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APPROXIMATION PROPERTIES OF A MODIFIED GAMMA TYPE OPERATOR

AJAY KUMAR

Abstract. This article presents a new sequence of Gamma-type operators that retains the test function $e_r(t) = t^r$, $r \in \mathbb{N}$. Initially, we derive the moment formulas for these operators. Later, we analyze the approximation properties using the standard and weighted modulus of smoothness and prove an asymptotic Voronovskaja-type theorem. Furthermore, we compare the convergence rate and error estimation of the proposed operators with existing ones that preserve test functions in various ways, using numerical examples.

1. Introduction

The classical Gamma operators were first introduced by Lupas and Müller in 1967 [17]. This is a well-known sequence of positive linear operators used for improving the approximation of a target function on the interval $[0, \infty)$. These classical Gamma operators are defined as:

$$G_n(f, x) = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^\infty e^{-xu} u^n f\left(\frac{n}{u}\right) du, \quad \text{for all } x \in \mathbb{R}^+ = (0, \infty), n \in \mathbb{N}. \tag{1.1}$$

The above operators not only maintain constants but also preserve linear functions. To achieve more precise approximations compared to the original operator (1.1), numerous researchers have proposed various modified versions of the classical Gamma operators, which are extensively discussed in the literature. For additional information, refer to the relevant sources [3, 5, 6, 11, 12, 14, 15, 19–24] and the references cited theirin.

Key words and phrases. Gamma operator, rate of convergence, modulus of continuity, Voronovskaja type theorem.

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A new modified form of the classical Gamma operators has been considered by Betus and Usta [4] in recent times, which is expressed as follows:

$$\widetilde{G}_n(f,x) = \frac{x^n}{\Gamma(n+1)} \int_0^\infty e^{-xu^{1/n}} f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right) du, \quad n \in \mathbb{N}. \tag{1.2}$$

The modified form of the classical Gamma operators preserves the test functions $e_0(t) = 1$ and $e_2(t) = t^2$, as noted in [4]. Furthermore, it has been observed in the same source that this modified form of the operators yields improved approximation results compared to the original operator (1.1).

In the literature, several researchers have proposed new constructions or modifications of operators for better approximation results. King [13] was the first to present a new construction of Bernstein operators that preserve the test functions $e_r = x^r$ for $r = 0, 2$. Acar et al. [2] introduced a generalized form of (1.1) that can reproduce exponential test functions and established some approximation properties for the considered sequence. Deveci et al. [7] defined a refinement of Gamma operators that preserves constants and functions of the form $e^{2\mu}$ for $\mu > 0$. Gupta and Agrawal [9] considered modified Post-Widder operators that preserve the test functions $e_r(t) = t^r$ for $r \in \mathbb{N}$ and discussed that these operators provide a better approximation for $r = 3$.

Based on the aforementioned discussion, we are motivated to modify the operator (1.2) such that it preserves the test function $e_r(t) = t^r$ for $r \in \mathbb{N}$. We shall start with the following expression:

$$G_{n,r}(f,x) = \frac{(b_{n,r}(x))^n}{\Gamma(n+1)} \int_0^\infty e^{-b_{n,r}(x)u^{1/n}} f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right) du,$$

where $b_{n,r}(x) \in \mathbb{R}^+$. Then,

$$\begin{aligned} G_{n,r}(t^r,x) = x^r &= \frac{(b_{n,r}(x))^n}{\Gamma(n+1)} \int_0^\infty e^{-b_{n,r}(x)u^{1/n}} \left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)^r du \\ &= \frac{\left(\sqrt{(n-1)(n-2)}\right)^r}{\Gamma(n+1)} (b_{n,r}(x))^n \int_0^\infty e^{-b_{n,r}(x)u^{1/n}} \frac{1}{u^{r/n}} du \\ &= \frac{\Gamma(n-r)}{\Gamma(n)} (b_{n,r}(x))^r \left(\sqrt{(n-1)(n-2)}\right)^r. \end{aligned}$$

Above implies that

$$b_{n,r}(x) = \left(\frac{\Gamma(n)}{\Gamma(n-r)}\right)^{1/r} \frac{x}{\sqrt{(n-1)(n-2)}} = \frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}, \tag{1.3}$$

where the rising factorial is given by $(-n+1)_r = (-n+1)(-n+2)\cdots(-n+r)$, $(-n+1)_1 = -n+1$ and $(-n+1)_0 = 1$. Therefore, using (1.3), the modified form of the operator $G_{n,r}(f,x)$ for $r \in \mathbb{N}$ can be written as:

$$G_{n,r}(f,x) = \frac{1}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n$$

$$\times \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}} f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)du. \tag{1.4}$$

This modified form of the operator $G_{n,r}(f,x)$, which is defined by (1.4), preserves the test function $e_r(t) = t^r$ for $r \in \mathbb{N}$, as well as the constant function. Notably, if $r = 2$, the generalized operator (1.4) reduces to the original operator (1.2). In this paper, we aim to investigate the approximation properties of the modified Gamma operator defined by (1.4).

2. Auxiliary Results

The following lemma provides a general expression for moments of the proposed operators.

Lemma 2.1. *Let $x \in \mathbb{R}^+$ and $e_m(t) = t^m$, $m = 0, 1, 2, \ldots$ Then, for $r \in \mathbb{N}$, we have*

$$\begin{aligned} G_{n,r}(e_m,x) =& \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{(n-1)(n-2)(n-3)\cdots(n-m)}x^m \\ =& \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{\Pi_{i=1}^m(n-i)}x^m, \quad m \in \mathbb{N}\cup\{0\}. \end{aligned}$$

Proof. From (1.4), we have

$$G_{n,r}(e_m,x) = \frac{1}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n \tag{2.1}$$

$$\times \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}} \left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)^m du.$$

Let $\alpha u^{1/n} = t$, where $\alpha = \frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}$. Substituting $du = \frac{n}{\alpha}\left(\frac{t}{\alpha}\right)^{n-1}dt$ and $\frac{1}{u^{m/n}} = \left(\frac{\alpha}{t}\right)^m$ in (2.1), we get

$$\begin{aligned} G_{n,r}(e_m,x) &= \frac{(\alpha)^n\{(n-1)(n-2)\}^{m/2}}{\Gamma(n+1)}\int_0^\infty e^{-t}\left(\frac{\alpha}{t}\right)^m \frac{n}{\alpha}\left(\frac{t}{\alpha}\right)^{n-1}dt \\ &= \frac{\alpha^m\{(n-1)(n-2)\}^{m/2}}{\Gamma(n)}\int_0^\infty e^{-t}t^{(n-m)-1}dt \\ &= \frac{\alpha^m\{(n-1)(n-2)\}^{m/2}}{\Gamma(n)}\Gamma(n-m) \\ &= \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{(n-1)(n-2)(n-3)\cdots(n-m)}x^m = \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{\Pi_{i=1}^m(n-i)}x^m. \end{aligned}$$

Thus, the lemma is completed. □

Remark 2.1. Note that from Lemma 2.1, it is evident that when $r = m$, the operator (1.4) preserves the test functions $e_r(x) = x^r$ for $r \in \mathbb{N} \cup \{0\}$. If we set $r = 2$, the resulting operator (1.4) reduces to the operator (1.2) and preserves both the constant function and the test function x^2.

Lemma 2.2. *Let us define the central moment for $m \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$ as follows: $\mu_m^{G_{n,r}}(x) = G_{n,r}((t-x)^m, x)$, $n > m$, then*

$$\mu_1^{G_{n,r}}(x) = \left[\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} - 1\right]x,$$

$$\mu_2^{G_{n,r}}(x) = \left[\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - 2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\right]x^2,$$

$$\mu_4^{G_{n,r}}(x) = \Bigg[\frac{\{(-1)^r(-n+1)_r\}^{4/r}}{(n-1)(n-2)(n-3)(n-4)} - \frac{4\{(-1)^r(-n+1)_r\}^{3/r}}{(n-1)(n-2)(n-3)} + \frac{6\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - \frac{4\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\Bigg]x^4,$$

$$\mu_6^{G_{n,r}}(x) = \Bigg[\frac{\{(-1)^r(-n+1)_r\}^{6/r}}{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)} - \frac{6\{(-1)^r(-n+1)_r\}^{5/r}}{(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{15\{(-1)^r(-n+1)_r\}^{4/r}}{(n-1)(n-2)(n-3)(n-4)} - \frac{20\{(-1)^r(-n+1)_r\}^{3/r}}{(n-1)(n-2)(n-3)} + \frac{15\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - \frac{6\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\Bigg]x^6.$$

Proof. The proof of this lemma can be obtained through straightforward computation using (1.4) and Lemma 2.1. We omit the details of the proof. □

Lemma 2.3. *For $f \in C[0,\infty)$, we have $\|G_{n,r}(f)\| \leq \|f\|$.*

Proof. Using (1.4) and Lemma 2.1, we can obtain the following expression:

$$\begin{aligned}|G_{n,r}(f,x)| \leq &\frac{1}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n \\ &\times \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}}\left|f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)\right|du, \\ \leq &\frac{\|f\|}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}}du \\ = &\|f\|.\end{aligned}$$

Therefore, we have completed the proof. □

3. Convergence Properties of $G_{n,r}$

Let $C_B[0,\infty)$ denote the space of all real-valued uniformly continuous and bounded functions on $[0,\infty)$ equipped with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. For $f \in C_B[0,\infty)$ and $\delta > 0$, the n-th order modulus of continuity is defined as

$$\omega_n(f,\delta) = \sup_{0\le h\le\delta} \sup_{x\in[0,\infty)} |\Delta_h^n f(x)|, \quad n \in \mathbb{N},$$

where Δ denotes the forward difference operator. When $n = 1$, we obtain the usual modulus of continuity, which is denoted by $\omega(f,\delta)$.

Theorem 3.1. *If $f \in C_B[0,\infty)$, then for every $x \in [0,\infty)$, we have*

$$|G_{n,r}(f,x) - f(x)| \le 2\omega(f, er(r)),$$

where $\delta = er(r) = \sqrt{\mu_2^{G_{n,r}}(x)}$ is the error function for $r = 1,2,3,\dots$

Proof. Let $x \in [0,\infty)$ and $r \in \mathbb{N}$. In view of the fact that $G_{n,r}(1;x) = 1$, we have

$$|G_{n,r}(f;x) - f(x)| = |G_{n,r}(f;x) - G_{n,r}(f(x);x)| \le G_{n,r}\left(|f(t) - f(x)|;x\right).$$

Now, using the property of modulus of continuity $|f(t) - f(x)| \le \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right)$ in the above inequality, we have

$$|G_{n,r}(f;x) - f(x)| \le \omega(f;\delta)\left(\frac{G_{n,r}((t-x)^2;x)}{\delta^2} + 1\right).$$

By choosing $\delta = \sqrt{\mu_2^{G_{n,r}}(x)}$, we get the desired result. □

Thus for different preservation of the operators $G_{n,r}$, i.e. $r = 1,2,3$, we have

$$|G_{n,1}(f;x) - f(x)| \le 2\omega\left(f; \frac{x}{\sqrt{(n-2)}}\right),$$

$$|G_{n,2}(f;x) - f(x)| \le 2\omega\left(f; x\sqrt{2\frac{\sqrt{(n-1)} - \sqrt{(n-2)}}{\sqrt{(n-2)}}}\right),$$

$$|G_{n,3}(f;x) - f(x)| \le 2\omega\left(f; x\sqrt{\left[\frac{(n-3)^2}{(n-1)(n-2)}\right]^{1/3} - 2\left[\frac{(n-2)(n-3)}{(n-1)^2}\right]^{1/3} + 1}\right).$$

Theorem 3.2. *Let $f \in C_B[0,\infty)$. Then,*

$$|G_{n,r}(f;x) - f(x)| \le M\omega_2(f, \sqrt{\zeta_{n,r}}) + \omega\left(f, \left|\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - x\right|\right),$$

where M is a positive constant and

$$\zeta_{n,r} = \left[\frac{(2n-3)\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)^2(n-2)} - 4\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 2\right]x^2, \quad n \ne 1,2.$$

Proof. Let us begin with the auxiliary operators $G^*_{n,r} : C_B[0,\infty) \to C_B[0,\infty)$ defined by

$$G^*_{n,r}(f,x) = G_{n,r}(f;x) - f\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x\right) + f(x). \tag{3.1}$$

Let $h \in C^2_B[0,\infty)$, $C^2_B[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$ and $x, t \in [0,\infty)$. In view of Taylor series expansion, we have

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-\theta)h''(\theta)d\theta.$$

Applying the operator $G^*_{n,r}$ on both sides of the above equation and using Lemma 2.2, we have

$$\begin{aligned}|G^*_{n,r}(h,x) - h(x)| \leq & \left|G^*_{n,r}\left(\int_x^t (t-\theta)h''(\theta)d\theta, x\right)\right| \\ \leq & \left|G_{n,r}\left(\int_x^t (t-\theta)h''(\theta)d\theta, x\right)\right| \\ & + \left|\int_x^{\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x} \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - \theta\right)h''(\theta)d\theta\right| \\ \leq & \mu_2^{G_{n,r}}(x)\|h''\| \\ & + \left|\int_x^{\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x} \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - \theta\right)d\theta\right|\|h''\|,\end{aligned}$$

(3.2)

$$|G^*_{n,r}(h,x) - h(x)| \leq \left[\mu_2^{G_{n,r}}(x) + \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - x\right)^2\right]\|h''\| := \zeta_{n,r}\|h''\|.$$

With the help of Lemma 2.3 and (3.1), we get

$$\|G^*_{n,r}(f,x)\| \leq \|G_{n,r}(f,x)\| + 2\|f\| \leq 3\|f\|, \quad f \in C_B[0,\infty). \tag{3.3}$$

Using (3.1), (3.2) and (3.3), we have

$$\begin{aligned}|G_{n,r}(f,x) - f(x)| \leq & |G^*_{n,r}(f-h,x) - (f-h)(x)| + |G^*_{n,r}(h,x) - h(x)| \\ & + \left|f\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x\right) - f(x)\right| \\ \leq & 4\|f-g\| + \zeta_{n,r}\|h''\| + \left|f\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x\right) - f(x)\right| \\ \leq & M\{\|f-h\| + \zeta_{n,r}\|h''\|\} \\ & + \omega\left(f, \left|\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - x\right|\right).\end{aligned}$$

Taking the infimum in the last step of the above inequality and using Peetre's K-functional, which is defined as

$$K_2(f,\beta)=\inf_{h\in C_B^2[0,\infty)}\{\|f-h\|+\beta\|h''\|:h\in C_B^2[0,\infty)\},$$

we obtain

$$|G_{n,r}(f,x)-f(x)|\leq K_2(f,\zeta_{n,r})+\omega\bigg(f,\bigg|\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x-x\bigg|\bigg).$$

Then by using Lorentz-DeVore property [8] $K_2(f,\beta)\leq M\omega_2(f,\sqrt{\beta})$, $\beta>0$, we can conclude our desired proof. □

Remark 3.1. For $r\in\mathbb{N}$ and sufficiently large n, from Theorem 3.2 one can easily verify that the operator $G_{n,r}(f,x)\to f(x)$ as $\zeta_{n,r}\to 0$.

We observe that at $r=1$, Theorem 3.2 reduces to the result as follows.

Corollary 3.1. *Let $f\in C_B[0,\infty)$ and $x\in[0,\infty)$. Then,*

$$|G_{n,r}(f,x)-f(x)|\leq M\omega\bigg(f,\frac{x}{\sqrt{n-2}}\bigg),$$

where M is a positive constant.

4. Weighted Modulus of Continuity

Let us define the weighted space of real-valued functions $f:[0,\infty)\to\mathbb{R}$ with the property $|f(x)|\leq M_f\phi(x)$ by $B_\phi[0,\infty)=\{f:[0,\infty)\to\mathbb{R}:|f(x)|\leq M_f\phi(x),x\in[0,\infty)\}$, where M_f is a positive constant depending on f but independent of x and a weight function $\phi(x)=1+x^2$, which is continuous on $\mathbb{R}$.

Let $C_\phi[0,\infty)=C[0,\infty)\cap B_\phi[0,\infty)$ and by $C_\phi^J[0,\infty)$, we denote the subspace of all continuous functions $f\in C_\phi[0,\infty)$ for which $\lim_{x\to\infty}\frac{|f(x)|}{1+x^2}=J_f$, exists and finite, where J_f is a constant depending on f. Then, for each $f\in C_\phi[0,\infty)$, the weighted modulus of continuity is defined as (see [1])

$$\Omega(f,\delta)=\sup_{|h|<\delta,x\in(0,\infty)}\frac{|f(x+h)-f(x)|}{1+x^2+h^2+h^2x^2}.$$

The next result is a quantitative Voronovskaja type asymptotic formula.

Theorem 4.1. *Let $f''\in C_\phi^J[0,\infty)$ and $x>0$. Then, we have*

$$\begin{aligned}&\bigg|G_{n,r}(f,x)-f(x)-\bigg(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}-1\bigg)xf'(x)\\&-\bigg(\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)}-2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}+1\bigg)x^2f''(x)\bigg|\\&\leq 16(1+x^2)\Omega\bigg(f'',\bigg(\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\bigg)^{1/4}\bigg)\mu_2^{G_{n,r}}(x),\end{aligned}$$

where $\mu_2^{G_{n,r}}(x)$ *and* $\mu_6^{G_{n,r}}(x)$ *are defined in Lemma 2.2.*

Proof. Let $f'' \in C_\phi^J[0,\infty)$ and $x \in (0,\infty)$. Then, by Taylor's expansion, we have

$$f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + h(x,t)(t-x)^2,$$

where $h(x,t) := \frac{f''(\xi)-f''(x)}{2}$, is a continuous function and $\xi \in (x,t)$. Now, applying $G_{n,r}$ on both sides of the above equation, we have

$$\begin{aligned} G_{n,r}(f(t)-f(x),x) =& G_{n,r}\Big((t-x)f'(x),x\Big) + G_{n,r}\left(\frac{(t-x)^2}{2!}f''(x),x\right) \\ &+ G_{n,r}\Big(h(x,t)(t-x)^2,x\Big). \end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned} &\Bigg|G_{n,r}(f,x) - f(x) - \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} - 1\right)xf'(x) \\ &- \left(\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - 2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\right)x^2 f''(x)\Bigg| \\ \le& G_{n,r}(|h(x,t)|(t-x)^2,x). \end{aligned}$$

In view of the inequality $|\xi - x| \le |x-t|$ and by simple computation, we can write

$$|h(t,x)| \le 8(1+x^2)\left(1 + \frac{(t-x)^4}{\delta^4}\right)\Omega(f'',\delta).$$

In view of Lemma 2.2, we obtain

$$\begin{aligned} G_{n,r}(|h(x,t)|(t-x)^2,x) &\le 8(1+x^2)\Omega(f'',\delta)\left\{\mu_2^{G_{n,r}}(x) + \frac{\mu_6^{G_{n,r}}(x)}{\delta^4}\right\} \\ &= 8(1+x^2)\Omega(f'',\delta)\left\{1 + \frac{1}{\delta^4}\cdot\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\right\}\mu_2^{G_{n,r}}(x). \end{aligned}$$

Choosing $\delta = \left(\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\right)^{1/4}$, we have

$$\begin{aligned} &\Bigg|G_{n,r}(f,x) - f(x) - \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} - 1\right)xf'(x) \\ &- \left(\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - 2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\right)x^2 f''(x)\Bigg| \\ \le& 16(1+x^2)\Omega\left(f'', \left(\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\right)^{1/4}\right)\mu_2^{G_{n,r}}(x), \end{aligned}$$

as desired. □

Remark 4.1. For $r \in \mathbb{N}$ and fixed $x \in [0, \infty)$, we observe that

$$\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)} \to 0, \quad \text{as } n \to \infty,$$

which guarantees the convergence of Theorem 4.1.

Let $[0, \alpha]$, $\alpha \geq 0$, be the closed interval. The standard modulus of continuity is denoted by $\omega_\alpha(f, \delta)$ and defined as

$$\omega_\alpha(f, \delta) = \sup_{|t-x| \leq \delta, t, x \in [0,\alpha]} |f(t) - f(x)|.$$

It is also clear that, for any $f \in C_B[0, \infty)$, the modulus of continuity $\omega_\alpha(f, \delta) \to 0$ as $\delta \to 0$.

Theorem 4.2. *Let $f \in C_B[0, \alpha]$ and $\alpha > 0$. Then, the following inequality satisfies*

$$|G^*_{n,r}(f, x) - f(x)| \leq 4M_f(1 + x^2)\delta_n^2(x) + 2\omega_{\alpha+1}(f; \delta_n(x)),$$

where

$$\delta_n^2(x) = \left[\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - 2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\right] x^2$$

and M_f is a constant depending on f.

Proof. From [10], for all $0 \leq x \leq \alpha$ and $t > \alpha + 1$, we have

$$|f(t) - f(x)| \leq 4M_f(1 + x^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{\alpha+1}(f; \delta_n(x)), \quad \delta > 0.$$

Applying the operator $G^*_{n,r}$ and Cauchy-Schwartz inequality on both sides of the above equation, we have

$$\begin{aligned} |G^*_{n,r}(f, x) - f(x)| \leq & 4M_f(1 + x^2)G^*_{n,r}((t - x)^2, x) \\ & + \left(1 + \frac{\sqrt{G^*_{n,r}((t - x)^2, x)}}{\delta}\right)\omega_{\alpha+1}(f; \delta_n(x)), \\ = & 4M_f(1 + x^2)\delta_n^2(x) + \left(1 + \frac{\delta_n(x)}{\delta}\right)\omega_{\alpha+1}(f; \delta_n(x)). \end{aligned}$$

By choosing $\delta = \delta_n(x)$, we get the desired result. □

Remark 4.2. For $r = 1$ and $x \in [0, \alpha]$, from Theorem 4.2, we have

$$|G^*_{n,1}(f, x) - f(x)| \leq 4M_f(1 + x^2)\frac{x^2}{n-2} + 2\omega_{\alpha+1}\left(f; \frac{x}{\sqrt{n-2}}\right), \quad n > 2.$$

Also, for $x \in [0, \alpha]$ and $r = 2$, Theorem 4.2 reduced to Theorem 4 of [4] as

$$|G^*_{n,2}(f, x) - f(x)| \leq 4M_f(1 + x^2)x^2\left[2 - 2\sqrt{\frac{n-2}{n-1}}\right]$$

$$+ 2\omega_{\alpha+1}\left(f; \sqrt{\left[2 - 2\sqrt{\frac{n-2}{n-1}}\right]x^2}\right), \quad n > 1.$$

It is also observed that $G^*_{n,1}(f,x)$ and $G^*_{n,2}(f,x)$ converges to $f(x)$, as $n \to \infty$.

5. Point-wise Estimates

This section is dedicated to some point-wise estimates of the rate of convergence for the modified Gamma operator $G^*_{n,r}$ defined in (1.4). First, we define the relationship between the local smoothness of f and local approximation.

Let $\eta \in (0,1]$ and $S \subset [0,\infty)$. A function $f \in C_B[0,\infty)$ is in $L_M(\eta)$ on S, if it satisfies the following condition

$$|f(t) - f(x)| \leq M|t-x|^\eta, \quad t \in [0,\infty), x \in S,$$

where M is a constant depending on f and η.

Theorem 5.1. *Let $f \in C_B[0,\infty) \cap L_M(\eta)$. Then, we have*

$$|G^*_{n,r}(f,x) - f(x)| \leq M\left(\left\{\mu_2^{G_{n,r}}(x)\right\}^{\frac{\eta}{2}} + 2d^\eta(x,S)\right),$$

where $\mu_2^{G_{n,r}}(x)$ is defined in Lemma 2.2 and $d(x,S)$ is a distance function from x to S and defined as

$$d(x,S) = \inf\{|t-x| : t \in S\}.$$

Proof. Let $\bar{S}$ be the closure of S in $[0,\infty)$. Then there exists at least one point s_0 in $\bar{S}$ such that $d(x,S) = |x - s_0|$. By the monotonicity property of $G^*_{n,r}$, we get

$$\begin{aligned}
|G^*_{n,r}(f,x) - f(x)| &\leq G^*_{n,r}(|f(t) - f(s_0)|, x) + G^*_{n,r}(|f(x) - f(s_0)|, x) \\
&\leq M\left(G^*_{n,r}\left(|t-s_0|^\eta, x\right) + |x-s_0|^\eta\right) \\
&\leq M\left(G^*_{n,r}\left(|t-x|^\eta, x\right) + 2|x-s_0|^\eta\right).
\end{aligned}$$

Thus, applying Hölder's inequality with $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$, we have

$$|G^*_{n,r}(f,x) - f(x)| \leq M\left(\left\{G^*_{n,r}((t-x)^2, x)\right\}^{\frac{\eta}{2}} + 2|x-s_0|^\eta\right).$$

Finally, using Lemma 2.2, we obtain the desired result. □

Next, we discuss the local direct estimation for the operator (1.4), with the help of Lipschitz-type maximal function of order η defined by B. Lenze [16] as

$$\tilde{\omega}_\eta(f,x) = \sup_{t \neq x, t \in [0,\infty)} \frac{|f(t) - f(x)|}{|t-x|^\eta}, \quad x \in [0,\infty) \text{ and } \eta \in (0,1]. \tag{5.1}$$

Theorem 5.2. *Let $f \in C_B[0,\infty)$ and $0 < \eta \leq 1$. Then, for all $x \in [0,\infty)$, we have*

$$|G^*_{n,r}(f,x) - f(x)| \leq \tilde{\omega}_\eta(f,x)\left\{\mu_2^{G_{n,r}}(x)\right\}^{\frac{\eta}{2}},$$

where $\mu_2^{G_{n,r}}(x)$ is defined in Lemma 2.2.

Proof. In view of (5.1), we have

$$|G^*_{n,r}(f,x)-f(x)| \le \tilde{\omega}_\eta(f,x)G^*_{n,r}\Big(|t-x|^\eta, x\Big).$$

Using Hölder's inequality in the above equation, we obtain

$$|G^*_{n,r}(f,x)-f(x)| \le \tilde{\omega}_\eta(f,x)G^*_{n,r}\left(|t-x|^2, x\right)^{\frac{\eta}{2}} \le \tilde{\omega}_\eta(f,x)\left\{\mu_2^{G_{n,r}}(x)\right\}^{\frac{\eta}{2}}.$$

Thus, the theorem is completed. □

Let us consider the Lipschitz-type space with two parameters defined in [18], for any $u,v>0$, such that

$$L_M^{u,v}(\eta)=\left\{f\in C[0,\infty): |f(t)-f(x)| \le \frac{M|t-x|^\eta}{(ux^2+vx+t)^{\frac{\eta}{2}}},\ x,t\in(0,\infty)\right\},$$

where M is a positive constant and $0<\eta\le 1$.

Theorem 5.3. *For $f\in L_M^{u,v}(\eta)$. Then, for all $x>0$, we have*

$$|G^*_{n,r}(f,x)-f(x)| \le M\left(\frac{\mu_2^{G_{n,r}}(x)}{ux^2+vx}\right)^{\frac{\eta}{2}}.$$

Proof. The proof of this theorem is divided into two parts. In the first part, we prove our theorem for $\eta=1$. Then, for $f\in L_M^{u,v}(1)$ and $x\in(0,\infty)$, we have

$$\begin{aligned}|G^*_{n,r}(f,x)-f(x)| &\le G^*_{n,r}(|f(t)-f(x)|,x) \le MG^*_{n,r}\left(\frac{|t-x|}{(ux^2+vx+t)^{\frac{1}{2}}},x\right)\\ &\le \frac{M}{(ux^2+vx)^{\frac{1}{2}}}G^*_{n,r}\left(|t-x|,x\right).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we obtain

$$|G^*_{n,r}(f,x)-f(x)| \le \frac{M}{(ux^2+vx)^{\frac{1}{2}}}\left(G^*_{n,r}(t-x)^2,x\right)^{\frac{1}{2}} \le M\left(\frac{\mu_2^{G_{n,r}}(x)}{ux^2+vx}\right)^{\frac{1}{2}}.$$

Thus the result holds for $\eta=1$.

Now, we prove the result for $0<\eta<1$. Then, for $x\in(0,\infty)$ and $f\in L_M^{u,v}(\eta)$, we obtain

$$\begin{aligned}|G^*_{n,r}(f,x)-f(x)| &\le G^*_{n,r}(|f(t)-f(x)|a,x) \le MG^*_{n,r}\left(\frac{|t-x|^\eta}{(ux^2+vx+t)^{\frac{\eta}{2}}},x\right)\\ &\le \frac{M}{(ux^2+vx)^{\frac{\eta}{2}}}G^*_{n,r}\left(|t-x|^\eta,x\right).\end{aligned}$$

Applying Hölder's inequality by taking $p=\frac{2}{\eta}$ and $q=\frac{2}{2-\eta}$, we obtain

$$|G^*_{n,r}(f,x)-f(x)| \le \frac{M}{(ux^2+vx)^{\frac{\eta}{2}}}G^*_{n,r}\Big((t-x)^2,x\Big)^{\eta/2}.$$

Finally, using Lemma 2.2, we get the desired result. □

Remark 5.1. At particular $r = 2$, Theorem 5.2 and Theorem 5.3 are reduced to the Theorem 6 and Theorem 7 respectively of [4].

6. Numerical Experiments

In this section, we provide numerical examples to verify the approximation properties of the modified Gamma operator (1.4) with different preservation of test functions. The implementation is carried out in Mathematica.

Example 1. Consider the test function $f(x) = x^8 + 8x + 2$ on the interval $[1, 2]$. Figure 1 indicates that the operator $G_{n,r}(f, x)$ approaches the function $f(x)$ faster as the value of r increases. Moreover, Figure 2 suggests that the error function $E_{n,r}(f, x) = |G_{n,r}(f, x) - f(x)|$ tends to the x-axis, which means that the error decreases as the value of r increases.

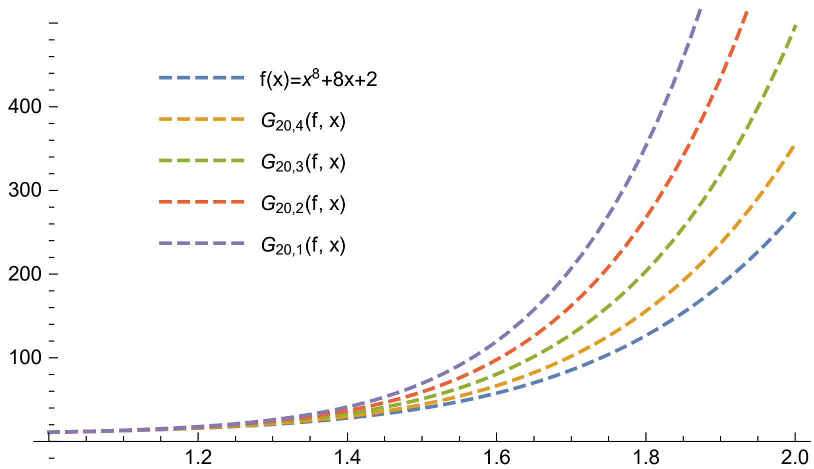

Figure 1. Convergence of $G_{n,r}(f; x)$ to $f(x) = x^8 + 8x + 2$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[1, 2]$.

Example 2. Consider the test function $f(x) = xe^{-x}$ on the interval $[0, 4]$. Figure 3 indicates that the operator $G_{n,r}(f, x)$ approaches the function $f(x)$ faster as the value of r increases. Moreover, Figure 4 suggests that the error function $E_{n,r}(f, x) = |G_{n,r}(f, x) - f(x)|$ tends to the x-axis, which means that the error decreases as the value of r increases.

Therefore, based on the aforementioned observations and figures, we can conclude that the approximation gets better as the value of r increases. Further, we can also notice that the operator (1.4) produces better convergence over the operator (1.2) for $r \geq 2$.

