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ON GENERALIZED COMMUTATIVE JACOBSTHAL QUATERNIONS

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Abstract. In this paper, we introduce and study generalized commutative Jacobsthal quaternions and their one-parameter generalization. We present some fundamental properties of them, among others the Binet formula, Catalan, Cassini, d'Ocagne and Vajda identities. Moreover, we give the generating functions and summation formulas for these numbers.

1. Introduction

Quaternions were introduced in 1843 by W. Hamilton for representing vectors in the space as follows. A quaternion q is a hyper-complex number represented by an equation $q = a + bi + cj + dk$, where a, b, c, d are real numbers and i, j, k are standard orthonormal basis in $\mathbb{R}^3$, which satisfy the quaternion multiplication rules

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j \quad \text{and} \quad ij = -ji = k.$$

From the above rules, it immediately follows that multiplication of quaternions is not commutative. For these reasons, it is not easy to study the problems of quaternion algebra, however Hamiltonian quaternions were modified or generalized in this direction so that commutative property in multiplication is possible, see [14].

Let $\mathbb{H}^c_{\gamma\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3, \tag{1.1}$$

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where quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta, \tag{1.2}$$

$$e_1e_2 = e_2e_1 = e_3, \quad e_2e_3 = e_3e_2 = \beta e_1, \quad e_3e_1 = e_1e_3 = \alpha e_2, \tag{1.3}$$

and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$. The generalized commutative quaternions were introduced in [16]. They generalize elliptic quaternions ($\alpha < 0$, $\beta = 1$), parabolic quaternions ($\alpha = 0$, $\beta = 1$), hyperbolic quaternions ($\alpha > 0$, $\beta = 1$), bicomplex numbers ($\alpha = -1$, $\beta = -1$), complex hyperbolic numbers ($\alpha = -1$, $\beta = 1$) and hyperbolic complex numbers ($\alpha = 1$, $\beta = -1$).

Similarly to classical quaternions, we can define the conjugate of the generalized commutative quaternion $\mathbf{x}$ as $\overline{\mathbf{x}} = x_0 - x_1e_1 - x_2e_2 - x_3e_3$ and norm $N(\mathbf{x})$ of $\mathbf{x}$ of the form $N(\mathbf{x}) = \mathbf{x} \cdot \overline{\mathbf{x}} = \overline{\mathbf{x}} \cdot \mathbf{x}$. By simple calculations we obtain $N(\mathbf{x}) = x_0^2 - x_1^2\alpha - x_2^2\beta - x_3^2\alpha\beta - 2x_2x_3\beta e_1 - 2x_1x_3\alpha e_2 - 2x_1x_2e_3$.

Note that associative quaternions can be divided into two families, the first family of noncommutative quaternions, for example Hamiltonian, hyperbolic, split quaternions and the second family of commutative quaternions including in particular generalized Serge quaternions, dual quaternions, see [14].

The theory of quaternions is complemented by Fibonacci type sequences, i.e., sequences defined by a second-order linear homogeneous recurrence relations with real coefficients, see [8, 10, 15, 17]. In this paper, we use the Jacobsthal sequence for studying commutative quaternions. Let us recall necessary definitions.

The Jacobsthal sequence $\{J_n\}$ was introduced by Horadam [9]. The first ten terms of the sequence are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171$. This sequence is defined by the following recurrence relation

$$J_n = J_{n-1} + 2J_{n-2}, \quad \text{for } n \geq 2, \tag{1.4}$$

with initial conditions $J_0 = 0$, $J_1 = 1$. The Binet formula of this sequence has the form

$$J_n = \frac{2^n - (-1)^n}{3}, \quad \text{for } n \geq 0.$$

The Jacobsthal numbers are a special case of Horadam numbers W_n defined by the recurrence

$$W_n = pW_{n-1} - qW_{n-2}, \quad \text{for } n \geq 2,$$

with fixed real numbers W_0 and W_1 and $p, q \in \mathbb{Z}$.

Fibonacci type numbers and their generalizations have applications in the theory of hypercomplex numbers, see for example [1, 7, 12, 18].

Many authors have studied some generalizations of the recurrence of the Jacobsthal sequence, see [5, 6, 11] and generalizations of hypercomplex Jacobsthal numbers, see [3,13]. In [2], a one-parameter generalization of the Jacobsthal numbers was introduced. We recall this generalization.

Let $n \geq 0$, $r \geq 0$ be integers. The nth r-Jacobsthal number $J_{r,n}$ is defined as follows

$$J_{r,n} = 2^r J_{r,n-1} + (2^r + 4^r)J_{r,n-2}, \quad \text{for } n \geq 2, \tag{1.5}$$

with $J_{r,0} = 1$, $J_{r,1} = 1 + 2^{r+1}$.

For $r = 0$, we have $J_{0,n} = J_{n+2}$. By (1.5) we obtain

$$\begin{aligned}
J_{r,0} &= 1,\\
J_{r,1} &= 2 \cdot 2^r + 1,\\
J_{r,2} &= 3 \cdot 4^r + 2 \cdot 2^r,\\
J_{r,3} &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r,\\
J_{r,4} &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r,\\
J_{r,5} &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r.
\end{aligned} \tag{1.6}$$

The r-Jacobsthal numbers have a graph interpretation, they can be used for the counting of independent sets of special classes of graphs, see [2]. We will recall some properties of the r-Jacobsthal numbers.

