}''(1) + 18\mathbb{A}'(1) + 7\mathbb{A}(1)) + e^{-\frac{nx}{b_n}}(6D''(1) + 18D'(1) + 7D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}x^2 \\ &+ \frac{b_n^3}{n^3}\left(\frac{e^{\frac{nx}{b_n}}(4\mathbb{A}'''(1) + 18\mathbb{A}''(1) + 14\mathbb{A}'(1) + \mathbb{A}(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right. \\ &\left. - \frac{e^{-\frac{nx}{b_n}}(4D'''(1) + 18D''(1) + 14D'(1) + D(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right)x \\ &+ \frac{b_n^4}{n^4}\left(\frac{e^{\frac{nx}{b_n}}(\mathbb{A}^{iv}(1) + 6\mathbb{A}'''(1) + 7\mathbb{A}''(1) + \mathbb{A}'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right. \\ &\left. + \frac{e^{-\frac{nx}{b_n}}(D^{iv}(1) + 6D'''(1) + 7D''(1) + D'(1))}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\right).\end{aligned}$$

*Proof.* From the generating functions of the Appell polynomials of class $\mathbb{A}^{(2)}$ presented by (1.3), we obtain

$$\begin{aligned}
\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right) =&\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}},\\
\sum_{k=0}^{+\infty} kp_k\left(\frac{nx}{b_n}\right) =&\frac{n}{b_n}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}}-D(1)e^{-\frac{nx}{b_n}}\right)x+\mathbb{A}'(1)e^{\frac{nx}{b_n}}+D'(1)e^{-\frac{nx}{b_n}},\\
\sum_{k=0}^{+\infty} k^2p_k\left(\frac{nx}{b_n}\right) =&\frac{n^2}{b_n^2}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}\right)x^2\\
&+\frac{n}{b_n}\left((\mathbb{A}(1)+2\mathbb{A}'(1))e^{\frac{nx}{b_n}}-(D(1)+2D'(1))e^{-\frac{nx}{b_n}}\right)x\\
&+\mathbb{A}'(1)e^{\frac{nx}{b_n}}+D'(1)e^{-\frac{nx}{b_n}}+\mathbb{A}''(1)e^{\frac{nx}{b_n}}+D''(1)e^{-\frac{nx}{b_n}},\\
\sum_{k=0}^{+\infty} k^3p_k\left(\frac{nx}{b_n}\right) =&\frac{n^3}{b_n^3}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}}-D(1)e^{-\frac{nx}{b_n}}\right)x^3\\
&+\frac{n^2}{b_n^2}\left((3\mathbb{A}'(1)+3\mathbb{A}(1))e^{\frac{nx}{b_n}}+(3D'(1)+3D(1))e^{-\frac{nx}{b_n}}\right)x^2\\
&+\frac{n}{b_n}\left((3\mathbb{A}''(1)+6\mathbb{A}'(1)+\mathbb{A}(1))e^{\frac{nx}{b_n}}\right.\\
&\left.-(3D''(1)+6D'(1)+D(1))e^{-\frac{nx}{b_n}}\right)x\\
&+e^{\frac{nx}{b_n}}(\mathbb{A}'''(1)+3\mathbb{A}''(1)+\mathbb{A}'(1))+e^{-\frac{nx}{b_n}}(D'''(1)+3D''(1)+D'(1)),\\
\sum_{k=0}^{+\infty} k^4p_k\left(\frac{nx}{b_n}\right) =&\frac{n^4}{b_n^4}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}\right)x^4\\
&+\frac{n^3}{b_n^3}\left((4\mathbb{A}'(1)+6\mathbb{A}(1))e^{\frac{nx}{b_n}}-(4D'(1)+6D(1))e^{-\frac{nx}{b_n}}\right)x^3\\
&+\frac{n^2}{b_n^2}\left((6\mathbb{A}''(1)+18\mathbb{A}'(1)+7\mathbb{A}(1))e^{\frac{nx}{b_n}}\right.\\
&\left.+(6D''(1)+18D'(1)+7D(1))e^{-\frac{nx}{b_n}}\right)x^2\\
&+\frac{n}{b_n}\left((4\mathbb{A}'''(1)+18\mathbb{A}''(1)+14\mathbb{A}'(1)+\mathbb{A}(1))e^{\frac{nx}{b_n}}\right.\\
&\left.-(4D'''(1)+18D''(1)+14D'(1)+D(1))e^{-\frac{nx}{b_n}}\right)x\\
&+e^{\frac{nx}{b_n}}\left((\mathbb{A}^{iv}(1)+6\mathbb{A}'''(1)+7\mathbb{A}''(1)+\mathbb{A}'(1))\right)\\
&+e^{-\frac{nx}{b_n}}\left((D^{iv}(1)+6D'''(1)+7D''(1)+D'(1))\right).
\end{aligned}$$

Given these equalities, we get the intended outcomes. □

**Lemma 2.2.** *The operators* (2.1) *confirm:*

$$\mathcal{B}_n^*((e_1-x);x) = \frac{-2D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}x + \frac{b_n}{n}\left(\frac{\mathbb{A}'(1)e^{\frac{nx}{b_n}}+D'(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right),$$

$$\begin{aligned}\mathcal{B}_n^*((e_1-x)^2;x) =& \frac{4D'(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}x^2 + \frac{b_n}{n}\left(\frac{\mathbb{A}(1)e^{\frac{nx}{b_n}}-(4D'(1)+D(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x \\ &+ \frac{b_n^2}{n^2}\left(\frac{(\mathbb{A}'(1)+\mathbb{A}''(1))e^{\frac{nx}{b_n}}+(D'(1)+D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right),\end{aligned}$$