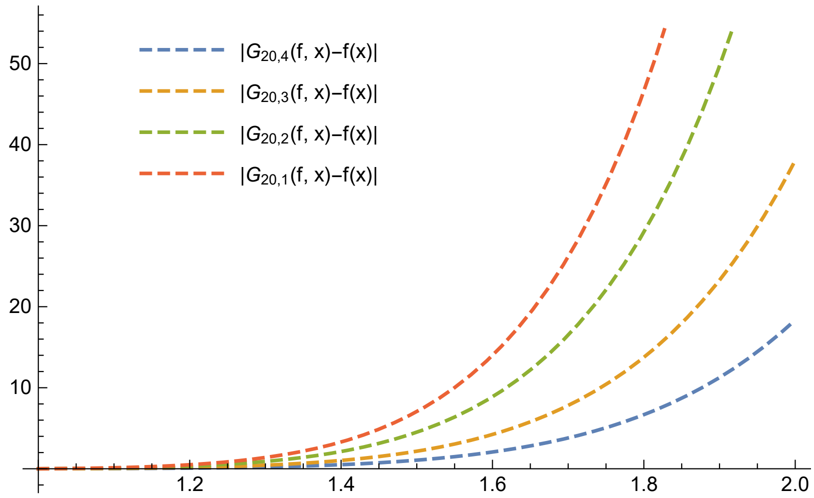

FIGURE 2. Error estimation of the operator $G_{n,r}(f;x)$ to the function $f(x) = x^8 + 8x + 2$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[1, 2]$.

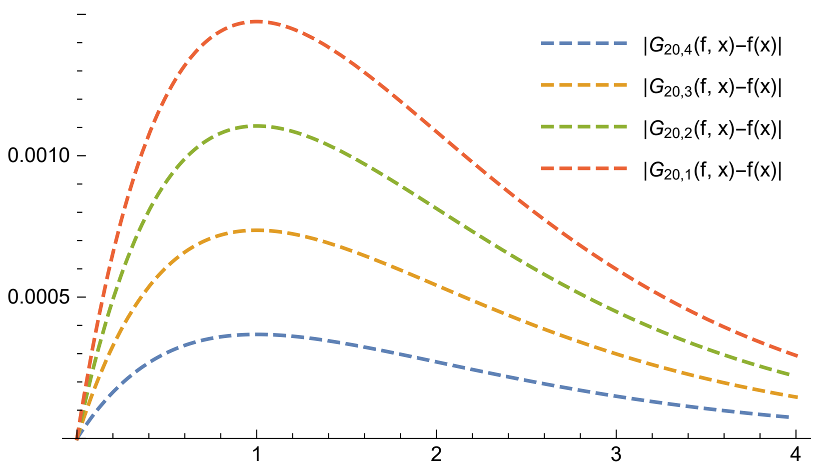

FIGURE 3. Convergence of $G_{n,r}(f;x)$ to $f(x) = xe^{-x}$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[0, 4]$.

7. CONCLUSION

The aim of the present study was to construct a modified sequence of Gamma-type operators, which preserves the test function $e_r(t) = t^r$, $r \in \mathbb{N}$. These newly defined Gamma-type operators play a crucial role in encompassing existing Gamma-type operators and facilitating the definition of new ones that can yield improved approximation results under suitable conditions. To establish that these newly defined operators constitute an approximation process, we also present some of their fundamental properties. Finally, we offer numerical experiments to validate our theoretical findings.

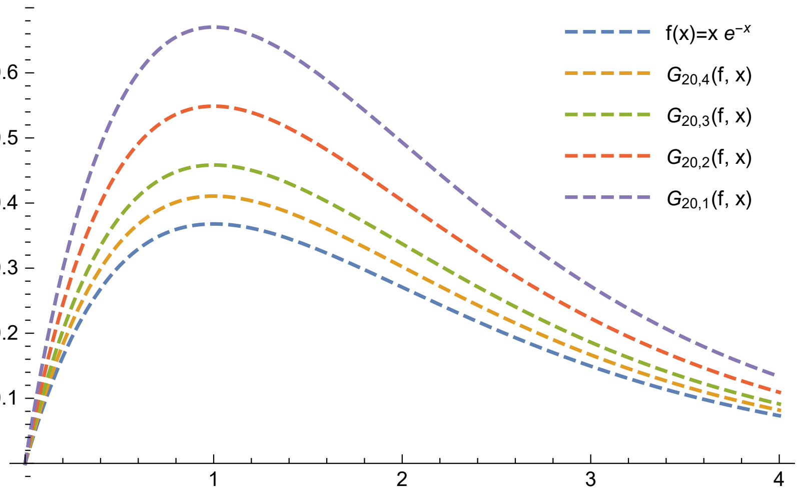

FIGURE 4. Error estimation of the operator $G_{n,r}(f;x)$ to the function $f(x) = xe^{-x}$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[0, 4]$.

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NONLOCAL NEUTRAL FUNCTIONAL SEQUENTIAL DIFFERENTIAL EQUATIONS WITH CONFORMABLE FRACTIONAL DERIVATIVE

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ABSTRACT. In this paper, we investigate the existence, uniqueness, and stability results of second-order neutral evolution differential equations within the framework of sequential conformable derivatives with nonlocal conditions. Utilizing Krasnoselskii's fixed-point theorem, we establish results concerning the existence of at least one solution, while the uniqueness of the solution is derived using Banach's fixed-point theorem. The final section is devoted to an example that illustrates the applicability of our findings.

1. INTRODUCTION

Differential equations with nonlocal conditions are essential in various scientific fields, including engineering and physics. Numerous researchers have explored the theory of these equations concerning different types of derivatives. Hernández [6] studied the second-order Cauchy problem with nonlocal conditions for the classical derivative. Recently, fractional differential equations have gained popularity in modeling various problems in biology, chemistry, and other applied areas [8, 9, 11–17]. In [18], Shur et al. treated a fractional Cauchy problem of order $\alpha \in (1,2)$ with non-local conditions using the Caputo fractional derivative. Their study primarily focused on the results of the existence and uniqueness of mild solutions. For physical interpretations of the non-local conditions, we refer to references [2, 3, 10].

Key words and phrases. Conformable derivative, cosine family of linear operators, fixed point theorem, nonlocal conditions.
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The authors in [4] investigated the existence and regularity of solutions for some partial differential equations with nonlocal conditions in the α-norm for the following problem:

$$\begin{cases} \frac{d}{dt}\left(y(t) - \mathcal{F}\left(t, y(h_1(t))\right)\right) = -B\left(y(t) - \mathcal{F}\left(t, y(h_1(t))\right)\right) + \mathcal{G}\left(t, y(h_2(t))\right), & t \in [0, a], \\ y(0) = y_0 + \varphi(y), \end{cases}$$

where $B : D(B) \subset Y \to Y$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space Y. The functions $\mathcal{F}$, $\mathcal{G}$, φ, h_1 and h_2 are continuous functions.

In [5], the authors considered the following fractional conformable problem:

$$\begin{cases} \frac{d^\beta}{dt^\beta}\left(y(t) - \mathcal{F}\left(t, y(h_1(t))\right)\right) = -B\left(y(t) - \mathcal{F}\left(t, y(h_1(t))\right)\right) + \mathcal{G}\left(t, y(h_2(t))\right), & t \in [0, a], \\ y(0) = y_0 + \varphi(y), \end{cases}$$

where $\frac{d^\beta}{dt^\beta}$ is the conformable fractional derivative of $\beta \in (0, 1)$. B is a sectorial operator which generates a strongly analytic semigroup $(T(t))$ on a Banach space Y. The functions $\mathcal{F}$, $\mathcal{G}$, φ, h_1 and h_2 are continuous functions.

Motivated by the previously mentioned publications, we are interested in a related problem: the second-order sequential Cauchy problem with non-local conditions, using the conformable derivative. More precisely, we are interested in second-order sequential conformable differential equations characterized by the following non-local conditions:

(1.1)
$$\begin{cases} \frac{d^\beta}{dt^\beta}\left[\frac{d^\beta}{dt^\beta}\left(y(t) - \mathcal{F}\left(t, y(h_1(t))\right)\right)\right] = B\left(y(t) - \mathcal{F}\left(t, y(h_1(t))\right)\right) + \mathcal{G}\left(t, y(h_2(t))\right), & t \in [0, a], \\ y(0) = y_0 + \varphi(y), \\ \frac{d^\beta}{dt^\beta}\left(y(0) - \mathcal{F}\left(0, y(h_1(0))\right)\right) = y_1 + \psi(y), \end{cases}$$

where $\frac{d^\beta}{dt^\beta}$ is conformable fractional derivative of order β. The operator B is the infinitesimal generator of a family of cosines $\{C(t), S(t)\}_{t\in\mathbb{R}}$ on a Banach space $(Y, \|\cdot\|)$. y_0 and y_1 are two elements in the Banach Y. The expression $\mathcal{C} = \mathcal{C}\left([0, a], Y\right)$ denotes Banach space of continuous functions y with the norm $|y| = \sup\{\|y(t)\|, t \in [0, a]\}$. The functions $\mathcal{F} : [0, a] \times \mathcal{C} \to Y$, $\mathcal{G} : [0, a] \times Y \to Y$, $\varphi : \mathcal{C} \to Y$, $\psi : \mathcal{C} \longrightarrow Y$, h_1 and h_2 are continuous functions.

This paper is summarized as follows. In Section 2, we review some tools related to the conformable derivative as well as some needed results. Section 3 will be devoted to the statements and the proof of the main results. In Section 4, as application, we investigate a second-order sequential conformal partial differential equation with a non-local condition.

2. Preliminaries

We begin by recalling some fundamental concepts of conformable calculus [7].

Definition 2.1. The conformable derivative of order β for a function $\mathcal{H}$ is defined as follows:

$$\frac{d^\beta \mathcal{H}(t)}{dt^\beta} = \lim_{\varepsilon \to 0} \frac{\mathcal{H}(t + \varepsilon t^{1-\beta}) - \mathcal{H}(t)}{\varepsilon}, \qquad t > 0.$$

If this limit exists, we say that $\mathcal{H}$ is (β)-differentiable at t.

If $\mathcal{H}$ is (β)-differentiable and $\lim_{\varepsilon \to 0^+} \frac{d^\beta \mathcal{H}(t)}{dt^\beta}$ exists, we define

$$\frac{d^\beta \mathcal{H}(0)}{dt^\beta} = \lim_{\varepsilon \to 0^+} \frac{d^\beta \mathcal{H}(t)}{dt^\beta}.$$

The (β)-fractional integral of a function $\mathcal{H}$ is given by

$$I^\beta \mathcal{H}(t) = \int_0^t (t - \vartheta)^{\beta - 1} \mathcal{H}(\vartheta) d\vartheta.$$

Theorem 2.1. *If $\mathcal{H} \in D(I^\beta)$ is a continuous function, we obtain*

$$\frac{d^\beta (I^\beta \mathcal{H}(t))}{dt^\beta} = \mathcal{H}(t).$$

The Laplace transform associated with the conformable derivative is given by the following definition.

Definition 2.2 ([1])**.** The conformable fractional Laplace transform of $\mathcal{H}$ of order β is defined by

$$\mathcal{L}_\beta \left(\mathcal{H}\left(t\right)\right)(\lambda) = \int_0^{+\infty} t^{\beta-1} e^{-\lambda \frac{t^\beta}{\beta}} \mathcal{H}(t) dt.$$

The following proposition shows the effect of the fractional Laplace transform on the conformal derivative.

Proposition 2.1. *If $\mathcal{H}(t)$ is differentiable, we obtain*

$$\begin{aligned} I^\beta \left(\frac{d^\beta \mathcal{H}(t)}{dt^\beta} \right) =& \mathcal{H}(t) - \mathcal{H}(0), \\ \mathcal{L}_\beta \left(\frac{d^\beta \mathcal{H}(t)}{dt^\beta} \right)(\lambda) =& \lambda \mathcal{L}_\beta(\mathcal{H}(t))(\lambda) - \mathcal{H}(0). \end{aligned}$$

So let's recall certain results related to the theory of the cosine family [19].

Definition 2.3. A family $(C(\xi))_{\xi \in \mathbb{R}}$ of bounded linear operators on Y is defined as a strongly continuous family of cosines if and only if:

(a) $C(0) = I$;
(b) $C(\nu + \xi) + C(\nu - \xi) = 2C(\nu)C(\xi)$, for all $\xi, \nu \in \mathbb{R}$;
(c) $\xi \to C(\xi)y$ is continuous for each fixed $y \in Y$.

We define also the sine family by

$$S(\xi)y := \int_0^\xi C(\vartheta) y d\vartheta.$$

Definition 2.4. B is the infinitesimal generator of a strongly continuous cosine family $((C(\xi))_{\xi\in\mathbb{R}}, (S(\xi))_{\xi\in\mathbb{R}})$ on Y defined by:

$$\begin{aligned} D(B) =& \{y \in Y, \xi \to C(\xi)y \text{ is a twice continuously differentiable function}\},\\ Ay =& \frac{d^2C(0)y}{d\xi^2}. \end{aligned}$$

We end this section with the following results.

Proposition 2.2. *The following assertions are true.*

(a) *There exist constants $\omega \geq 0$ and $M \geq 1$ where*

$$|S(\xi) - S(\nu)| \leq M \left| \int_\nu^\xi \exp(\omega|\vartheta|)d\vartheta \right|, \quad \text{for all } \nu, \xi \in \mathbb{R}.$$

(b) *If $y \in Y$ and $\xi, \nu \in \mathbb{R}$, then $\int_\nu^\xi S(\vartheta)yd\vartheta \in D(B)$ and*

$$B\int_\nu^\xi S(\vartheta)yd\vartheta = C(\xi)y - C(\nu)y.$$

(c) *If $\xi \mapsto C(\xi)y$ is differentiable, hence $S(\xi)y \in D(B)$ and $\frac{dC(\xi)}{d\xi}y = BS(\xi)y$.*

(d) *For λ such that $Re(\lambda) > \omega$, we get*
- $\lambda^2 \in \rho(B)$, *($\rho(B)$: is the resolvent set of B),*
- $\lambda(\lambda^2 I - B)^{-1}y = \int_0^{+\infty} e^{-\lambda\xi}C(\xi)yd\xi$, $y \in Y$,
- $(\lambda^2 I - B)^{-1}y = \int_0^{+\infty} e^{-\lambda\xi}S(\xi)yd\xi$, $y \in Y$.

3. Main Results

Before presenting our main results, we introduce the following assumptions.

(H_1) The function $\mathcal{G}(t,\cdot) : Y \to Y$ is continuous, and for all $r > 0$, there exists a function $\mu_r \in L^\infty([0,a],\mathbb{R}^+)$ such that $\sup_{\|y\|\leq r} \|\mathcal{G}(t,y)\| \leq \mu_r(t)$ for all $t \in [0,a]$.

(H_2) The function $\mathcal{G}(\cdot,y) : [0,a] \to Y$ is continuous for all $y \in Y$.

(H_3) There exists a constant $l_1 > 0$ such that $\|\mathcal{F}(t,x) - \mathcal{F}(t,y)\| \leq l_1|x-y|$ for all $x,y \in \mathcal{C}$.

(H_4) There exists a constant $l_2 > 0$ such that $\|\varphi(x) - \varphi(y)\| \leq l_2|x-y|$ for all $x,y \in \mathcal{C}$.

(H_5) There exists a constant $l_3 > 0$ such that $\|\psi(x) - \psi(y)\| \leq l_3|x-y|$ for all $x,y \in \mathcal{C}$.

3.1. **Existence and uniqueness of the mild solution.** Using the fractional Laplace transform in equation (1.1), we get

$$
\begin{aligned}
&\mathcal{L}_\beta\left(\frac{d^\beta}{dt^\beta}\left[\frac{d^\beta}{dt^\beta}\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)\right]\right)(\lambda)\\
=&\lambda\left(\lambda^2-B\right)^{-1}\left(y\left(0\right)-\mathcal{F}\left(0,y(h_1(0))\right)\right)+\left(\lambda^2-B\right)^{-1}\frac{d^\beta}{dt^\beta}\left(y(0)-\mathcal{F}\left(0,y(h_1(0))\right)\right)\\
&+\left(\lambda^2-B\right)^{-1}\mathcal{L}_\beta(\mathcal{G}(t,y(h_2(t))))(\lambda)\\
=&\lambda\left(\lambda^2-B\right)^{-1}\left(y_0+\varphi(y)-\mathcal{F}\left(0,y(h_1(0))\right)\right)+\left(\lambda^2-B\right)^{-1}\left(y_1+\psi(y)\right)\\
&+\left(\lambda^2-B\right)^{-1}\mathcal{L}_\beta(\mathcal{G}(t,y(h_2(t))))(\lambda).
\end{aligned}
$$

According to the inverse fractional Laplace transform, we find the Duhamel's formula

$$
\begin{aligned}
y(t)=&\mathcal{F}\left(t,y(h_1(t))\right)+C\left(\frac{t^\beta}{\beta}\right)\left(y_0+\varphi(y)-\mathcal{F}(0,y(h_1(0)))\right)\\
&+S\left(\frac{t^\beta}{\beta}\right)\left(y_1+\psi(y)\right)+\int_0^t s^{\beta-1}S\left(\frac{t^\beta}{\beta}-\frac{s^\beta}{\beta}\right)\mathcal{G}(s,y(h_2(s)))ds.
\end{aligned}
$$

Definition 3.1. $y\in\mathcal{C}$ is a mild solution of problem (1.1) if the following assertion is true:

$$
\begin{aligned}
y(t)=&\mathcal{F}\left(t,y(h_1(t))\right)+C\left(\frac{t^\beta}{\beta}\right)\left(y_0+\varphi(y)-\mathcal{F}(0,y(h_1(0)))\right)\\
&+S\left(\frac{t^\beta}{\beta}\right)\left(y_1+\psi(y)\right)+\int_0^t s^{\beta-1}S\left(\frac{t^\beta}{\beta}-\frac{s^\beta}{\beta}\right)\mathcal{G}(s,y(h_2(s)))ds,\quad t\in[0,a].
\end{aligned}
$$

Theorem 3.1. *If $(S(t))_{t>0}$ is compact and (H_1)-(H_5) are satisfied, then, the Cauchy problem (1.1) has at least one mild solution provided that*

$$
l_1+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|l_3<1.
$$

Proof. Choosing

$$
r\geq\frac{\|\mathcal{F}(0,0)\|+\sup\limits_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|y_0\|+\|\varphi(0)\|+\|\mathcal{F}(0,0)\|\right)+\sup\limits_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\times\Delta}{1-l_1-\sup\limits_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)-\sup\limits_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|l_3},
$$

with

$$
\Delta=\|y_1\|+\|\psi(0)\|+\frac{a^\beta}{\beta}|\mu|_{L^\infty}.
$$

Let $B_r = \{y \in \mathcal{C}, |y| \leq r\}$, for $y \in B_r$, we define the operators P_1 and P_2 as follows

$$P_1(y(t)) = \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right)\Big(y_0 + \varphi(y) - \mathcal{F}(0, y(h_1(0)))\Big) + S\left(\frac{t^\beta}{\beta}\right)\Big(y_1 + \psi(y)\Big),$$

$$P_2(y(t)) = \int_0^t s^{\beta-1} S\left(\frac{t^\beta}{\beta} - \frac{s^\beta}{\beta}\right) \mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0, \tau].$$

By using assumptions (H_1)-(H_5), we prove that $P_1(y) + P_2(z) \in B_r$ for all $y, z \in B_r$. Moreover, the operator P_1 is a contraction on B_r.

We are going to prove that the operator P_2 is compact and continuous.

- Firstly, we show that P_2 is continuous.

Let $y_n \in B_r$ such that $y_n \to y$ in B_r. Therefore, by using (H_1), we have

$$\|s^{\beta-1}\left(\mathcal{G}(s, y_n((h_2(s))) - \mathcal{G}(s, y((h_2(s))))\right\| \leq 2\mu_r(s)s^{\beta-1}$$

and

$$\mathcal{G}(s, y_n((h_2(s))) \to \mathcal{G}(s, y((h_2(s))), \quad \text{as } n \to +\infty.$$

Also, we obtain

$$P_2(y_n(t)) - P_2(y(t)) = \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right)\Big[\mathcal{G}(s, y_n(h_2(s))) - \mathcal{G}(s, y((h_2(s)))\Big]ds,$$

for $t \in [0, a]$. Accordingly, we obtain

$$|P_2(y_n) - P_2(y)| \leq \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| \int_0^a s^{\beta-1}\|\mathcal{G}(s, y_n(h_2(s))) - \mathcal{G}(s, y((h_2(s)))\|ds.$$

By using the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty}|P_2(y_n) - P_2(y)| = 0.$$

- Secondly, we prove the compactness of P_2.

Claim 1. We show that $\{P_2(y(t)), y \in B_r\}$ is relatively compact in Y.

For $t \in]0, a[$, let $\varepsilon \in]0, t[$, and we define the operator P_2^ε by

$$P_2^\varepsilon(y(t)) = \int_0^{(t^\beta - \varepsilon^\beta)^{\frac{1}{\beta}}} s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right)\mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0, \tau], \text{ for all } y \in B_r.$$

The relative compactness of $\{P_2^\varepsilon(y(t)), y \in B_r\}$ in Y is guaranteed by the compactness of $(S(t))_{t>0}$. Using assumption (H_1), we have

$$\|P_2^\varepsilon(y(t)) - P_2(y(t))\| \leq |\mu_r|_{L^\infty([0,a],\mathbb{R}^+)} \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\frac{\varepsilon^\beta}{\beta}.$$

Then, we conclude that $\{P_2(y(t)), y \in B_r\}$ is relatively compact in Y. It is clear that the set $\{P_2(y(0)), y \in B_r\}$ is compact. Therefore, $\{P_2(y(t)), y \in B_r\}$ is relatively compact in Y for all $t \in [0, a]$.

Claim 2. We prove that $P_2(B_r)$ is equicontinuous.

Let $t_1, t_2 \in]0, a]$ such that $t_1 < t_2$, we have

$$P_2(y(t_2)) - P_2(y(t_1)) = \int_0^{t_1} s^{\beta-1} \left[S\left(\frac{t_2^\beta - s^\beta}{\beta}\right) - S\left(\frac{t_1^\beta - s^\beta}{\beta}\right)\right] \mathcal{G}(s, y(h_2(s)))ds$$
$$+ \int_{t_1}^{t_2} s^{\beta-1} S\left(\frac{t_2^\beta - s^\beta}{\beta}\right) \mathcal{G}(s, y(h_2(s)))ds.$$

Therefore, we obtain

$$\|P_2(y(t_2)) - P_2(y(t_1))\| \leq |\mu_r|_{L^\infty([0,a],\mathbb{R}^+)} \left[\frac{K}{\omega^2}\left(\exp\left(\frac{\omega t_2^\beta}{\beta}\right) - \exp\left(\frac{\omega t_1^\beta}{\beta}\right)\right)\right]$$
$$+ \sup_{t\in[0,a]} \left|S\left(\frac{t^\beta}{\beta}\right)\right| \left(\frac{t_2^\beta - t_1^\beta}{\beta}\right).$$

We conclude that the functions $P_2(y)$, $y \in B_r$, are equicontinuous at $t \in [0, a]$. Applying the Arzelà-Ascoli theorem, we establish that P_2 is a compact operator. Finally, the Krasnoselskii's fixed point theorem completes the proof. □

To establish the uniqueness of the mild solution, we need the following assumption.

(H_6) There exists a constant $l_4 > 0$ such that $\|\mathcal{G}(t, z) - \mathcal{G}(t, y)\| \leq l_4\|z - y\|$ for all $z, y \in Y$ and $t \in [0, a]$.

Theorem 3.2. *Assume that* (H_2)-(H_6) *hold. Then, the Cauchy problem* (1.1) *has a unique mild solution provided that*

$$l_1 + \sup_{t\in[0,a]} \left|C\left(\frac{t^\beta}{\beta}\right)\right| (l_1 + l_2) + \sup_{t\in[0,a]} \left|S\left(\frac{t^\beta}{\beta}\right)\right| \left(l_3 + l_4\frac{a^\beta}{\beta}\right) < 1.$$

Proof. We define the operator $P : \mathcal{C} \to \mathcal{C}$ by

$$P(y(t)) = \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right)\Big(y_0 + \varphi(y) - \mathcal{F}(0, y(h_1(0)))\Big)$$
$$+ S\left(\frac{t^\beta}{\beta}\right)\Big(y_1 + \psi(y)\Big) + \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right) \mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0, a].$$

Next, let $y, z \in \mathcal{C}$, we have

$$P(z(t)) - P(y(t)) = \mathcal{F}(t, z(h_1(t))) - \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right)(\varphi(z) - \varphi(y))$$
$$+ C\left(\frac{t^\beta}{\beta}\right)\Big(\mathcal{F}(0, y(h_1(0))) - \mathcal{F}(0, z(h_1(0)))\Big)$$
$$+ S\left(\frac{t^\beta}{\beta}\right)(\psi(z) - \psi(y))$$
$$+ \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right)\Big[\mathcal{G}(s, z(h_2(s))) - \mathcal{G}(s, y(h_2(s)))\Big]ds.$$

Accordingly, we obtain

$$\begin{aligned}&\|P(z(t))-P(y(t))\|\\ \leq&\left[l_1+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(l_3+l_4\frac{a^\beta}{\beta}\right)\right]|z-y|.\end{aligned}$$

Then, we get

$$|P(z)-P(y)|\leq\left[l_1+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(l_3+l_4\frac{a^\beta}{\beta}\right)\right]|z-y|.$$

Therefore, P has a unique fixed point in $\mathcal{C}$. ☐

3.2. **Continuous dependence of the mild solution.** Now, we will give some results concerning the continuous dependence of the mild solution.

Theorem 3.3. *Assume that the conditions of Theorem* 3.2 *are satisfied. Let* y_0, z_0, y_1, $z_1 \in Y$ *and denote by* y, z *the solutions associated with* (y_0, y_1) *and* (z_0, z_1), *respectively. Then, we have*

$$\begin{aligned}|z-y|\leq&\frac{\beta}{\beta-\beta l_1-\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\beta l_3+l_4a^\beta\right)-\beta\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)}\\ &\times\left[\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\|z_0-y_0\|+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\|z_1-y_1\|\right].\end{aligned}$$

Proof. For $t\in[0,a]$, we have

$$\begin{aligned}z(t)-y(t)=&\mathcal{F}(t,z(h_1(t)))-\mathcal{F}(t,y(h_1(t)))+S\left(\frac{t^\beta}{\beta}\right)(z_1-y_1+\psi(z)-\psi(y))\\ &+C\left(\frac{t^\beta}{\beta}\right)\Big(z_0-y_0+\varphi(z)-\varphi(y)+\mathcal{F}(0,y(h_1(0)))-\mathcal{F}(0,z(h_1(0)))\Big)\\ &+\int_0^t s^{\beta-1}S\left(\frac{t^\beta-s^\beta}{\beta}\right)\Big[\mathcal{G}(s,z(h_2(s)))-\mathcal{G}(s,y(h_2(s)))\Big]ds.\end{aligned}$$

Since we obtain

$$\begin{aligned}\|z(t)-y(t)\|\leq&l_1|z-y|+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_1-y_1\|+\left(l_3+l_4\frac{a^\beta}{\beta}\right)|z-y|\right)\\ &+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_0-y_0\|+(l_1+l_2)|z-y|\right).\end{aligned}$$

Accordingly, we show that

$$\begin{aligned}|z-y| \leq & l_1|z-y| + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_1-y_1\| + \left(l_3 + l_4\frac{a^\beta}{\beta}\right)|z-y|\right) \\ & + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_0-y_0\| + (l_1+l_2)\,|z-y|\right).\end{aligned}$$

Finally, we get the following estimation:

$$\begin{aligned}|z-y| \leq & \frac{\beta}{\beta-\beta l_1 - \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\beta l_3 + l_4 a^\beta\right) - \beta\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)} \\ & \times\left[\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\|z_0-y_0\| + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\|z_1-y_1\|\right].\end{aligned}$$

□

Theorem 3.4. *Assume that the conditions of Theorem* 3.2 *are satisfied. Let* $y_0, z_0, y_1, z_1 \in Y$ *and denote by* y, z *the solutions associated with* (y_0, y_1) *and* (z_0, z_1)*, respectively. Then, we have*

$$|z-y| \leq \frac{\left(\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\|z_1-y_1\| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\|z_0-y_0\|\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)}{1-\left(l_1+l_3\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)},$$

provided that

$$\left(l_1+l_3\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right) < 1.$$

Proof. For $t \in [0,a]$, we have

$$\begin{aligned}z(t)-y(t) =& F\left(t, z(h_1(t))\right) - F\left(t, y(h_1(t))\right) + S\left(\frac{t^\beta}{\beta}\right)\left(z_1-y_1+\psi(z)-\psi(y)\right) \\ & + C\left(\frac{t^\beta}{\beta}\right)\Big(z_0-y_0+\varphi(z)-\varphi(y)+F(0,y(h_1(0)))-F(0,z(h_1(0)))\Big) \\ & + \int_0^t s^{\beta-1}S\left(\frac{t^\beta-s^\beta}{\beta}\right)\Big[\mathcal{G}(s,z(h_2(s)))-\mathcal{G}(s,y(h_2(s)))\Big]ds.\end{aligned}$$

Then, we get

$$\begin{aligned}\|z(t)-y(t)\| \leq & l_1|z-y| + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_1-y_1\| + l_3|z-y|\right) \\ & + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_0-y_0\| + (l_1+l_2)\,|z-y|\right) \\ & + l_4\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\int_0^t s^{\beta-1}\|z(s)-y(s)\|ds.\end{aligned}$$

Therefore, we show that

$$|z-y| \leq \Bigg(l_1|z-y| + \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| (\|z_1-y_1\| + l_3|z-y|) + \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right) \right| (\|z_0-y_0\| + (l_1+l_2)|z-y|) \times \exp\left(l_4 \frac{a^\beta}{\beta} \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \right)\Bigg).$$

Finally, we conclude that

$$|z-y| \leq \frac{\left(\sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \|z_1-y_1\| + \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right) \right| \|z_0-y_0\| \right) \exp\left(l_4 \frac{a^\beta}{\beta} \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \right)}{1-\left(l_1 + l_3 \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| + \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right) \right| (l_1+l_2) \right) \exp\left(l_4 \frac{a^\beta}{\beta} \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \right)}. \quad \square$$

Remark 3.1. If we take

$$C_1 = \frac{\exp\left(l_4 \frac{a^\beta}{\beta} \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \right)}{1-\left(l_1 + l_3 \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| + \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right) \right| (l_1+l_2) \right) \exp\left(l_4 \frac{a^\beta}{\beta} \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \right)},$$

$$C_2 = \frac{\beta}{\beta - \beta l_1 - \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \left(\beta l_3 + l_4 a^\beta\right) - \beta \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right) \right| (l_1+l_2)},$$

we have that $C_1 < C_2$. Then, Theorem 3.4 is better than Theorem 3.3.

3.3. **Special case of nonlocal conditions.** Here, we study a special case of nonlocal conditions, this means that the functions ψ and φ are given by:

$$\varphi(y) = \sum_{i=1}^{n} c_i y(t_i) \quad \text{and} \quad \psi(y) = \sum_{i=1}^{n} b_i y(t_i),$$

where c_i, b_i, $i = 1, 2, \ldots, n$, are given constants and $0 < t_1 < t_2 < \cdots < t_n < a$.