Theorem 1.1 (Binet formula [2])**.** *Let $n \geq 0$, $r \geq 0$ be integers. Then, the nth r-Jacobsthal number is given by*

$$J_{r,n} = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}\gamma^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}\delta^n,$$

where

$$\gamma = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \delta = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Theorem 1.2 ([2])**.** *Let $n \geq 1$, $r \geq 0$ be integers. Then,*

$$\sum_{l=0}^{n-1} J_{r,l} = \frac{J_{r,n} + (2^r + 4^r)J_{r,n-1} - 2 - 2^r}{4^r + 2^{r+1} - 1}.$$

Theorem 1.3 (Convolution identity [2])**.** *Let n, m, r be integers such that $m \geq 2, n \geq 1$, $r \geq 0$. Then,*

$$J_{r,m+n} = 2^r J_{r,m-1}J_{r,n} + (4^r + 8^r)J_{r,m-2}J_{r,n-1}.$$

Theorem 1.4 ([2])**.** *The generating function of the sequence of r-Jacobsthal numbers has the following form*

$$f(t) = \frac{1 + (1 + 2^r)t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

2. Generalized Commutative Jacobsthal Quaternions

For $n \geq 0$, the nth generalized commutative Jacobsthal quaternion is defined by the following relation

$$gc\mathfrak{J}_n = J_n + J_{n+1}\, e_1 + J_{n+2}\, e_2 + J_{n+3}\, e_3, \tag{2.1}$$

where J_n is the nth Jacobsthal number and e_1, e_2, e_3 are quaternionic units which satisfy the rules (1.2) and (1.3).

Analogously, for $n \geq 0$ and $r \geq 0$, we define the nth generalized commutative r-Jacobsthal quaternion

$$gc\mathfrak{J}_{r,n} = J_{r,n} + J_{r,n+1}\, e_1 + J_{r,n+2}\, e_2 + J_{r,n+3}\, e_3, \tag{2.2}$$

where $J_{r,n}$ is the nth r-Jacobsthal number.

By (1.6) and (2.2), we obtain

$$\begin{aligned} gc\mathfrak{J}_{r,0} =& 1 + (2^{r+1}+1)e_1 + (3\cdot 4^r + 2^{r+1})e_2 + (5\cdot 8^r + 5\cdot 4^r + 2^r)e_3, \\ gc\mathfrak{J}_{r,1} =& 2^{r+1} + 1 + (3\cdot 4^r + 2^{r+1})e_1 + (5\cdot 8^r + 5\cdot 4^r + 2^r)e_2 \\ & + (8\cdot 16^r + 10\cdot 8^r + 3\cdot 4^r)e_3, \\ gc\mathfrak{J}_{r,2} =& 3\cdot 4^r + 2^{r+1} + (5\cdot 8^r + 5\cdot 4^r + 2^r)e_1 \\ & + (8\cdot 16^r + 10\cdot 8^r + 3\cdot 4^r)e_2 \\ & + (13\cdot 32^r + 20\cdot 16^r + 9\cdot 8^r + 4^r)e_3. \end{aligned} \tag{2.3}$$

Using the fact that $J_{0,n} = J_{n+2}$, we get $gc\mathfrak{J}_{0,n} = gc\mathfrak{J}_{n+2}$.

By the definition of the generalized commutative r-Jacobsthal quaternion, we obtain the following recurrence relation.

Proposition 2.1. *Let $n \geq 2$, $r \geq 0$ be integers. Then,*

$$gc\mathfrak{J}_{r,n} = 2^r gc\mathfrak{J}_{r,n-1} + (2^r + 4^r) gc\mathfrak{J}_{r,n-2},$$

where $gc\mathfrak{J}_{r,0}$, $gc\mathfrak{J}_{r,1}$ are given by (2.3).

In [16], the authors introduced generalized commutative Horadam quaternions

$$gc\mathcal{H}_n = W_n + W_{n+1}\, e_1 + W_{n+2}\, e_2 + W_{n+3}\, e_3,$$

where W_n is the nth Horadam number and e_1, e_2, e_3 are quaternionic units which satisfy the rules (1.2) and (1.3). It was proved the following result.

Theorem 2.1 (Binet formula for generalized commutative Horadam quaternions, [16])**.** *Let $n \geq 0$ be an integer. Then,*

$$gc\mathcal{H}_n = At_1^n \hat{t_1} + Bt_2^n \hat{t_2},$$

where

$$\begin{aligned} t_1 =& \frac{1}{2}\left(p - \sqrt{p^2 - 4q}\right), \quad t_2 = \frac{1}{2}\left(p + \sqrt{p^2 - 4q}\right), \\ \hat{t_1} =& 1 + t_1 e_1 + t_1^2 e_2 + t_1^3 e_3, \quad \hat{t_2} = 1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3, \\ A =& \frac{W_1 - W_0 t_2}{t_1 - t_2}, \quad B = \frac{W_0 t_1 - W_1}{t_1 - t_2}. \end{aligned}$$

By Theorem 2.1, we get the Binet formula for the generalized commutative r-Jacobsthal quaternions and for the generalized commutative Jacobsthal quaternions.