$$\begin{aligned}\mathcal{B}_n^*((e_1-x)^4;x) =& \left(\frac{-8\mathbb{A}(1)e^{\frac{nx}{b_n}}+8D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}+8\right)x^4 \\ &+ \frac{b_n}{n}\left(\frac{(-24D(1)-32D'(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x^3 \\ &+ \frac{b_n^2}{n^2}\left(\frac{3\mathbb{A}(1)e^{\frac{nx}{b_n}}+(11D(1)+48D'(1)+24D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x^2 \\ &+ \frac{b_n^3}{n^3}\left(\frac{(\mathbb{A}(1)+10\mathbb{A}'(1)+6\mathbb{A}''(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right. \\ &\left.-\frac{(D(1)+18D'(1)+30D''(1)+8D'''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)x \\ &+ \frac{b_n^4}{n^4}\left(\frac{(\mathbb{A}'(1)+7\mathbb{A}''(1)+6\mathbb{A}'''(1)+\mathbb{A}^{iv}(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right. \\ &\left.+\frac{(D'(1)+7D''(1)+6D'''(1)+D^{iv}(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right).\end{aligned}$$

*Proof.* By using the linearity of the $\mathcal{B}_n^*$,

$$\begin{aligned}\mathcal{B}_n^*((e_1-x);x) =& \mathcal{B}_n^*(e_1;x) - x\mathcal{B}_n^*(e_0;x), \\ \mathcal{B}_n^*((e_1-x)^2;x) =& \mathcal{B}_n^*(e_2;x) - 2x\mathcal{B}_n^*(e_1;x) + x^2\mathcal{B}_n^*(e_0;x), \\ \mathcal{B}_n^*((e_1-x)^4;x) =& \mathcal{B}_n^*(e_4;x) - 4x\mathcal{B}_n^*(e_3;x) + 6x^2\mathcal{B}_n^*(e_2;x) - 4x^3\mathcal{B}_n^*(e_1;x) \\ &+ x^4\mathcal{B}_n^*(e_0;x),\end{aligned}$$

we obtain the desired outcome of the lemma. □

**Theorem 2.1.** *Let*

$$E = \left\{ f : \frac{f(x)}{1+x^2} \text{ is convergent as } x \to +\infty \right\}$$

*and $\mathcal{B}_n^*$ be the operators given by* (2.1)*. Then for any $f \in C[0,+\infty) \cap E$, the following relation holds*

$$\lim_{n\to+\infty} \mathcal{B}_n^*(f;x) = f(x)$$

*uniformly on each compact subset of* $[0,+\infty)$.

*Proof.* According to Lemma 2.1, we get

$$\lim_{n\to+\infty} \mathcal{B}_n^*(e_i;x) = e_i(x), \quad i \in \{0,1,2\}.$$

In every compact subset of $[0,+\infty)$, the stated convergence is uniformly confirmed. The desired outcome is obtained by applying the Korovkin theorem [2]. □

## 3. Rate of Convergence

**Definition 3.1.** For any function $f \in \widetilde{C}[0,+\infty)$ and $\delta > 0$, the modulus of continuity $\omega(f,\delta)$ of the function $f$ [6] is identified by

$$\omega(f,\delta) = \sup_{\substack{x,y\in[0,+\infty) \\ |x-y|\le\delta}} |f(x)-f(y)|,$$

where the space of uniformly continuous functions is given by $\widetilde{C}[0,+\infty)$. Keep in mind that one can write

$$|f(x)-f(y)| \le \omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right), \tag{3.1}$$

for each $x \in [0,+\infty)$ and any $\delta > 0$.

**Theorem 3.1.** *If* $f \in \widetilde{C}[0,+\infty) \cap E$, $\mathcal{B}_n^*$ *operators affirm the following inequality:*

$$|\mathcal{B}_n^*(f;x) - f(x)| \le 2\omega\left(f, \sqrt{\vartheta_n(x)}\right),$$

*where*

$$\vartheta := \vartheta_n(x) = (\mathcal{B}_n^*(t-x)^2;x). \tag{3.2}$$

*Proof.* By implementing the triangle inequality and the widely recognized feature of $\omega(f,\delta)$, we obtain

$$\begin{aligned}|\mathcal{B}_n^*(f;x) - f(x)| &= \left|\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left(f\left(\frac{k}{n}b_n\right) - f(x)\right)\right| \\ &\le \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|f\left(\frac{k}{n}b_n\right) - f(x)\right|.\end{aligned}$$

By using the equation (3.1) we have

$$\begin{aligned}|\mathcal{B}_n^*(f;x) - f(x)| \le& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left(1 + \frac{1}{\delta}\left|\frac{k}{n}b_n - x\right|\right)\omega(f,\delta) \\ =&\omega(f,\delta)\left(\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\right. \\ &\left.+\frac{1}{\delta}\cdot\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n - x\right|\right).\end{aligned} \tag{3.3}$$

By applying Lemma 2.2 and considering the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n - x\right| &= \sum_{k=0}^{+\infty}\sqrt{p_k\left(\frac{nx}{b_n}\right)}\sqrt{p_k\left(\frac{nx}{b_n}\right)}\left|\frac{k}{n}b_n - x\right| \\ &\le \left(\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\right)^{\frac{1}{2}}\left(\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n - x\right|^2\right)^{\frac{1}{2}} \\ &= \left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)\left(\mathcal{B}_n^*((t-x)^2;x)\right)^{\frac{1}{2}}. \qquad (3.4)\end{aligned}$$