Proposition 3.1. *Assume that* (H_2), (H_3) *and* (H_6) *hold. Then, the fractional problem* (1.1) *has a unique mild solution provided that there exists* $\varepsilon_0 \in]0,1[$ *such that*

$$l_1 + \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right) \right| \left(\sum_{i=1}^{n} |c_i| + l_1 \right) + \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right| \sum_{i=1}^{n} |d_i| \leq \varepsilon_0.$$

Proof. Define the operator $P : \mathcal{C} \to \mathcal{C}$ by

$$P(y(t)) = \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right) (y_0 + \varphi(y) - \mathcal{F}(0, y(h_1(0)))) + S\left(\frac{t^\beta}{\beta}\right) \big(y_1 + \psi(y)\big) + \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right) \mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0,a].$$

Now, we define a new norm $|\cdot|_\beta$ in $\mathcal{C}$ by

$$|y|_\beta = \left| \exp\left(\frac{-\varepsilon(\cdot)^\beta}{\beta} \right) y \right|,$$

where

$$\varepsilon = \frac{l_4 \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right|}{\varepsilon_0 - l_1 - \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum_{i=1}^{n} |c_i| + l_1\right) - \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \sum_{i=1}^{n} |d_i|}.$$

Let $y, z \in \mathcal{C}$, and $t \in [0, a]$. Then,

$$\begin{aligned} P(z(t)) - P(y(t)) =& \mathcal{F}\left(t, z(h_1(t))\right) - \mathcal{F}\left(t, y(h_1(t))\right) + C\left(\frac{t^\beta}{\beta}\right)\left(\varphi\left(z\right) - \varphi\left(y\right)\right) \\ &+ C\left(\frac{t^\beta}{\beta}\right)\Big(\mathcal{F}\left(0, y\left(h_1\left(0\right)\right)\right) - \mathcal{F}(0, z(h_1(0)))\Big) \\ &+ S\left(\frac{t^\beta}{\beta}\right)\left(\psi(z) - \psi(y)\right) \\ &+ \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right)\Big[\mathcal{G}(s, z(h_2(s))) - \mathcal{G}(s, y(h_2(s)))\Big] ds. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|P(z(t)) - P(y(t))\| \leq& \Bigg[l_1 \exp\left(\frac{\varepsilon t^\beta}{\beta}\right) + \exp\left(\frac{\varepsilon t^\beta}{\beta}\right) \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum_{i=1}^{n} |c_i| + l_1\right) \\ &+ \exp\left(\frac{\varepsilon t^\beta}{\beta}\right) \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \sum_{i=1}^{n} |d_i| \\ &+ l_4 \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \int_0^t s^{\beta-1} \exp\left(\frac{\varepsilon s^\beta}{\beta}\right) ds \Bigg] |z - y|_\beta. \end{aligned}$$

Accordingly, we show that

$$\begin{aligned} |P(z) - P(y)|_\beta \leq& \Bigg[l_1 + \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum_{i=1}^{n} |c_i| + l_1\right) \\ &+ \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum_{i=1}^{n} |d_i| + \frac{l_4}{\varepsilon}\right) \Bigg] |z - y|_\beta. \end{aligned}$$

Hence, we conclude that

$$|P(z) - P(y)|_\beta \leq \varepsilon_0 |z - y|_\beta.$$

By using the contraction principle, we obtain the result. □

4. Application

Consider the fractional partial differential equation of the following form

$$
\begin{aligned}
&\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\cdot\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left(v(t,y)-\int_0^{\pi}c(t,y,\theta)v(\sin t,\theta)d\theta\right)\\
=&\frac{\partial^2(\cdot)}{\partial y^2}\left(v(t,y)-\int_0^{\pi}c(t,y,\theta)v(\sin t,\theta)d\theta\right)\\
&+\phi\left(t,\frac{\partial v(t,y)}{\partial y}\right)+\frac{|v(t,y)|}{1+|v(t,y)|}+\int_0^t\frac{|v(s,y)|}{1+|v(s,y)|}ds,
\end{aligned}
\tag{4.1}
$$

with the following nonlocal conditions:

$$
v(t,0)=v(t,\pi)=0 \quad\text{and}\quad v(0,y)=\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}v(0,y)=\sum_{i=1}^{n}c_i v(t_i,y),\quad y\in[0,\pi],
\tag{4.2}
$$

where $0<t_1<\cdots<t_n<1$ and $c_1,\dots,c_n$ are given real constants such that

$$
\sum_{i=1}^{n}|c_i|<\frac{4}{10}.
$$

Let $Y=L^2([0,\pi])$ and define the operator $B:Y\to Y$ by

$$
B=\frac{\partial^2(\cdot)}{\partial y^2}\quad\text{and}\quad D(B)=\{v\in H^2(0,\pi),v(\pi)=v(0)=0\}.
$$

The operator B is the infinitesimal generator of a family of cosines $\{C(t),S(t)\}_{t\in\mathbb{R}}$. Furthermore, we have $|C(t)|\leq 1$ and $|S(t)|\leq 1$ for all $t\in[0,1]$.

We consider the following functions:

$$
\begin{aligned}
z(t)(y)=&v(t,y),\quad \mathcal{F}\Big(t,z\Big)(\cdot)=\int_0^{\pi}c(t,\cdot,\theta)z(\theta)d\theta,\\
\mathcal{G}(t,z(t))=&\frac{|z(t)|}{1+|z(t)|}+\int_0^t\frac{|z(s)|}{1+|z(s)|}ds,\quad h_1(t)=\sin(t),\quad h_2(t)=t,
\end{aligned}
$$

and

$$
\varphi(z)=\psi(z)=\sum_{i=1}^{n}c_i z(t_i).
$$

We assume that the following condition hold: $c:[0,1]\times[0,\pi]\times[0,\pi]\to\mathbb{R}$ is continuous with $c(t,\cdot,0)=c(t,\cdot,\pi)=0$. Then, (4.1) and (4.2) become as follows:

$$
\begin{cases}
\frac{d^\beta}{dt^\beta}\left[\frac{d^\beta}{dt^\beta}\Big(z(t)-\mathcal{F}(t,z(h_1(t)))\Big)\right]=B\Big(z(t)-\mathcal{F}\Big(t,z(h_1(t))\Big)\Big)+\mathcal{G}\Big(t,z(h_2(t))\Big),\\
z(0)=\varphi(z),\\
\frac{d^\beta}{dt^\beta}\Big(z(0)-\mathcal{F}(0,z(h_1(0)))\Big)=\psi(z).
\end{cases}
\tag{4.3}
$$

Moreover, $\mathcal{F} : [0, 1] \times \mathcal{C} \to X$, we have

$$\begin{aligned}\|\mathcal{F}(t, z)\|^2 &\leq \int_0^\pi \left(\int_0^\pi c(t, \eta, \theta) z(\theta) d\theta\right)^2 d\eta \\ &\leq \left(\int_0^\pi \int_0^\pi c(t, \eta, \theta)^2 d\theta d\eta\right) \int_0^\pi z^2(\theta) d\theta \\ &\leq \sup_{0 \leq t \leq 1} \left(\int_0^\pi \int_0^\pi c(t, \eta, \theta)^2 d\theta d\eta\right) |z|^2.\end{aligned}$$

Finally, all the hypotheses of Proposition 3.1 are verified.

Therefore, the above fractional problem has a unique mild solution provided that

$$2 \left(\sup_{0 \leq t \leq 1} \left(\int_0^\pi \int_0^\pi c(t, \eta, \theta)^2 d\theta d\eta\right)\right)^{\frac{1}{2}} + \frac{4}{5} < 1.$$

5. Conclusion

In this manuscript, we have explored the existence of solutions for second-order conformal differential equations evolving sequentially with non-local conditions by utilizing Krasnoselskii's fixed-point theorem. Additionally, through the application of the Banach fixed-point theorem, we have established the uniqueness of the mild solution. Finally, we have presented a relevant example to demonstrate the practical application of our theoretical findings.

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ON A CLASS OF GENERALIZED CAPILLARITY PHENOMENA INVOLVING FRACTIONAL ψ-HILFER DERIVATIVE WITH $p(\cdot)$-LAPLACIAN OPERATOR

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Abstract. This research delves into a comprehensive investigation of a class of ψ-Hilfer generalized fractional nonlinear eigenvalue equation originated from a capillarity phenomenon with Dirichlet boundary conditions. The nonlinearity of the problem, in general, do not satisfies the Ambrosetti-Rabinowitz (AR) type condition. Using critical point theorem with variational approach and the (S_+) property of the operator, we establish the existence of positive solutions of our problem with respect to every positive parameter ξ in appropriate fractional ψ-Hilfer spaces. Our main results is novel and its investigation will enhance the scope of the literature on differential equation of fractional ψ-Hilfer generalized capillary phenomena.

1. Introduction

Differential equations are essential in present-day physics, engineering, and various scientific fields, as highlighted in [5]. The ongoing evolution in modern physics and mechanics, as discussed in [1,8–11,16], is leading to changes in traditional areas, which necessitates the development of new mathematical models (see [12–22]). For instance, a reevaluation of the study of capillary phenomena in current literature underscores the pressing need for such advancements. Capillary action, also referred to as capillarity, capillary motion, capillary effect, or wicking, is the ability of a liquid to flow through narrow spaces without external forces, including gravity, and sometimes even against them. This phenomenon can be observed in various situations, such as the upward

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flow of liquids between the bristles of a paintbrush, through thin tubes, within porous materials like paper and plaster, and certain non-porous substances like sand and liquefied carbon fiber, as well as within biological cells. Capillary action is propelled by the intermolecular forces between the liquid and the surrounding solid surfaces. When the diameter of a tube is sufficiently small, a combination of surface tension arising from cohesion within the liquid and adhesive forces between the liquid and the container's wall collaborates to propel the liquid upward. The ascent of water in narrow tubes, and the formation of liquid drops or bubbles, can be effectively analyzed using variational calculus. This method, which hinges on the energy-minimizing nature of observed equilibrium configurations, offers a comprehensive framework for addressing mathematical questions pertinent to diverse phenomena. Recently, the study of capillary phenomena has attracted increased attention, spurred not only by the intrigue surrounding naturally occurring events such as the motion of drops, bubbles, and waves, but also by its practical importance in various applied fields, including industrial, biomedical, pharmaceutical, and microfluidic systems. Given the vast scope of this subject, we will focus on select examples to illustrate key concepts for those interested (see [2, 27, 29, 39]).

The most recent examination of this problem involves applying the $p(\cdot)$-Laplacian and incorporating fractional derivatives in the time dimension. As an example, the authors in [26] employed Ricceri's variational principle, originally due to Bonanno and Molica Bisci, to establish the existence of at least one weak solution and infinitely many weak solutions for the Neumann problem derived from capillary phenomena.

$$\begin{cases} -\operatorname{div}\left(\left(1+\dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)+\alpha(x)|u|^{p(x)-2}u=\lambda f(x,u), & \text{in } \Omega,\\ \dfrac{\partial u}{\partial v}=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^1, ν is the outer unit normal to $\partial\Omega$, $\alpha \in L^\infty(\Omega)$, f is an L^1-Carathéodory function.

In [30], the authors studied the existence and multiplicity of solutions for nonlinear eigenvalue problems involving $p(x)$-Laplacian-like operators

$$\begin{cases} -\operatorname{div}\left(\left(1+\dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\Omega)$ and $p(x) > 2$, for all $x \in \Omega$, $\lambda > 0$ and f satisfies some growth condition and Ambrosetti-Rabinowitz type condition (AR).

An interesting question arises: Can capillary phenomena be practically applied using fractional derivatives operators? In one specific example, Pata describes how oxygen is conveyed from capillaries to the surrounding tissues. This process is modeled by a subdiffusion equation that incorporates two fractional derivatives in time. The problem is described as $\mathbf{D}_t^{\nu_1}\mathfrak{C} - \tau\mathbf{D}_t^{\nu_2}\mathfrak{C} = \operatorname{div}(\rho\nabla\mathfrak{C}) - k$, with $0 < \nu_2 < \nu_1 \leq 1$, $\mathfrak{C}$ is a function of space and time, representing the concentration of oxygen, τ is the time lag in concentration of oxygen along the capillary, k is the rate of consumption per volume of tissue, and is the diffusion coefficient of oxygen, which possibly dependent on $\mathfrak{C}$. In particular, the term $\mathbf{D}_t^{\nu_1}\mathfrak{C} - \tau\mathbf{D}_t^{\nu_2}\mathfrak{C}$ details the net diffusion of oxygen to all tissues and $\mathbf{D}_t^{\theta}$ stands for the usual Caputo fractional derivative of order $\theta \in (0,1)$ with respect to time. We refer to an other example [31] where the authors utilized a fractal structure to describe variational formulation of viscoelastic deformation problem in Capillary-Porous materials. There are various options for introducing fractional integro-differentiation operations, in particular, the Riemann-Liouville, Caputo, Grunwald-Letnikov approaches, and their various modifications. Today, it is necessary to account for the dynamic evolution in modern physics and mechanics when constructing mathematical models. In this context, inspired by the studies mentioned above, we aim to highlight such areas to enrich the theoretical knowledge of these problems. Therefore, in this paper, we utilize the generalized ψ-Hilfer fractional derivative to study a nonlinear eigenvalue equation. This equation, which originates from a capillarity phenomenon, includes Dirichlet boundary conditions of the following form:

$$\tag{1.3} \begin{cases} \mathrm{D}_T^{\gamma,\beta;\psi}\left(\left(1+\dfrac{|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}}\right)|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)-2}\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right) = \xi g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $g \in C(\bar{\Omega}\times\mathbb{R})$ is superlinear and do not satisfy the (AR)-condition, ξ is a positive parameter, $1 < p^- := \operatorname{ess\,inf}_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ := \operatorname{ess\,sup}_{x\in\bar{\Omega}} p(x) < +\infty$. The operators $\mathrm{D}_T^{\gamma,\beta;\psi}$ and $\mathrm{D}_{0^+}^{\gamma,\beta;\psi}$, defined in Section 2.2, are ψ-Hilfer fractional partial derivatives of order $\frac{1}{p(x)} < \gamma < 1$ and type $0 \leq \beta \leq 1$. For more applications of this type of operator, we refer to [25,32–38]. Our objective is to establish the existence of weak solutions to the problem as described by equation (1.3), using a critical point approach and various variational techniques. To achieve this, it is essential to impose specific assumptions on the nonlinear term $g(x,u)$.

(g_1) $g : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and

$$|g(x,u)| \leq c_1 + c_2|u|^{r(x)-1}, \quad \text{for all } (x,t) \in \Omega\times\mathbb{R},$$

where $r \in C(\bar{\Omega})$, $c_1, c_2 > 0$ and

$$p^+ < r^- \leq r(x) < p^\star(x) = \begin{cases} \dfrac{Np(x)}{N-\gamma p(x)}, & \text{if } \gamma p(x) < N, \\ +\infty, & \text{if } \gamma p(x) \geq N, \end{cases} \quad \text{for all } x \in \Omega.$$

(g_2) The following limit holds uniformly for a.e. $x \in \Omega$

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{|t|^{p^+}} = +\infty, \quad \text{where } G(x,t) = \int_0^t g(x,\tau)d\tau.$$

(g_3) $g(x,t) = o\left(|t|^{p^+-1}\right)$, $t \to 0$, for $x \in \Omega$ uniformly.

(g_4) There exists a constant $c_3 > 0$ such that

$$\overline{G}(x,t) \leq \overline{G}(x,s) + c_3,$$

for any $x \in \Omega, 0 < t < s$ or $s < t < 0$, where $\overline{G}(x,t) := tg(x,t) - 2p^+G(x,t)$.

We note that condition (g_4) can be derived from the following condition:

(g_4)$'$ there exists $u_0 > 0$ such that $t \mapsto \frac{g(x,t)}{|t|^{2p^+-1}}$ is increasing in $t \geq t_0$ and decreasing in $t \leq -t_0$.

An example of a function that satisfies conditions (g_1)-(g_4) and does not satisfy the (AR)-condition is as follows:

$$g(x,u) = 2p^+|u|^{2p^+-2}u\log(|u|+1).$$

Definition 1.1. Let X be a real Banach space and $\Upsilon \in C^1(X,\mathbb{R})$. We say that Υ satisfies the Cerami condition at the level c or $(C)_c$ for short, if any sequence $u_n \subset X$ such that $\Upsilon(u_n) \to c$ and $(1 + \|u_n\|)\,\Upsilon'(u_n) \to 0$ called a $(C)_c$ sequence, has a convergent subsequence.

Our approach to proving the existence results for problems (1.3) relies on proving that our operator is the type (S_+) and employing recent critical points theorems, for the convenience of the readers, we recall the critical theorems in [3]. It is important to note that these theorems play a crucial role in our strategy. Let us recall the following.

Theorem 1.1 ([3])**.** *Let X be a real Banach space and let $E \in C^1(X,\mathbb{R})$ satisfy the (C_c) condition for any $c > 0$, $E(0) = 0$ and the following conditions hold.*

(i) *There exist two positive constants ρ and R such that $E(u) \geq R$ for any $u \in X$ with $\|u\| = \rho$.*

(ii) *There exists a function $\xi \in X$ such that $\|\xi\| > \rho$ and $E(\xi) < 0$.*

Then, the functional E has a critical value $c \geq R$, i.e., there exists $u \in X$ such that $E'(u) = 0$ and $E(u) = c$.

Remark 1.1. In the case where $\beta \to 1$ and $\psi(x) = x$, our problem (1.3) reduces to the integer case (for more details, refer to [32]). For this reason, we observe that our problem generalizes many papers in the literature.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of variable exponent Lebesgue spaces and ψ-fractional derivative spaces. Moving on to Section 3, we present the existing solutions to problems (1.3), along with their corresponding proofs.

2. Preliminary

In this section we collect preliminary concepts of the theory of variable exponent Lebesgue space, classical and fractional ψ-Hilfer derivative space (see [4,6,7,23–25,28]).

2.1. **Variable exponent Lebesgue space.** In the following, we define

$$C(\bar{\Omega}) = \Big\{ s \in C(\Omega) : \ 1 < s^- \leq s^+ < +\infty \Big\},$$

where

$$s^- := \inf_{x\in\bar{\Omega}} s(x) \quad \text{and} \quad s^+ := \sup_{x\in\bar{\Omega}} s(x).$$

Denote by $\mathbf{U}(\Omega)$ the set of all measurable real-valued functions defined in Ω. For any $s \in C^+(\Omega)$, we denote the variable exponent Lebesgue space by

$$L^{s(x)}(\Omega) = \left\{ u \in \mathbf{U}(\Omega) : \int_\Omega |u(x)|^{s(x)} dx < +\infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{s(x)}} = \inf \left\{ \lambda > 0 : \ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{s(x)} dx \leq 1 \right\},$$

then, the variable exponent Lebesgue space $\Big(L^{s(x)}(\Omega), \|\cdot\|_{L^{s(x)}}\Big)$ becomes a Banach space.

We have the following generalized Hölder inequality

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \ 2\|u\|_{L^{s(x)}} \|v\|_{L^{\bar{s}(x)}},$$

for $u \in L^{s(x)}(\Omega)$, $v \in L^{\bar{s}(x)}(\Omega)$ such that $\frac{1}{s(x)} + \frac{1}{\bar{s}(x)} = 1$.

At this point, let define the following map $\sigma_{s(x)} : L^{s(x)}(\Omega) \to \mathbb{R}$ by

$$\sigma_{s(x)}(u) = \int_\Omega |u(x)|^{s(x)} dx.$$

Then, we can see the important relationship between the norm $||\cdot||_{L^{s(x)}}$ and the corresponding modular function $\sigma_{s(x)}(\cdot)$ given in the next proposition.

Proposition 2.1 ([25]). *If u and $(u_n)_{n\in\mathbb{N}} \in L^{s(x)}(\Omega)$, we have*

(i) $\|u\|_{L^{s(x)}} < 1 \ (= 1, > 1)$ *if and only if* $\sigma_{s(x)}(u) < 1 \ (= 1, > 1)$;

(ii) $\|u\|_{L^{s(x)}} > 1,$ *then* $\|u\|^{s^-}_{L^{s(x)}} \leq \sigma_{s(x)}(u) \leq \|u\|^{s^+}_{L^{s(x)}}$;

(iii) $\|u\|_{L^{s(x)}} < 1,$ *then* $\|u\|^{s^+}_{L^{s(x)}} \leq \sigma_{s(x)}(u) \leq \|u\|^{s^-}_{L^{s(x)}}$;

(iv) $\lim\limits_{n\to+\infty} \|u_n - u\|_{L^{s(x)}} = 0,$ *if and only if* $\lim\limits_{n\to+\infty} \sigma_{s(x)}(u_n - u) = 0.$

2.2. **ψ-Hilfer fractional derivative space.** Let $A := [c, d]$, $-\infty \leq c < d \leq +\infty$, $n-1 < \gamma < n$, $n \in \mathbb{N}$, $\mathbf{f}$, $\psi \in C^n(A, \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in A$.

• The left-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_c^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_c^x \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)dy. \tag{2.1}$$

• The right-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_d^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_x^d \psi'(y)(\psi(y) - \psi(x))^{\gamma-1}\mathbf{f}(y)dy. \tag{2.2}$$

• The left-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$$\mathrm{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\beta(n-\gamma);\psi}\Big(\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{I}_c^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

• The right-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$$\mathrm{D}_d^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\beta(n-\gamma);\psi}\Big(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{I}_d^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

Choosing $\beta \to 1$, we obtain ψ-Caputo fractional derivatives left-sided and right-sided, given by

$$\mathrm{D}_c^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_c^{(n-\gamma);\psi}\Big(\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{f}(x), \tag{2.3}$$

$$\mathrm{D}_d^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_d^{(n-\gamma);\psi}\Big(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{f}(x). \tag{2.4}$$

Remark 2.1. The ψ-Hilfer fractional derivatives defined as above can be written in the following form

$$\mathrm{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\mu-\gamma;\psi}\mathrm{D}_c^{\gamma;\psi}\mathbf{f}(x) \quad \text{and} \quad \mathrm{D}_d^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\mu-\gamma;\psi}\mathrm{D}_d^{\gamma;\psi}\mathbf{f}(x),$$

with $\mu = \gamma + \beta(n - \gamma)$ and $\mathbf{I}_c^{\mu-\gamma;\psi}$, $\mathbf{I}_d^{\mu-\gamma;\psi}$, $\mathrm{D}_c^{\gamma;\psi}$ and $\mathrm{D}_d^{\gamma;\psi}$ as defined in (2.1), (2.2), (2.3) and (2.4).

In this paper we take $\Omega = A_1 \times \cdots \times A_N = [c_1, d_1] \times \cdots \times [c_N, d_N]$ where $0 < c_i < d_i$ for all $i \in \mathbb{N}$, $0 < \gamma_1, \ldots, \gamma_N < 1$. Consider also $\psi(\cdot)$ be an increasing and positive monotone function on $(c_1, d_1), \ldots, (c_N, d_N)$, having a continuous derivative $\psi'(\cdot)$ on $(c_1, d_1], \ldots, (c_N, d_N]$.

• The ψ-Riemann-Liouville fractional partial integral of order γ of N-variables $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_N)$ is defined by

$$\mathbf{I}_{c,x}^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_{A_1}\int_{A_2} \cdots \int_{A_N} \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)dy,$$

with $\psi'(y)(\psi(x)-\psi(y))^{\gamma-1} = \psi'(y_1)(\psi(x_1)-\psi(y_1))^{\gamma_1-1}\cdots\psi'(y_N)(\psi(x_N)-\psi(y_N))^{\gamma_N-1}$ and $\Gamma(\gamma) = \Gamma(\gamma_1)\Gamma(\gamma_2)\cdots\Gamma(\gamma_N)$, $x_i = x_1x_2\cdots x_N$ and $dy_i = dy_1dy_2\cdots dN$, for all $i \in \{1,2,\dots,N\}$.

• The ψ-Hilfer fractional partial derivative of N-variables of order γ and type β, $0 \le \beta \le 1$, is defined by

$$\mathrm{D}_{c,x_i}^{\gamma,\beta;\psi}\mathbf{f}(x_i) = \mathbf{I}_{c,x_i}^{\beta(n-\gamma);\psi}\Big(\frac{1}{\psi'(x_i)}\cdot\frac{\partial^N}{\partial x_i}\Big)\mathbf{I}_{c,x_i}^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x_i),$$

with $\partial x_i = \partial x_1, \partial x_2, \dots, \partial x_N$ and $\psi'(x_i) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$ for all $i \in \{1,2,\dots,N\}$. Analogously, it is defined $\mathrm{D}_{d,x_i}^{\gamma,\beta;\psi}(\cdot)$.

Now that we have all the necessary tools, we are ready to commence our study. To facilitate this, we define the working space $\mathbb{H}_{p(x)}^{\gamma,\beta,\psi}(\Omega)$ as follow

$$\mathbb{H}_{p(x)}^{\gamma,\beta,\psi}(\Omega) = \mathbb{H} := \Big\{u \in L^{p(x)}(\Omega) : |\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u| \in L^{p(x)}(\Omega)\Big\},$$

equipped with the norm $\|u\|_{\mathbb{H}} = ||u||_{L^{p(x)}} + ||\mathrm{D}_{0^+}^{\gamma,\beta,\psi}u||_{L^{p(x)}}$.

Proposition 2.2 ([33]). *Let $0 < \gamma \le 1$, $0 \le \beta \le 1$ and $1 < p(x)$. The ψ-Hilfer fractional derivative space $\mathbb{H}_{p(x)}^{\gamma,\beta,\psi}(\Omega)$ is a reflexive and separable Banach space.*

Remark 2.2. We can define $\mathbb{H}(\Omega) := \mathbb{H}_{p(x),0}^{\gamma,\beta,\psi}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $\mathbb{H}_{p(x)}^{\gamma,\beta,\psi}(\Omega)$ which can be renormed by the equivalent norm $\|u\| := \||\mathrm{D}_{0^+}^{\gamma,\beta,\psi}u|\|_{L^{p(x)}}$. This space is a separable and reflexive Banach space [37].

Proposition 2.3 ([37]). *Let Ω a Lipschitz bounded domain in Ω. Let $p \in C^0(\overline{\Omega})$. If $r : \overline{\Omega} \to (1,+\infty)$ such that $1 \le r(x) < p^\star(x)$. Then, the embedding $\mathbb{H}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is compact.*

2.3. **The (S_+) property.** In this subsection, we prove the (S_+) property of the operator

$$\mathcal{L} := \mathrm{D}_T^{\gamma,\beta;\psi}\left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)-2} + \frac{|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)-2}}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}}\right).$$

For this, let consider the following functional:

$$\mathcal{A}(u) = \int_\Omega \frac{1}{p(x)}\left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)} + \sqrt{1+|\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}\right)dx, \quad \text{for all } u \in \mathbb{H}(\Omega).$$

Note that $\mathcal{A} \in C^1(\mathbb{H}(\Omega),\mathbb{R})$ and the derivative operator of $\mathcal{A}$ in weak sense $\mathcal{A}' : \mathbb{H}(\Omega) \to (\mathbb{H}(\Omega))^*$ is such that

$$\langle \mathcal{A}'(u), v\rangle = \int_\Omega \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)-2} + \frac{|\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)-2}}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}}\right)\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\mathrm{D}_{0^+}^{\gamma,\beta;\psi}v dx,$$

for all $u, v \in \mathbb{H}(\Omega)$.

Proposition 2.4. *The mapping $\mathcal{A}' : \mathbb{H}(\Omega) \to (\mathbb{H}(\Omega))^*$ is a convex, bounded homeomorphism and strictly monotone operator, and is a mapping of type (S_+).*

Proof. It is clear that $\mathcal{A}$ is a continuous, bounded, and strictly monotone operator. Considering that $\mathcal{A}'$ is a continuous, bounded and strictly monotone operator, if $u_n \rightharpoonup u$ and $\overline{\lim}_{n\to+\infty}(\mathcal{A}'(u_n)-\mathcal{A}'(u),u_n-u)\leq 0$, then $\lim\limits_{n\to+\infty}(\mathcal{A}'(u_n)-\mathcal{A}'(u),u_n-u)=0$. According to Fatou lemma, we have

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx \geq \int_\Omega \frac{1}{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}dx,$$

and note that $\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}\geq |\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}$ for all $u\in\mathbb{H}(\Omega)$, then we get

$$\begin{aligned}&\underline{\lim}_{n\to+\infty}\int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}\right)dx\\ &\geq \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|^{2p(x)}}\right)dx.\end{aligned}\tag{2.5}$$

With $u_n \rightharpoonup u$, we have

$$\lim_{n\to+\infty}(\mathcal{A}'(u_n),u_n-u)=\lim_{n\to+\infty}(\mathcal{A}'(u_n)-\mathcal{A}'(u),u_n-u)=0.\tag{2.6}$$

Moreover, we also have

$$\begin{aligned}&(\mathcal{A}'(u_n),u_n-u)\\ &=\int_\Omega\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-2}+\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)-2}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}\right)\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\mathrm{D}_{0^+}^{\gamma,\beta;\psi}(u_n-u)dx\\ &=\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-2}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\,dx\\ &\quad+\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx-\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)-2}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx\\ &\geq\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-1}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|dx\\ &\quad+\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx-\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)-1}|\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx:=I_1+I_2-I_3.\end{aligned}$$

Note that

$$\begin{aligned}I_1&=\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-1}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|dx\\ &\geq\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left(\frac{p(x)-1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}+\frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|\right)dx\\ &\geq\int_\Omega\frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|dx,\end{aligned}$$

$$
\begin{aligned}
I_2 &= \int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx = \int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} \cdot \frac{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx \\
&= \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx \\
&= \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \left(1 - \frac{1}{1+\underbrace{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}_{\geq c_1 > 0}}\right) dx \\
&\geq \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \left(1 - \frac{1}{1+c_1}\right) dx \geq c_2 \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= -\int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)-1} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx \\
&\geq -\int_\Omega \left(\frac{p(x)-1}{p(x)} \cdot \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)-1}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} + \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| \right) dx.
\end{aligned}
$$

Then,

$$
\begin{aligned}
I_2 + I_3 \geq& \int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx - \int_\Omega \frac{p(x)-1}{p(x)} \cdot \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx \\
&- \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx \\
\geq& \int_\Omega \frac{1}{p(x)} \cdot \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)-1}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx} - \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx \\
\geq& c_2 \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx \\
\geq& c_2 \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} dx \\
\geq& c_3 \left(\int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} dx \right).
\end{aligned}
$$

Therefore,

$$
(\mathcal{A}'(u_n), u_n - u)
$$

$$\geq \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} dx - \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx$$

$$+ c_3 \left(\int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} dx \right). \tag{2.7}$$

Based on (2.5)–(2.7), we deduce

$$\lim_{n\to+\infty} \int_\Omega \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \right) dx$$

$$= \int_\Omega \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} \right) dx. \tag{2.8}$$

Following (2.8), it can be inferred that the integrals of the family of functions

$$\left\{ \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \right) \right\}$$

posses absolutely equicontinuity on Ω. Given that

$$\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} + c_3 \frac{1}{p(x)} \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|} \right|^{2p(x)}$$

$$\leq L_1 \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} \right)$$

$$+ L_2 \left(c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} \right).$$

The integrals of family

$$\left\{ \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} + c_3 \frac{1}{p(x)} \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|} \right|^{2p(x)} \right\}$$

are also absolutely equicontinuous on Ω. Hence,

$$\lim_{n\to+\infty} \int_\Omega \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_\mathrm{n}(x) - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|^{p(x)} \right.$$

$$\left. + c_3 \frac{1}{p(x)} \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_\mathrm{n}(x)\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|} \right|^{2p(x)} \right) dx = 0. \tag{2.9}$$

According to (2.9) we have

$$\lim_{n\to+\infty} \int_\Omega \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n(x) - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|^{p(x)} dx = 0 \tag{2.10}$$

and

$$\lim_{n\to+\infty} \int_\Omega \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n(x)\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|} \right|^{2p(z)} dx = 0. \tag{2.11}$$

By Proposition 2.1, along with (2.10) and (2.11), we have $u_n \to u$, i.e., $\mathcal{L}$ is of type (S_+). By the strictly monotonicity, $\mathcal{L}$ is an injection. On the other hand, since

$$\lim_{\|u\|\to+\infty} \frac{1}{\|u\|} (\mathcal{A}'(u), u) = \lim_{\|u\|\to+\infty} \frac{1}{\|u\|} \int_\Omega \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)} + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(u)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} x\right|^{2p(x)}}} \right) dx$$
$$= + \infty,$$

$\mathcal{A}'$ is coercive. Therefore, $\mathcal{A}'$ is a surjection. Hence, $\mathcal{A}'$ has an inverse mapping $(\mathcal{A}')^{-1}$: $(\mathbb{H}(\Omega))^* \to \mathbb{H}(\Omega)$. Subsequently the continuity of $(\mathcal{A}')^{-1}$ is sufficient to ensure $\mathcal{A}'$ to be a bomeomorphism.

If $g_n, g \in (\mathbb{H}(\Omega))^*$, $g_n \to g$, let $u_n = (\mathcal{A}')^{-1}(g_n)$ and $u = (\mathcal{A}')^{-1}(g)$. Then $\mathcal{A}'(u_n) = g_n$, $\mathcal{A}'(u) = g$. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}(\Omega)$. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$. Since $g_n \to g$, then

$$\lim_{n\to+\infty} (\mathcal{A}'(u_n) - \mathcal{A}'(u_0), u_n - u_0) = \lim_{n\to+\infty} (g_n, u_n - u_0) = 0.$$

Given that $\mathcal{A}'$ is of type (S_+) and $u_n \to u_0$, we deduce that $u_n \to u$, and therefore, $\mathcal{A}'$ is continuous. □

Definition 2.1. We say that $u \in \mathbb{H}(\Omega)$ is a weak solution of (1.3), if for every $\mu \in \mathbb{H}(\Omega)$, the following holds:

$$\int_\Omega \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)-2}\, \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u + \frac{|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)-2}\, \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}}} \right) \mathrm{D}_{0^+}^{\gamma,\beta;\psi} \mu(x) dx$$
$$= \xi \int_\Omega g(x,u)\mu(x)dx.$$

3. Main Results

The primary outcome established in this paper is formulated as follows.