Corollary 2.1 (Binet formula for generalized commutative r-Jacobsthal quaternions)**.** *Let $n \geq 0$, $r \geq 0$ be integers. Then,*

$$gc\mathfrak{J}_{r,n} = C_1\underline{\gamma}\gamma^n + C_2\underline{\delta}\delta^n, \tag{2.4}$$

where

$$\begin{aligned}
\gamma =& 2^{r-1} + \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \quad \delta = 2^{r-1} - \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r},\\
\underline{\gamma} =& 1 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3, \quad \underline{\delta} = 1 + \delta e_1 + \delta^2 e_2 + \delta^3 e_3,\\
C_1 =& \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} + 3\cdot 2^r + 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}, \quad C_2 = \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} - 3\cdot 2^r - 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}.
\end{aligned}$$

Corollary 2.2 (Binet formula for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$ be an integer. Then,*

$$gc\mathfrak{J}_n = \frac{1}{3}\Big[2^n(1 + 2e_1 + 4e_2 + 8e_3) - (-1)^n(1 - e_1 + e_2 - e_3)\Big].$$

Proof. By Corollary 2.1, for $r = 0$, we have $C_1 = \frac{4}{3}$, $C_2 = -\frac{1}{3}$, $\gamma = 2$, $\delta = -1$, and

$$\begin{aligned}
gc\mathfrak{J}_{0,n} &= \frac{4}{3}\cdot 2^n(1 + 2e_1 + 4e_2 + 8e_3) - \frac{1}{3}(-1)^n(1 - e_1 + e_2 - e_3)\\
&= \frac{1}{3}\cdot 2^{n+2}(1 + 2e_1 + 4e_2 + 8e_3) - \frac{1}{3}(-1)^{n+2}(1 - e_1 + e_2 - e_3)\\
&= gc\mathfrak{J}_{n+2}.
\end{aligned}$$

□

3. Properties of the Generalized Commutative r-Jacobsthal Quaternions

In this section, we give some identities such as Catalan, Cassini, d'Ocagne, and Vajda identities for the generalized commutative r-Jacobsthal quaternions. Moreover, we present convolution identity, a summation formula, and generating function for the generalized commutative r-Jacobsthal quaternions. In particular, we get analogous results for the generalized commutative Jacobsthal quaternions.

Theorem 3.1 (General bilinear index-reduction formula for the generalized commutative r-Jacobsthal quaternions)**.** *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then,*

$$gc\mathfrak{J}_{r,a}\cdot gc\mathfrak{J}_{r,b} - gc\mathfrak{J}_{r,c}\cdot gc\mathfrak{J}_{r,d} = -\frac{(1+2^r)^2}{4\cdot 2^r + 5\cdot 4^r}\underline{\gamma}\underline{\delta}\left(\gamma^a\delta^b + \gamma^b\delta^a - \gamma^c\delta^d - \gamma^d\delta^c\right).$$

Proof. By formula (2.4), we get

$$\begin{aligned}
&gc\mathfrak{J}_{r,a}\cdot gc\mathfrak{J}_{r,b} - gc\mathfrak{J}_{r,c}\cdot gc\mathfrak{J}_{r,d}\\
=&(C_1\underline{\gamma}\gamma^a + C_2\underline{\delta}\delta^a)(C_1\underline{\gamma}\gamma^b + C_2\underline{\delta}\delta^b) - (C_1\underline{\gamma}\gamma^c + C_2\underline{\delta}\delta^c)(C_1\underline{\gamma}\gamma^d + C_2\underline{\delta}\delta^d)\\
=&C_1^2\underline{\gamma}^2(\gamma^{a+b} - \gamma^{c+d}) + C_2^2\underline{\delta}^2(\delta^{a+b} - \delta^{c+d}) + C_1C_2\underline{\gamma}\underline{\delta}(\gamma^a\delta^b - \gamma^c\delta^d + \gamma^b\delta^a - \gamma^d\delta^c).
\end{aligned}$$

Using the fact that $a + b = c + d$ and $C_1C_2 = -\frac{(1+2^r)^2}{4\cdot 2^r + 5\cdot 4^r}$, we get the result. □

Moreover, by simple calculations, we get

$$\begin{aligned}\underline{\gamma\delta} = \underline{\delta\gamma} = &1 + \gamma\delta\alpha + \gamma^2\delta^2\beta + \gamma^3\delta^3\alpha\beta + (\gamma+\delta)(1+\gamma^2\delta^2\beta)e_1 \\ &+ (\gamma^2+\delta^2)(1+\gamma\delta\alpha)e_2 + (\gamma^3+\delta^3+\gamma\delta(\gamma+\delta))e_3.\end{aligned}$$