From the inequalities (3.3) and (3.4), we find that

$$|\mathcal{B}_n^*(f;x) - f(x)| \le \left(1 + \frac{1}{\delta}\left(\mathcal{B}_n^*((t-x)^2;x)\right)^{\frac{1}{2}}\right)\omega(f,\delta), \qquad (3.5)$$

where $\vartheta_n(x)$ is given by (3.2). We get the desired result in inequality (3.5) by selecting $\delta = \sqrt{\vartheta_n(x)}$. □

**Lemma 3.1.** *For $0 < \alpha \le 1$ and $t_1, t_2 \in [0,+\infty)$, let us present the following function class:*

$$Lip_M^{(\alpha)} = \{f : |f(t_1) - f(t_2)| \le M|t_1 - t_2|^\alpha\}.$$

**Theorem 3.2.** *Suppose that $f \in Lip_M^{(\alpha)}$. Then, we attain*

$$|\mathcal{B}_n^*(f;x) - f(x)| \quad \le \quad M[\mathcal{B}_n^*((t-x)^2;x)]^{\frac{\alpha}{2}}.$$

*Proof.* Since $f \in Lip_M^{(\alpha)}$, we obtain

$$\begin{aligned}|\mathcal{B}_n^*(f;x) - f(x)| &= |\mathcal{B}_n^*(f(t) - f(x);x)| \\ &\le \mathcal{B}_n^*(|f(t) - f(x)|;x) \le M\mathcal{B}_n^*(|t-x|^\alpha;x).\end{aligned}$$

Finally, from Hölder inequality we deduce the following expression

$$\begin{aligned}\mathcal{B}_n^*(|t-x|^\alpha;x) =& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left|\frac{k}{n}b_n - x\right|^\alpha \\ =& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty}\left[p_k\left(\frac{nx}{b_n}\right)^{\frac{2-\alpha}{2}}\right]\left[p_k\left(\frac{nx}{b_n}\right)^{\frac{\alpha}{2}}\right]\left|\frac{k}{n}b_n - x\right|^\alpha \\ \le& \frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)^{\frac{2-\alpha}{2}} \\ &\times\left(\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\right)^{\frac{2-\alpha}{2}} \\ &\times\left(\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}\right)^{\frac{\alpha}{2}} \\ &\times\left(\frac{1}{\mathbb{A}(1)e^{\frac{nx}{b_n}} + D(1)e^{-\frac{nx}{b_n}}}\sum_{k=0}^{+\infty} p_k\left(\frac{nx}{b_n}\right)\left(\frac{k}{n}b_n - x\right)^2\right)^{\frac{\alpha}{2}}\end{aligned}$$

$$= (\mathcal{B}_n^*(1;x))^{\frac{2-\alpha}{2}} (\mathcal{B}_n^*((t-x)^2;x))^{\frac{\alpha}{2}}.$$

Thus, we achieve the desired results. □

**Definition 3.2.** The second modulus of continuity of the function $f \in C[a,b]$ is identified by

$$\omega_2(f,\delta) := \sup_{0<t\leq\delta} ||f(\cdot+2t) - 2f(\cdot+t) + f(\cdot)||,$$

where $||f|| = \max_{x\in[a,b]} |f(x)|$.

**Lemma 3.2** (Gavrea and Rasa [10])**.** *Suppose that we have the sequence of positive linear operators $g \in C^2[0,a]$ and $(\mathcal{B}_n^*)_{n\geq 0}$ with $\mathcal{B}_n^*(e_0;x) = 1$. Then,*

$$|\mathcal{B}_n^*(g;x) - g(x)| \leq ||g'||\sqrt{\mathcal{B}_n^*\left((t-x)^2;x\right)} + \frac{1}{2}||g''||\mathcal{B}_n^*\left((t-x)^2;x\right).$$

*For $f \in C[a,b]$, the second-order Steklov function of $f$ [24] is identified by*

$$f_h(x) := \frac{1}{h}\int_{-h}^{h}\left(1-\frac{|t|}{h}\right) f(h;x+t)dt, \quad x\in[a,b],$$

*where $f(h;\cdot) : [a-h,b+h] \to \mathbb{R}$, $h>0$, by*

$$f(h;x) = \begin{cases} P_-(x), & a-h \leq x \leq a, \\ f(x), & a \leq x \leq b, \\ P_+(x), & b < x \leq b+h, \end{cases}$$

*and $P_-, P_+$ are the linear best approximations to $f$ given piecewisely.*

**Lemma 3.3** (Zhuk [24])**.** *Let $f_h$ where $f \in [c,d]$ and $h \in \left(0, \frac{c-d}{2}\right)$ be the second-order Steklov function attached to $f$. Then, the inequalities*

(i) $||f_h - f|| \leq \frac{3}{4}\omega_2(f,h)$;

(ii) $||f_h''|| \leq \frac{3}{2h^2}\omega_2(f,h)$;

*hold.*

**Theorem 3.3.** *Assume that $f$ is a continuous function on $[0,+\infty)$. Then, we attain*

$$|\mathcal{B}_n^*(f;x) - f(x)| \leq \frac{3}{4}(2+a+h^2)\omega_2(f,h) + \frac{2}{a}h^2||f||,$$

*where*

$$h := h_n(x) = (\mathcal{B}_n^*((t-x)^2;x))^{\frac{1}{4}}$$

*and $\omega_2(f;h)$ is the second order modulus of continuity.*

*Proof.* Let $f_h$ be the second-order Steklov function of the function $f$. So, in view of $\mathcal{B}_n^*(1;x) = 1$, we attain

$$\begin{aligned} |\mathcal{B}_n^*(f;x) - f(x)| &\leq \mathcal{B}_n^*(|f-f_h|;x) + |\mathcal{B}_n^*(f_h;x) - f_h(x)| + |f_h(x) - f(x)| \\ &\leq 2||f-f_h|| + |\mathcal{B}_n^*(f_h;x) - f_h(x)|. \end{aligned} \tag{3.6}$$