Theorem 3.1. *Assume that* (g_1)-(g_4) *are satisfied. Then, the problem* (1.3) *has at least one nontrivial weak solution for all* $\xi > 0$.

Let us introduce the energy functional $\mathbb{E}_\xi : \mathbb{H}(\Omega) \to \mathbb{R}$ associated to problem (1.3), which is defined as follow

$$\mathbb{E}_\xi(u) = \int_\Omega \frac{1}{p(x)} \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)} + \sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}} \right) dx - \xi \int_\Omega G(x,u)dx. \tag{3.1}$$

Keep in mind that $\mathbb{E}_\xi \in C^1\big(\mathbb{H}(\Omega), \mathbb{R}\big)$ and it is noteworthy that the critical points of $\mathbb{E}_\xi$ correspond to weak solutions of (1.3) and its Gateaux derivative is

$$\langle \mathbb{E}_\xi'(u), u\rangle = \int_\Omega \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)} + \frac{|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}}} \right) dx - \xi \int_\Omega u g(x,u)dx. \tag{3.2}$$

Mountain-pass geometry.

Next, we illustrate that the energy function follows the mountain pass geometry.

Lemma 3.1. *Given that the conditions* (g_1)-(g_3) *hold true. Then we have the following assertions.*

(i) *There exists* $\omega \in \mathbb{H}(\Omega)$*,* $\omega > 0$ *such that* $\mathbb{E}_\xi(t\omega) \to -\infty$ *as* $t \to +\infty$*.*

(ii) *There exist* $e > 0$ *and* $\eta > 0$ *such that* $\mathbb{E}_\xi(u) \geq \eta$ *for any* $u \in \mathbb{H}(\Omega)$ *with* $\|u\| = e$*.*

Proof. (i) By applying (g_2), it can be inferred that for all $K > 0$, there exist $C_K > 0$, such that

$$G(x,t) \geq K|t|^{p^+} - C_K, \quad \text{for all } x \in \Omega, \text{ for all } t \in \mathbb{R}.$$

Let $\omega \in \mathbb{H}(\Omega)$ with $\omega > 0$, then from last inequality, one has

$$\begin{aligned}
\mathbb{E}_\xi(t\omega) &= \int_\Omega \frac{1}{p(x)}\Big(t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + \sqrt{1 + t^{2p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{2p(x)}}\Big)dx - \xi\int_\Omega G(x,t\omega)dx \\
&\leq \int_\Omega \frac{1}{p(x)}\left(t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + 1 + t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)}\right)dx - \xi\int_\Omega G(x,t\omega)dx \\
&\leq \int_\Omega \frac{1}{p(x)}\left(t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + t^{p(x)} + t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)}\right)dx - \xi\int_\Omega G(x,t\omega)dx \\
&\leq t^{p^+}\left[\int_\Omega \frac{1}{p(x)}\left(2|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + 1\right)dx - \xi K\int_\Omega \omega^{p^+}dx\right] + \xi C_K|\Omega|, \qquad (3.3)
\end{aligned}$$

where $t > 1$ and $|\Omega|$ denotes the Lebesgue measure of Ω. Then, from (3.3), if K is large enough such that

$$\int_\Omega \frac{1}{p(x)}\left(2\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega\right|^{p(x)} + 1\right)dx - \xi K\int_\Omega \omega^{p^+}dx < 0,$$

thus we get

$$\mathbb{E}_\xi(t\omega) \to -\infty, \quad \text{as } t \to +\infty.$$

This concludes the proof of (i).

(ii) According to Proposition 2.3, there exists constant $c_3 > 0$ such that

$$\|u\|_{L^{p^+}} \leq c_3\|u\| \quad \text{and} \quad \|u\|_{L^{r(x)}} \leq c_3\|u\|. \qquad (3.4)$$

It follows from (g_1) and (g_3), that for all given $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that

$$G(x,t) \leq \frac{\varepsilon}{p^+}|t|^{p^+} + C_\varepsilon|t|^{r(x)}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \qquad (3.5)$$

Therefore, for $u \in \mathbb{H}(\Omega)$ with $\|u\| < 1$, we have from (3.4) and (3.5)

$$\begin{aligned}
\mathbb{E}_\xi(u) &= \int_\Omega \frac{1}{p(x)}\Big(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)} + \sqrt{1 + |\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}\Big)dx - \xi\int_\Omega G(x,u)dx \\
&\geq \int_\Omega \frac{1}{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}dx - \frac{\xi}{p^+}\varepsilon\int_\Omega |u|^{p^+}dx - \xi C_\varepsilon\int_\Omega |u|^{r(x)}dx
\end{aligned}$$

$$\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\xi}{p^+}\varepsilon c_3^{p^+}\|u\|^{p^+} - \xi C_\varepsilon c_3^{r^-}\|u\|^{r^-}$$

$$\geq \frac{1}{p^+}\left(1 - \xi\varepsilon c_3^{p^+}\right)\|u\|^{p^+} - \xi C_\varepsilon c_3^{r^-}\|u\|^{r^-}. \tag{3.6}$$

This implies the existence of $e > 0$ and $\eta > 0$ such that $\mathbb{E}_\xi(u) \geq \eta > 0$ for every $u \in \mathbb{H}(\Omega)$ and $\|u\| = e$. This completes the proof of (ii). □

Lemma 3.2. *Assume that* (g_1)-(g_3) *hold and* $0 < \xi_0 < \mu_0$, *then* $\mathbb{E}_\xi$ *possesses uniform mountain-pass geometric structure around* $u = 0$ *for* $\xi \in [\xi_0, \mu_0]$, *i.e., there is an* $\varrho \in \mathbb{H}(\Omega)$ *such that* $\mathbb{E}_{\xi_0}(\varrho) < 0$ *for any* $\xi \in [\xi_0, \mu_0]$, *and there are* $\rho > 0, v > 0$ *such that* $\mathbb{E}_\xi(u) \geq v$ *for any* $\xi \in [\xi_0, \mu_0]$ *and* $u \in \mathbb{H}(\Omega)$ *with* $\|u\| = \rho$.

Proof. Fix $\varepsilon > 0$ small enough such that $(1 - \mu_0\varepsilon c_3^{p^+}) \geq \frac{1}{2}$, then by (3.6), one has

$$\mathbb{E}_\xi(u) \geq \frac{1}{2p^+}\|u\|^{p^+} - \mu_0 C_\varepsilon c_3^{\alpha^-}\|u\|^{r^-}, \quad \text{when } \|u\| < 1,\ \xi_0 \leq \xi \leq \mu_0.$$

Hence, there exist $\rho = \rho(\mu_0, \varepsilon) > 0$ and $v = v(\mu_0, \varepsilon) > 0$ such that

$$\mathbb{E}_\xi(u) \geq v, \quad \|u\| = \rho, \quad \text{for all } \xi \in [\xi_0, \mu_0].$$

Furthermore, from Lemma 3.1 (i), we can choose $\varrho = t_0\omega \in \mathbb{H}(\Omega)$ with t_0 large enough such that $\mathbb{E}_{\xi_0}(\varrho) < 0$. Then for any $0 < \xi_0 < \xi$, we have

$$\mathbb{E}_\xi(\varrho) < \mathbb{E}_{\xi_0}(\varrho) < 0, \quad \xi_0 \leq \xi \leq \mu_0.$$

This implies that

$$\mathbb{E}_\xi(\varrho) < 0, \quad \xi_0 \leq \xi \leq \mu_0.$$ □

The boundedness of Cerami sequence.

Lemma 3.3. *Assume that* (g_1)-(g_4) *are satisfied. Then the functional* $\mathbb{E}_\xi$ *satisfies the* (C_c) *condition for any* $c > 0$.

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{H}(\Omega)$ is a (C_c) sequence for $\mathbb{E}_\xi$, that is,

$$\mathbb{E}_\xi(u_n) \to c > 0, \quad (1 + \|u_n\|)\,\mathbb{E}'_\xi(u_n) \to 0, \quad \text{as } n \to +\infty.$$

This indicates that

$$c = \mathbb{E}_\xi(u_n) + o(1), \quad \langle \mathbb{E}'_\xi(u_n), u_n\rangle = o(1), \tag{3.7}$$

where $o(1) \to 0$ as $n \to +\infty$.

Claim. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}(\Omega)$. Let's assume the contrary. By considering a subsequence if needed, we can assume that $\|u_n\| \to +\infty$ as $n \to +\infty$. Let define

$$w_n = \frac{u_n}{\|u_n\|}, \quad n \geq 1.$$

We can assume that

$$\left\{\begin{array}{l} w_n \to w \ \text{ strongly in } \ L^{r(x)}(\Omega), \\ w_n \to w \ \text{ strongly in } L^{p^+}(\Omega), \\ w_n(x) \to w(x) \ \text{ a.e in } \Omega, \\ w_n \rightharpoonup w \ \text{ weakly in } \ \mathbb{H}(\Omega). \end{array}\right. \tag{3.8}$$

Let $\tilde{\Omega} := \{x \in \Omega : w(x) \neq 0\}$. Then, in $\tilde{\Omega}$, one has

$$\lim_{n\to+\infty} w_n(x) = \lim_{n\to+\infty} \frac{u_n(x)}{\|u_n\|} = w(x) \neq 0,$$

this implies that

$$|u_n(x)| \to +\infty, \quad \text{ a.e. in } \ \tilde{\Omega}. \tag{3.9}$$

Furthermore, based on (g_2), we obtain

$$\lim_{n\to+\infty} \frac{G\left(x, u_n(x)\right)}{|u_n(x)|^{p^+}} = +\infty, \quad x \in \tilde{\Omega}.$$

This implies that

$$\lim_{n\to+\infty} \frac{G\left(x, u_n(x)\right)}{|u_n(x)|^{p^+}} |w_n(x)|^{p^+} = +\infty, \quad x \in \tilde{\Omega}. \tag{3.10}$$

Due to the assumptions (g_2), there exists a positive constant K such that

$$\frac{G(x,t)}{|t|^{p^+}} > 1, \tag{3.11}$$

for any $x \in \Omega$ and $t \in \mathbb{R}$ with $|t| \geq K$. Since $G(x,t)$ is continuous on $\overline{\Omega} \times [-K, K]$, there exists a positive constant c_4 such that

$$|G(x,t)| \leq c_4, \tag{3.12}$$

for all $(x,t) \in \overline{\Omega} \times [-K, K]$. Therefore, from (3.11) and (3.12), we can see that there is a constant $c_5 > 0$ such that for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$, we have

$$G(x,t) \geq c_5,$$

which signifies that

$$\frac{G\left(x, u_n(x)\right) - c_5}{\|u_n\|^{p^+}} \geq 0.$$

This implies that

$$\frac{G\left(x, u_n(x)\right)}{|u_n(x)|^{p^+}} |w_n(x)|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}} \geq 0. \tag{3.13}$$

Using (3.7), we derive that

$$
\begin{aligned}
c &= \mathbb{E}_\xi(u_n) + o(1),\\
&= \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}\right) dx - \xi\int_\Omega G(x,u_n)\,dx + o(1),\\
&\geq \frac{2}{p^+}\|u_n\|^{p^-} - \xi\int_\Omega G(x,u_n)\,dx + o(1).
\end{aligned}
$$

Hence, we can see that

$$
\int_\Omega G(x,u_n)\,dx \geq \frac{2}{\xi p^+}\|u_n\|^{p^-} - \frac{c}{\xi} + o(1) \to +\infty, \quad \text{as } n\to+\infty. \tag{3.14}
$$

Similarly, from (3.7), we also have

$$
\begin{aligned}
c &= \mathbb{E}_\xi(u_n) + o(1),\\
&= \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}\right) dx - \xi\int_\Omega G(x,u_n)\,dx + o(1),\\
&\leq \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + 1 + \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)}\right) dx - \xi\int_\Omega G(x,u_n)\,dx + o(1),\\
&\leq \frac{2}{p^-}\|u_n\|^{p^+} + \frac{1}{p^-}|\Omega| - \xi\int_\Omega G(x,u_n)\,dx + o(1).
\end{aligned}
$$

Therefore, it follows from (3.14) that

$$
\|u_n\|^{p^+} \geq \frac{p^-}{2}c - \frac{|\Omega|}{2} + \frac{p^-}{2}\xi\int_\Omega G(x,u_n)\,dx - o(1) > 0, \tag{3.15}
$$

for n large enough.

• Let prove that $\left|\tilde{\Omega}\right| = 0$. Indeed, if $\left|\tilde{\Omega}\right| \neq 0$, then according to (3.10), (3.13), (3.15) and Fatou's Lemma, one has

$$
\begin{aligned}
+\infty &= \int_{\tilde{\Omega}} \liminf_{n\to+\infty} \frac{G(x,u_n(x))}{|u_n(x)|^{p^+}}|w_n(x)|^{p^+}\,dx - \int_{\tilde{\Omega}} \limsup_{n\to+\infty} \frac{c_5}{\|u_n\|^{p^+}}dx\\
&= \int_{\tilde{\Omega}} \liminf_{n\to+\infty} \left[\frac{G(x,u_n(x))}{|u_n(x)|^{p^+}}|w_n(x)|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}}\right] dx\\
&\leq \liminf_{n\to+\infty} \int_{\tilde{\Omega}} \left[\frac{G(x,u_n(x))}{|u_n(x)|^{p^+}}|w_n(x)|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}}\right] dx\\
&\leq \liminf_{n\to+\infty} \int_{\Omega} \left[\frac{G(x,u_n(x))}{|u_n(x)|^{p^+}}|w_n(x)|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}}\right] dx\\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{G(x,u_n(x))}{|u_n(x)|^{p^+}}|w_n(x)|^{p^+}\,dx - \limsup_{n\to+\infty} \int_\Omega \frac{c_5}{\|u_n\|^{p^+}}dx\\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{G(x,u_n(x))}{\|u_n\|^{p^+}}dx
\end{aligned}
$$

$$\leq \liminf_{n\to+\infty} \frac{1}{\frac{p^- c}{2} - \frac{|\Omega|}{2} + \frac{\xi p^-}{2}\int_\Omega G\left(x,u_n\right)dx - o(1)} \int_\Omega G\left(x,u_n(x)\right)dx. \tag{3.16}$$

Hence, combining (3.14) and (3.16), we conclude that

$$+\infty \leq \frac{2}{\xi p^-}.$$

This leads to a contradiction. This demonstrates that $\left|\tilde{\Omega}\right| = 0$ and then $w(x) = 0$ a.e. in Ω. Since $\mathbb{E}_\xi\left(tu_n\right)$ is continuous in $t \in [0,1]$, for each n there exists $t_n \in [0,1]$, $n = 1,2,\dots$, such that

$$\mathbb{E}_\xi\left(t_n u_n\right) := \max_{t\in[0,1]} \mathbb{E}_\xi\left(tu_n\right). \tag{3.17}$$

We can see that, for $t_n > 0$ we have

$$\mathbb{E}_\xi\left(t_n u_n\right) \geq c_\xi > 0 = \mathbb{E}_\xi(0).$$

- If $t_n < 1$, then $\frac{d}{dt}\mathbb{E}_\xi\left(tu_n\right)\Big|_{t=t_n} = 0$, which implies $\left\langle \mathbb{E}'_\xi\left(t_n u_n\right), t_n u_n\right\rangle = 0$.
- If $t_n = 1$, then, according to (3.7), we have $\left\langle \mathbb{E}'_\xi\left(u_n\right), u_n\right\rangle = o(1)$.

Hence,

$$\left\langle \mathbb{E}'_\xi\left(t_n u_n\right), t_n u_n\right\rangle = o(1). \tag{3.18}$$

Let $\{s_k\}_{k\in\mathbb{N}}$ be a positive sequence of real numbers such that $s_k > 1$ for any k and $\lim_{k\to+\infty} s_k = +\infty$. Then,

$$\|s_k w_n\| = \left\| s_k \frac{u_n}{\|u_n\|}\right\| = s_k > 1, \quad \text{for any } k \text{ and } n.$$

Let fix k. Since $w_n \to 0$ in $L^{r(x)}(\Omega)$, and $w_n(x) \to 0$ a.e. $x \in \Omega$ as $n \to +\infty$. Then, from (g_1) and the Lebesgue dominated convergence theorem we can deduce that

$$\lim_{n\to+\infty} \int_\Omega G\left(x, s_k w_n\right)dx = 0. \tag{3.19}$$

From the fact that $\|u_n\| \to +\infty$ as $n \to +\infty$. Therefore, we have

$$\|u_n\| > s_k \quad \text{or} \quad 0 < \frac{s_k}{\|u_n\|} < 1, \quad \text{for } n \text{ large enough.}$$

Thus, by (3.19), we can infer that

$$\begin{aligned}
\mathbb{E}_\xi\left(t_n u_n\right) \geq & \mathbb{E}_\xi\left(\frac{s_k}{\|u_n\|}u_n\right) = \mathbb{E}_\xi\left(s_k w_n\right) \\
& = \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} s_k w_n\right|^{p(x)} + \sqrt{1 + \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} s_k w_n\right|^{2p(x)}}\right)dx \\
& \quad - \xi \int_\Omega G\left(x, s_k w_n\right)dx \\
& \geq \int_\Omega \frac{2}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} s_k w_n\right|^{p(x)} dx - \xi \int_\Omega G\left(x, s_k w_n\right)dx
\end{aligned}$$

$$\geq\frac{2s_k^{p^-}}{p^+}\left\|w_n\right\|^{p^-}-\xi\int_\Omega G\left(x,s_kw_n\right)dx=\frac{2s_k^{p^-}}{p^+}. \tag{3.20}$$

From (3.20), as we let $n,k\to+\infty$, we obtain

$$\lim_{n\to+\infty}\mathbb{E}_\xi\left(t_nu_n\right)=+\infty. \tag{3.21}$$

Due to (g_4) and (3.7), for sufficiently large n, one has

$$\begin{aligned}
\mathbb{E}_\xi\left(t_nu_n\right)=&\mathbb{E}_\xi\left(t_nu_n\right)-\frac{1}{2p^+}\langle\mathbb{E}_\xi'\left(t_nu_n\right),t_nu_n\rangle+o(1)\\
=&\int_\Omega\frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{2p(x)}}\right)dx\\
&-\frac{1}{2p^+}\int_\Omega\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{p(x)}+\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{2p(x)}}}\right)dx\\
&-\xi\int_\Omega G\left(x,t_nu_n\right)dx+\frac{\xi}{2p^+}\int_\Omega g\left(x,t_nu_n\right)t_nu_ndx+o(1)\\
=&\int_\Omega\left[\frac{1}{p(x)}\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{2p(x)}}-\frac{1}{2p^+}\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{2p(x)}}}\right]dx\\
&+\frac{\xi}{2p^+}\int_\Omega\Big[g\left(x,t_nu_n\right)t_nu_n-2p^+G\left(x,t_nu_n\right)\Big]dx\\
&+\int_\Omega\left[\frac{1}{p(x)}-\frac{1}{2p^+}\right]\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}t_nu_n\right|^{p(x)}dx+o(1)\\
&\leq\int_\Omega\left[\frac{1}{p(x)}\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}-\frac{1}{2p^+}\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}\right]dx\\
&+\frac{\xi}{2p^+}\int_\Omega\Big[g\left(x,u_n\right)u_n-2p^+G\left(x,u_n\right)+c_3\Big]dx\\
&+\int_\Omega\left[\frac{1}{p(x)}-\frac{1}{2p^+}\right]\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx+o(1)\\
=&\frac{1}{2p^+}\int_\Omega\left[\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}+\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}\right]dx\\
&+\int_\Omega\frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}\right)dx\\
&-\frac{\xi}{2p^+}\int_\Omega g\left(x,u_n\right)u_ndx-\xi\int_\Omega G\left(x,u_n\right)dx+\frac{\xi}{2p^+}c_3|\Omega|+o(1)
\end{aligned}$$

$$=\mathbb{E}_{\xi}\left(u_{n}\right)+\frac{1}{2p^{+}}\left\langle\mathbb{E}_{\xi}^{\prime}\left(u_{n}\right),u_{n}\right\rangle+\frac{\xi}{2p^{+}}c_{3}|\Omega|+o(1)\rightarrow c+\frac{\xi}{2p^{+}}c_{3}|\Omega|, \tag{3.22}$$

as $n\rightarrow+\infty$. By combining (3.21) and (3.22), we arrive at a contradiction. This establishes that the sequence $\{u_{n}\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}(\Omega)$. Thus, from (3.8) and Hölder inequality, we have

$$\begin{aligned}\Big|\int_{\Omega}g\left(x,u_{n}\right)\left(u_{n}-u\right)dx\Big|\leq\int_{\Omega}\left|g\left(x,u_{n}\right)\right|\left|u_{n}-u\right|dx\leq C\int_{\Omega}\Big|1+\left|u_{n}\right|^{r(x)-1}\Big|\left|u_{n}-u\right|dx\\ \leq 2C\|u_{n}-u\|_{L^{r(x)}}\Big\|1+\left|u_{n}\right|^{r(x)-1}\Big\|_{L^{r'(x)}}\rightarrow 0,\quad\text{as }n\rightarrow+\infty,\end{aligned}$$

which implies that

$$\lim_{n\rightarrow+\infty}\int_{\Omega}g\left(x,u_{n}\right)\left(u_{n}-u\right)dx=0. \tag{3.23}$$

However, according to (3.7), we have

$$\left\langle\mathbb{E}_{\xi}^{\prime}\left(u_{n}\right),u_{n}-u\right\rangle\rightarrow 0,\qquad\text{as }n\rightarrow+\infty. \tag{3.24}$$

By combining (3.23) and (3.24), we obtain

$$\left\langle\mathcal{A}\left(u_{n}\right),u_{n}-u\right\rangle=\xi\int_{\Omega}g\left(x,u_{n}\right)\left(u_{n}-u\right)dx+\left\langle\mathbb{E}_{\xi}^{\prime}\left(u_{n}\right),u_{n}-u\right\rangle\longrightarrow 0,\qquad\text{as }n\rightarrow+\infty.$$

According to Proposition 2.4, we conclude that $u_{n}\rightarrow u$ in $\mathbb{H}(\Omega)$. This establishes that $\mathbb{E}_{\xi}(u)$ satisfies the (C_{c}) condition on $\mathbb{H}(\Omega)$. ☐

Conclusion of the proof of Theorem 3.1**.** Indeed, based on Lemmas 3.2 and 3.3, the functional $\mathbb{E}_{\xi}$ satisfies all conditions of the mountain pass Theorem 1.1. Therefore, the functional $\mathbb{E}_{\xi}$ has a critical value $c\geq v>0$. Thus problem (1.3) has at least one nontrivial weak solution in $\mathbb{H}(\Omega)$.

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SOME APPLICATIONS RELATED TO ADMISSIBLE FUNCTIONS FOR HIGHER-ORDER DERIVATIVES OF MEROMORPHIC MULTIVALENT FUNCTIONS

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Abstract. In the present manuscript, we obtain some differential subordination and superordination results for higher-order derivatives of meromorphic multivalent functions in the punctured unit disk by investigating appropriate families of admissible functions. These results are applied to obtain differential sandwich results.

1. Introduction

We denote by Σ_p the family of all functions f of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and multivalent in the punctured unit disk

$$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

A function $f \in \Sigma_p$ is meromorphic multivalent starlike if $f(z) \neq 0$ and

$$-\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in U^*),$$

and $f \in \Sigma_p$ is meromorphic multivalent convex if $f'(z) \neq 0$ and

$$-\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in U^*).$$

Key words and phrases. Differential subordination, differential superordination, multivalent functions, meromorphic functions, higher-order derivatives, admissible functions.

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Upon differentiating both sides of (1.1) j-times with respect to z, we obtain

$$f^{(j)}(z) = \frac{(-1)^j (p+j-1)!}{(p-1)!} z^{-p-j} + \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} a_n z^{n-j} \quad (p, j \in \mathbb{N};\ p > j).$$

Let $\mathcal{H}(U)$ be the collection of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ be the sub-collection of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

with $\mathcal{H} = \mathcal{H}[1, 1]$.

Let f and g be members of $\mathcal{H}(U)$. The function f is said to be subordinate to g, or (equivalently) g is said to be superordinate to f, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$. In such a case, we write $f \prec g$ or $f(z) \prec g(z)$, $z \in U$. Further, if the function g is univalent in U, then we have the following equivalence (see [5])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 1.1 ([6])**.** Let $F, h \in \mathcal{H}(U)$ and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. If F is analytic in U and satisfies the following (second-order) differential subordination:

$$\phi\left(F(z), zF'(z), z^2F''(z); z\right) \prec h(z), \tag{1.2}$$

then F is called a solution of the differential subordination (1.2). The univalent function q is called a dominant of the solutions of the differential subordination or more simply a dominant if $F(z) \prec q(z)$ for all F satisfying (1.2). A dominant $\check{q}$ that satisfies $\check{q}(z) \prec q(z)$ for all dominants q of (1.2) is said to be the best dominant.

Definition 1.2 ([7])**.** Let $F, h \in \mathcal{H}(U)$ and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. If F and

$$\phi\left(F(z), zF'(z), z^2F''(z); z\right)$$

are univalent in U for $\zeta \in \bar{U}$ and satisfy the following (second-order) differential superordination:

$$h(z) \prec \phi\left(F(z), zF'(z), z^2F''(z); z\right), \tag{1.3}$$

then F is called a solution of the differential superordination (1.3). An analytic function q is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $q(z) \prec F(z)$ for all F satisfying (1.3). A univalent subordinant $\check{q}$ that satisfies $q(z) \prec \check{q}(z)$ for all subordinants q of (1.3) is said to be the best subordinant.

Definition 1.3 ([6])**.** Denote by Q the set consisting of all functions q that are analytic and injective on $\bar{U} \backslash E(q)$, where

$$E(q) = \left\{\xi \in \partial U : \lim_{z \to \xi} q(z) = \infty\right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial U \backslash E(q)$.

Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition 1.4 ([6])**.** Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}$. The family of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; z) \notin \Omega$, whenever

$$r = q(\xi), \quad s = k\xi q'(\xi) \quad \text{and} \quad \operatorname{Re}\left\{\frac{t}{s} + 1\right\} \geq k\operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$z \in U$, $\xi \in \partial U \backslash E(q)$ and $k \geq n$.

We simply write $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

Definition 1.5 ([6])**.** Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The family of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; \xi) \in \Omega$, whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \operatorname{Re}\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\xi \in \partial U$ and $m \geq n \geq 1$.

In particular, we write $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$.

In our investigations we shall need the following lemmas.

Lemma 1.1 ([6])**.** *Let $\psi \in \Psi_n[\Omega, q]$, with $q(0) = a$. If $F \in \mathcal{H}[a, n]$ satisfies*

$$\psi\left(F(z), zF'(z), z^2F''(z); z\right) \in \Omega,$$

then $F(z) \prec q(z)$.

Lemma 1.2 ([6])**.** *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $F \in Q(a)$ and*

$$\psi\left(F(z), zF'(z), z^2F''(z); z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\psi\left(F(z), zF'(z), z^2F''(z); z\right) : z \in U, \zeta \in \bar{U}\right\}$$

implies $q(z) \prec F(z)$.

In recent years, several authors obtained many interesting results in differential subordination and superordination, such as Seoudy [12], Wanas and Srivastava [19], Lupas and Catas [4] and others (see, for example, [1–3, 8–11, 13–18, 20]). In this investigation, we consider certain suitable families of admissible functions and derive some differential subordination and superordination properties for higher-order derivatives of meromorphic multivalent functions.

2. Subordination Results

Definition 2.1. Let Ω be a set in C and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_j[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; z) \notin \Omega$, whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi)}{q(\xi)}, \; q(\xi) \neq 0 \quad \text{and} \quad \text{Re}\left\{\frac{w + v^2}{v}\right\} \geq k\text{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

where $z \in U$, $\xi \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem 2.1. *Let $\phi \in \Phi_j[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

$$\begin{aligned}\text{(2.1)} \qquad &\left\{\phi\left(\frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1,\right.\right.\\ &\left.\left.\frac{z^2 f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) : z \in U\right\} \subset \Omega,\end{aligned}$$

then

$$\frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Proof. Define the function F by

$$F(z) = \frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}. \tag{2.2}$$

Then, the function F is analytic in U. After some calculation, we have

$$\frac{zF'(z)}{F(z)} = \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1. \tag{2.3}$$

Further computations show that

$$\begin{aligned}\text{(2.4)} \quad \frac{z^2 F''(z)}{F(z)} + \frac{zF'(z)}{F(z)} - \left(\frac{zF'(z)}{F(z)}\right)^2 =& z\left[\frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1\right]'\\ =& \frac{z^2 f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2.\end{aligned}$$

Now, we define the transforms from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s}{r}, \quad w = \frac{r(t+s) - s^2}{r^2}.$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi\left(r, \frac{s}{r}, \frac{r(t+s) - s^2}{r^2}; z\right). \tag{2.5}$$

The proof will make use of Lemma 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.5) that

(2.6)
$$\psi\left(F(z), zF'(z), z^2F''(z); z\right) = \phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1,\right.$$
$$\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right).$$

Therefore, (2.1) becomes $\psi\left(F(z), zF'(z), z^2F''(z); z\right) \in \Omega$.

To complete the proof, we next show that the admissibility condition for $\phi \in \Phi_j[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.4.

Note that

$$\frac{t}{s} + 1 = \frac{w + v^2}{v},$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, $F(z) \prec q(z)$ or equivalently

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$ □

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi_j[h(U), q]$ is written as $\Phi_j[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_j[h, q]$. If $f \in \Sigma_p$ satisfies*

(2.7)
$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1,\right.$$
$$\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \prec h(z),$$

then

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

By taking $\phi(u, v, w; z) = u + \frac{v}{\beta u + \gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 2.2, we state the following corollary.

Corollary 2.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U, with $h(0) = 1$, and*

$$\text{Re}\,\{\beta h(z) + \gamma\} > 0.$$

If $f \in \Sigma_p$ satisfies

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} + \frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z)+(p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2+\gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)} \prec h(z),$$

then

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of q on ∂U is not known.

Corollary 2.2. *Let $\Omega \in \mathbb{C}$ and q be univalent in U with $q(0)=1$. Let $\phi \in \Phi_j[h, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \in \Omega,$$

then

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Proof. Theorem 2.1 yields

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_\rho(z).$$

The result is now deduced from the fact that $q_\rho(z) \prec q(z)$. □

Theorem 2.3. *Let h and q be univalent in U with $q(0)=1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(1) *$\phi \in \Phi_j[h, q_\rho]$ for some $\rho \in (0,1)$;*

(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_j[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.7), *then*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Proof. Case (1). By applying Theorem 2.1, we obtain

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_\rho(z).$$

Since $q_\rho(z) \prec q(z)$, we deduce

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Case (2). Let

$$F(z) = \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \quad \text{and} \quad F_\rho(z) = F(\rho z).$$

Then,

$$\phi\left(F_\rho(z), zF'_\rho(z), z^2F''_\rho(z); \rho z\right) = \phi\left(F(\rho z), zF'(\rho z), z^2F''(\rho z); \rho z,\right) \in h_\rho(U).$$

By using Theorem 2.1 and the comment associated with

$$\phi\left(F(z), zF'(z), z^2F''(z); w(z)\right) \in \Omega,$$

where w is any function mapping U into U, with $w(z) = \rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $F(z) \prec q(z)$. Therefore,

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z). \qquad \square$$

The next result gives the best dominant of the differential subordination (2.7).

Theorem 2.4. *Let h be univalent in U and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)}{q(z)}, \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left(\frac{zq'(z)}{q(z)}\right)^2; z\right) = h(z) \tag{2.8}$$

has a solution q, with $q(0) = 1$, and satisfies one of the following conditions:

(1) *$q \in Q_1$ and $\phi \in \Phi_j[h, q]$;*

(2) *q is univalent in U and $\phi \in \Phi_j[h, q_\rho]$ for some $\rho \in (0, 1)$;*

(3) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_j[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.7), *then*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z)$$

and q is the best dominant.