Using the equalities

$$\begin{aligned}\gamma+\delta &= 2^r,\\ \gamma-\delta &= \sqrt{4\cdot 2^r + 5\cdot 4^r},\\ \gamma\delta &= -(4^r+2^r),\\ \gamma^2+\delta^2 &= (\gamma+\delta)^2 - 2\gamma\delta = 3\cdot 4^r + 2^{r+1},\\ \gamma^3+\delta^3 &= (\gamma+\delta)^3 - 3\gamma\delta(\gamma+\delta) = 4\cdot 8^r + 3\cdot 4^r,\end{aligned}$$

we have

$$\begin{aligned}\underline{\gamma\delta} = \underline{\delta\gamma} = &1 - (4^r+2^r)\alpha + (4^r+2^r)^2\beta - (4^r+2^r)^3\alpha\beta \\ &+ 2^r(1+(4^r+2^r)^2\beta)e_1 + (2^{r+1}+3\cdot 4^r)(1-(4^r+2^r)\alpha)e_2 \\ &+ (2\cdot 4^r + 3\cdot 8^r)e_3.\end{aligned} \tag{3.1}$$

It is easily seen that for special values of a, b, c, d, by Theorem 3.1, we get new identities for generalized commutative r-Jacobsthal quaternions:

- for $a = b = n$, $c = n - m$ and $d = n + m$ Catalan identity,
- for $a = b = n$, $c = n - 1$ and $d = n + 1$ Cassini identity,
- for $a = n$, $b = m + 1$, $c = n + 1$ and $d = m$ d'Ocagne identity,
- for $a = m + k$, $b = n - k$, $c = m$ and $d = n$ Vajda identity.

Corollary 3.1 (Catalan identity for generalized commutative r-Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq m$. Then,*

$$gc\mathfrak{J}_{r,n}^2 - gc\mathfrak{J}_{r,n-m}\cdot gc\mathfrak{J}_{r,n+m} = -\frac{(-4^r-2^r)^n(1+2^r)^2}{4\cdot 2^r + 5\cdot 4^r}\underline{\gamma\delta}\left(2 - \left(\frac{\gamma}{\delta}\right)^m - \left(\frac{\delta}{\gamma}\right)^m\right),$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

Corollary 3.2. (*Cassini identity for generalized commutative r-Jacobsthal quaternions*) *Let $n \geq 1$, $r \geq 0$ be integers. Then,*

$$gc\mathfrak{J}_{r,n}^2 - gc\mathfrak{J}_{r,n-1}\cdot gc\mathfrak{J}_{r,n+1} = (-4^r-2^r)^{n-1}(1+2^r)^2\underline{\gamma\delta},$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

Corollary 3.3. (*d'Ocagne identity for the generalized commutative r-Jacobsthal quaternions*) *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq m$. Then,*

$$\begin{aligned}&gc\mathfrak{J}_{r,n}\cdot gc\mathfrak{J}_{r,m+1} - gc\mathfrak{J}_{r,n+1}\cdot gc\mathfrak{J}_{r,m}\\ =&\frac{(1+2^r)^2\sqrt{4\cdot 2^r+5\cdot 4^r}}{4\cdot 2^r+5\cdot 4^r}(-4^r-2^r)^m\underline{\gamma\delta}\left(\gamma^{n-m}-\delta^{n-m}\right),\end{aligned}$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

Corollary 3.4 (Vajda identity for the generalized commutative r-Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, $k \geq 0$, $r \geq 0$ be integers such that $n \geq k$ and $n \geq m$. Then,*

$$\begin{aligned} & gc\mathfrak{J}_{r,m+k} \cdot gc\mathfrak{J}_{r,n-k} - gc\mathfrak{J}_{r,m} \cdot gc\mathfrak{J}_{r,n} \\ &= -\frac{(1+2^r)^2(-4^r-2^r)^m}{4 \cdot 2^r + 5 \cdot 4^r} \underline{\gamma\delta} \left(\delta^{n-m} \left[\left(\frac{\gamma}{\delta} \right)^k - 1 \right] + \gamma^{n-m} \left[\left(\frac{\delta}{\gamma} \right)^k - 1 \right] \right), \end{aligned}$$

where $\underline{\gamma\delta}$ is given by (3.1)*.*

In particular, by Theorem 3.1, we obtain the following identities for generalized commutative Jacobsthal quaternions.

Corollary 3.5 (Catalan identity for generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, be integers such that $n \geq m$. Then,*

$$\begin{aligned} & gc\mathfrak{J}_n^2 - gc\mathfrak{J}_{n-m} \cdot gc\mathfrak{J}_{n+m} \\ =& \frac{1}{9}(-2)^{n-m} \left((-2)^m - 1\right)^2 (1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3). \end{aligned}$$

Corollary 3.6 (Cassini identity for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 1$ be an integer. Then,*

$$gc\mathfrak{J}_n^2 - gc\mathfrak{J}_{n-1} \cdot gc\mathfrak{J}_{n+1} = (-2)^{n-1}(1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3).$$

Corollary 3.7 (d'Ocagne identity for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$ be integers. Then,*

$$\begin{aligned} & gc\mathfrak{J}_n \cdot gc\mathfrak{J}_{m+1} - gc\mathfrak{J}_{n+1} \cdot gc\mathfrak{J}_m \\ =& \frac{1}{3}(-2)^m \left(2^{n-m} - (-1)^{n-m}\right) (1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3). \end{aligned}$$

Corollary 3.8 (Vajda identity for the generalized commutative Jacobsthal quaternions)**.** *Let $n \geq 0$, $m \geq 0$, $k \geq 0$ be integers such that $n \geq k$. Then,*

$$\begin{aligned} & gc\mathfrak{J}_{m+k} \cdot gc\mathfrak{J}_{n-k} - gc\mathfrak{J}_m \cdot gc\mathfrak{J}_n \\ &= -\frac{1}{9}(-2)^m \left((-1)^{n-m} \left[(-2)^k - 1 \right] + 2^{n-m} \left[\left(-\frac{1}{2} \right)^k - 1 \right] \right) \\ & \quad \times (1 - 2\alpha + 4\beta - 8\alpha\beta + (1+4\beta)e_1 + (5-10\alpha)e_2 + 5e_3). \end{aligned}$$

Now, we give the convolution identity for the generalized commutative r-Jacobsthal quaternions.