Given $f_h \in C^2[0,a]$ and Lemma 3.2, it is evident that

$$|\mathcal{B}_n^*(f_h;x)-f_h(x)| \le ||f_h'(x)||\sqrt{\mathcal{B}_n^*((t-x)^2;x)}+\frac{1}{2}||f_h''(x)||\mathcal{B}_n^*((t-x)^2;x). \tag{3.7}$$

The Landau inequality is defined from [8] as follows

$$||f'|| \le 2||f||^{\frac{1}{2}}||f''||^{\frac{1}{2}}.$$

Additionally, by combining Lemma 3.3 with the Landau inequality,

$$||f_h'|| \le \frac{2}{a}||f_h||+\frac{a}{2}||f_h''|| \le \frac{2}{a}||f||+\frac{3a}{4h^2}\omega_2(f,h),$$

where $h=(\mathcal{B}_n^*((t-x)^2;x))^{\frac{1}{4}}$,

$$|\mathcal{B}_n^*(f_h;x)-f_h(x)| \le \frac{2}{a}||f||h^2+\frac{3a}{4}\omega_2(f,h)+\frac{3}{4}h^2\omega_2(f,h).$$

Now we use the inequality (3.7) in (3.6). Then with the help of Lemma 3.2

$$|\mathcal{B}_n^*(f;x)-f(x)| \le \frac{2}{a}||f||h^2+\frac{3}{4}(a+2+h^2)\omega_2(f,h)$$

is obtained and the proof is done. □

## 4. Weighted Approximation

To calculate the rate of convergence of the unbounded function described on $[0,+\infty)$, we require weighted spaces. Here, we work on the features of approximations of newly generated operators $\mathcal{B}_n^*$ on weighted spaces of exponentially growing functions on $[0,+\infty)$. At the beginning, we review the weighted spaces' notations. Let $\rho(x) = (1+x^2)$ be the weighted function and $R_f$ be a positive constant.

$$B_\rho([0,+\infty)) = \{f:[0,+\infty)\to\mathbb{R} \mid |f(x)| \le R_f\rho(x)\},$$

is a linear normed space equipped with $||f|| = \sup_{x\in[0,+\infty)}\frac{|f(x)|}{\rho(x)}$,

$$C_\rho([0,+\infty)) = \{f \in B_\rho([0,+\infty)) \mid f \text{ is continuous}\},$$

$$C_\rho^*([0,+\infty)) = \left\{f \in C_\rho([0,+\infty)) \mid \lim_{x\to+\infty}\frac{f(x)}{\rho(x)} < +\infty\right\}.$$

The relationship between these spaces can be expressed as follows: $C_\rho^*([0,+\infty)) \subset C_\rho([0,+\infty)]) \subset B_\rho([0,+\infty))$.

If $f$ is not uniformly continuous on $[0,+\infty)$, then $\omega(f,\delta)$ does not tendency 0 as $\delta\to 0$. Therefore, Gadjieva and Dogru [9] defined the weighted modulus of continuity as follows in 1998:

$$\Omega(f;\delta) = \sup_{x\ge 0,|h|\le\delta}\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}.$$

Yuksel and Ispir [23] identified the weighted modulus of continuity in 2006:

$$\Omega(f;\delta) = \sup_{x\ge 0}\sup_{0<|h|\le\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2},$$

where $f \in C_\rho^*[0,+\infty)$. We will give the properties of $\Omega(\cdot,\cdot)$ in the following lemma.

**Lemma 4.1** ([23]). *If $f \in C_\rho^*[0,+\infty)$, then*

(i) *$\Omega(f,x)$ is monotone increasing function of $\delta$;*
(ii) *$\lim_{\delta\to 0^+}\Omega(f,x)=0$;*
(iii) *for any $\lambda\in[0,+\infty)$, $\Omega(f,\lambda x)\le(1+\lambda)\Omega(f,x)$.*

*Using the weighted modulus of continuity, we will now find the rate of convergence for $f \in C_\rho^*[0,+\infty)$.*

**Theorem 4.1.** *If $f \in C_\rho^*[0,+\infty)$, then*

$$\sup_{x\in[0,+\infty)}\frac{|\mathcal{B}_n^*(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}}\le 2\left(2+M_0^*(n)+\sqrt{M_1^*(n)}\right)\Omega\left(f,\sqrt{M_0^*(n)}\right),$$

*where*

$$\begin{aligned}M_0^*(n)=&\frac{4D'(1)e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}+\frac{1}{2}\cdot\frac{b_n}{n}\left(\frac{\mathbb{A}(1)e^{\frac{nx}{b_n}}-(4D'(1)+D(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)\\&+\frac{b_n^2}{n^2}\left(\frac{(\mathbb{A}'(1)+\mathbb{A}''(1))e^{\frac{nx}{b_n}}+(D'(1)+D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right),\end{aligned}\tag{4.1}$$

$$\begin{aligned}M_1^*(n)=&\left(\frac{16D(1)e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)+\frac{3\sqrt{3}}{16}\cdot\frac{b_n}{n}\left(\frac{(-24D(1)-32D'(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)\\&+\frac{1}{4}\cdot\frac{b_n^2}{n^2}\left(\frac{3\mathbb{A}(1)e^{\frac{nx}{b_n}}+(11D(1)+48D'(1)+24D''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)\\&+\frac{3\sqrt{3}}{16}\cdot\frac{b_n^3}{n^3}\left(\frac{(\mathbb{A}(1)+10\mathbb{A}'(1)+6\mathbb{A}''(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right.\\&\left.-\frac{(D(1)+18D'(1)+30D''(1)+8D'''(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right)\\&+\frac{b_n^4}{n^4}\left(\frac{(\mathbb{A}'(1)+7\mathbb{A}''(1)+6\mathbb{A}'''(1)+\mathbb{A}^{(iv)}(1))e^{\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right.\\&\left.+\frac{(D'(1)+7D''(1)+6D'''(1)+D^{(iv)}(1))e^{-\frac{nx}{b_n}}}{\mathbb{A}(1)e^{\frac{nx}{b_n}}+D(1)e^{-\frac{nx}{b_n}}}\right).\end{aligned}\tag{4.2}$$