Proof. It follows from Theorems 2.2 and 2.3, that q is a dominant of (2.7). Since q satisfies (2.8), it is also a solution of (2.7), then q will be dominated by all dominants. Thus, q is the best dominant of (2.7). $\square$

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.1, the family of admissible functions $\Phi_j[\Omega, q]$ denoted by $\Phi_j[\Omega, M]$ can be expressed in the following form.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M>0$. The family of admissible functions $\Phi_j[\Omega, M]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ such that

$$\phi\left(1+Me^{i\theta}, \frac{kM}{M+e^{-i\theta}}, \frac{kM+Le^{-i\theta}}{M+e^{-i\theta}}-\left(\frac{kM}{M+e^{-i\theta}}\right)^2; z\right)\notin\Omega, \tag{2.9}$$

whenever $z\in U$, $\theta\in\mathbb{R}$, $\operatorname{Re}\{Le^{-i\theta}\}\geq k(k-1)M$ for all θ and $k\geq 1$.

Corollary 2.3. *Let $\phi\in\Phi_j[\Omega, M]$. If $f\in\Sigma_p$ satisfies*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\right.$$
$$\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)\in\Omega,$$

then

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

When $\Omega=q(U)=\{w:|w-1|<M\}$, the family $\Phi_j[\Omega, M]$ is simply denoted by $\Phi_j[M]$, then Corollary 2.3 takes the following form.

Corollary 2.4. *Let $\phi\in\Phi_j[M]$. If $f\in\Sigma_p$ satisfies*

$$\left|\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\right.\right.$$
$$\left.\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)-1\right|<M.$$

Then,

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

Example 2.1. If $M>0$ and $f\in\Sigma_p$ satisfies

$$\left|\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2-p-j+1\right|<M,$$

then

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

This implication follows from Corollary 2.4 by taking $\phi(u,v,w;z)=w-v+1$.

Example 2.2. If $M>0$ and $f\in\Sigma_p$ satisfies

$$\left|\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-2\right|<\frac{M}{M+1},$$

then

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

This implication follows from Corollary 2.3 by taking $\phi(u,v,w;z)=v$ and $\Omega=h(U)$ where $h(z)=\frac{M}{M+1}z$, $M>0$. To apply Corollary 2.3, we need to show that $\phi\in\Phi_j[\Omega,M]$, that is the admissibility condition (2.9) is satisfied follows from

$$\left|\phi\left(1+Me^{i\theta},\frac{kM}{M+e^{-i\theta}},\frac{kM+Le^{-i\theta}}{M+e^{-i\theta}}-\left(\frac{kM}{M+e^{-i\theta}}\right)^2;z\right)\right|=\frac{kM}{M+1}\geq\frac{M}{M+1},$$

for $z\in U$, $\theta\in\mathbb{R}$ and $k\geq 1$.

3. Superordination Results

In this section, we derive differential superordination. For this purpose the family of admissible functions given in the following definition will be required.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{H}$. The class of admissible functions $\Phi'_j[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ that satisfy the admissibility condition: $\phi(u,v,w;\xi)\in\Omega$, whenever

$$u=q(z),\quad v=\frac{zq'(z)}{mq(z)},\ q(z)\neq 0\quad\text{and}\quad \operatorname{Re}\left\{\frac{w+v^2}{v}\right\}\leq\frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

where $z\in U$, $\xi\in\partial U$ and $m\geq 1$.

Theorem 3.1. *Let $\phi\in\Phi'_j[\Omega,q]$. If $f\in\Sigma_p$,*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}\in Q_1$$

and

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\right.$$
$$\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right)$$

is univalent in U, then

$$\Omega\subset\left\{\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\right.\right.$$
$$\left.\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right):z\in U\right\}\tag{3.1}$$

implies

$$q(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}.$$

Proof. Let F defined by (2.2) and $\psi\left(F(z), zF'(z), z^2F''(z); z\right)$ defined by (2.6). Since $\phi \in \Phi'_j\left[\Omega, q\right]$, from (2.6) and (3.1), we have

$$\Omega \subset \left\{\psi\left(F(z), zF'(z), z^2F''(z); z\right) : z \in U\right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_j\left[\Omega, q\right]$ is equivalent to the admissibility condition for ψ as given in Definition 1.5. Hence, $\psi \in \Psi'\left[\Omega, q\right]$ and by Lemma 1.2, $q(z) \prec F(z)$ or equivalently

$$q(z) \prec \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}. \qquad \square$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi'_j\left[h(U), q\right]$ is written as $\Phi'_j\left[h, q\right]$. The following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let $\phi \in \Phi'_j\left[h, q\right]$, $q \in \mathcal{H}$ and h be analytic in U. If $f \in \Sigma_p$, $\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in Q_1$ and*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)$$

is univalent in U, then

$$h(z) \prec \phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \tag{3.2}$$

implies

$$q(z) \prec \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}.$$

By taking $\phi(u, v, w; z) = u + \frac{v}{\beta u + \gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 3.2, we state the following corollary.

Corollary 3.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U with $h(0) = 1$. Suppose that the differential equation $q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$ has univalent solution q that satisfies $q(0) = 1$ and $q(z) \prec h(z)$. If $f \in \Sigma_p$,*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in \mathcal{H} \cap Q_1$$

and

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}+\frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z)+(p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2+\gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)}$$

is univalent in U, then

$$h(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}+\frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z)+(p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2+\gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)}$$

implies

$$q(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}.$$

The next result gives the best subordinant of the differential superordination (3.2).

Theorem 3.3. *Let h be analytic in U and $\phi:\mathbb{C}^3\times U\to\mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z),\frac{zq'(z)}{q(z)},\frac{z^2q''(z)}{q(z)}+\frac{zq'(z)}{q(z)}-\left(\frac{zq'(z)}{q(z)}\right)^2;z\right)=h(z)$$

has a solution $q\in Q_1$. If $\phi\in\Phi_j'[h,q]$, $f\in\Sigma_p$, $\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}\in Q_1$ and

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right)$$

is univalent in U, then

$$h(z)\prec\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right)$$

implies

$$q(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}$$

and q is the best subordinant.

Proof. The proof is similar to that of Theorem 2.4 and is omitted. ◻

4. Sandwich Results

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich result.

Theorem 4.1. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent in U, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_j[h_2, q_2] \cap \Phi'_j[h_1, q_1]$. If $f \in \Sigma_p$,*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in \mathcal{H} \cap Q_1$$

and

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_2(z).$$

By combining Corollary 2.1 and Corollary 3.1, we obtain the following sandwich result.

Corollary 4.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h_1, h_2 be convex in U with $h_1(0) = h_2(0) = 1$. Suppose that the differential equations $q_1(z) + \frac{zq_1'(z)}{\beta q_1(z)+\gamma} = h_1(z)$, $q_2(z) + \frac{zq_2'(z)}{\beta q_2(z)+\gamma} = h_2(z)$ have a univalent solutions q_1 and q_2, respectively, that satisfies $q_1(0) = q_2(0) = 1$ and $q_1(z) \prec h_1(z)$, $q_2(z) \prec h_2(z)$. If $f \in \Sigma_p$,*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in \mathcal{H} \cap Q_1$$

and

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} + \frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z) + (p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2 + \gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)}$$

is univalent in U*, then*

$$\begin{aligned} h_1(z) \prec & \frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \\ & + \frac{(-1)^{j-1}(p+j-2)! \left[z f^{(j)}(z) + (p+j-1) f^{(j-1)}(z) \right]}{\beta (p-1)! z^{p+j-1} \left(f^{(j-1)}(z) \right)^2 + \gamma (-1)^{j-1}(p+j-2)! f^{(j-1)}(z)} \\ \prec & h_2(z) \end{aligned}$$

implies

$$q_1(z) \prec \frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_2(z).$$

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SOLVING THE FRACTIONAL SCHRÖDINGER EQUATION WITH SINGULAR POTENTIAL BY MEANS OF THE FOURIER TRANSFORM

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Abstract. The focus of this paper is on the study of fractional Schrödinger's equations with δ-like potential and initial data, which have both time-fractional and space-fractional components. We employ the Fourier transform to prove the existence-uniqueness theorems. Additionally, we give the association with the classical solution.

1. Introduction

The main focus of this paper is on the investigation of the fractional Schrödinger equation that involves distributional potentials. Specifically, we consider the Cauchy problem defined as follows:

$$\begin{cases} i\partial_t^{\alpha} u(t,x) + (-\Delta)^s u(t,\ x) + q(x)u(t,x) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ u(0,\ x) = u_0(x). \end{cases} \tag{1.1}$$

Here, α lies in the interval $(0,1)$, ∂_t^{α} represents the time-fractional Caputo derivative, $(-\Delta)^s$ denotes the space-fractional Laplacian, and $q(x)$ denotes the singular potential. The value of s is assumed to be greater than 0.

Colombeau algebra, also known as generalized functions or nonlinear generalized functions, is a mathematical concept developed by French mathematician Jean-Francois Colombeau in the 1980's [12]. The idea behind Colombeau algebra is to create a space of functions that is larger than the space of distributions but still

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contains it as a subset. Initially, Colombeau introduced his algebra as a tool to study the singularities of solutions to nonlinear partial differential equations. Later on, the theory was extended to include global analysis and differential geometry. Colombeau algebra has applications in a variety of fields, including mathematical physics, theoretical mechanics, and fluid mechanics (see [14]). The development of Colombeau algebra was motivated by the need to extend the theory of distributions, introduced by Laurent Schwartz in the 1950's [16], to include functions that are not distributions but still have some properties of distributions. Colombeau's approach was to define generalized functions as equivalence classes of smooth functions that are equal up to a set of measure zero. This allowed him to extend the algebraic and topological properties of the space of smooth functions to the space of generalized functions. Since its inception, Colombeau algebra has been the subject of extensive research, leading to numerous generalizations and applications. One of the main challenges in the development of the theory has been to find a suitable notion of convergence for sequences of generalized functions. This led to the introduction of the concept of the natural topology, which has been extensively studied and used in applications. Overall, Colombeau algebra has become an important tool in the study of singularities and nonlinear partial differential equations. It has also led to new insights in the theory of distributions and has opened up new avenues for research in other areas of mathematics. One of Stojanović's notable contributions is her work on extending the notion of Colombeau algebra of generalized functions to fractional derivatives [17]. In [18] Stojanović studied the fractional Schrödinger equation was first introduced by Laskin in quantum mechanics. Motivated by the previous paper, and also [5, 13, 19] and reference therein, we will studied the existence and uniqness of fractional Schrödinger equation in a suitable spaces.

The paper is structured as follows. Section 2 provides a review of fundamental spaces and their inclusion into Colombeau algebras type. The main result is presented in Section 3. Finally, we conclude the paper by discussing the implications and offering perspectives for future research.

2. Basic Spaces

In this section, we will discuss various concepts related to the Colombeau algebra type and its properties.

2.1. **Generalized Fractional Spaces.** Let $r > 0$, the fractional Sobolev space H^r is defined as:

$$H^r = \Big\{u \in L^2(\mathbb{R}) \mid \|u\|_r = \|u\|_{L^2} + \|(-\Delta)^{\frac{r}{2}}\|_{L^2} < +\infty\Big\}.$$

We denote by $\|\cdot\|_\alpha$ the norm defined by

$$\|u\|_\alpha = \|u(t,\cdot)\|_{L^2} + \|\partial_t^\alpha u(t,\cdot)\|_{L^2} + \|(-\Delta)^{\frac{r}{2}}(t,\cdot)\|_{L^2}.$$

Let Ω be an open subset of $\mathbb{R}^n$. We denote $\mathcal{E}(\Omega)$ the set of all $\mathcal{C}^\infty$ maps from Ω into $\mathbb{R}$. The set of moderate functions is defined as

$$\mathcal{E}_s(\Omega) = \Big\{ R \in \mathcal{E}(\Omega) \mid (\forall D)(\exists N) \|R\|_{H^r} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right) \Big\}.$$

The set of negligible functions is defined by

$$\mathcal{N}_s(\Omega) = \Big\{ R \in \mathcal{E}_s(\Omega) \mid (\forall D)(\forall q \in \mathbb{N}) \|R\|_{H^r} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{q}\right) \Big\}.$$

Then, our space $\mathcal{G}_{H^r}(\Omega) = \mathcal{E}_s(\Omega)/\mathcal{N}_s(\Omega)$ of simplified generalized functions on Ω is the quotient algebra. In the same we define the algebra type Colombeau $\mathcal{G}_{L^\infty}$ as a factor algebra given by the quotient $\mathcal{E}_\infty/\mathcal{N}_\infty$, where

$$\mathcal{E}_\infty = \Big\{ (R_\epsilon) \mid (\forall D)(\exists N) \sup_{x\in\Omega} \|DR_\epsilon(x)\|_{L^\infty} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right) \Big\}$$

and

$$\mathcal{N}_\infty = \Big\{ (R_\epsilon) \mid (\forall D)(\forall q) \sup_{x\in\Omega} \|DR_\epsilon(x)\|_{L^\infty} = \mathcal{O}_{\epsilon\to 0}(\epsilon^{q}) \Big\}.$$

Now, let $D_{L^\infty}(\Omega)$ the set of all $\mathcal{C}^\infty$ functions on Ω, globally bounded on Ω as well as all its derivatives, then to f associate $f_\epsilon = f$. This given the following inclusion $D_{L^\infty} \subset \mathcal{G}_\infty$. Let f be a function in the space $L^\infty(R^d)$, then to f associate $f_\epsilon = f * \rho_\epsilon$ with a chosen $\rho_\epsilon(t) = \epsilon^{-d}\rho\left(\frac{t}{\epsilon}\right)$, where $\rho \in \mathcal{D}(\mathbb{R}^d)$ and $\int \rho = 1$. For any given mollifier ρ this gives an inclusion $L^\infty(\mathbb{R}^d) \subset \mathcal{G}_\infty$. More generally let T be a distribution in D'_{L^∞}, i.e., T is a finite sum of derivatives of functions in $L^\infty(\mathbb{R}^d)$. To T associate $T_\epsilon = T * \rho_\epsilon$ as above. For given ρ as above this gives an inclusion of $D'_{L^\infty} \subset \mathcal{G}_{H^r}$. Similarly, one has an inclusion of $\mathcal{E}'$ space of all distributions with compact support into $\mathcal{G}_{H^r}$.

2.2. **Regularized Laplace-fractional operator.** In this section, we regularize the fractional Laplace operator as described in reference [1], but this time we use a scaling function. The Laplace-Fractional operator is given by

$$(-\triangle)^{\frac{r}{2}} f(x) = \frac{-\Gamma[\frac{r-1}{2}]}{\pi^{\frac{2-r}{2}} 2^{2-r} \Gamma[\frac{2-r}{2}]} \int \frac{\triangle f(\xi)}{|x-\xi|^{r-1}} d\xi.$$

Note that

$$(-\triangle)^{\frac{r}{2}} f(x) = \frac{\eta}{t^{r-1}} * \Delta f(t),$$

where $\eta = \frac{-\Gamma[(r-1)/2)]}{\pi^{(2-r)/2} 2^{2-r} \Gamma[(2-r)/2]}$. Now using the following regularization

$$(-\tilde{\Delta})^{\frac{r}{2}} f(t) = \frac{\eta}{t^{r-1}} * \Delta f(t) * \rho_{h(\epsilon)}(t),$$

where $h : [0,1] \to [0,1]$ is a scaling function, for more information see [11].

Proposition 2.1. *For each* $(u_\epsilon) \in \mathcal{E}_s$, $\left((-\tilde{\Delta})^{\frac{r}{2}} u_\epsilon\right) \in \mathcal{E}_s$.

Proof. Through the beginning of the section, for all derivative D we have

$$D\Big((-\tilde{\Delta})^{\frac{r}{2}}u_\epsilon\Big) = \frac{\eta}{t^{r-1}} * \Delta f(t) * D\rho_{h(\epsilon)}(t),$$

which proves the result. □

Proposition 2.2. *We have the following result*

$$(-\tilde{\Delta})^{\frac{r}{2}} f \approx_{L^2} (-\Delta)^{\frac{r}{2}} f.$$

Proof. Since $\rho_{h(\epsilon)} \to \delta$, by applying The dominated convergence theorem, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left|(-\tilde{\triangle})^{\frac{r}{2}} f_\epsilon(x) - (-\triangle)^{\frac{r}{2}} f_\epsilon(x)\right| &= \eta \left|(-\tilde{\triangle})^{\frac{r}{2}} f_\epsilon(x) - (-\triangle)^{\frac{r}{2}} f_\epsilon(x)\right| \\ &= \eta \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} \triangle f_\epsilon(x) t^{1-r} |\rho_{h(\epsilon)}(t) - \delta(t)| \to 0,\end{aligned}$$

but, f_ϵ is a function with compact support, then

$$\left\|\triangle f_\epsilon(x) t^{1-r}\left(\rho_{h(\epsilon)}(t) - \delta(t)\right)\right\|_{L^2} \to 0.$$ □

We regularize the Caputo fractional derivative in the same way, we put

$$\tilde{D^\alpha} u_\epsilon(t) = D^\alpha u_\epsilon * \rho_{h(\epsilon)}(t).$$

In the same we can prove that $\tilde{D^\alpha} u_\epsilon \approx D^\alpha u_\epsilon$.

In what remains we note D^α and $(-\triangle)^r$ in the place of $\tilde{D^\alpha}$ and $\tilde{\triangle}^\alpha$.

3. Main Results

The objective of this section is to establish the existence, uniqueness, and continuity of the problem (1.1). We begin by formulating our problem for each representative solution u of equation (1.1).

Now let's consider the approximate problem.

$$\begin{cases} i\partial_t^\alpha u_\epsilon(t,x) + (-\Delta)_\epsilon^s u_\epsilon(t,x) + q_\epsilon(x) u_\epsilon(t,x) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ u_\epsilon(0,x) = u_{0\epsilon}(x). \end{cases} \tag{3.1}$$

3.1. **Existence and uniqueness.** We provide the following definition for the concept of a generalized solution.

Definition 3.1. A solution (3.2) to the problem is a generalized function u which belongs to the $\mathcal{G}_{H^r}$ such that for each representant u_ϵ of u satisfy the problem (1.1).

Proposition 3.1 ([14])**.** *A moderate function (u_ϵ), is negligible if and only if the following condition is satisfied:*

$$\|u_\epsilon\|_{L^\infty} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^m\right), \quad \textit{for all } n \in \mathbb{N}.$$

The main results are presented in the following theorem.

Theorem 3.1. *If $u_0 \in \mathcal{G}_{H_r}$ and $q \in G_{L^\infty}$, then for all $T > 0$ the problem* (1.1) *has a unique solution in $\mathcal{G}\left([0,T] \times H_r\right)$.*

Proof. **Existence.** We have

$$i\partial_t^\alpha u_\epsilon(t,\ x) + (-\tilde{\Delta}_\epsilon)^r u_\epsilon(t,\ x) + q_\epsilon(x) u_\epsilon(t,\ x) = 0.$$

After application of the Fourier transformation in this equation we get

$$i\partial_t \hat{u}_\epsilon(t,\xi) + |\xi|^{2r}\hat{u}_\epsilon(t,\xi) = \hat{f}_\epsilon(t,\xi),$$

u_ϵ and f_ϵ with respect to the spatial variable x and $f_\epsilon(t,x) = -p_\epsilon(x)u_\epsilon(t,x)$, where $\widehat{u}_\epsilon, \widehat{f}_\epsilon$, denote the Fourier transforms.

Now using Duhamel's principle, we get the following representation of the solution to the Cauchy problem

$$\begin{aligned}\hat{u}_\epsilon(t,\xi) =& \hat{u}_{0\epsilon} e^{-i|\xi|^{2r}e^{\frac{i\alpha\pi}{2}}t} e^{i|\xi|^{2r}t^{1-\alpha}} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{i|\xi|^{2r}e^{\frac{i\alpha\pi}{2}}(t-s)} e^{i|\xi|^{2r}(t-s)^{1-\alpha}} \hat{f}_\epsilon(s)ds.\end{aligned} \tag{3.2}$$

Further, we can find two numbers a and b such that

$$\|(-\Delta)^r u_\epsilon\|_{L^2} \le a\|u_\epsilon\|_{L^2} \quad \text{and} \quad \|\partial^\alpha u_\epsilon\|_{L^2} \le b\|u_\epsilon\|_{L^2},$$

which implies the estimate

$$\|\hat{u}_\epsilon\|_{L^2} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right),$$

for some $N \in \mathbb{N}$.

Using the Plancherel-Parseval formula, we can write

$$\|u_\epsilon\|_{L^2} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

Then,

$$\|u_\epsilon\| = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

As we know the Fractional Laplace $(-\Delta)^s$ can be written as a convolution of

$$\frac{-\Gamma[(r-1)/2)]}{*^{d-2+2s}\pi^{(2-r)/2}2^{2-r}\Gamma[(2-r)/2]}$$

and $\Delta u(t,\cdot)$, which is permutable with any integer derivative D. Thus,

$$i\partial_t \hat{Du}_\epsilon(t,\xi) + |\xi|^{2r}\hat{Du}_\epsilon(t,\xi) = \hat{Df}_\epsilon(t,\xi).$$

By the same method, we can prove that for each derivative D

$$\|Du_\epsilon\| = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

Then, for some $N \in \mathbb{N}$, that is (u_ϵ) is moderate, it follows that the classe u is a solution of the problem.

Uniqueness. Let u and v be two solutions of the problem (1.1). Put $U = u - v$, it is clear that $u_0 = v_0$.

Let's go to the Fourier transform,

$$\hat{U}_\epsilon = \left(\hat{U}_{0\epsilon}\right) e^{-i|\xi|^{2r} e^{\frac{i\alpha\pi}{2}} e^{i|\xi|^{2r}t^{1-\alpha}}} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{i|\xi|^{2r} e^{\frac{i\alpha\pi}{2}}(t-s)} e^{i|\xi|^{2r}(t-s)^{1-\alpha}} q_\epsilon \hat{U}_\epsilon(s) ds, \tag{3.3}$$

which implies that

$$\|\hat{U}_\epsilon\|_{L^2} \le \|\hat{U}_{0\epsilon}\|_{L^2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|q_\epsilon\|_{L^\infty} \|\hat{U}_\epsilon\|_{L^2} ds.$$

Gronwall's lemma and Plancherel-Parseval ensure that U_ϵ has bound negligible functions. We use [14, (1.2.3) page 11], and we find that $(U_\epsilon) \in \mathcal{N}_s\left([0,T] \times H^r\right)$. □

3.2. **Association.** In this section we will prove the association with the classical solution to the problem (1.1). First, we will define the meaning of association.

Definition 3.2. A function $f \in \mathcal{G}(\mathbb{R})$ is considered to have an "associated distribution", denoted as $f \approx u$, if for every representative $f(\varphi_\epsilon, y)$ of f and $\psi(y) \in \mathcal{D}(\mathbb{R})$, there exists a natural number q such that for any $\varphi(y) \in \mathcal{A}_q(\mathbb{R})$, we have:

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} f(\varphi_\epsilon, y)\psi(y)dy = \langle u, \psi \rangle.$$

Then, we give the following result.

Theorem 3.2. *Let $q \in L^\infty(\mathbb{R}^d)$. Assume that $u_0 \in H^r(\mathbb{R}^d)$ the Cauchy problem*

$$\begin{cases} i\partial^\alpha u_t(t,x) + (-\Delta)^r u(t,x) + q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3.4}$$

has a unique solution $u \in \mathcal{C}^1\left([0,T] : L^2(\mathbb{R}^d)\right) \cap \mathcal{C}\left([0,T] : H^r(\mathbb{R}^d)\right)$.

Proof. To prove the existence and uniqueness of a solution, we can use the theory of linear evolution equations.

We consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}u = -(-\Delta)^r u - q(x)u. \tag{3.5}$$

The fractional Sobolev space $H^r(\mathbb{R}^d)$ is the natural domain of the operator $(-\Delta)^r$. It is a reflexive Banach space, and we can prove that $\mathcal{L}$ is a closed operator from $H^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. By applying theorem of Lumer-Phillips [15], the operator $\mathcal{L}$ generates a strongly continuous semigroup on $L^2(\mathbb{R}^d)$. Moreover, the semigroup satisfies the properties of positivity, contractivity and boundedness. This means that for each $t \ge 0$, there exists a linear operator $S(t)$ such that $\int_0^\infty S(\left(\frac{t^\alpha}{\theta^\alpha}\right) t)u_0 d\theta$ is the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(t,x) = \mathcal{L}u(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{3.6}$$

for the argument see [21]. □

Theorem 3.3. *The classical solution given by Theorem* 3.2 *is associated with the solution of the problem* (3.1).

Proof. Let u be the classical solution to

$$\begin{cases} i\partial^\alpha u_t(t,x) + (-\Delta)^r u(t,x) + q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases}$$

We have $u(t,\cdot) \in H^r(\mathbb{R}^d)$ for all $t \in [0,T]$, and let $[(u_\epsilon)_\epsilon]$ be the solution of (3.2). It satisfies

$$\begin{cases} i\partial^\alpha u_\epsilon(t,x) + (-\Delta)^r u_\epsilon(t,x) + q_\epsilon(x)u_\epsilon(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u_\epsilon(0,x) = u_{0,\epsilon}(x). \end{cases}$$

Let us denote by $U_\epsilon(t,x) := u(t,x) - u_\epsilon(t,x)$. It solves

$$\begin{cases} i\partial^\alpha U_\epsilon(t,x) + (-\Delta)^r U_\epsilon(t,x) + q_\epsilon(x)U_\epsilon(t,x) = p_\epsilon(t,x), & (t,x) \in (0,\ T) \times \mathbb{R}^d, \\ U_\epsilon(0,x) = (u_0 - u_{0,\epsilon})(x), \end{cases}$$

where $p_\epsilon(t,x) = (q_\epsilon(x) - q(x))u(t,x)$.

Using Duhamel's principle and similar arguments as in Theorem 3.2, we get the estimate

$$\|U_\epsilon(t,\cdot)\|_{L^2} \leq \|u_0 - u_{0,\epsilon}\|_{L^2} + \frac{1}{\gamma(\alpha)} \int_0^T T^{\alpha-1} \|g_\epsilon(s,\cdot)\|_{L^2} ds,$$

where $g_\epsilon = p_\epsilon - q_\epsilon u$, which implies that

$$\|U_\epsilon(t,\cdot)\|_{L^2} \leq \|u_0 - u_{0,\epsilon}\|_{L^2} + \frac{T^\alpha}{\Gamma(\alpha+1)} \|p_\epsilon\|_{L^\infty} + \frac{T^\alpha}{\Gamma(\alpha)} \int_0^T \|U_\epsilon(s,\cdot)\|_{L^2} ds.$$

Now, use the Gronwal's lemma, we obtain

$$\|U_\epsilon(t,\cdot)\|_{L^2} \leq \left(\|U_{0,\epsilon}\|_{L^2} + \frac{T^\alpha}{\Gamma(\alpha+1)} \|p_\epsilon\|_{L^\infty} \right) \exp \frac{T^{\alpha+1}}{\Gamma(\alpha)}.$$

When $\epsilon \to 0$, the right hand side of the last inequality tends to 0, since $\|p_\epsilon\|_{L^\infty} \to 0$ and $\|U_{0,\epsilon}\|_{L^2} \to 0$. Hence, $U \approx 0$. □

4. Conclusion

In this paper, we utilize the Fourier transform on an arbitrary representative to establish the existence and uniqueness of a generalized fractional Schrödinger equation. The utilization of Gronwall's lemma and the Plancherel-Parseval formula plays a crucial role in achieving this objective. In the future, we plan to further investigate this type of equation through numerical simulations.

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A NOTE ON DISCRETE CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. We introduce the concept of $D_{w,p}$-classical orthogonal polynomials, where $D_{w,p}$ is the lowering operator given by $D_{w,p} := \frac{\tau_{-w}-\tau_{-p}}{w-p}$, $w, p \in \mathbb{C}$, with $\tau_{-w}f(x) := f(x+w)$. We conclude that these polynomials are the shifted discrete classical orthogonal polynomials.

1. Introduction

An orthogonal sequence of polynomials $\{p_n\}_{n\geq 0}$ is called classical if $\{p_n'\}_{n\geq 0}$ is also orthogonal. This characterization is essentially the Hahn-Sonine characterization (see [11, 19]) of the classical orthogonal polynomials. In [12], Hahn proved similar characterizations for orthogonal sequences of polynomials p_n such that $D_w p_n$ or $H_q p_n$ $(n \geq 1)$ are again orthogonal sequences. Here, D_w is the difference operator and H_q is the q-difference operator given, respectively, by $D_w f(x) = \frac{f(x+w)-f(x)}{w}$, $w \neq 0$ and $H_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}$, $q \neq 1$. Note that differentiation, difference, and q-difference are lowering operators as they reduce the degree of a polynomial by exactly one.

The concept of O-classical orthogonal polynomials, where O is an operator on the space of polynomials, has been studied by many authors in the literature (see [1–14]).

The aim of the present paper is to pick up orthogonal sequences of polynomials under a lowering operator denoted by $D_{w,p}$, where $D_{w,p}f(x) := \frac{f(x+w)-f(x+p)}{w-p}$, generalizing the difference operator $D_w f(x) := \frac{f(x+w)-f(x)}{w}$ (see [1]).

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The structure of this paper is as follows. In Section 2, we present the terminology and basic definitions that will be used later on. In Section 3, we give some properties of the $D_{w,p}$-classical orthogonal polynomials.

2. Preliminaries

Let $\mathcal{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathcal{P}'$ its dual space, whose elements are *forms*. We denote by $\langle u, p\rangle$ the action of $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle,\ n \geq 0$, the moments of u.

Let us introduce some useful operations in $\mathcal{P}'$. For any $u \in \mathcal{P}',\ g \in \mathcal{P},\ a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$, we let $Du = u'$, gu, $h_a u$ and $\tau_b u$, be the forms defined by duality [15]

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle gu, f\rangle := \langle u, gf\rangle, \quad f \in \mathcal{P}$$

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle, \quad \langle \tau_{-b} u, f\rangle := \langle u, \tau_b f\rangle = \langle u, f(x-b)\rangle, \quad f \in \mathcal{P}.$$

A form u is called normalized if it satisfies $(u)_0 = 1$. We assume that the forms used in this paper are normalized.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$ with $n,\ m \geq 0$. Note that u_0 is said to be the canonical functional associated with the MPS $\{P_n\}_{n\geq 0}$.

Let us recall the following result.

Lemma 2.1 ([15,16])**.** *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent:*

(i) $\langle u, P_{m-1}\rangle \neq 0,\ \langle u, P_n\rangle = 0,\ n \geq m;$

(ii) *exist* $\lambda_\nu \in \mathbb{C},\ 0 \leq \nu \leq m-1,\ \lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$

The form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m}, \quad r_n \neq 0, n, m \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then called an *orthogonal* sequence of monic polynomials (MOPS) with respect to u. Note that $u = (u)_0 u_0 = u_0$. When u is regular, let F be a polynomial such that if $Fu = 0$, then $F = 0$ (see [18]).

Proposition 2.1 ([15,16])**.** *Let $\{P_n\}_{n\geq 0}$ be an MPS with* $\deg P_n = n,\ n \geq 0$, *and let $\{u_n\}_{n\geq 0}$ be its dual sequence. The following statements are equivalent.*

(i) $\{P_n\}_{n\geq 0}$ *is orthogonal with respect to* u_0.