Theorem 3.2. *Let $m \geq 2, n \geq 1$, $r \geq 0$ be integers. Then,*

(3.2)
$$\begin{aligned} 2gc\mathfrak{J}_{r,m+n} =& 2^r gc\mathfrak{J}_{r,m-1} \cdot gc\mathfrak{J}_{r,n} + (4^r + 8^r) gc\mathfrak{J}_{r,m-2} \cdot gc\mathfrak{J}_{r,n-1} \\ & + J_{r,m+n} - \alpha J_{r,m+n+2} - \beta J_{r,m+n+4} - 2gc\mathfrak{J}_{r,m+n+3}\, e_3 + \alpha\beta J_{r,m+n+6}. \end{aligned}$$

Proof. By simple calculations we get

$$\begin{aligned}
&gc\mathfrak{J}_{r,m-1} \cdot gc\mathfrak{J}_{r,n}\\
=&J_{r,m-1}J_{r,n} + J_{r,m-1}J_{r,n+1}e_1 + J_{r,m-1}J_{r,n+2}e_2 + J_{r,m-1}J_{r,n+3}e_3\\
&+ J_{r,m}J_{r,n}e_1 + \alpha J_{r,m}J_{r,n+1} + J_{r,m}J_{r,n+2}e_3 + \alpha J_{r,m}J_{r,n+3}e_2\\
&+ J_{r,m+1}J_{r,n}e_2 + J_{r,m+1}J_{r,n+1}e_3 + \beta J_{r,m+1}J_{r,n+2} + \beta J_{r,m+1}J_{r,n+3}e_1\\
&+ J_{r,m+2}J_{r,n}e_3 + \alpha J_{r,m+2}J_{r,n+1}e_2 + \beta J_{r,m+2}J_{r,n+2}e_1 + \alpha\beta J_{r,m+2}J_{r,n+3}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&gc\mathfrak{J}_{r,m-2} \cdot gc\mathfrak{J}_{r,n-1}\\
=&J_{r,m-2}J_{r,n-1} + J_{r,m-2}J_{r,n}e_1 + J_{r,m-2}J_{r,n+1}e_2 + J_{r,m-2}J_{r,n+2}e_3\\
&+ J_{r,m-1}J_{r,n-1}e_1 + \alpha J_{r,m-1}J_{r,n} + J_{r,m-1}J_{r,n+1}e_3 + \alpha J_{r,m-1}J_{r,n+2}e_2\\
&+ J_{r,m}J_{r,n-1}e_2 + J_{r,m}J_{r,n}e_3 + \beta J_{r,m}J_{r,n+1} + \beta J_{r,m}J_{r,n+2}e_1\\
&+ J_{r,m+1}J_{r,n-1}e_3 + \alpha J_{r,m+1}J_{r,n}e_2 + \beta J_{r,m+1}J_{r,n+1}e_1 + \alpha\beta J_{r,m+1}J_{r,n+2}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&2^r\, gc\mathfrak{J}_{r,m-1} \cdot\ gc\mathfrak{J}_{r,n} + (4^r + 8^r) gc\mathfrak{J}_{r,m-2} \cdot\ gc\mathfrak{J}_{r,n-1}\\
=&2^r J_{r,m-1}J_{r,n} + (4^r + 8^r)\, J_{r,m-2}J_{r,n-1}\\
&+ [2^r\, J_{r,m-1}J_{r,n+1} + (4^r + 8^r)\, J_{r,m-2}J_{r,n} + 2^r\, J_{r,m}J_{r,n} + (4^r + 8^r)\, J_{r,m-1}J_{r,n-1}]e_1\\
&+ [2^r\, J_{r,m-1}J_{r,n+2} + (4^r + 8^r)\, J_{r,m-2}J_{r,n+1} + 2^r\, J_{r,m+1}J_{r,n} + (4^r + 8^r)\, J_{r,m}J_{r,n-1}]e_2\\
&+ [2^r\, J_{r,m-1}J_{r,n+3} + (4^r + 8^r)\, J_{r,m-2}J_{r,n+2} + 2^r\, J_{r,m+2}J_{r,n} + (4^r + 8^r)\, J_{r,m+1}J_{r,n-1}]e_3\\
&+ \alpha(2^r\, J_{r,m}J_{r,n+1} + (4^r + 8^r)\, J_{r,m-1}J_{r,n}) + \beta(2^r\, J_{r,m+1}J_{r,n+2} + (4^r + 8^r)\, J_{r,m}J_{r,n+1})\\
&+ \alpha[2^r\, J_{r,m}J_{r,n+3} + (4^r + 8^r)\, J_{r,m-1}J_{r,n+2} + 2^r\, J_{r,m+2}J_{r,n+1} + (4^r + 8^r)\, J_{r,m+1}J_{r,n}]e_2\\
&+ [2^r\, J_{r,m+1}J_{r,n+1} + (4^r + 8^r)\, J_{r,m}J_{r,n} + 2^r\, J_{r,m}J_{r,n+2} + (4^r + 8^r)\, J_{r,m-1}J_{r,n+1}]e_3\\
&+ \beta[2^r\, J_{r,m+2}J_{r,n+2} + (4^r + 8^r)\, J_{r,m+1}J_{r,n+1} + 2^r J_{r,m+1}J_{r,n+3} + (4^r + 8^r)J_{r,m}J_{r,n+2}]e_1\\
&+ \alpha\beta(2^r J_{r,m+2}J_{r,n+3} + (4^r + 8^r)J_{r,m+1}J_{r,n+2}).
\end{aligned}$$