*Proof.* Based on Lemma 4.1 and the description of the weighted modulus of continuity, we get

$$\begin{aligned}|f(t)-f(x)|&\le\left(1+(x+|t-x|^2)\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f,\delta)\\&\le 2(1+x^2)\left(1+(t-x)^2)\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f,\delta).\end{aligned}$$

Furthermore, applying $\mathcal{B}_n^*$ for both sides, we obtain

$$|\mathcal{B}_n^*(f;x)-f(x)| \leq 2(1+x^2) \times \left(1+\mathcal{B}_n^*\left((t-x)^2;x\right)+\mathcal{B}_n^*\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\right)\Omega(f,\delta).$$

By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\mathcal{B}_n^*(f;x)-f(x)| \leq& 2(1+x^2)\Big(1+\mathcal{B}_n^*\left((t-x)^2;x\right)\\ &+\frac{1}{\delta}\sqrt{\mathcal{B}_n^*\left((t-x)^2;x\right)}\\ &+\frac{1}{\delta}\sqrt{\mathcal{B}_n^*\left((t-x)^4;x\right)}\sqrt{\mathcal{B}_n^*\left((t-x)^2;x\right)}\Big)\Omega(f,\delta).\end{aligned}$$

From Lemma 2.2, we can write $\mathcal{B}_n^*\left((t-x)^2;x\right) \leq M_0^*(n)(1+x^2)$ and $\mathcal{B}_n^*\left((t-x)^4;x\right) \leq M_1^*(n)(1+x^2)^2$. Choosing $\delta = M_0^*(n)$, we have

$$\begin{aligned}|\mathcal{B}_n^*(f;x)-f(x)| \leq& 2(1+x^2)\Big(1+M_0^*(n)(1+x^2)+(1+x^2)^{\frac{1}{2}}\\ &+\sqrt{M_1^*(n)}(1+x^2)^{\frac{3}{2}}\Omega(f,\delta)\Big)\\ \leq& 2(1+x^2)^{\frac{5}{2}}\left(2+M_0^*(n)+\sqrt{M_1^*(n)}\right)\Omega(f,\delta).\end{aligned}$$

Finally, we achieve the desired result

$$\sup_{x\in[0,+\infty)}\frac{|\mathcal{B}_n^*(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2+M_0^*(n)+\sqrt{M_1^*(n)}\right)\Omega\left(f,\sqrt{M_0^*(n)}\right).$$

Here $M_0^*(n)$ and $M_1^*(n)$ are given by (4.1) and (4.2), respectively. □

## 5. Numerical Example

5.1. **Gould-Hopper polynomials.** Gould-Hopper polynomials [11] have the generating form

$$e^{ht^{d+1}}\exp(xt) = \sum_{k=0}^{+\infty} g_k^{d+1}(x,h)\frac{t^k}{k!},$$

and detailed representations can be obtained by

$$g_k^{d+1}(x,h) = \sum_{s=0}^{[\frac{k}{d+1}]}\frac{k!}{s!(k-(d+1)s)!}h^s x^{k-(d+1)s}. \tag{5.1}$$

The Gould-Hopper polynomials $g_k^{d+1}(x,h)$ are a set of Hermite type $d$-orthogonal polynomials [7]. $d$-orthogonality is introduced by Maroni [17] and Van [20]. By selecting the Appell polynomials of class $\mathbb{A}^{(2)}$ as follows

$$\mathbb{A}(t) = e^{ht^{d+1}}, \quad D(t) = 0.$$

Gould-Hopper polynomials can be obtained. Assuming $h \geq 0$, all of the restraint $D(1) \geq 0$, $\mathbb{A}(1) > 0$ and $p_k(x) > 0$ are satisfied for all values of $k = 0, 1 \ldots.$ Using generating functions in (5.1), we obtain the series of operators in their obvious form, which includes Gould-Hopper polynomials $\mathcal{B}_n^G$

$$\mathcal{B}_n^G(f;x) = e^{-\frac{nx}{b_n}-h} \sum_{k=0}^{+\infty} \frac{g_k^{d+1}(\frac{nx}{b_n},h)}{k!} f\left(\frac{k}{n}b_n\right).$$

**Lemma 5.1.** *For the operators $\mathcal{B}_n^G$ one has*

$$\begin{aligned}
\mathcal{B}_n^G(e_0;x) =& 1,\\
\mathcal{B}_n^G(e_1;x) =& x+\frac{b_n}{n}h(d+1),\\
\mathcal{B}_n^G(e_2;x) =& x^2+\frac{b_n}{n}(1+2h(d+1))x+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h),\\
\mathcal{B}_n^G(e_3;x) =& x^3+\frac{b_n}{n}3(h(d+1)+1)+\frac{b_n^2}{n^2}(3d(d+1)h+3(d+1)^2h^2+6h(d+1)+1))\\
&+\frac{b_n^3}{n^3}((d+1)^3h^3+3d(d+1)^2h^2+(d+1)d(d-1)h+3d(d+1)h\\
&+3(d+1)^2h^2+h(d+1)),\\
\mathcal{B}_n^G(e_4;x) =& x^4+\frac{b_n}{n}(4h(d+1)+6)x^3\\
&+\frac{b_n^2}{n^2}(6d(d+1)h+6(d+1)^2h^2+18h(d+1)+7)x^2\\
&+\frac{b_n^3}{n^3}(4(d+1)^3h^3+12d(d+1)^2h^2+4(d+1)d(d-1)h+18d(d+1)h\\
&+18(d+1)^2h^2+14h(d+1)+1)x+\frac{b_n^4}{n^4}((d+1)^4h^4+6(d+1)^3dh^3\\
&+3(d+1)^2d(2d-1)h^2+(d+1)^2d(d-1)h^2+(d-2)(d+1)d(d-1)h\\
&+6(d+1)^3h^3+18d(d+1)^2h^2+6(d+1)d(d-1)h+7(d+1)dh\\
&+7(d+1)^2h^2+h(d+1)).
\end{aligned}$$