(ii) *For all* $n \geq 0$

$$u_n = \langle u_0, P_n^2\rangle^{-1} P_n u_0.$$

(iii) $\{P_n\}_{n\geq 0}$ *satisfies the three-term recurrence relation*

$$(\text{TTRR}) \ : \begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \end{cases}$$

where $\beta_n = \langle u_0, xP_n^2\rangle\langle u_0, P_n^2\rangle^{-1},\ n \geq 0$ *and* $\gamma_{n+1} = \langle u_0, P_{n+1}^2\rangle\langle u_0, P_n^2\rangle^{-1} \neq 0,\ n \geq 0.$

If $\{P_n\}_{n\geq 0}$ is a MOPS with respect to the regular form u_0, then $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) = a^{-n}P_n(ax),\ n \geq 0,\ a \neq 0$, is a MOPS with respect to the regular form $\tilde{u}_0 = h_{a^{-1}}u_0$, and satisfies [16]

$$\begin{cases} \tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x), \quad n \geq 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}\beta_n$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

Recall the operator

$$(D_{w,p}f)(x) := \frac{f(x+w) - f(x+p)}{w-p}, \quad f \in \mathcal{P}, w, p \in \mathbb{C}.$$

The transposition ${}^tD_{w,p}$ of $D_{w,p}$ is $-D_{-w,-p}$, with a slight abuse of notation which is harmless. Thus,

$$\langle D_{-w,-p}u, f\rangle = -\langle u, D_{w,p}f\rangle, \quad u \in \mathcal{P}', f \in \mathcal{P}, w, p \in \mathbb{C}.$$

Note that $D_{w,0}$ reduces to the operator D_w where $(D_wf)(x) = \frac{f(x+w)-f(x)}{w}$ (see [1]).

Lemma 2.2. *The following formulas hold*

$$(D_{w,p}fg)(x) = (\tau_{-p}f)(x)(D_{w,p}g)(x) + (\tau_{-w}g)(x)(D_{w,p}f)(x), \quad f, g \in \mathcal{P}, \tag{2.1}$$

$$(D_{w,p}f(\tau_w g))(x) = (\tau_{-p}f)(x)(D_{w,p}(\tau_w g))(x) + g(x)(D_{w,p}f)(x), \quad f, g \in \mathcal{P}, \tag{2.2}$$

$$(\tau_{-w}fg)(x) = (\tau_{-w}f)(x)(\tau_{-w}g)(x), \quad f, g \in \mathcal{P}, \tag{2.3}$$

$$(\tau_{-w}gu) = (\tau_{-w}g)(\tau_{-w}u), \quad g \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.4}$$

$$D_{-w,-p}(gu) = (\tau_w g)(D_{-w,-p}u) + (D_{-w,-p}g)(\tau_p u), \quad g \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.5}$$

$$\begin{aligned} &(\tau_b \circ D_{w,p})(f) = (D_{w,p} \circ \tau_b)(f), \\ &(\tau_b \circ D_{w,p})(u) = (D_{w,p} \circ \tau_b)(u), \quad f \in \mathcal{P},\ u \in \mathcal{P}',\ b \in \mathbb{C}, \end{aligned} \tag{2.6}$$

$$(h_a \circ D_{w,p})(u) = (aD_{aw,ap} \circ h_a)(u), \quad u \in \mathcal{P}',\ a \in \mathbb{C} \setminus \{0\}, \tag{2.7}$$

$$(h_a \circ D_{w,p})(f) = (a^{-1}D_{a^{-1}w,a^{-1}p} \circ h_a)(f), \quad f \in \mathcal{P},\ a \in \mathbb{C} \setminus \{0\}. \tag{2.8}$$

The relations (2.1)–(2.4) are evident. Further, we have

$$\begin{aligned} \langle D_{-w,-p}(gu), f\rangle = &- \langle u, g(D_{w,p}f)\rangle = -\langle u, D_{w,p}(f(\tau_w g)) - (\tau_{-p}f)D_{w,p}(\tau_w g)\rangle \\ &\text{(from (2.2))} \\ =&\langle(\tau_w g)(D_{-w,-p}u) + \tau_p((D_{w,p}\tau_w g)u), f\rangle, \end{aligned}$$

but

$$\begin{aligned} \tau_p\Big((D_{w,p}\tau_w g)u\Big) =&\Big(\tau_p \circ D_{w,p} \circ \tau_w g\Big)(\tau_p u) \quad \text{(from(2.3))} \\ =&(D_{-w,-p}g)(\tau_p u) \quad \text{(following the definitions.)} \end{aligned}$$

Hence, we have (2.5).

The proofs of (2.6)–(2.8) follow easily from the definitions.

Now, consider $\{P_n\}_{n\geq 0}$ as above in Section 1 and let

$$\widehat{P}_n(x) = \frac{1}{n+1}(D_{w,p}P_{n+1})(x), \quad n \geq 0. \tag{2.9}$$

Denoting by $\{\widehat{u}_n\}_{n\geq 0}$ the dual sequence of $\{\widehat{P}_n\}_{n\geq 0}$, we have the following result.

Lemma 2.3.

$$D_{-w,-p}(\widehat{u}_n) = -(n+1)u_{n+1}, \quad n \geq 0. \tag{2.10}$$

Indeed, from the definition $\langle \widehat{u}_n, \widehat{P}_m\rangle = \delta_{n,m}$, $n, m \geq 0$, we have $-\langle D_{-w,-p}(\widehat{u}_n), P_{m+1}\rangle = (m+1)\delta_{n,m}$, therefore

$$\begin{aligned} \langle D_{-w,-p}(\widehat{u}_n), P_m\rangle =& 0, \quad m \geq n+2, n \geq 0,\\ \langle D_{-w,-p}(\widehat{u}_n), P_{n+1}\rangle =& -(n+1), \quad n \geq 0. \end{aligned}$$

By virtue of Lemma 2.1,

$$D_{-w,-p}(\widehat{u}_n) = \sum_{\mu=0}^{n+1} \lambda_{n,\mu}u_\mu.$$

However, $\langle D_{-w,-p}(\widehat{u}_n), P_\mu\rangle = \lambda_{n,\mu}$, $0 \leq \mu \leq n+1$ and $\lambda_{n,\mu} = 0$, $0 \leq \mu \leq n$, $\lambda_{n,n+1} = -(n+1)$, $n \geq 0$. Hence, we have (2.10).

Let ϕ and ψ be two polynomials with ϕ monic, and $\deg\phi = t$, $\deg\psi = q \geq 1$. We suppose that the pair (ϕ,ψ) is admissible, i.e., when $q = t-1$, writing $\psi(x) = a_q x^q + \cdots$, then a_q is not a positive integer.

Definition 2.1. A form u is called $D_{w,p}$-semi-classical when it is regular and satisfies

$$D_{-w,-p}(\phi u) + \psi u = 0, \tag{2.11}$$

where the pair (ϕ,ψ) is admissible. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called $D_{w,p}$-semi-classical.

Lemma 2.4. *Consider the sequence $\{\tilde{P}_n\}_{n\geq 0}$ obtained by shifting P_n, i.e.,*

$$\tilde{P}_n(x) = a^{-n}P_n(ax+b) = a^{-n}(h_a \circ \tau_{-b}P_n)(x), \quad n \geq 0, a \neq 0.$$

If u_0 satisfies (2.11)*, then $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$ fulfils the equation*

$$D_{-wa^{-1},-pa^{-1}}(\tilde{\phi}\tilde{u}_0) + \tilde{\psi}\tilde{u}_0 = 0, \tag{2.12}$$

where $\tilde{\phi}(x) = a^{-t}\phi(ax+b)$ and $\tilde{\psi}(x) = a^{1-t}\psi(ax+b)$.

We need the following formulas, which are easy to prove.

$$\begin{cases} g(\tau_b u) = \tau_b((\tau_{-b}g)u), \quad g \in \mathcal{P}, u \in \mathcal{P}', b \in \mathbb{C},\\ g(h_a u) = h_a((h_a g)u), \quad g \in \mathcal{P}, u \in \mathcal{P}', a \in \mathbb{C}\backslash\{0\}. \end{cases} \tag{2.13}$$

Let $u_0 = (\tau_b \circ h_a)\tilde{u}_0$ and $v = h_a\tilde{u}_0$. From (2.13) we have

$$\begin{aligned} \psi u_0 =& \psi(\tau_b v) = \tau_b((\tau_{-b}\psi)v)\\ =& \tau_b((\tau_{-b}\psi)(h_a\tilde{u}_0)) = (\tau_b \circ h_a)(h_a \circ \tau_{-b}\psi)\tilde{u}_0 = (\tau_b \circ h_a)(\psi(ax+b)\tilde{u}_0). \end{aligned}$$

Further, by using (2.13) and (2.7) we get

$$\begin{aligned} D_{-w,-p}(\phi u_0) =& D_{-w,-p}(\phi(\tau_b v)) = D_{-w,-p}(\tau_b((\tau_{-b}\phi)v)) \\ =& \tau_b D_{-w,-p}\Big((\tau_{-b}\phi)(h_a\tilde{u}_0)\Big) = \tau_b D_{-w,-p}\Big(h_a\left((h_a\circ\tau_{-b}\phi)\,\tilde{u}_0\right)\Big) \\ =& a^{-1}(\tau_b\circ h_a)D_{-wa^{-1},-pa^{-1}}\Big(\phi(ax+b)\tilde{u}_0\Big). \end{aligned}$$

Equation (2.11) becomes

$$(\tau_b\circ h_a)\Big(D_{-wa^{-1},-pa^{-1}}\Big(\phi(ax+b)\tilde{u}_0\Big) + a\psi(ax+b)\tilde{u}_0\Big) = 0.$$

Hence, we have the desired result.

Regarding general semi-classical sequences, we have the following statement that we give for the sake of completeness [17, 18].

Proposition 2.2. *For any monic polynomial ϕ and any orthogonal sequence $\{P_n\}_{n\geq 0}$, the following statements are equivalent.*

a) *There exists an integer $s \geq 0$ such that*

$$\begin{aligned} \phi(x)\widehat{P}_n(x) =& \sum_{\nu=n-s}^{n+t}\lambda_{n,\nu}P_\nu(x), \quad n\geq s, \\ \lambda_{n,n-s} \neq& 0, \quad n\geq s+1. \end{aligned}$$

b) *There exists a polynomial ψ, $deg\psi = q \geq 1$ such that*

$$D_{-w,-p}(\phi u_0) + \psi u_0 = 0, \tag{2.14}$$

where the pair (ϕ,ψ) is admissible.

Remark 2.1. (a) We also have the following statement: the form u_0 is $D_{w,p}$-semi-classical if and only if the sequence $\{\widehat{P}_n\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to ϕu_0.

(b) When $\{P_n\}_{n\geq 0}$ is orthogonal, it fulfils the standard recurrence relation

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1}\neq 0,\ n\geq 0. \end{cases}$$

Likewise, when $\{\widehat{P}_n\}_{n\geq 0}$ is orthogonal ($s = 0$), it fulfils the recurrence relation

$$\begin{cases} \widehat{P}_0(x) = 1,\ \widehat{P}_1(x) = x - \widehat{\beta}_0, \\ \widehat{P}_{n+2}(x) = (x-\widehat{\beta}_{n+1})\widehat{P}_{n+1}(x) - \widehat{\gamma}_{n+1}\widehat{P}_n(x), \quad \widehat{\gamma}_{n+1}\neq 0,\ n\geq 0. \end{cases}$$

3. The $D_{w,p}$-Classical Orthogonal Polynomials

When $s = 0$, the sequence $\{P_n\}_{n\geq 0}$ is called $D_{w,p}$-classical (discrete classical orthogonal polynomials), moreover, we have the more accurate following statements.

Proposition 3.1. *For any orthogonal sequence $\{P_n\}_{n\geq 0}$, the following statements are equivalent.*

a) *The sequence $\{P_n\}_{n\geq 0}$ is $D_{w,p}$-classical.*

b) *The sequence* $\{\widehat{P}_n\}_{n\geq 0}$ *is orthogonal.*

c) *There are two polynomials,* ϕ *which is monic with degree at most* 2*, and* ψ *with degree* 1*, along with a sequence* $\{\lambda_n\}_{n\geq 0}$*, where each* λ_n *is nonzero for* $n \geq 0$*, such that*

$$\phi(x)(D_{w,p} \circ D_{-w,-p}P_{n+1})(x) - \psi(x)(D_{-w,-p}P_{n+1})(x) + \lambda_n P_{n+1}(x) = 0, \quad n \geq 0.$$

Proof. a) $\Rightarrow$ b). From (2.14) and Lemma 2.2, we have

$$\begin{aligned}
\langle u_0, \phi P_m \widehat{P}_n\rangle =& \frac{1}{n+1}\langle P_m\phi u_0, D_{w,p}P_{n+1}\rangle \\
=& -\frac{1}{n+1}\langle D_{-w,-p}(P_m\phi u_0), P_{n+1}\rangle \\
=& -\frac{1}{n+1}\langle (\tau_w P_m)D_{-w,-p}(\phi u_0) + (D_{-w,-p}P_m)\tau_p(\phi u_0), P_{n+1}\rangle \\
=& \frac{1}{n+1}\langle (\tau_w P_m)\psi u_0 - (D_{-w,-p}P_m)\tau_p(\phi u_0), P_{n+1}\rangle \\
=& \frac{1}{n+1}\left\langle u_0, ((\tau_w P_m)\psi P_{n+1} - \phi\tau_{-p}((D_{-w,-p}P_m))P_{n+1})\right\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle \phi u_0, P_m\widehat{P}_n\rangle =& 0, \quad 0 \leq m \leq n-1, n \geq 1, \\
\langle \phi u_0, (\widehat{P}_n)^2\rangle =& \frac{1}{n+1}\left(\psi'(0) - \frac{1}{2}\phi''(0)n\right)\langle u_0, P_{n+1}^2\rangle \neq 0, \quad n \geq 0,
\end{aligned}$$

since (ϕ, ψ) is admissible.

b) $\Rightarrow$ c). From (2.10) and the assumptions,

$$D_{-w,-p}(\widehat{P}_n\widehat{u}_0) = -\mathcal{X}_n P_{n+1}u_0, \quad n \geq 0, \tag{3.1}$$

with

$$\mathcal{X}_n = (n+1)\frac{\langle \widehat{u}_0, \widehat{P}_n^2\rangle}{\langle u_0, P_{n+1}^2\rangle}, \quad n \geq 0.$$

For $n = 0$ in (3.1), we obtain

$$D_{-w,-p}(\widehat{u}_0) = -\gamma_1^{-1}P_1 u_0. \tag{3.2}$$

In accordance with Lemma 2.2, we have

$$D_{-w,-p}(\widehat{P}_n\widehat{u}_0) = (\tau_w\widehat{P}_n)(D_{-w,-p}\widehat{u}_0) + (D_{-w,-p}\widehat{P}_n)(\tau_p\widehat{u}_0),$$

therefore, on account of (3.2),

$$-\mathcal{X}_0 P_1(\tau_w\widehat{P}_n)u_0 + (D_{-w,-p}\widehat{P}_n)(\tau_p\widehat{u}_0) = -\mathcal{X}_n P_{n+1}u_0, \quad n \geq 0. \tag{3.3}$$

Putting $n = 1$, we get

$$\tau_p(\widehat{u}_0) = \gamma_1^{-1}\kappa\phi u_0, \tag{3.4}$$

where $\kappa\phi(x) = P_1(x)(\tau_w \widehat{P}_1)(x) - 2\widehat{\gamma_1}\gamma_2^{-1}P_2(x)$ (ϕ monic). So, Equations (3.3), (3.4) and the regularity of u_0 imply

$$\phi(x)(D_{-w,-p}\widehat{P}_n)(x) - \psi(x)(\tau_w \widehat{P}_n)(x) + \gamma_1\kappa^{-1}\mathfrak{X}_n P_{n+1}(x) = 0, \quad n \geq 0,$$

with $\psi(x) = \kappa^{-1}P_1(x)$. Comparing the degrees, we obtain

$$\frac{1}{2}\phi''(0)n - \psi'(0) + \gamma_1\kappa^{-1}\mathfrak{X}_n = 0, \quad n \geq 0,$$

which means that the pair (ϕ, ψ) is admissible. Finally, we have the desired second-order difference equation with $\lambda_n = \gamma_1\kappa^{-1}(n+1)\mathfrak{X}_n,\ n \geq 0$. In fact, we also have proved that b) $\Rightarrow$ c).

c) $\Rightarrow$ a). From the given equation, we get

$$\langle u_0, \phi(D_{w,p} \circ D_{-w,-p}P_{n+1}) - \psi(D_{-w,-p}P_{n+1})\rangle = 0, \quad n \geq 0.$$

Hence

$$\langle D_{w,p}\Big(D_{-w,-p}(\phi u_0) + \psi u_0\Big), P_{n+1}\rangle = 0, \quad n \geq 0.$$

Since $\Big\langle D_{w,p}\Big(D_{-w,-p}(\phi u_0) + \psi u_0\Big), 1\Big\rangle = 0$, we get

$$D_{w,p}\Big(D_{-w,-p}(\phi u_0) + \psi u_0\Big) = 0.$$

Hence, (2.14) where the pair (ϕ, ψ) is admissible on account of $\lambda_n \neq 0,\ n \geq 0$. $\square$

Remark 3.1. (a) In the case $s = 0$, when the pair (ϕ, ψ) is not admissible, then the solution u of (2.11) is not regular. In other words, when the solution u of (2.11) is regular, then the pair (ϕ, ψ) is necessarily admissible.

(b) Necessarily, we have

$$\begin{aligned}\kappa\phi(x) =& (1 - 2\widehat{\gamma}_1\gamma_2^{-1})x^2 + (2\widehat{\gamma}_1\gamma_2^{-1}(\beta_0 + \beta_1) - \beta_0 - \widehat{\beta}_0 - w)x \\ &+ \beta_0(\widehat{\beta}_0 + w) - 2\widehat{\gamma}_1\gamma_2^{-1}(\beta_0\beta_1 - \gamma_1), \\ k\psi(x) =& P_1(x).\end{aligned}$$

Proposition 3.2. *If $\{P_n\}_{n\geq 0}$ is $D_{w,p}$-classical, the sequence $\{\widehat{P}_n\}_{n\geq 0}$ is $D_{w-p,0}$-classical and we have*

$$D_{p-w,0}(\phi_1\widehat{u}_0) + \psi_1\widehat{u}_0 = 0, \tag{3.5}$$

with $\phi_1(x) = (\tau_{-w}\phi)(x)$ and $\psi_1(x) = \Big(\kappa^{-1}(\tau_{-p}P_1) - (D_{w,p}\phi)\Big)(x)$.

Proof. When $\{P_n\}_{n\geq 0}$ is $D_{w,p}$-classical, we have (3.2)

$$D_{-w,-p}(\widehat{u}_0) = -\gamma_1^{-1}P_1u_0.$$

Multiplying this equation by ϕ, we get

$$\phi D_{-w,-p}(\widehat{u}_0) = -\gamma_1^{-1}\phi P_1u_0.$$

But by (2.5), $\phi D_{-w,-p}(\widehat{u}_0) = D_{-w,-p}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) - \Big(D_{-w,-p} \circ \tau_{-w}\phi\Big)\tau_p\widehat{u}_0$, so we have

$$D_{-w,-p}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) - \Big(D_{-w,-p} \circ \tau_{-w}\phi\Big)\tau_p\widehat{u}_0 = -\gamma_1^{-1}\phi P_1u_0$$

and

$$\tau_{-p} \circ D_{-w,-p}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) - (D_{w,p}\phi)\widehat{u}_0 = -\gamma_1^{-1}\tau_{-p}(P_1)\tau_{-p}\Big(\phi u_0\Big).$$

By (3.4), we have $\widehat{u}_0 = \gamma_1^{-1}\kappa\tau_{-p}\Big(\phi u_0\Big)$, and so we now get

$$D_{-w+p,0}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) + \Big(\kappa^{-1}(\tau_{-p}P_1) - (D_{w,p}\phi)\Big)\widehat{u}_0 = 0.$$

This completes the proof. □

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THE CURVELET TRANSFORM ON FUNCTION SPACES

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Abstract. In this paper, we delve into the comprehensive exploration of the continuous curvelet transform (CCT), an advanced iteration of the continuous wavelet transform. Renowned for its applications in diverse mathematical realms such as signal analysis, image processing, and seismic exploration, the CCT holds significant promise. Our focus is on an in-depth examination of the CCT's properties within function spaces, i.e., in Sobolev spaces $H^s(\mathbb{R}^2)$, $W^{m,p}(\mathbb{R}^2)$, the weighted Sobolev space $W^{m,p}_\kappa(\mathbb{R}^2)$, the generalized Sobolev space $H^\omega_w(\mathbb{R}^2)$, Besov space $B^{\alpha,q}_p(\mathbb{R}^2)$, weighted Besov space $B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, Hardy space $H^p(\mathbb{R}^2)$ and $BMO(\mathbb{R}^2)$ space. Through investigation, we uncover valuable insights into the continuity and boundedness of the CCT within these function spaces.

1. Introduction

In higher dimensions, wavelets struggle to handle discontinuities along curves due to poor orientation management. To address this limitation, Candés and Donoho [1, 2] introduced the curvelet transform.

Curvelets are efficient tools for managing discontinuities along curves. The curvelet transforms has been used in a variety of applications during the last two decades. Starck et al. have shown applications of the CCT in image de-noising [3], astronomical image representation [4], and color image enhancement [5], while Choi et al. [6] and Nencini et al. have examined image fusion using the CCT [7]. Jero et al. accomplished ECG steganography with the CCT [8]. Dong et al. studied image fusion methods based on the CCT [9], whereas Singh et al. recently studied watermarking techniques utilizing the CCT [10]. However, literature on the theoretical aspects of curvelet

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transforms in spaces like function spaces, Bochner spaces, and quaternion spaces has been lacking.

Roopkumar and collaborators identified this research gap, extending the concept of curvelet transforms to tempered Boehmians [11], Boehmians [12], tempered distributions [13], and studying curvelet transforms on periodic distributions [14]. Additionally, Akila and Roopkumar explored the quaternionic version of curvelet transforms [15]. While the curvelet transform traditionally uses the Fourier transform for intermediate calculations, associating the linear canonical transform with curvelet transforms has been suggested to yield better results [16–18].

Recently, Khan [19] examined the properties of the linear canonical curvelet transform in the quaternionic domain. In 2-D, the localization operator and wavelet multipliers have been studied in the context of linear canonical curvelet transforms by Catana et al. [20], while Starck et al. [21] explored curvelets on the sphere with applications in astronomy. Similar studies for the second-generation curvelet transform have been conducted by Chan et al. [22]. Sharma et al. [23] have shown that a partial differential equation can be solved numerically by curvelet transforms.

Building upon this existing literature, we aim to address the following unanswered problems.

- Can the concept of curvelet transform be extended to Sobolev spaces and weighted Sobolev space?
- Can we extend the concept of curvelet transform to generalized Sobolev spaces?
- Furthermore, can we extend the concept of curvelet transform to Besov space, Hardy space and BMO spaces?

These problems are essential for further study of the topological properties of the functions or signals and the curvelet transforms. The discontinuous signals can be approximated using mollifiers in these function spaces, and then their respective applications can be examined. In this paper, we have addressed these problems and obtained important inequalities.

This paper addresses questions on curvelet transforms through a structured exploration organized into four sections. Section 1 introduces curvelet transforms and reviews relevant literature. Section 2 delves into generalized Sobolev spaces, enriching the theoretical foundation. Furthermore, the curvelet transforms extended to Sobolev and weighted Sobolev spaces, exploring continuity. In Section 3, the continuous extension of the curvelet transform to Besov space and weighted Besov space is discussed. In Section 4 and 5, the continuity of curvelet transform in Hardy and BMO space is discussed. Following that, Section 6 wraps up the work by summarizing major findings and contributions. This systematic approach offers a clear understanding of curvelet transforms in diverse mathematical contexts.

Definition 1.1 (The Fourier Transform)**.** The Fourier transform of a function $f \in L^1(\mathbb{R}^2)$ is defined by

$$\mathscr{F}(f)(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^2} f(\mathbf{t}) e^{-i\langle \mathbf{t},\boldsymbol{\xi}\rangle} d\mathbf{t}.$$

Definition 1.2 (The Continuous curvelet transform)**.** Consider two functions $W : (0,+\infty) \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ supported in $(0.5, 2)$ and $(-1, 1)$, respectively, and satisfying the following conditions:

$$\int_{1/2}^{2} W^2(r)\,\frac{dr}{r} = 1, \tag{1.1}$$

$$\int_{-1}^{1} V^2(t)\,dt = 1. \tag{1.2}$$

The functions W and V are called radial window and angular window respectively. The conditions given in equations (1.1) and (1.2) are admissibility conditions for radial and angular windows. A basic curvelet $\gamma_{a,\mathbf{0},0}$ $\gamma_{a,\mathbf{0},0}$ is defined by:

$$\hat{\gamma}_{a,\mathbf{0},0}(r,\omega) = W(ar)\,V(\omega/\sqrt{a})\,a^{3/4}, \quad 0 < a < a_0.$$

The family of curvelets is defined by

$$\gamma_{a,\mathbf{b},\theta}(\mathbf{x}) = \gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{x} - \mathbf{b})),$$

where a is positive scaling parameter, $\theta \in [0, 2\pi)$ is rotation parameter and $\mathbf{b} \in \mathbb{R}^2$ is translation parameter. For $\mathbf{u} \equiv (u_1, u_2) \in \mathbb{R}^2$, the rotation operator $R_\theta(\mathbf{u}) = (u_1 \cos\theta - u_2 \sin\theta, u_1 \sin\theta + u_2 \cos\theta)$. The continuous curvelet transform (CCT) of a function $f \in L^2(\mathbb{R}^2)$ is defined as follows [2]

$$\begin{aligned}(\Gamma_\gamma f)(a,\mathbf{b},\theta) &= \langle f, \gamma_{a,\mathbf{b},\theta}\rangle \\ &= \int_{\mathbb{R}^2} f(\mathbf{t})\overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{t})}d\mathbf{t}, \quad 0 < a < a_0 < \pi^2,\ \mathbf{b} \in \mathbb{R}^2, \theta \in [0, 2\pi).\end{aligned}$$

Here, the choice of coarsest fixed scale $a_0 \le \pi^2$ is essential for derivation of reconstruction formula (see [12]).

Theorem 1.1. ([1, Theorem 1, p. 167]). *For a function $f \in L^2(\mathbb{R}^2)$, with $\hat{f}(\xi) = 0$ for all $\xi < \frac{2}{a_0}$, the reconstruction formula is given by*

$$f(\boldsymbol{x}) = \int (\Gamma_\gamma f)(a, \boldsymbol{b}, \theta)\,\gamma_{a,\boldsymbol{b},\theta}\,\frac{da}{a^3}\,d\boldsymbol{b}\,d\theta,$$

which is valid for high frequency. Parseval formula for functions having high-frequency is given by

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \int |(\Gamma_\gamma f)(a, \boldsymbol{b}, \theta)|^2\,\frac{da}{a^3}\,d\boldsymbol{b}\,d\theta.$$

Theorem 1.2. *If $f \in L^2(\mathbb{R}^2)$, then the following results hold.*

(a) (*Linearity*) $(\Gamma_\gamma(Af + Bg))(a, \boldsymbol{b}, \theta) = A(\Gamma_\gamma f)(a, \boldsymbol{b}, \theta) + B(\Gamma_\gamma g)(a, \boldsymbol{b}, \theta)$*, where A and B are scalars.*

(b) (*Shifting*) $(\Gamma_\gamma T_{\boldsymbol{c}} f)(a, \boldsymbol{b}, \theta) = (\Gamma_\gamma f)(a, \boldsymbol{b}\text{-}\boldsymbol{c}, \theta)$*, where $T_{\boldsymbol{c}} f(\boldsymbol{t}) = f(\boldsymbol{t}\text{-}\boldsymbol{c})$, for $\boldsymbol{t},\boldsymbol{c} \in \mathbb{R}^2$.*

Example 1.1. For the Dirac delta function, we can find the following:

(a) $(\Gamma_\gamma \delta)(a, \mathbf{b}, \theta) = \overline{\gamma}_{a,\mathbf{b},\theta}(\mathbf{0})$;

(b) $(\Gamma_\gamma T_{\mathbf{c}}\delta)(a,\mathbf{b},\theta) = \overline{\gamma}_{a,\mathbf{b},\theta}(\mathbf{c})$.

Example 1.2. The CCT of $f(\mathbf{t}) = 1$ is $\overline{\hat{\gamma}}_{a,\mathbf{b},\theta}(\mathbf{0})$.

2. The Continuous Curvelet Transform on Sobolev Space

Let us recall the basic definitions which are required for Sobolev space on $\mathbb{R}^2$.

Definition 2.1 ([24])**.** A distribution is a continuous linear functional defined on test function space $\mathscr{D}(\mathbb{R}^2) := \{\phi \in C_K^\infty(\mathbb{R}^2) : \phi(x) \in \mathbb{C}\}$, where $C_K^\infty(\mathbb{R}^2)$ denotes the space of infinitely differentiable functions having compact support K. The collection of such distributions form linear space and is denoted by $\mathscr{D}'(\mathbb{R}^2)$. If for each multi-index α and $\phi_j \in C^\infty(\mathbb{R}^2)$, the $D^\alpha \phi_j \to 0$ uniformly on every compact subset of $\mathbb{R}^2$, then the sequence $\{\phi_j\}_{j\in\mathbb{N}}$ is said to be a convergent sequence on $C^\infty(\mathbb{R}^2)$ with limit 0. The space of such convergent sequences is denoted by $\mathscr{E}(\mathbb{R}^2)$. The collection of compactly supported distributions is denote by $\mathscr{E}'(\mathbb{R}^2)$.

Definition 2.2. Let $\phi \in C^\infty(\mathbb{R}^2)$ be a rapidly decreasing function with

$$\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\phi) = \sup_{\mathbf{x}\in\mathbb{R}^2} |\mathbf{x}^{\boldsymbol{\alpha}} D^{\boldsymbol{\beta}}\phi(\mathbf{x})| < +\infty \,\text{for all multi-indices } \boldsymbol{\alpha} = (\alpha_1,\alpha_2) \text{ and} \boldsymbol{\beta} = (\beta_1,\beta_2).$$

The collection of such functions ϕ is called Schwartz space and is denoted by $S(\mathbb{R}^2)$. The continuous linear functionals on $S(\mathbb{R}^2)$ are tempered distributions and the space of tempered distributions is denoted by $S'(\mathbb{R}^2)$.

Definition 2.3 (The Sobolev space $H^s(\mathbb{R}^2)$)**.** The space containing all such tempered distributions f, i.e., $f \in S'(\mathbb{R}^2)$ having property:

$$\left(1+|\boldsymbol{\eta}|^2\right)^{s/2} \mathscr{F}\{f\}(\boldsymbol{\eta}) \in L^2(\mathbb{R}^2), \quad \text{for all } s \in \mathbb{R},$$

is called Sobolev space and it is denoted by $H^s(\mathbb{R}^2)$. The inner-product on $H^s(\mathbb{R}^2)$ is defined by

$$\langle f,g\rangle_{H^s} = \int_{\mathbb{R}^2} (1+|\boldsymbol{\eta}|^2)^s \mathscr{F}\{f\}(\boldsymbol{\eta}) \overline{\mathscr{F}\{g\}(\boldsymbol{\eta})}\, d\boldsymbol{\xi}.$$

The norm induced by above inner-product is given by

$$||f||_{H^s} = \left(\int_{\mathbb{R}^2} (1+|\boldsymbol{\eta}|^2)^s |\mathscr{F}\{f\}(\boldsymbol{\eta})|^2 \, d\boldsymbol{\eta}\right)^{\frac{1}{2}} < +\infty.$$

Let us recall the definition of generalized Sobolev space defined in [25, 26].

Definition 2.4 (Generalized Sobolev space)**.** The collection of continuous real-valued functions ω on $\mathbb{R}^2$ satisfying the following conditions:

(i) $0 = \omega(\mathbf{0}) \le \omega(\boldsymbol{\xi}+\boldsymbol{\eta}) \le \omega(\boldsymbol{\xi}) + \omega(\boldsymbol{\eta})$;

(ii) $\displaystyle\int_{\mathbb{R}^2} \frac{\omega(\boldsymbol{\xi})\, d\boldsymbol{\xi}}{(1+|\boldsymbol{\xi}|)^3} < +\infty$;

(iii) $a + b\log(1+|\boldsymbol{\xi}|) \le \omega(\boldsymbol{\xi})$, $a \in \mathbb{R}$, $b \in (0,+\infty)$, is denoted by $\mathscr{M}$.

The set $\mathscr{M}_c$ consists of all $\omega \in \mathscr{M}$ such that $\omega(\boldsymbol{\xi}) = \sigma(|\boldsymbol{\xi}|)$, where σ concave on $[0, +\infty)$.