Using Theorem 1.3, we get

$$\begin{aligned}
&2^r\, gc\mathfrak{J}_{r,m-1} \cdot\ gc\mathfrak{J}_{r,n} + (4^r + 8^r)\, gc\mathfrak{J}_{r,m-2} \cdot\ gc\mathfrak{J}_{r,n-1}\\
=&J_{r,m+n} + 2[J_{r,m+n+1}e_1 + J_{r,m+n+2}e_2 + J_{r,m+n+3}e_3]\\
&+ \alpha J_{r,m+n+2} + \beta J_{r,m+n+4} + 2J_{r,m+n+3}e_3\\
&+ 2\alpha J_{r,m+n+4}e_2 + 2\beta J_{r,m+n+5}e_1 + \alpha\beta J_{r,m+n+6}\\
=&2\, gc\mathfrak{J}_{r,m+n} - J_{r,m+n} + 2(J_{r,m+n+3} + J_{r,m+n+4}e_1 + J_{r,m+n+5}e_2\\
&+ J_{r,m+n+6}e_3)e_3 + \alpha J_{r,m+n+2} + \beta J_{r,m+n+4} - \alpha\beta J_{r,m+n+6}\\
=&2\, gc\mathfrak{J}_{r,m+n} + 2gc\mathfrak{J}_{r,m+n+3}\, e_3 - J_{r,m+n} + \alpha J_{r,m+n+2}\\
&+ \beta J_{r,m+n+4} - \alpha\beta J_{r,m+n+6}.
\end{aligned}$$

Hence, we get the result. □

Remark 3.1. The formula (3.2) can be written in the following way

$$\begin{aligned} gc\mathfrak{J}_{r,m+n} =& 2^r gc\mathfrak{J}_{r,m-1} \cdot gc\mathfrak{J}_{r,n} + (4^r + 8^r) gc\mathfrak{J}_{r,m-2} \cdot gc\mathfrak{J}_{r,n-1} \\ & - e_1 gc\mathfrak{J}_{r,m+n+1} - e_2 gc\mathfrak{J}_{r,m+n+2} - e_3 gc\mathfrak{J}_{r,m+n+3}. \end{aligned}$$

Corollary 3.9. *Let $m \geq 1$, $n \geq 1$ be integers. Then,*

$$\begin{aligned} 2gc\mathfrak{J}_{m+n} =& gc\mathfrak{J}_{m-1} \cdot gc\mathfrak{J}_{n} + 2gc\mathfrak{J}_{m} \cdot gc\mathfrak{J}_{n-1} \\ & + J_{m+n} - \alpha J_{m+n+2} - \beta J_{m+n+4} - 2gc\mathfrak{J}_{m+n+3}\, e_3 + \alpha\beta J_{m+n+6}. \end{aligned}$$

The next theorem presents a summation formula for the generalized commutative r-Jacobsthal quaternions.

Theorem 3.3. *Let $n \geq 1$, $r \geq 0$ be integers. Then,*

$$\begin{aligned} \sum_{l=1}^{n} gc\mathfrak{J}_{r,l} =& \frac{gc\mathfrak{J}_{r,n+1} + (2^r + 4^r)\, gc\mathfrak{J}_{r,n} - (1 + e_1 + e_2 + e_3)(1 + 3 \cdot 2^r + 4^r)}{4^r + 2^{r+1} - 1} \\ & - (2^{r+1} + 1)e_1 - (2^{r+2} + 3 \cdot 4^r + 1)e_2 - (1 + 5 \cdot 2^r + 8 \cdot 4^r + 5 \cdot 8^r)e_3. \end{aligned}$$

Proof. By Theorem 1.2, we get $\sum_{l=1}^{n} J_{r,l} = \frac{J_{r,n+1} + (2^r + 4^r)J_{r,n} - 1 - 3\cdot 2^r - 4^r}{4^r + 2^{r+1} - 1}$. Hence, we have