**Lemma 5.2.** *For every $x\in[0,+\infty)$, the operators $\mathcal{B}_n^G$ confirm:*

$$\begin{aligned}
\mathcal{B}_n^G((t-x);x) =& \frac{b_n}{n}h(d+1),\\
\mathcal{B}_n^G((t-x)^2;x) =& \frac{b_n}{n}x+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h),\\
\mathcal{B}_n^G((t-x)^4;x) =& \frac{b_n^4}{n^4}\Big[(d+1)^4h^4+6(d+1)^3dh^3+3(d+1)^2d(2d-1)h^2\\
&+(d+1)^2d(d-1)h^2+(d-2)(d+1)d(d-1)h+6(d+1)^3h^3\\
&+18d(d+1)^2h^2+6(d+1)d(d-1)h+7d(d+1)h\\
&+7(d+1)^2h^2+h(d+1)\Big]+\frac{b_n^3}{n^3}\Big[4(d+1)^3h^3+6(d+1)dh
\end{aligned}$$

$$+6(d+1)^2h^2+10(d+1)h-4(d+1)^3h^3+1\Big]\,x+\frac{b_n^2}{n^2}3x^2.$$

**Theorem 5.1** (Voronovskaja-type theorem)**.** *Let $f\in C^2[0,a]$. Then, one has*

$$\lim_{n\to+\infty}\frac{n}{b_n}[\mathcal{B}_n^G(f;x)-f(x)]=h(d+1)f'(x)+\frac{xf''(x)}{2}.$$

*Proof.* In light of the function $f$'s Taylor formula, we determine

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+(t-x)^2\eta(t;x), \tag{5.2}$$

where $\eta(t;x)\in C[0,a]$ and $\lim_{t\to x}\eta(t;x)=0$. Implementing $\mathcal{B}_n^G$ to the both sides of (5.2), we attain

$$\mathcal{B}_n^G(f;x)=f(x)+f'(x)\mathcal{B}_n^G(t-x;x)+\frac{f''(x)}{2}\mathcal{B}_n^G((t-x)^2;x)+\mathcal{B}_n^G((t-x)^2\eta(t;x)). \tag{5.3}$$

According to Lemmas 5.1–5.2, (5.3) becomes

(5.4)

$$\mathcal{B}_n^G(f;x)=f(x)+f'(x)\frac{b_n}{n}h(d+1)+\frac{f''(x)}{2}\left[\frac{b_n}{n}x+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h\right]+I,$$

where

$$I:=e^{-\frac{nx}{b_n}-h}\sum_{k=0}^{+\infty}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right).$$

Let's now take the sum $I$ as follows:

$$\begin{aligned}I=&e^{-\frac{nx}{b_n}-h}\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|\le\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right)\\&+e^{-\frac{nx}{b_n}-h}\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right).\end{aligned} \tag{5.5}$$

From the continuity of function $\eta$, it results that for all $\epsilon>0$, there exists a positive $\delta$ such that if $\left|\left(\frac{k}{n}b_n\right)-x\right|\le\delta$, then $\left|\eta\left(\frac{k}{n}b_n;x\right)\right|<\epsilon$. Moreover, we can type $\left|\eta\left(\frac{k}{n}b_n;x\right)\right|<M$ for $\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta$ because the function $\eta$ is limited. Given these facts, (5.5) implies

$$I\le\epsilon\mathcal{B}_n^G((t-x)^2;x)+Me^{-\frac{nx}{b_n}-h}\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\eta\left(\frac{k}{n}b_n;x\right).$$

Taking into account the fact

$$\sum_{\left|\left(\frac{k}{n}b_n\right)-x\right|>\delta}\frac{g_k^{d+1}\left(\frac{n}{b_n}x,h\right)}{k!}\left(\frac{k}{n}b_n-x\right)^2\le\frac{1}{\delta^2}\mathcal{B}_n^G((t-x)^4;x)$$

in the last inequality, we attain

$$I \leq \epsilon \mathcal{B}_n^G((t-x)^2;x) + \frac{M}{\delta^2}\mathcal{B}_n^G((t-x)^4;x). \tag{5.6}$$

Substituting the inequality (5.6) in the equality (5.4), then from Lemma 5.2, we get

$$\begin{aligned}
\mathcal{B}_n^G(f;x) - f(x) \leq & f'(x)\frac{b_n}{n}h(d+1) \\
& + \left(\epsilon + \frac{f''(x)}{2}\right)\left[\frac{b_n}{n}x + \frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h\right] \\
& + \frac{M}{\delta^2}\left(\frac{b_n^4}{n^4}\left[(d+1)^4h^4 + 6(d+1)^3dh^3 + 3(d+1)^2d(2d-1)h^2\right.\right. \\
& +(d+1)^2d(d-1)h^2 + (d+4)(d+1)d(d-1)h + 6(d+1)^3h^3 \\
& \left. +18d(d+1)^2h^2 + 7d(d+1)h + 7(d+1)^2h^2 + h(d+1)\right] \\
& +\frac{b_n^3}{n^3}\left[4(d+1)^3h^3 + 6(d+1)dh + 6(d+1)^2h^2 + 10h(d+1)\right. \\
& \left.\left. -4(d+1)^3h^3 + 1\right]x + \frac{b_n^2}{n^2}3x^2\right).
\end{aligned}$$