Definition 2.5 ([27])**.** For $\omega \in \mathscr{M}_c$, the Bjorck-space $S_\omega(\mathbb{R}^2)$ is the set of all functions $\phi \in L^1(\mathbb{R}^2)$ such that $\phi, \hat{\phi} \in C^\infty$ and for each multi-indices $\boldsymbol{\alpha}$ and each non-negative number λ

$$p_{\boldsymbol{\alpha},\lambda}(\phi) = \sup_{\mathbf{x}} e^{\lambda\omega(\mathbf{x})} \left|D^{\boldsymbol{\alpha}}\phi(\boldsymbol{x})\right| < +\infty$$

and

$$\pi_{\boldsymbol{\alpha},\lambda}(\phi) = \sup_{\boldsymbol{\xi}} e^{\lambda\omega(\boldsymbol{\xi})} \left|D^{\boldsymbol{\alpha}}\hat{\phi}(\boldsymbol{\xi})\right| < +\infty.$$

The dual of S_ω is denoted by S'_ω, the elements of which are called ultradistributions. We may found its various properties in [25].

Now, we consider a continuous weight function w on $\mathbb{R}^2$ with the following properties. There exist $\lambda > 0$ and $C, D, E > 0$, such that, for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^2$, $t \in \mathbb{R}$, $|t| < 1$ and $\omega \in \mathscr{M}_c$

$$\begin{aligned} w(\boldsymbol{\xi}) &\leq C e^{\lambda\omega(\boldsymbol{\xi})}, \\ w(\boldsymbol{\xi}+\boldsymbol{\eta}) &< D\left(w(\boldsymbol{\xi}) + w(\boldsymbol{\eta})\right), \\ w(t\,\boldsymbol{\xi}) &< E\, w(\boldsymbol{\xi}). \end{aligned} \tag{2.1}$$

Definition 2.6 (Generalized Sobolev space $H^\omega_w(\mathbb{R}^2)$ [28])**.** The generalized Sobolev space $H^\omega_w(\mathbb{R}^2)$ is defined as the set of all ultradistributions $f \in S'_\omega$ such that

$$||f||^2_{H^\omega_w(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\hat{f}(\boldsymbol{\xi})|^2 w(\boldsymbol{\xi}) d\boldsymbol{\xi} < +\infty.$$

Theorem 2.1. *Let $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. Then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : H^\omega_w \to H^\omega_w$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{H^\omega_w(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f||^2_{H^\omega_w(\mathbb{R}^2)}.$$

Proof. Since, $|\mathscr{F}\left(\left(\Gamma_\gamma f\right)(a,\cdot,\theta)\right)(\boldsymbol{\xi})|^2 = \left|\hat{f}(\boldsymbol{\xi})\right|^2 \left|\hat{\gamma}_{a,\mathbf{0},\theta}(\boldsymbol{\xi})\right|^2$. Therefore,

$$\begin{aligned} ||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{H^\omega_w(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \left|\mathscr{F}\left((\Gamma_\gamma f)(a,\cdot,\theta)\right)(\boldsymbol{\xi})\right|^2 w(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^2} \left|\hat{f}(\boldsymbol{\xi})\right|^2 \left|\overline{\hat{\gamma}}_{a,\mathbf{0},\theta}(\boldsymbol{\xi}))\right|^2 w(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ &\leq ||\gamma_{a,\mathbf{0},0}||^2_{L_1(\mathbb{R}^2)} \int_{\mathbb{R}^2} \left|\hat{f}(\boldsymbol{\xi})\right|^2 w(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ &= ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f||^2_{H^\omega_w(\mathbb{R}^2)}. \end{aligned}$$

□

Corollary 2.1. *If $\gamma_{a,\mathbf{0},0}$, $\phi_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$ and $f, g \in H^\omega_w(\mathbb{R}^2)$, then for fixed $a > 0$, the following estimate holds*

$$||(\Gamma_\gamma f)(a,\cdot,\theta) - (\Gamma_\phi g)(a,\cdot,\theta)||^2_{H^\omega_w(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0} - \phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f||^2_{H^\omega_w(\mathbb{R}^2)} + ||\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\, ||f - g||^2_{H^\omega_w(\mathbb{R}^2)}.$$

Since, the spaces $H^\omega_w(\mathbb{R}^2)$ reduce to Sobolev space $H^s(\mathbb{R}^2)$ for weight function $w(\boldsymbol{\xi}) = (1 + |\boldsymbol{\xi}|^2)^s$, $s \in \mathbb{R}$. Therefore, we have the following result for space $H^s(\mathbb{R}^2)$.

Theorem 2.2. *Let $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. If $f \in S'(\mathbb{R}^2)$, then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{H^s(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)} ||f||^2_{H^s(\mathbb{R}^2)}.$$

Corollary 2.2. *If $\phi_{a,\mathbf{0},0}, \gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$, then for the curvelet transforms $(\Gamma_\gamma f)$ and $(\Gamma_\phi g)$ with admissible and curvelets $\phi_{a,\mathbf{0},0}, \gamma_{a,\mathbf{0},0}$ and $f, g \in H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, the following estimate holds*

$$||(\Gamma_\gamma f)(a,\cdot,\theta) - (\Gamma_\phi g)(a,\cdot,\theta)||_{H^s(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0} - \phi_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)}\, ||f||_{H^s(\mathbb{R}^2)} + ||\phi_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)}\, ||f - g||_{H^s(\mathbb{R}^2)}.$$

Definition 2.7 (The Sobolev space $W^{m,p}(\mathbb{R}^2)$ [27])**.** Let $1 \leq p \leq \infty$ and $m \in \mathbb{N} \cup \{0\}$. The Sobolev space $W^{m,p}(\mathbb{R}^2)$ is defined by

$$W^{m,p}(\mathbb{R}^2) = \left\{ f \in \mathscr{D}'(\mathbb{R}^2) : D^{\boldsymbol{\alpha}} f \in L^p(\mathbb{R}^2) \text{ for all } |\boldsymbol{\alpha}| \leq m \right\}$$

and equipped with the norm

$$||f||_{W^{m,p}(\mathbb{R}^2)} = \left(\sum_{|\boldsymbol{\alpha}| \leq m} ||D^{\boldsymbol{\alpha}} f||^p_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < +\infty,$$

and $||f||_{W^{m,\infty}} = \sup_{|\boldsymbol{\alpha}| \leq m} ||D^{\boldsymbol{\alpha}} f||_{L^\infty(\mathbb{R}^2)}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, and α_1, α_2 are non-negative integers, and partial derivatives $D^{\boldsymbol{\alpha}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}$ in distributional sense.

Definition 2.8 (The weighted L^p space [29, 30])**.** Let κ be a weight function, i.e., a non-negative locally integrable function. For $1 \leq p < +\infty$, the weighted $L^p_\kappa(\mathbb{R}^2)$ space is defined as the set of all measurable functions f on $\mathbb{R}^2$ such that

$$\|f\|_{L^p_\kappa(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^p \, \kappa(\mathbf{x}) d^2\mathbf{x} \right)^{1/p} < +\infty.$$

Theorem 2.3 (Weighted Young's Inequality [31])**.** *Suppose κ be a weight function for which there exists another weight function w such that*

$$\kappa(\boldsymbol{x} + \boldsymbol{y}) \leq C\, w(\boldsymbol{x})\, \kappa(\boldsymbol{y}), \quad \textit{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2, \tag{2.2}$$

here C is a constant. Let $f \in L^p_\kappa(\mathbb{R}^2)$, $g \in L^1_{w^{1/p}}(\mathbb{R}^2)$, $1 < p < +\infty$. Then, we have the following inequality

$$\|f * g\|_{L^p_\kappa(\mathbb{R}^2)} \le C\,||f||_{L^p_\kappa(\mathbb{R}^2)}\,||g||_{L^1_{w^{1/p}}(\mathbb{R}^2)}. \tag{2.3}$$

Definition 2.9 (The weighted Sobolev space $W^{m,p}_\kappa(\mathbb{R}^2)$ [30])**.** Let m be a non-negative integer and $1 \le p < +\infty$. The weighted Sobolev space $W^{m,p}_\kappa(\mathbb{R}^2)$ is defined as the set of all $f \in \mathscr{D}'(\mathbb{R}^2)$ with distributional derivatives $D^{\boldsymbol{\alpha}} f \in L^p_\kappa(\mathbb{R}^2)$ for $|\boldsymbol{\alpha}| \le m$. The norm of f in $W^{m,p}_\kappa(\mathbb{R}^2)$ is defined as

$$\|f\|_{W^{m,p}_\kappa(\mathbb{R}^2)} = \left(\sum_{|\boldsymbol{\alpha}|\le m} \|D^{\boldsymbol{\alpha}} f\|^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{1/p}.$$

Theorem 2.4. *Suppose that κ, w are weight functions that satisfy* (2.2)*. If $\gamma_{a,\mathbf{0},0} \in L^1_{w^{1/p}}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : W^{m,p}_\kappa(\mathbb{R}^2) \to W^{m,p}_\kappa(\mathbb{R}^2)$$

is continuous and

$$\begin{aligned} ||(\Gamma_\gamma f)(a,\cdot,\theta)||_{W^{m,p}_\kappa(\mathbb{R}^2)} &= \left(\sum_{|\boldsymbol{\alpha}|\le m} ||D^{\boldsymbol{\alpha}}_{\boldsymbol{b}}(\Gamma_\gamma f)(a,\boldsymbol{b},\theta)||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &\le C\,||\gamma_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||f||_{W^{m,p}_\kappa(\mathbb{R}^2)}. \end{aligned}$$

Proof. Since $f \in W^{m,p}_\kappa(\mathbb{R}^2)$, therefore, for all $|\boldsymbol{\alpha}| \le m$, $D^{\boldsymbol{\alpha}} f \in L^p_\kappa(\mathbb{R}^2)$. Using weighted Young's inequality, we have

$$\begin{aligned} ||D^{\boldsymbol{\alpha}}_{\mathbf{b}}(\Gamma_\gamma f)(a,\mathbf{b},\theta)||^p_{L^p_\kappa(\mathbb{R}^2)} &= ||D^{\boldsymbol{\alpha}}_{\mathbf{b}}(f(\cdot) * \overline{\gamma_{a,\mathbf{0},\theta}(\cdot)})(\mathbf{b})||^p_{L^p_\kappa(\mathbb{R}^2)} \\ &= ||\left(\overline{\gamma}_{a,\mathbf{0},\theta}(\cdot) * D^{\boldsymbol{\alpha}}_{\mathbf{b}}\, f(\cdot)\right)(\mathbf{b})\,||^p_{L^p_\kappa(\mathbb{R}^2)} \\ &\le C^p\,||\gamma_{a,\mathbf{0},0}||^p_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||D^{\boldsymbol{\alpha}}_{\mathbf{b}} f\,||^p_{L^p_\kappa(\mathbb{R}^2)} \end{aligned}$$

and hence,

$$\begin{aligned} ||(\Gamma_\gamma f)(a,\cdot,\theta)||_{W^{m,p}_\kappa(\mathbb{R}^2)} &= \left(\sum_{|\boldsymbol{\alpha}|\le m} ||D^{\boldsymbol{\alpha}}_{\mathbf{b}}(\Gamma_\gamma f)(a,\mathbf{b},\theta)||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &\le \left(\sum_{|\boldsymbol{\alpha}|\le m} C^p\,||\gamma_{a,\mathbf{0},0}||^p_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||D^{\boldsymbol{\alpha}}_{\mathbf{b}} f||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &\le C\,||\gamma_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\left(\sum_{|\boldsymbol{\alpha}|\le m} ||D^{\boldsymbol{\alpha}}_{\mathbf{b}} f||^p_{L^p_\kappa(\mathbb{R}^2)}\right)^{\frac{1}{p}} \\ &= C\,||\gamma_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)}\,||f||_{W^{m,p}_\kappa(\mathbb{R}^2)}. \end{aligned}$$

□

Corollary 2.3. *Suppose that κ, w are weight functions that satisfy* (2.2). *If $\gamma_{a,\mathbf{0},0}, \phi_{a,\mathbf{0},0} \in L^1_{w^{1/p}}(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$ and $f, g \in W^{m,p}_{\kappa}(\mathbb{R}^2)$, then, for fixed $a > 0$, the following estimate holds*

$$\begin{aligned}&||(\Gamma_\gamma f)(a,\cdot,\theta) - (\Gamma_\phi g)(a,\cdot,\theta)||_{W^{m,p}_\kappa(\mathbb{R}^2)}\\ \leq & C\left(||\gamma_{a,\mathbf{0},0} - \phi_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)} ||f||_{W^{m,p}_\kappa(\mathbb{R}^2)} + ||\phi_{a,\mathbf{0},0}||_{L^1_{w^{1/p}}(\mathbb{R}^2)} ||f-g||_{W^{m,p}_\kappa(\mathbb{R}^2)}\right).\end{aligned}$$

The space $W^{m,p}_\kappa(\mathbb{R}^2)$ reduces to Sobolev space $W^{m,p}(\mathbb{R}^2)$ for $\kappa = 1$. Hence, we have the following result for $W^{m,p}(\mathbb{R}^2)$ space.

Theorem 2.5. *Let $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : W^{m,p}(\mathbb{R}^2) \to W^{m,p}(\mathbb{R}^2)$$

is a continuous map and

$$\begin{aligned}||(\Gamma_\gamma f)(a,\cdot,\theta)||_{W^{m,p}(\mathbb{R}^2)} &= \left(\sum_{|\boldsymbol{\alpha}|\leq m} ||D^{\boldsymbol{\alpha}}_{\boldsymbol{b}}(\Gamma_\gamma f)(a,\boldsymbol{b},\theta)||^p_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{p}}\\ &\leq ||\gamma_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)} ||f||_{W^{m,p}(\mathbb{R}^2)}.\end{aligned}$$

3. Curvelet Transform on Besov space

Russian Mathematician, Oleg Vladimirovich Besov has defined a new Banach space $B^{\alpha,q}_p$ with quasi-norm to study the regularity and smoothness of functions in 1961. To commemorate 'O. V. Besov', the function space is known as Besov space. This function space has many applications in study of PDEs, fluid dynamics and quantum mechanics. Some existing works related to wavelet analysis in Besov space can be found in [24, 29, 31–33].

Let us recall the definition of notions related to Besov space. The modulus of smoothness for the function $f \in L^p(\mathbb{R}^2)$ is defined by $w_p(f,\mathbf{h}) = \|f(\cdot + \mathbf{h}) - f(\cdot)\|_{L^p(\mathbb{R}^2)}$, where $0 \neq \mathbf{h} \in \mathbb{R}^2$.

Definition 3.1 (Besov space)**.** For $1 \leq p, q \leq +\infty$ and $\alpha \in (0,1)$, the Besov space $B^{\alpha,q}_p(\mathbb{R}^2)$ is defined as

$$B^{\alpha,q}_p(\mathbb{R}^2) = \left\{f \in L^p(\mathbb{R}^2) : \int_{\mathbb{R}^n} [\omega_p(f,\mathbf{h})]^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} < +\infty\right\},$$

for $1 \leq p < +\infty$ and for $q = +\infty$

$$B^{\alpha,\infty}_p(\mathbb{R}^2) = \left\{f \in L^p(\mathbb{R}^2) : |\mathbf{h}|^{-\alpha}\omega_p(f,\mathbf{h}) \in L^\infty(\mathbb{R}^2)\right\},$$

where $|\mathbf{h}|$ is an Euclidean norm of $\mathbf{h} \in \mathbb{R}^2$. The space $B_p^{\alpha,q}(\mathbb{R}^2)$ is Banach space with norms

$$\|f\|_p^{\alpha,q} = \|f\|_{L^p(\mathbb{R}^2)} + \left(\int_{\mathbb{R}^n} [\omega_p(f,\mathbf{h})]^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} \right)^{\frac{1}{q}}, \quad \text{for } q < +\infty,$$

$$\|f\|_p^{\alpha,\infty} = \|f\|_{L^p(\mathbb{R}^2)} + \||\mathbf{h}|^{-\alpha}\omega_p(f,\mathbf{h})\|_\infty, \quad \text{for } q = +\infty.$$

For $f \in L^p_\kappa(\mathbb{R}^2)$, the modulus of smoothness is defined as:

$$w_{p,\kappa}(f,\mathbf{h}) = \|f(\cdot + \mathbf{h}) - f(\cdot)\|_{L^p_\kappa(\mathbb{R}^2)},$$

where κ is a weight function and $\mathbf{h}$ is non-zero element of $\mathbb{R}^2$.

Definition 3.2 (Weighted Besov space)**.** For $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$, the weighted Besov space $B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, $0 < \alpha < 1$, is defined as

$$B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2) = \left\{ f \in L^p_\kappa(\mathbb{R}^2) : \int_{\mathbb{R}^2} (w_{p,\kappa}(f,\mathbf{h}))^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} < \infty \right\}, \quad \text{for all } 1 \leq q < +\infty,$$

and

$$B^{\alpha,\infty}_{p,\kappa}(\mathbb{R}^2) = \left\{ f \in L^p_\kappa(\mathbb{R}^2) : |\mathbf{h}|^{-\alpha} w_{p,\kappa} \in L^\infty\left(\mathbb{R}^2\right) \right\}, \quad \text{for } q = +\infty.$$

It is easy to see that the space $B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, $1 \leq q < +\infty$, is a Banach space associated with the norm defined by

$$||f||_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)} = ||f||_{L^p_\kappa(\mathbb{R}^2)} + \left(\int_{\mathbb{R}^2} (w_{p,\kappa}(f,\mathbf{h}))^q \frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} \right)^{\frac{1}{q}},$$

and if $q = +\infty$,

$$||f||_{B^{\alpha,\infty}_{p,\kappa}(\mathbb{R}^2)} = ||f||_{L^p_\kappa(\mathbb{R}^2)} + \left\| |\mathbf{h}|^{-\alpha} w_{p,\kappa}(f,\mathbf{h}) \right\|_\infty.$$

Theorem 3.1. *Let* $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. *If* $f \in B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$, *then, for fixed* $a > 0$, *the curvelet transform*

$$\Gamma_\gamma : B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2) \to B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||_{L^1(\mathbb{R}^2)} ||f||_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}.$$

Proof. By change of variable, we have

$$\begin{aligned}(\Gamma_\gamma f)(a,\mathbf{b},\theta) &= \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{t} - \mathbf{b}))} d\mathbf{t} \\ &= \int_{\mathbb{R}^2} f(\mathbf{u} + \mathbf{b}) \overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})} d\mathbf{u}.\end{aligned}$$

Now, smoothness function for curvelet transform is given by:

$$\begin{aligned}
&\omega_{p,\kappa}(\Gamma_\gamma f)(a,\cdot,\theta),\mathbf{h})\\
=&\|(\Gamma_\gamma f)(a,\cdot+\mathbf{h},\theta)-(\Gamma_\gamma f)(a,\cdot,\theta)\|_{L^p_\kappa(\mathbb{R}^2)}\\
=&\left(\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2}(f(\mathbf{u}+\mathbf{b}+\mathbf{h})-f(\mathbf{u}+\mathbf{b}))\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}d\mathbf{u}\right|^p\kappa(\mathbf{x})d\mathbf{b}\right)^{\frac{1}{p}}\\
\le&\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|(f(\mathbf{u}+\mathbf{b}+\mathbf{h})-f(\mathbf{u}+\mathbf{b}))\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\right|^p\kappa(\mathbf{x})d\mathbf{b}\right)^{\frac{1}{p}}d\mathbf{u}\\
=&\int_{\mathbb{R}^2}\left|\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\right|\left(\int_{\mathbb{R}^2}|(f(\mathbf{u}+\mathbf{b}+\mathbf{h})-f(\mathbf{u}+\mathbf{b}))|^p\kappa(\mathbf{x})d\mathbf{b}\right)^{\frac{1}{p}}d\mathbf{u}\\
\le&\|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)}\,\omega_{p,\kappa}(f,\mathbf{h}).
\end{aligned}$$

Hence, for $q<+\infty$, we have

$$\left(\int_{\mathbb{R}^2}[\omega_p(\Gamma_\gamma f)(a,\mathbf{0},\theta),\mathbf{h})]^q\frac{d\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}}\right)^{\frac{1}{q}}\le\|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)}\|f\|_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}.\qquad\square$$

Corollary 3.1. *If $\gamma_{a,\mathbf{0},0}$, $\phi_{a,\mathbf{0},0}\in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$ and $f,g\in B^{\alpha,q}_p(\mathbb{R}^2)$, then for fixed $a>0$, the following estimate holds*

$$\begin{aligned}
||(\Gamma_\gamma f)(a,\cdot,\theta)-(\Gamma_\phi g)(a,\cdot,\theta)||^2_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}\le&\,||\gamma_{a,\mathbf{0},0}-\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f||^2_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}\\
&+||\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f-g||^2_{B^{\alpha,q}_{p,\kappa}(\mathbb{R}^2)}.
\end{aligned}$$

For $\kappa(x)=1$, the weighted Besov space reduces to the Besov space, yielding the following theorem.

Theorem 3.2. *Let $\gamma_{a,\mathbf{0},0}\in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$. If $f\in B^{\alpha,q}_p(\mathbb{R}^2)$, then, for fixed $a>0$, the curvelet transform*

$$\Gamma_\gamma: B^{\alpha,q}_p(\mathbb{R}^2)\to B^{\alpha,q}_p(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}\le\,||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}||f||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}.$$

Corollary 3.2. *If $\gamma_{a,\mathbf{0},0}$, $\phi_{a,\mathbf{0},0}\in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$ and $f,g\in B^{\alpha,q}_p(\mathbb{R}^2)$, then for fixed $a>0$, the following estimate holds*

$$\begin{aligned}
||(\Gamma_\gamma f)(a,\cdot,\theta)-(\Gamma_\phi g)(a,\cdot,\theta)||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}\le&\,||\gamma_{a,\mathbf{0},0}-\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}\\
&+||\phi_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}\,||f-g||^2_{B^{\alpha,q}_p(\mathbb{R}^2)}.
\end{aligned}$$

4. Curvelet Transform on Hardy space

In the early 20th century, Hardy and Littlewood's work on Hardy spaces was primarily focused on understanding the properties of analytic functions and their behavior on the boundary of the domain of analyticity. Their collaboration resulted in the development of the classical Hardy spaces, H^p, which are defined as spaces of analytic functions for which the p-norm of the function is finite on certain domains, such as the unit disk in the complex plane. This space has application in PDEs, Harmonic analysis, function spaces and operator theory (see [30, 34–36]).

Definition 4.1 (Hardy space)**.** Hardy Space $H^p(\mathbb{R}^2)$ is defined as the space of all functions $f \in L^p(\mathbb{R}^2)$ such that

$$\|f\|_{H^p(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \sup_{t>0} |(f * \varphi_t)(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}, \tag{4.1}$$

where $\varphi_t = t^{-n}\varphi\left(\frac{\mathbf{x}}{t}\right)$, $t > 0$, $\mathbf{x} \in \mathbb{R}^2$, and φ be a function in the Schwartz space such that $\int_{\mathbb{R}^2} \varphi(\mathbf{x})d\mathbf{x} \neq 0$.

Theorem 4.1. *Let $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. If $f \in H^p(\mathbb{R}^2)$, then, for fixed $a > 0$, the curvelet transform*

$$\Gamma_\gamma : H^p(\mathbb{R}^2) \to H^p(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a, \cdot, \theta)||^2_{H^p(\mathbb{R}^2)} \leq ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}||f||^2_{H^p(\mathbb{R}^2)}.$$

Proof. Invoking change of variable in the definition of curvelet transform, we have

$$\begin{aligned}((\Gamma_\gamma f)(a, \cdot, \theta) * \varphi_t)(\mathbf{b}) &= \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})} \left(\int_{\mathbb{R}^2} f(\mathbf{u}{+}\mathbf{b}{-}\mathbf{x})\varphi_t(\mathbf{x})d\mathbf{x}\right) d\mathbf{u}\\ &= \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})}(f * \varphi_t)(\mathbf{u} + \mathbf{b})d\mathbf{u}.\end{aligned}$$

Therefore, the application of Minkowski inequality yields

$$\begin{aligned}\|(\Gamma_\gamma f)(a, \cdot, \theta)\|_{H^p} &= \left(\int_{\mathbb{R}^2} \sup_{t>0} |((\Gamma_\gamma f)(a, \cdot, \theta) * \varphi_t)(\mathbf{b})|^p d\mathbf{b}\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}^2} \sup_{t>0} \left|\int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{u}))}(f * \varphi_t)(\mathbf{u} + \mathbf{b})d\mathbf{u}\right|^p d\mathbf{b}\right)^{\frac{1}{p}}\\ &\leq \int_{\mathbb{R}^2} \left(\sup_{t>0} \int_{\mathbb{R}^2} \left|\overline{\gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{u}))}(f * \varphi_t)(\mathbf{u} + \mathbf{b})\right|^p d\mathbf{b}\right)^{\frac{1}{p}} d\mathbf{u}\\ &= \int_{\mathbb{R}^2} |\overline{\gamma_{a,\mathbf{0},0}(R_\theta \mathbf{u})}| \left(\int_{\mathbb{R}^2} \sup_{t>0} |(f * \varphi_t)(\mathbf{u} + \mathbf{b})|^p d\mathbf{b}\right)^{\frac{1}{p}} d\mathbf{u}\\ &\leq \|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)}\|f\|_{H^p(\mathbb{R}^2)}.\end{aligned}$$

□

5. Curvelet Transfrom on BMO Space

The "Bounded Mean Oscillation space" (BMO space) was defined by F. John and L. Nirenberg in 1961 and it is dual space of Hardy space H^1.

Definition 5.1. The space BMO($\mathbb{R}^2$) is defined as the space of all functions $f \in L^1_{loc}(\mathbb{R}^2)$ such that

$$\|f\|_{BMO(\mathbb{R}^2)} = \sup_{B\subset\mathbb{R}^2} \frac{1}{|B|}\int_B |f - f_B|\, d\mathbf{x} < +\infty,$$

where the supremum is taken over all disk B in $\mathbb{R}^2$, and f_B is the mean value of the function f on B defined by $f_B = \frac{1}{|B|}\int_B f(\mathbf{y})d\mathbf{y}$ for each disk $B \subset \mathbb{R}^2$.

Theorem 5.1. *Let* $\gamma_{a,\mathbf{0},0} \in L^1(\mathbb{R}^2) \bigcap L^2(\mathbb{R}^2)$. *If* $f \in BMO(\mathbb{R}^2)$. *Then, for fixed* $a > 0$, *the curvelet transform*

$$\Gamma_\gamma : BMO(\mathbb{R}^2) \to BMO(\mathbb{R}^2)$$

is continuous and

$$||(\Gamma_\gamma f)(a,\cdot,\theta)||^2_{BMO(\mathbb{R}^2)} \le ||\gamma_{a,\mathbf{0},0}||^2_{L^1(\mathbb{R}^2)}||f||^2_{BMO(\mathbb{R}^2)}.$$

Proof. For an arbitrary disk B contained in $\mathbb{R}^2$, we have

$$\begin{aligned}\int_B(\Gamma_\gamma f)(a,\mathbf{b},\theta)d\mathbf{b} &\le \int_{\mathbb{R}^2}\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\left(\int_B f(\mathbf{u}+\mathbf{b})d\mathbf{b}\right)d\mathbf{u}\\ &= \int_{\mathbb{R}^2}\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}\left(\int_Q f(\mathbf{y})d\mathbf{y}\right)d\mathbf{u},\end{aligned}$$

where $Q = \mathbf{u} + B$. Since, $Q \subset \operatorname{supp}\gamma_{a,\mathbf{0},0} + B \subseteq \mathbb{R}^2$ is compact set and $f \in L^1_{loc}(\mathbb{R}^2)$. It follows that

$$\int_B((\Gamma_\gamma f)(a,\cdot,\theta) * \varphi_t)(\mathbf{b})d\mathbf{b} \le K\int_{\mathbb{R}^2}\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})d\mathbf{u} = K\|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)} < \infty,$$

and hence, $(\Gamma_\gamma f)(a,\cdot,\theta) \in L^1_{loc}(\mathbb{R}^2)$. Using Fubini's theorem, we have

$$\Gamma_{f_B}(a,\cdot,\theta) = \int_{\mathbb{R}^2}\left(\frac{1}{B}\int_B f(\mathbf{u}+\mathbf{b})\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}d\mathbf{b}\right)d\mathbf{u} = \int_{\mathbb{R}^2} f_Q\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}d\mathbf{u}.$$

Applying Minkowski's inequality, we obtain

$$\begin{aligned}\|(\Gamma_\gamma f)(a,\cdot,\theta)\|_{BMO(\mathbb{R}^2)} &= \sup_{B\subset\mathbb{R}^2}\frac{1}{B}\int_B|(\Gamma_\gamma f)(a,\mathbf{b},\theta) - \Gamma_{f_B}(a,\mathbf{b},\theta)|d\mathbf{b}\\ &\le \sup_{B\subset\mathbb{R}^2}\frac{1}{B}\int_B\left(\int_{\mathbb{R}^2}|(f(\mathbf{u}+\mathbf{b}) - f_Q)\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}|d\mathbf{u}\right)d\mathbf{b}\\ &= \int_{\mathbb{R}^2}|\overline{\gamma_{a,\mathbf{0},0}(R_\theta\mathbf{u})}|\left(\sup_{Q\subset\mathbb{R}^2}\frac{1}{Q}\int_Q|f(\mathbf{y}) - f_Q|d\mathbf{y}\right)d\mathbf{u}\\ &\le \|\gamma_{a,\mathbf{0},0}\|_{L^1(\mathbb{R}^2)}\|f\|_{BMO(\mathbb{R}^2)}.\end{aligned}$$

□

6. Conclusion

The questions posed in the introduction have been addressed and answered affirmatively in this paper, contributing to the existing literature on curvelet transforms. The research gap identified in the introduction has been successfully bridged through the continuous extension of the curvelet transform to functional spaces, i.e., Sobolev space, weighted Sobolev space, generalized Sobolev space, Besov space, weighted

Besov space, Hardy space and BMO space. The continuity of curvelet transform in these spaces provides the basis for applications like solution of partial differential equations in these spaces.

The theorems presented for the aforementioned spaces provide valuable insights into the behaviour of the curvelet transform across different functional domains. These results offer bounds that enhance our understanding of the curvelet transform's applicability and effectiveness in diverse mathematical spaces.

The successful extension of the curvelet transform to these spaces opens avenues for exploring its applications in a broader range of mathematical and scientific disciplines. Future research may delve deeper into the implications and potential advancements stemming from this extended framework, paving the way for innovative applications and theoretical developments in the field.

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SINGLE-VALUED NEUTROSOPHIC SET WITH HYBRID NUMBER INFORMATION: AN INTRODUCTORY STUDY

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Abstract. In this paper, we introduce the concept of hybrid single-valued neutrosophic number, whose basic units are truth, falsity and indeterminacy memberships, and their properties are investigated. Then, we give the hybrid single-valued neutrosophic whose coefficients are consecutive Fibonacci and Lucas. Especially for consecutive coefficient Fibonacci and Lucas hybrid single-valued neutrosophic numbers, fundamental properties and identities such as Tagiuri, d'Ocagne, Catalan, and Cassini are given. We obtain the Binet formula and generating function formula for these numbers. Moreover, we give some sums of the consecutive coefficient Fibonacci and Lucas hybrid single-valued neutrosophic numbers.

1. Introduction

Number sequences arise in many different theoretical and applied areas, as well as in mathematical modeling of all the problems where there is a kind of invariance to shift in terms of space or of time. As in the computation of spline functions, signal and image processing, queueing theory, time series, analysis, polynomial and power series computations and many other areas, typical problems modelled by number sequences are the numerical solution of certain differential and integral equations. Ide and Renault [16] investigated integer sequences H satisfying the Fibonacci recurrence relation $H_n = H_{n-1} + H_{n-2}$ that also have the property that $H \equiv a^n$, $n \in \mathbb{N}$, for some modulus m. Sacco [25] investigated the complex relationship between fractal structures and cultural evolution using the Fibonacci time series model. Bhattacharya and Kumar [2] took over some of the popular technical analysis methodologies based

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on Fibonacci sequences and also advanced a theoretical rationale as to why security prices may be seen to follow such sequences. The Fibonacci sequence has delighted mathematics and scientists alike for centuries because of its beauty and tendency to appear in unexpected places such as the Pascal triangle, fractal types, graph theory, computer algorithms, geometry, stock market analysis, and graphic design. Music, finance, engineering, art, geostatistics, architecture, signaling, physics, and biology are some of the main fields of study. In Conti's [6] study, some mathematical and historical properties of Fibonacci numbers were shown, focused on their applications in art, music, and geometry. Falcon and Plaza [10] introduced a general Fibonacci sequence by studying the recursive application of two geometrical transformations used in the well-known four-triangle longest-edge partition. Keçilioğlu and Akkuş [19] obtained the Fibonacci and Lucas octonions and gave the generating function and Binet formula for these octonions. Vajda [28] brought together in his book some of the studies on the theory and applications of Fibonacci and Lucas numbers. They are unique in that they are emerging in other areas as well.