$$\begin{aligned} \sum_{l=1}^{n} gc\mathfrak{J}_{r,l} =& \sum_{l=1}^{n} (J_{r,l} + J_{r,l+1}e_1 + J_{r,l+2}e_2 + J_{r,l+3}e_3) \\ =& \sum_{l=1}^{n} J_{r,l} + \sum_{l=1}^{n} J_{r,l+1}e_1 + \sum_{l=1}^{n} J_{r,l+2}e_2 + \sum_{l=1}^{n} J_{r,l+3}e_3 \\ =& \frac{1}{4^r + 2^{r+1} - 1} \Big[J_{r,n+1} + (2^r + 4^r)J_{r,n} - 1 - 3 \cdot 2^r - 4^r \\ & + (J_{r,n+2} + (2^r + 4^r)J_{r,n+1} - 1 - 3 \cdot 2^r - 4^r)e_1 \\ & + (J_{r,n+3} + (2^r + 4^r)J_{r,n+2} - 1 - 3 \cdot 2^r - 4^r)e_2 \\ & + (J_{r,n+4} + (2^r + 4^r)J_{r,n+3} - 1 - 3 \cdot 2^r - 4^r)e_3 \Big] \\ & - J_{r,1}e_1 - (J_{r,1} + J_{r,2})e_2 - (J_{r,1} + J_{r,2} + J_{r,3})e_3. \end{aligned}$$

By simple calculations, we obtain

$$\begin{aligned} \sum_{l=1}^{n} gc\mathfrak{J}_{r,l} =& \frac{1}{4^r + 2^{r+1} - 1} \Big[J_{r,n+1} + J_{r,n+2}e_1 + J_{r,n+3}e_2 + J_{r,n+4}e_3 \\ & + (2^r + 4^r)(J_{r,n} + J_{r,n+1}e_1 + J_{r,n+2}e_2 + J_{r,n+3}e_3) \\ & - (1 + 3 \cdot 2^r + 4^r)(1 + e_1 + e_2 + e_3) \Big] - (2^{r+1} + 1)e_1 \\ & - (2^{r+2} + 3 \cdot 4^r + 1)e_2 - (1 + 5 \cdot 2^r + 8 \cdot 4^r + 5 \cdot 8^r)e_3 \\ =& \frac{gc\mathfrak{J}_{r,n+1} + (2^r + 4^r)\, gc\mathfrak{J}_{r,n} - (1 + e_1 + e_2 + e_3)(1 + 3 \cdot 2^r + 4^r)}{4^r + 2^{r+1} - 1} \\ & - (2^{r+1} + 1)e_1 - (2^{r+2} + 3 \cdot 4^r + 1)e_2 - (1 + 5 \cdot 2^r + 8 \cdot 4^r + 5 \cdot 8^r)e_3. \end{aligned}$$

□

Corollary 3.10. *Let $n \geq 1$ be an integer. Then,*

$$\sum_{l=1}^{n} gc\mathfrak{J}_l = \frac{1}{2}(gc\mathfrak{J}_{n+2} - gc\mathfrak{J}_2).$$

Using the following identities ([4])

$$\sum_{l=0}^{n} J_{2l} = \frac{1}{3}(2J_{2n+1} - n - 2), \tag{3.3}$$

$$\sum_{l=0}^{n} J_{2l+1} = \frac{1}{3}(2J_{2n+2} + n + 1), \tag{3.4}$$

we get the next results for the generalized commutative Jacobsthal quaternions.

Theorem 3.4. *Let $n \geq 1$ be an integer. Then,*

(i) $\sum_{l=1}^{n} gc\mathfrak{J}_{2l} = \frac{1}{3}\left(2gc\mathfrak{J}_{2n+1} - n(2gc\mathfrak{J}_2 - gc\mathfrak{J}_3) - 2gc\mathfrak{J}_1\right);$

(ii) $\sum_{l=1}^{n} gc\mathfrak{J}_{2l-1} = \frac{1}{3}\left(2gc\mathfrak{J}_{2n} + n(2gc\mathfrak{J}_2 - gc\mathfrak{J}_3) - 2gc\mathfrak{J}_0\right);$

(iii) $\sum_{l=1}^{k} gc\mathfrak{J}_{n+l} = \frac{1}{2}\left(gc\mathfrak{J}_{n+k+2} - gc\mathfrak{J}_{n+1}\right).$

Proof. (i)

$$\begin{aligned}\sum_{l=1}^{n} gc\mathfrak{J}_{2l} =& J_2 + J_4 + \cdots + J_{2n} + (J_3 + J_5 + \cdots + J_{2n+1})e_1 \\ &+ (J_4 + J_6 + \cdots + J_{2n+2})e_2 + (J_5 + J_7 + \cdots + J_{2n+3})e_3.\end{aligned}$$

Using twice (3.3) and (3.4), we have

$$\begin{aligned}\sum_{l=1}^{n} gc\mathfrak{J}_{2l} =& \frac{1}{3}\Big(2J_{2n+1} - n - 2) + (2J_{2n+2} + n - 2)e_1 \\ &+ (2J_{2n+3} - n - 6)e_2 + (2J_{2n+4} + n - 10)e_3\Big) \\ =& \frac{1}{3}(2(J_{2n+1} + J_{2n+2}e_1 + J_{2n+3}e_2 + J_{2n+4}e_3) - n(1 - e_1 + e_2 - e_3) \\ &- 2(1 + e_1 + 3e_2 + 5e_3)) = \frac{1}{3}(2gc\mathfrak{J}_{2n+1} - n(2gc\mathfrak{J}_2 - gc\mathfrak{J}_3) - 2gc\mathfrak{J}_1).\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{l=1}^{n} gc\mathfrak{J}_{2l-1} =& J_1 + J_3 + \cdots + J_{2n-1} + (J_2 + J_4 + \cdots + J_{2n})e_1 \\ &+ (J_3 + J_5 + \cdots + J_{2n+1})e_2 + (J_4 + J_6 + \cdots + J_{2n+2})e_3.\end{aligned}$$