Equivalently, we can type

(5.7)

$$\begin{aligned}
\mathcal{B}_n^G(f;x) - f(x) = & \mathcal{O}\left(\frac{b_n}{n}\right)\left(f'(x)h(d+1)\right. \\
& + \left(\epsilon + \frac{f''(x)}{2}\right)\left[x + \frac{b_n}{n}h(d+1)(1+d+(d+1)h\right] \\
& + \frac{M}{\delta^2}\left[\frac{b_n^3}{n^3}[(1+d)^4h^4 + 6(d+1)^3dh^3 + 3(d+1)^2d(2d-1)h^2\right. \\
& +(d+1)^2d(d-1)h^2 + (d-2)(d+1)d(d-1)h + 6(d+1)^3h^3 \\
& +18d(d+1)^2h^2 + 6(d+1)d(d-1)h + 7d(d+1)h + 7(d+1)^2h^2 \\
& +h(d+1)] + \frac{b_n^2}{n^2}\left[4(d+1)^3h^3 + 6(d+1)dh + 6(d+1)^2h^2\right. \\
& \left.\left.\left. +10h(d+1) - 4(d+1)^3h^3 + 1\right]x + \frac{b_n}{n}3x^2\right]\right).
\end{aligned}$$

Rewriting (5.7) as

$$\lim_{n\to+\infty}\frac{n}{b_n}[\mathcal{B}_n^G(f;x) - f(x)] = h(d+1)f'(x) + \frac{xf''(x)}{2},$$

after applying limits for $n \to +\infty$ completes the proof. □

**Theorem 5.2.** *If $f \in C_\rho^*([0,+\infty))$, then*

$$\sup_{x\in[0,+\infty)} \frac{|\mathcal{B}_n^G(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2+M_0^G(n)+\sqrt{M_1^G(n)}\right)\Omega\left(f,\sqrt{M_0^G(n)}\right),$$

*where*

$$\begin{aligned} M_0^G(n) =& \frac{b_n}{2n}+\frac{b_n^2}{n^2}h(d+1)(1+d+(d+1)h),\\ M_1^G(n) =& \frac{b_n^4}{n^4}\Big[(1+d)^4h^4+6(d+1)^3dh^3+3(d+1)^2d(2d-1)h^2+(d+1)^2d(d-1)h^2\\ &+(d-2)(d+1)d(d-1)h+6(d+1)^3h^3+18d(d+1)^2h^2+6(d+1)d(d-1)h\\ &+7d(d+1)h+7(d+1)^2h^2+h(d+1)\Big]+\frac{3\sqrt{3}}{16}\cdot\frac{b_n^3}{n^3}\Big[4(d+1)^3h^3+6(d+1)dh\\ &+6(d+1)^2h^2+10h(d+1)-4(d+1)^3h^3+1\Big]+\frac{3}{4}\cdot\frac{b_n^2}{n^2}. \end{aligned}$$

*Example* 5.1. By taking $f(x)=\frac{x^2}{1+x^3}$, $b_n=n^{\frac{1}{3}}$ and $d=0.5$, we obtain the error approximation of the Chlodowsky variation of Szász-type operators including Gould-Hopper polynomials by using weighted modulus of continuity as we see in 5.1.

| $n$ | $h_1=1.5$ | $h_2=2$ | $h_3=3$ |
|---|---|---|---|
| 10 | 1.6880 | 2.0285 | 2.7854 |
| $10^2$ | 0.4665 | 0.5228 | 0.6487 |
| $10^3$ | 0.1791 | 0.1865 | 0.2049 |
| $10^4$ | 0.0792 | 0.0800 | 0.0822 |
| $10^5$ | 0.0365 | 0.0365 | 0.0368 |
| $10^6$ | 0.0169 | 0.0169 | 0.0169 |
| $10^7$ | 0.0078 | 0.0078 | 0.0078 |
| $10^8$ | 0.0036 | 0.0036 | 0.0036 |

TABLE 1. Error of $\mathcal{B}_n^G(f;x)$ by using weighted modulus of continuity for $d=0.5$.

5.2. **Hermite polynomials of variance $\varepsilon$.** If $\mathbb{A}(t)=e^{\frac{-\varepsilon t^2}{2}}$, then $R_n(x)=H_n^{(\varepsilon)}(x)$ is the Hermite polynomials of variance $\varepsilon$ which have the obvious presentation

$$H_n^{(\varepsilon)}(x)=\sum_{k=0}^{[\frac{n}{2}]}\left(-\frac{\varepsilon}{2}\right)^k\frac{1}{k!(n-2)!}x^{n-2k},$$

where $[\cdot]$ denotes the integer part. Denote by $\mathcal{B}_n^H$ the Chlodowsky variant of Szász-type operators containing the Hermite polynomials. Then,

$$\mathcal{B}_n^H(f;x) = e^{\frac{\varepsilon}{2}-\frac{n}{b_n}x}\sum_{k=0}^{+\infty} H_n^{(\varepsilon)}(x) f\left(\frac{k}{n}b_n\right).$$

For the operators $\mathcal{B}_n^H$, $(i),(ii)$ and assumptions (2.2) are confirmed with the assumption $\varepsilon \leq 0$.

**Lemma 5.3.** *We have the following results*

$$\begin{aligned}
\mathcal{B}_n^H(e_0;x) =& 1,\\
\mathcal{B}_n^H(e_1;x) =& x - \frac{b_n}{n}\varepsilon,\\
\mathcal{B}_n^H(e_2;x) =& x^2 + \frac{b_n}{n}(1-2\varepsilon)x + \frac{b_n^2}{n^2}(-2\varepsilon+\varepsilon^2),\\
\mathcal{B}_n^H(e_3;x) =& x^3 + \frac{b_n}{n}(-3\varepsilon+3)x^2 + \frac{b_n^2}{n^2}(-9\varepsilon+3\varepsilon^2+1)x + \frac{b_n^3}{n^3}(6\varepsilon^2-\varepsilon^3-4\varepsilon),\\
\mathcal{B}_n^H(e_4;x) =& x^4 + \frac{b_n}{n}(-4\varepsilon+6)x^3 + \frac{b_n^2}{n^2}(-24\varepsilon+6\varepsilon^2+7)x^2\\
&+ \frac{b_n^3}{n^3}(-4\varepsilon^3+18\varepsilon^2-20\varepsilon+1)x + \frac{b_n^4}{n^4}(\varepsilon^4-12\varepsilon^3+28\varepsilon^2-7\varepsilon-1).
\end{aligned}$$

**Lemma 5.4.** *For every $x \in [0,+\infty)$, the operators $\mathcal{B}_n^H$ satisfy*

$$\begin{aligned}
\mathcal{B}_n^H((t-x);x) &= -\frac{b_n}{n}\varepsilon,\\
\mathcal{B}_n^H((t-x)^2;x) &= \frac{b_n^2}{n^2}(-2\varepsilon+\varepsilon^2) + \frac{b_n}{n}x,\\
\mathcal{B}_n^H((t-x)^4;x) &= \frac{b_n^4}{n^4}(-8\varepsilon+28\varepsilon^2-12\varepsilon^3+\varepsilon^4) + \frac{b_n^3}{n^3}(1-16\varepsilon+6\varepsilon^2)x + \frac{b_n^2}{n^2}3x^2.
\end{aligned}$$

**Lemma 5.5.** *We have the following results*

$$\lim_{n\to+\infty}\frac{n}{b_n}\mathcal{B}_n^H(t-x:x) = -\varepsilon,$$

$$\lim_{n\to+\infty}\frac{n}{b_n}\mathcal{B}_n^H((t-x)^2:x) = x,$$

$$\lim_{n\to+\infty}\frac{n^2}{b_n^2}\mathcal{B}_n^H((t-x)^4:x) = 3x^2.$$

**Theorem 5.3.** *Let $f \in C^2[0,a]$ and $x \in [0,+\infty)$, we obtain*

$$\lim_{n\to+\infty}\frac{n}{b_n}[\mathcal{B}_n^H(f;x) - f(x)] = -\varepsilon f'(x) + \frac{xf''(x)}{2},$$

*uniformly in each compact subset of $[0,+\infty)$.*

**Theorem 5.4.** *If* $f \in C^*_\rho([0,+\infty))$*, then*

$$\sup_{x\in[0,+\infty)} \frac{|\mathcal{B}_n^H(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2+M_0^H(n)+\sqrt{M_1^H(n)}\right)\Omega\left(f,\sqrt{M_0^H(n)}\right),$$

*where*

$$M_0^H(n) = \frac{b_n^2}{n^2}(-2\varepsilon+\varepsilon^2)+\frac{b_n}{2n},$$
$$M_1^H(n) = \frac{b_n^4}{n^4}(-8\varepsilon+28\varepsilon^2-12\varepsilon^3+\varepsilon^4)+\frac{3\sqrt{3}}{16}\cdot\frac{b_n^3}{n^3}(1-16\varepsilon+6\varepsilon^2)+\frac{3}{4}\cdot\frac{b_n^2}{n^2}.$$

*Example* 5.2. By taking $f(x)=\frac{x^2}{1+x^3}$ and $b_n = n^{\frac{1}{4}}$, we can see the error estimation of the Chlodowsky variation of Szász-type operators including Hermite polynomials by the help of weighted modulus of continuity in Table 5.2.

| $n$ | $\varepsilon_1 = -0.002$ | $\varepsilon_2 = 1.5$ | $\varepsilon_3 = 2.5$ |
|---|---|---|---|
| 10 | 0.6874 | 0.5950 | 0.8060 |
| $10^2$ | 0.2916 | 0.2846 | 0.3022 |
| $10^3$ | 0.1250 | 0.1245 | 0.1259 |
| $10^4$ | 0.0532 | 0.0532 | 0.0533 |
| $10^5$ | 0.0225 | 0.0225 | 0.0225 |
| $10^6$ | 0.0095 | 0.0095 | 0.0095 |
| $10^7$ | 0.0040 | 0.0040 | 0.0040 |
| $10^8$ | 0.0016 | 0.0016 | 0.0016 |

TABLE 2. Error estimation of $\mathcal{B}_n^H(f;x)$ by using weighted modulus of continuity.

## 6. Conclusion

In our current research, we introduce the Chlodowsky variant of generalized Szász-type operators, a novel sequence of operators containing the Appell polynomials of class $\mathbb{A}^{(2)}$. The test functions and central moments of the operators were attained. Moreover, the rate of convergence is obtained through the use of the modulus of continuity by means of Steklov function. Then a Voronovskaya-type theorem for the quantitative asymtotic approximation is given. In the last section, it is shown that the Chlodowsky variant of generalized Szász-type operators Appell polynomials of class $\mathbb{A}^{(2)}$ reduce Gould-Hopper polynomials and Hermite polynomials under special choices.

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