By (3.3) and (3.4) we get

$$\sum_{l=1}^{n} gc\mathfrak{J}_{2l-1} = \frac{1}{3}\Big(2J_{2n} + n + (2J_{2n+1} - n - 2)e_1$$

$$\begin{aligned}&+ (2J_{2n+2} + n - 2)e_2 + (2J_{2n+3} - n - 6)e_3\Big)\\
=&\frac{1}{3}(2(J_{2n} + J_{2n+1}e_1 + J_{2n+2}e_2 + J_{2n+3}e_3) + n(1 - e_1 + e_2 - e_3)\\
&- 2(e_1 + e_2 + 3e_3)) = \frac{1}{3}(2gc\mathfrak{J}_{2n} + n(2gc\mathfrak{J}_2 - gc\mathfrak{J}_3) - 2gc\mathfrak{J}_0).\end{aligned}$$

(iii) By Corollary 3.10, we get

$$\begin{aligned}\sum_{l=1}^{k} gc\mathfrak{J}_{n+l} =&\sum_{l=1}^{n+k} gc\mathfrak{J}_l - \sum_{l=1}^{n-1} gc\mathfrak{J}_l\\
=&\frac{1}{2}(gc\mathfrak{J}_{n+k+2} - gc\mathfrak{J}_2) - \frac{1}{2}(gc\mathfrak{J}_{n+1} - gc\mathfrak{J}_2)\\
=&\frac{1}{2}\left(gc\mathfrak{J}_{n+k+2} - gc\mathfrak{J}_{n+1}\right).\end{aligned}$$

□

At the end, we give the generating function for the generalized commutative r-Jacobsthal quaternions and the generalized commutative Jacobsthal quaternions.

Theorem 3.5. *The generating function for the generalized commutative r-Jacobsthal quaternions has the following form*

$$f(t) = \frac{gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

Proof. Let

$$f(t) = gc\mathfrak{J}_{r,0} + gc\mathfrak{J}_{r,1}t + gc\mathfrak{J}_{r,2}t^2 + \cdots + gc\mathfrak{J}_{r,n}t^n + \cdots$$

be the generating function of the generalized commutative r-Jacobsthal quaternions. Then,

$$\begin{aligned}2^r t f(t) =&2^r gc\mathfrak{J}_{r,0}t + 2^r gc\mathfrak{J}_{r,1}t^2 + 2^r gc\mathfrak{J}_{r,2}t^3\\
&+ \cdots + 2^r gc\mathfrak{J}_{r,n-1}t^n + \cdots,\\
(2^r + 4^r)t^2 f(t) =&(2^r + 4^r)\, gc\mathfrak{J}_{r,0}t^2 + (2^r + 4^r)\, gc\mathfrak{J}_{r,1}t^3\\
&+ (2^r + 4^r)\, gc\mathfrak{J}_{r,2}t^4 + \cdots\\
&+ (2^r + 4^r)\, gc\mathfrak{J}_{r,n-2}t^n + \cdots.\end{aligned}$$

By Proposition 2.1, we get

$$\begin{aligned}f(t) - 2^r t f(t) - (2^r + 4^r)t^2 f(t) =&\, gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t\\
&+ (gc\mathfrak{J}_{r,2} - 2^r gc\mathfrak{J}_{r,1} - (2^r + 4^r)\, gc\mathfrak{J}_{r,0})t^2 + \cdots\\
=&\, gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t.\end{aligned}$$

Thus,

$$f(t) = \frac{gc\mathfrak{J}_{r,0} + (gc\mathfrak{J}_{r,1} - 2^r gc\mathfrak{J}_{r,0})t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

Using equality (2.3), we obtain

$$\begin{aligned} gc\mathfrak{J}_{r,0} =& 1 + (2^{r+1} + 1)e_1 + (3 \cdot 4^r + 2^{r+1})e_2 \\ &+ (5 \cdot 8^r + 5 \cdot 4^r + 2^r)e_3, \\ gc\mathfrak{J}_{r,1} - 2^r\, gc\mathfrak{J}_{r,0} =& 2^r + 1 + (4^r + 2^r)e_1 + (2 \cdot 8^r + 3 \cdot 4^r + 2^r)e_2 \\ &+ (3 \cdot 16^r + 5 \cdot 8^r + 2 \cdot 4^r)e_3. \end{aligned}$$

□

Corollary 3.11. *The generating function for the generalized commutative Jacobsthal quaternions has the following form*

$$f(t) = \frac{e_1 + e_2 + 3e_3 + (1 + 2e_2 + 2e_3)t}{1 - t - 2t^2}.$$

Concluding Remarks

In this paper, we introduced and studied generalized commutative Jacobsthal quaternions and their one-parameter generalization - generalized commutative r-Jacobsthal quaternions. The Jacobsthal sequence can also be generalized using Jacobsthal polynomials. It will be interesting to continue this research by defining r-Jacobsthal polynomials and then examining generalized r-Jacobsthal commutative quaternion polynomials.

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