Fibonacci and Lucas numbers are defined by the following recurrence relations

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0, \\ L_0 &= 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0, \end{aligned}$$

respectively. Besides, the nth Fibonacci and Lucas numbers are formulized as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \geq 0,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ [20].

Complex numbers, Hyperbolic numbers, Dual numbers and Hybrid numbers arise in many areas such as velocity analysis, coordinate transformation, displacement analysis, matrix modeling, rigid body dynamics, static analysis, mechanics, dynamic analysis, transformation, kinematics, physics, biology, mathematics, and geometry [9,13,15,27].

Horadam [14] introduced the concept, the complex Fibonacci numbers $C = F_n + iF_{n+1}$ where $F_n \in \mathbb{R}$, $i^2 = -1$ and F_n, nth Fibonacci numbers. Fjelstad and Gal [11] defined the hyperbolic numbers $H = a + jb$ where $a, b \in \mathbb{R}$, $j^2 = 1$ and $j \neq \mp 1$. Clifford [5] described the dual numbers $D = a + \varepsilon b$ where $a, b \in \mathbb{R}$, $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. Özdemir [24] presented the hybrid numbers $J = a + ib + \varepsilon c + jd$, where $a, b, cd \in \mathbb{R}$, $i^2 = -1$, $\varepsilon^2 = 0$, $\varepsilon \neq 0$ and $j^2 = 1$, $j \neq \mp 1$. The hybrid numbers are a number system. The hybrid numbers of the form $H = a + ib + \varepsilon c + jd$, where a, b, c, d are real numbers and i, ε, j are basic units with the properties form Table 1.

The journey that started with inferring meaning and mathematical results from uncertainty has continued with intuitive uncertainty, including membership and non-membership states. It has reached the present day by adding indeterminacy to its current journey. Neutrosophic Logic is a nascent field of study in which each proposition is estimated to have a rate of truth in a subset of T, an uncertainty rate

Table 1. The multiplication properties of basic units.

·	i	ε	j
i	-1	1-j	ε+i
ε	j+1	0	-ε
j	-ε-i	ε	1

in a subset of I, and a falsity rate in a subset of F. In [26], M being a space of points, a single-valued neutrosophic set A on a non-empty set M is characterized by a truth-membership function $T_A : M \to [0,1]$, an indeterminacy-membership function $I_A : M \to [0,1]$ and a falsity-membership function $F_A : M \to [0,1]$. Thus, $A = \Big\{ \Big\langle x, T_A(x), I_A(x), F_A(x) \Big\rangle \mid x \in M \Big\}$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ for all $x \in M$. Obviously, every ordinary neutrosophic fuzzy set have the form

$$N = \Big\{ \Big\langle x, T_A(x), I_A(x), F_A(x) \Big\rangle \mid x \in M \Big\},$$
$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 1 \text{ (or } \leq 2 \text{ or } \leq 3).$$

The concept of fuzzy subsets was introduced by Zadeh [30] and later applied in various branches of mathematics. Zadeh's expansion principle allowed us to describe arithmetical operations between fuzzy numbers by expanding classical ones. Kandasamy and Smarandache [18] defined the standard form of the neutrosophic complex number. Dubois and Prade [8] drew attention to their arithmetic properties. Buckley [4] took the first steps from fuzzy real numbers to complex fuzzy numbers. Zhang [12] introduced a new definition for fuzzy complex numbers and obtained some results. Moura et al. [22] proposed to extend the real fuzzy numbers to quaternions fuzzy numbers and to investigate their properties. Yormaz et al. [29] defined the basic operations of fuzzy split quaternions and gave some geometric properties of this fuzzy quaternion.

Fuzzy, intuitionistic, and neutrosophic set to decision-making have been widely applied in many fields in recent years and receive increasing attention, like engineering, linguistics, medical treatment, statistics, multicriteria, experimental sciences, and so on. They [1] investigated the suitability of combining the intuitionistic hesitant fuzzy set and set pair analysis theories in multi-attribute decision making and obtained the hybrid model named intuitionistic hesitant fuzzy connection number set. In the work of Broumi et al. [3] a new concept of complex fermatean neutrosophic graph was established, and various basic graph ideas such as the order, size, degree, and total degree of a vertex were introduced. Derrac [7] evaluated the most relevant approaches, applications, and theoretical studies on fuzzy nearest neighbor classification and stated various defining features to create a complete taxonomy. In[17], Jian et al. focused on the global Mittag-Leffler boundedness for fractional-order fuzzy quaternion-valued neural networks with linear threshold neurons. Liu and Wen [21] gave a new distance measure based on the distance of interval numbers in interval-valued intuitionistic fuzzy sets. Ngan et al. [23] generalized and expanded the utility of complex intuitionistic fuzzy sets using the space of quaternion numbers. The purposed study received composite features and conveyed multi-dimensional fuzzy information via the functions of real membership, imaginary membership, real non-membership, and imaginary non-membership. In [31], Zulqarnain et al. aimed to develop cosine and set theoretic similarity measures for the generalized multipolar neutrosophic soft set. Hybrid single-valued neutrosophic numbers provide a way to extend the neutrosophic set theory based on number fields. The field of hybrid numbers is another fundamental number field that we cannot ignore, so we continue to extend the hybrid single-valued neutrosophic numbers theory based on number fields.

In the following sections, the hybrid single-valued neutrosophic number is defined; the definition of a number whose basic units contain neutrosophic set elements has not been studied so far. Hybrid single-valued neutrosophic numbers, whose coefficients consist of consecutive Fibonacci and Lucas numbers, are defined. In this work, a variety of algebraic properties of the consecutive coefficient Fibonacci and Lucas hybrid single-valued neutrosophic numbers are presented in a unified manner. Some identities will be given for the consecutive coefficient Fibonacci and Lucas hybrid single-valued neutrosophic numbers such as Binet's formula, generating function formula, Tagiuri's identity, d'Ocagne's identity, Catalan's identity, Cassini's identity, and some sum formulas.

2. The Hybrid Neutrosophic Numbers

In this part, analogies with complex numbers, hyperbolic numbers, and dual neutrosophic numbers will be used to construct complex fuzzy numbers, hyperbolic intuitionistic numbers, and dual neutrosophic numbers. Additionally, the definition of hybrid single-valued neutrosophic numbers, which are created when these three numbers are combined, will be provided. Neutrosophic numbers, whose constituents exhibit the characteristics of well-known number sequences, appear to have a variety of uses.

Definition 2.1. The complex fuzzy numbers are defined by

$$\mathbb{C} = a + ib,$$

where a, b are real numbers and $i^2 = T_A(x)$, $T_A(x) \in [0, 1]$.

Definition 2.2. The hyperbolic intuitionistic numbers are defined by

$$\mathbb{H} = a + jb,$$

where a, b are real numbers and $j^2 = F_A(x)$, $j \neq 1$ and $F_A(x) \in [0, 1]$.

Definition 2.3. The dual neutrosophic numbers are defined by

$$\mathbb{D} = a + \varepsilon b,$$

where a, b are real numbers and $\varepsilon^2 = I_A(x)$, $\varepsilon \neq 0$ and $I_A(x) \in [0, 1]$.

TABLE 2. Some properties of complex fuzzy numbers, hyperbolic intuitionistic numbers, and dual neutrosophic numbers.

	Complex fuzzy numbers	Hyperbolic intuitionistic numbers	Dual neutrosophic numbers
Property	$\mathbb{C}$=a+ib, $i^2 = T_A(x)$	$\mathbb{H}$=a+jb, $j^2 = F_A(x)$	$\mathbb{D}$=a+εb, $\varepsilon^2 = I_A(x)$
Conjugate	$\overline{\mathbb{C}}$=a-ib	$\overline{\mathbb{H}}$=a-jb	$\overline{\mathbb{D}}$=a-εb
Norm	$\vert\mathbb{C}\vert=\sqrt{\vert a^2 - T_A(x)b^2\vert}$	$\vert\mathbb{H}\vert=\sqrt{\vert a^2 - F_A(x)b^2\vert}$	$\vert\mathbb{D}\vert=\sqrt{\vert a^2 - I_A(x)b^2\vert}$
Geometry	Euclidean geometry	Lorentzian geometry	Galilean geometry
Rotation type	Elliptic rotation	Hyperbolic rotation	Parabolic rotation

Definition 2.4. The hybrid neutrosophic numbers are defined by

$$\mathfrak{H} = a + ib + \varepsilon c + jd,$$

where a, b, c, d are real numbers and i, ε, j are hybrid number units which satisfy the equalities

$$\begin{aligned} i^2 =& T_A(x), \quad \varepsilon^2 = I_A(x), \quad j^2 = F_A(x), \\ \varepsilon \neq& 0, \quad j \neq 1, \quad ij = -ji = \varepsilon + i, \end{aligned}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x) \in [0,1]$ are the truth-membership, the indeterminacy-membership and the falsity-membership values of the single-valued neutrosophic set and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 1$ (or ≤ 2 or ≤ 3).

The hybrid neutrosophic numbers of the form $\mathfrak{H} = a + ib + \varepsilon c + jd$, where a, b, c, d are real numbers and i, ε, j are basic units.

TABLE 3. The multiplication properties of basic units.

$\cdot$	i	ε	j
i	$T_A(x)$	1-j	ε+i
ε	j+1	$I_A(x)$	-ε
j	-ε-i	ε	$F_A(x)$

Definition 2.5. The consecutive coefficient Fibonacci hybrid single-valued neutrosophic numbers ($\mathfrak{H}^{\mathfrak{F}}$) are defined by

$$\mathfrak{H}^{\mathfrak{F}}{}_n = F_n + iF_{n+s} + \varepsilon F_{n+2s} + jF_{n+3s}, \tag{2.1}$$

where F_n is nth Fibonacci numbers and i, ε, j are hybrid units which satisfy the equalities

$$\begin{aligned} i^2 =& T_A(x), \quad \varepsilon^2 = I_A(x), \quad j^2 = F_A(x), \\ \varepsilon \neq& 0, \quad j \neq 1, \quad ij = -ji = \varepsilon + i, \end{aligned}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x) \in [0,1]$ are the truth-membership, the indeterminacy-membership and the falsity-membership values of the single-valued neutrosophic set and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 1$ (or ≤ 2 or ≤ 3).

Remark 2.1. The consecutive coefficient Fibonacci hybrid single-valued neutrosophic numbers whose terms are in the form a decreasing sequence are called consecutive coefficient Gaussian Fibonacci hybrid single-valued neutrosophic numbers. The consecutive coefficient Gaussian Fibonacci hybrid single-valued neutrosophic numbers ${\mathfrak{G}^{\mathfrak{F}}}_n$ are defined by

$$ {\mathfrak{G}^{\mathfrak{F}}}_n = F_n + iF_{n-s} + \varepsilon F_{n-2s} + jF_{n-3s}. \tag{2.2} $$

Remark 2.2. Since the Lucas sequence is also obtained from the roots of the characteristic equation of the Fibonacci sequence, similar hybrid numbers can be defined in the Lucas sequence by

$$ \begin{aligned} {\mathfrak{H}^{\mathfrak{L}}}_n &= L_n + iL_{n+s} + \varepsilon L_{n+2s} + jL_{n+3s}, \\ {\mathfrak{G}^{\mathfrak{L}}}_n &= L_n + iL_{n-s} + \varepsilon L_{n-2s} + jL_{n-3s}. \end{aligned} $$

Remark 2.3. The recurrence relation between Fibonacci and Lucas numbers also applies to consecutive coefficient Fibonacci and Lucas hybrid single-valued neutrosophic numbers.

$$ \begin{aligned} {\mathfrak{H}^{\mathfrak{F}}}_{n+2} &= {\mathfrak{H}^{\mathfrak{F}}}_{n+1} + {\mathfrak{H}^{\mathfrak{F}}}_n, \\ {\mathfrak{H}^{\mathfrak{L}}}_n &= {\mathfrak{H}^{\mathfrak{F}}}_{n+1} + {\mathfrak{H}^{\mathfrak{F}}}_{n-1}, \\ {\mathfrak{H}^{\mathfrak{L}}}_n &= {\mathfrak{H}^{\mathfrak{F}}}_{n+2} - {\mathfrak{H}^{\mathfrak{F}}}_{n-2}, \\ {\mathfrak{H}^{\mathfrak{L}}}_{n+2} &= {\mathfrak{H}^{\mathfrak{L}}}_{n+1} + {\mathfrak{H}^{\mathfrak{L}}}_n, \\ 5{\mathfrak{H}^{\mathfrak{F}}}_n &= {\mathfrak{H}^{\mathfrak{L}}}_{n+1} + {\mathfrak{H}^{\mathfrak{L}}}_{n-1}, \\ 5{\mathfrak{H}^{\mathfrak{L}}}_n &= {\mathfrak{H}^{\mathfrak{L}}}_{n+2} - {\mathfrak{H}^{\mathfrak{L}}}_{n-2}. \end{aligned} $$

The same equations apply to consecutive coefficient Gaussian Fibonacci and Lucas hybrid single-valued neutrosophic numbers.

Theorem 2.1. *The Binet formula for the consecutive coefficient Fibonacci hybrid single-valued neutrosophic number is*

$$ {\mathfrak{H}^{\mathfrak{F}}}_n = \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta}, \tag{2.3} $$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\alpha^* = 1 + i\alpha^s + \varepsilon\alpha^{2s} + j\alpha^{3s}$ *and* $\beta^* = 1 + i\beta^s + \varepsilon\beta^{2s} + j\beta^{3s}$.

Proof. We have

$$ {\mathfrak{H}^{\mathfrak{F}}}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n+s} - \beta^{n+s}}{\alpha - \beta} + \varepsilon\frac{\alpha^{n+2s} - \beta^{n+2s}}{\alpha - \beta} + j\frac{\alpha^{n+3s} - \beta^{n+3s}}{\alpha - \beta}. $$

Then, some basic calculations are made and the desired result can be easily archieved. □

The Binet formula consecutive coefficient Lucas hybrid single-valued neutrosophic number is verified similarly.

Theorem 2.2. *The generating function formula for the consecutive coefficient Fibonacci hybrid single-valued neutrosophic number is*

$$\sum_{i=0}^{+\infty} \mathfrak{H}^{\mathfrak{F}}{}_i t^i = \frac{\mathfrak{H}^{\mathfrak{F}}{}_0 + (\mathfrak{H}^{\mathfrak{F}}{}_1 - \mathfrak{H}^{\mathfrak{F}}{}_0)t}{1-t-t^2}. \tag{2.4}$$

Proof. Let $h(t)$ function be the generating function formula of the consecutive coefficient Fibonacci hybrid single-valued neutrosophic number. We have

$$h(t) = \sum_{i=0}^{+\infty} \mathfrak{H}^{\mathfrak{F}}{}_i t^i = \mathfrak{H}^{\mathfrak{F}}{}_0 + \mathfrak{H}^{\mathfrak{F}}{}_1 t + \mathfrak{H}^{\mathfrak{F}}{}_2 t^2 + \cdots + \mathfrak{H}^{\mathfrak{F}}{}_n t^n + \cdots .$$

Then, $th(t) = \sum_{i=0}^{+\infty} \mathfrak{H}^{\mathfrak{F}}{}_i t^{i+1}$ and $t^2 h(t) = \sum_{i=0}^{+\infty} \mathfrak{H}^{\mathfrak{F}}{}_i t^{i+2}$. After the necessary calculations, the statement of the theorem follows

$$\sum_{i=0}^{+\infty} \mathfrak{H}^{\mathfrak{F}}{}_i t^i = \frac{\mathfrak{H}^{\mathfrak{F}}{}_0 + (\mathfrak{H}^{\mathfrak{F}}{}_1 - \mathfrak{H}^{\mathfrak{F}}{}_0)t}{1-t-t^2}. \qquad \square$$

The generating function formula consecutive coefficient Lucas hybrid single-valued neutrosophic number is verified similarly.

Remark 2.4. Some special sequences well-known for the Fibonacci sequence have also been calculated for the consecutive coefficient Fibonacci hybrid single-valued neutrosophic numbers. The proofs of these equations are omitted. $\mathfrak{H}^{\mathfrak{F}}{}_n$ be the nth consecutive coefficient Fibonacci hybrid single-valued neutrosophic number such that $n \geq 1$ integer. Then, the following equalities hold.

(a) Tagiuri's Identity:

$$\begin{aligned}
\mathfrak{H}^{\mathfrak{F}}{}_{m+r}\mathfrak{H}^{\mathfrak{F}}{}_{n-r} - \mathfrak{H}^{\mathfrak{F}}{}_m\mathfrak{H}^{\mathfrak{F}}{}_n =& (-1)^m F_r F_{n-m-r} + (-1)^{m+s} F_r F_{n-m-r} T_A(x) \\
&+ (-1)^{m+2s} F_r F_{n-m-r} I_A(x) + (-1)^{m+3s} F_r F_{n-m-r} F_A(x) \\
&+ i(-1)^m F_r \Big[F_{n-m+s-r} + (-1)^s F_{n-m-s-r}\Big] \\
&+ \varepsilon(-1)^m F_r \Big[F_{n-m+2s-r} + (-1)^{2s} F_{n-m-2s-r}\Big] \\
&+ j(-1)^m F_r \Big[F_{n-m+3s-r} + (-1)^{3s} F_{n-m-3s-r}\Big] \\
&+ i\varepsilon\Big[(-1)^{m+s} F_r F_{n-m+s-r}\Big] \\
&+ ij(-1)^{m+s} F_r \Big[F_{n-m+2s-r} - F_{n-m-2s-r}\Big] \\
&+ \varepsilon i\Big[(-1)^{m+2s} F_r F_{n-m-s-r}\Big] + \varepsilon j\Big[(-1)^{m+2s} F_r F_{n-m+s-r}\Big] \\
&+ j\varepsilon\Big[(-1)^{m+3s} F_r F_{n-m-s-r}\Big].
\end{aligned}$$

(b) d'Ocagne's Identity:

$$\mathfrak{H}^{\mathfrak{F}}{}_{m-1}\mathfrak{H}^{\mathfrak{F}}{}_{n+1} - \mathfrak{H}^{\mathfrak{F}}{}_m\mathfrak{H}^{\mathfrak{F}}{}_n = (-1)^m F_{n-m+1} + (-1)^{m+s} F_{n-m+1} T_A(x)$$

$$
\begin{aligned}
&+(-1)^{m+2s}F_{n-m+1}I_A(x)+(-1)^{m+3s}F_{n-m+1}F_A(x)\\
&+i(-1)^m\Big[F_{n-m+s+1}+(-1)^sF_{n-m-s+1}\Big]\\
&+\varepsilon(-1)^m\Big[F_{n-m+2s+1}+(-1)^{2s}F_{n-m-2s+1}\Big]\\
&+j(-1)^m\Big[F_{n-m+3s+1}+(-1)^{3s}F_{n-m-3s-r}\Big]\\
&+i\varepsilon\Big[(-1)^{m+s}F_{n-m+s+1}\Big]\\
&+ij(-1)^{m+s}\Big[F_{n-m+2s+1}-F_{n-m-2s+1}\Big]\\
&+\varepsilon i\Big[(-1)^{m+2s}F_{n-m-s+1}\Big]+\varepsilon j\Big[(-1)^{m+2s}F_{n-m+s+1}\Big]\\
&+j\varepsilon\Big[(-1)^{m+3s}F_{n-m-s+1}\Big].
\end{aligned}
$$

(c) Catalan's Identity:

$$
\begin{aligned}
\mathfrak{H}^{\mathfrak{F}}{}_{n+r}\mathfrak{H}^{\mathfrak{F}}{}_{n-r}-\mathfrak{H}^{\mathfrak{F}}{}_{n}\mathfrak{H}^{\mathfrak{F}}{}_{n}=&(-1)^nF_rF_{-r}+(-1)^{n+s}F_rF_{-r}T_A(x)\\
&+(-1)^{n+2s}F_rF_{-r}I_A(x)+(-1)^{n+3s}F_rF_{-r}F_A(x)\\
&+i(-1)^nF_r\Big[F_{s-r}+(-1)^sF_{-s-r}\Big]\\
&+\varepsilon(-1)^nF_r\Big[F_{2s-r}+(-1)^{2s}F_{-2s-r}\Big]\\
&+j(-1)^nF_r\Big[F_{3s-r}+(-1)^{3s}F_{-3s-r}\Big]+i\varepsilon\Big[(-1)^{n+s}F_rF_{s-r}\Big]\\
&+ij(-1)^{n+s}F_r\Big[F_{2s-r}-F_{-2s-r}\Big]+\varepsilon i\Big[(-1)^{n+2s}F_rF_{-s-r}\Big]\\
&+\varepsilon j\Big[(-1)^{n+2s}F_rF_{s-r}\Big]+j\varepsilon\Big[(-1)^{n+3s}F_rF_{-s-r}\Big].
\end{aligned}
$$

(d) Cassini's Identity:

$$
\begin{aligned}
\mathfrak{H}^{\mathfrak{F}}{}_{n+1}\mathfrak{H}^{\mathfrak{F}}{}_{n-1}-\mathfrak{H}^{\mathfrak{F}}{}_{n}\mathfrak{H}^{\mathfrak{F}}{}_{n}=&(-1)^n+(-1)^{n+s}T_A(x)\\
&+(-1)^{n+2s}I_A(x)+(-1)^{n+3s}F_A(x)\\
&+i(-1)^n\Big[F_{s-1}+(-1)^sF_{-s-1}\Big]\\
&+\varepsilon(-1)^n\Big[F_{2s-1}+(-1)^{2s}F_{-2s-1}\Big]\\
&+j(-1)^n\Big[F_{3s-1}+(-1)^{3s}F_{-3s-1}\Big]+i\varepsilon\Big[(-1)^{n+s}F_{s-1}\Big]\\
&+ij(-1)^{n+s}\Big[F_{2s-1}-F_{-2s-1}\Big]+\varepsilon i\Big[(-1)^{n+2s}F_{-s-1}\Big]\\
&+\varepsilon j\Big[(-1)^{n+2s}F_{s-1}\Big]+j\varepsilon\Big[(-1)^{n+3s}F_{-s-1}\Big].
\end{aligned}
$$

Remark 2.5. The identities Tagiuri, d'Ocagne, Catalan and Cassini can be calculated by the same method for the consecutive coefficient Lucas hybrid single-valued neutrosophic numbers.

3. Sums of the Consecutive Coefficient Fibonacci Hybrid Single-Valued Neutrosophic Numbers

In this section, we present some results concercing sums of terms of the consecutive coefficient Fibonacci hybrid single-valued neutrosophic number. Some known equations about Fibonacci numbers will be given in the following lemma.

Lemma 3.1. *Let F_n, $n \geq 0$, be the Fibonacci number. We have*

$$\begin{aligned}
\sum_{m=0}^{n} F_{m+s} =& F_{n+s+2} - F_{s+1}, \\
\sum_{m=0}^{n} F_{m+2s} =& F_{n+2s+2} - F_{2s+1}, \\
\sum_{m=0}^{n} F_{m+3s} =& F_{n+3s+2} - F_{3s+1}, \\
\sum_{m=0}^{n} F_{2m+s} =& F_{2n+s+1} - F_{s-1}, \\
\sum_{m=0}^{n} F_{2m+2s} =& F_{2n+2s+1} - F_{2s-1}, \\
\sum_{m=0}^{n} F_{2m+3s} =& F_{2n+3s+1} - F_{3s-1}, \\
\sum_{m=0}^{n} F_{2m+s+1} =& F_{2n+s+2} - F_{s}, \\
\sum_{m=0}^{n} F_{2m+2s+1} =& F_{2n+2s+2} - F_{2s}, \\
\sum_{m=0}^{n} F_{2m+3s+1} =& F_{2n+3s+2} - F_{3s}.
\end{aligned}$$

Proof. We give

$$\begin{aligned}
F_s &= F_{s-1} + F_{s-2}, \\
F_{s+1} &= F_s + F_{s-1}, \\
F_{s+2} &= F_{s+1} + F_s, \\
\cdots &= \cdots \\
F_{s+n} &= F_{s+n-1} + F_{s+n-2}.
\end{aligned}$$

Taking the sum of the equalities above, we obtained

$$\sum_{m=0}^{n} F_{m+s} = F_{n+s+2} - F_{s+1}.$$

Two other statements of the theorem are verified similarly. □

Some known equations for Lucas numbers can also be shown in a similar way.

Theorem 3.1. *Let $\mathfrak{H}^{\mathfrak{F}}{}_n$, $n \geq 0$, be the consecutive coefficient Fibonacci hybrid single-valued neutrosophic number. Then,*

$$\begin{aligned}\sum_{m=0}^{n} \mathfrak{H}^{\mathfrak{F}}{}_m =&(F_{n+2}-1)+i(F_{n+s+2}-F_{s+1})+\varepsilon(F_{n+2s+2}-F_{2s+1})\\&+j(F_{n+3s+2}-F_{3s+1}),\\ \sum_{m=0}^{n} \mathfrak{H}^{\mathfrak{F}}{}_{2m} =&(F_{2n+1}-1)+i(F_{2n+s+1}-F_{s-1})+\varepsilon(F_{2n+2s+1}-F_{2s-1})\\&+j(F_{2n+3s+1}-F_{3s-1}),\end{aligned}$$

$$\sum_{m=0}^{n} \mathfrak{H}^{\mathfrak{F}}{}_{2m+1} =(F_{2n+2}+1)+i(F_{2n+s+2}-F_{s})+\varepsilon(F_{2n+2s+2}-F_{2s})+j(F_{2n+3s+2}-F_{3s}).$$

Proof. We have

$$\begin{aligned}\mathfrak{H}^{\mathfrak{F}}{}_0 &= F_0+iF_s+\varepsilon F_{2s}+jF_{3s},\\ \mathfrak{H}^{\mathfrak{F}}{}_1 &= F_1+iF_{1+s}+\varepsilon F_{1+2s}+jF_{1+3s},\\ \mathfrak{H}^{\mathfrak{F}}{}_2 &= F_2+iF_{2+s}+\varepsilon F_{2+2s}+jF_{2+3s},\\ \cdots &= \cdots\\ \mathfrak{H}^{\mathfrak{F}}{}_n &= F_n+iF_{n+s}+\varepsilon F_{n+2s}+jF_{n+3s}.\end{aligned}$$

Taking the sum of the equalities above, we obtained

$$\sum_{m=0}^{n} \mathfrak{H}^{\mathfrak{F}}{}_m = (F_{n+2}-1)+i(F_{n+s+2}-F_{s+1})+\varepsilon(F_{n+2s+2}-F_{2s+1})+j(F_{n+3s+2}-F_{3s+1}).$$

Two other statements of the theorem are verified similarly. □

The sum values in the above theorem can be shown in a similar way for the consecutive coefficient Lucas hybrid single-valued neutrosophic numbers.

Theorem 3.2. *For $n \geq 0$, let $\mathfrak{H}^{\mathfrak{F}}{}_n$ and $\mathfrak{H}^{\mathfrak{L}}{}_n$ be the consecutive coefficient Fibonacci and Lucas hybrid single-valued neutrosophic numbers. Then,*

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}\mathfrak{H}^{\mathfrak{F}}{}_m &= F_{2n}+iF_{2n+s}+\varepsilon F_{2n+2s}+jF_{2n+3s},\\ \sum_{m=0}^{n}\binom{n}{m}\mathfrak{H}^{\mathfrak{L}}{}_m &= L_{2n}+iL_{2n+s}+\varepsilon L_{2n+2s}+jL_{2n+3s}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}\mathfrak{H}^{\mathfrak{F}}{}_m =& \sum_{m=0}^{n}\binom{n}{m}F_m + i\sum_{m=0}^{n}\binom{n}{m}F_{m+s}+\varepsilon\sum_{m=0}^{n}\binom{n}{m}F_{m+2s}\\&+j\sum_{m=0}^{n}\binom{n}{m}F_{m+3s},\end{aligned}$$

and we obtained

$$\sum_{m=0}^{n} \binom{n}{m} \mathfrak{H}^{\mathfrak{F}}{}_{m} = F_{2n} + iF_{2n+s} + \varepsilon F_{2n+2s} + jF_{2n+3s}.$$

The other statement of the theorem is verified similarly. □

4. Conclusion

This study presents the hybrid single-valued neutrosophic number, the consecutive coefficient Fibonacci and Lucas hybrid single-valued neutrosophic numbers. We obtained these new numbers not defined in the literature before. We have given a comprehensive introductory study of hybrid neutrosophic numbers as a guide. Since this study includes some new results, it contributes to literature by providing essential information concerning these new numbers. Hybrid numbers are an excellent mathematical tool in a number of different fields, and hybrid neutrosophic numbers may be a new mathematical tool for processing ambiguous information. Therefore, we hope that these new numbers and properties that we have found will offer a new perspective to the researches. As a future direction, we plan to study other identities and properties of number sequences by increasing the diversity of these number sequences. Researchers who are far from neutrosophic studies may think at first glance that the scope of this paper is limited to neutrosophics, but since Neutrosophy is a nice fuzzy generalization, we offer a working area for all fuzzy types, and we will continue to work with picture fuzzy, type fuzzy sets, extensions, and algebraic properties. We think that it would be useful to look at studies such as quaternion-valued fuzzy (intuitionistic fuzzy) and quaternionic fuzzy (intuitionistic fuzzy) from a hybridistic studies approach.

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ON STATISTICAL SUMMABILITY IN NEUTROSOPHIC SOFT NORMED LINEAR SPACES

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Abstract. In the present paper, we define the notions of statistical convergence and statistical Cauchy sequence in neutrosophic soft normed linear spaces and study some of their properties. We provide examples of a statistical Cauchy sequence that is not statistically convergent and give a useful characterisation of statistical convergence in these spaces.

1. Introduction

The concept of the statistical convergence was explored by Fast [9] and linked with summability theory by Schoenberg [11].

For any set $\mathcal{K} \subseteq \mathbb{N}$, the natural density of $\mathcal{K}$ is defined by

$$\delta(\mathcal{K}) = \lim_{\mathfrak{n}} \frac{1}{\mathfrak{n}} |\{\kappa \leq \mathfrak{n} : \kappa \in \mathcal{K}\}|$$

provided the limit exists. Further, a number sequence $\mathfrak{u} = (\mathfrak{u}_\kappa)$ is said to be statistical convergent to $\mathfrak{u}_0$ if for each $\mathcal{E} > 0$

$$\lim_{\mathfrak{n}} \frac{1}{\mathfrak{n}} |\{\kappa \leq \mathfrak{n} : |\mathfrak{u}_\kappa - \mathfrak{u}_0| \geq \mathcal{E}\}| = 0,$$

i.e., $\delta(\mathcal{K}_\mathcal{E}) = 0$, where $\mathcal{K}_\mathcal{E} = \{\kappa \leq \mathfrak{n} : |\mathfrak{u}_\kappa - \mathfrak{u}_0| \geq \mathcal{E}\}$. We write, in this case $\mathcal{S} - \lim_{\kappa} \mathfrak{u}_\kappa = \mathfrak{u}_0$. Subsequently, the idea is developed by several authors including Maddox [10], Fridy [12], Conner [13], Šalát [32] and many others.

Key words and phrases. Statistical convergence, statistical Cauchy, soft sets, neutrosophic soft normed linear spaces.

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Many problems arising in the areas of science and engineering cannot fit into the framework of classical sets due to complications of uncertainty. As a result, to address these problems, we primarily rely on three approaches: use of probability, interval-based theory, and fuzzy set theory. Among these, fuzzy sets emerge as the most suitable mathematical tool for handling such problems.

The notion of a fuzzy set was initially defined by Zadeh [16] as a generalization of a crisp set with the help of a membership function to deal with those problems that cannot be modeled in the framework of crisp sets. But there are situations which can not be covered by fuzzy sets and therefore we need to extend the idea of fuzzy set. Actually, one drawback of fuzzy sets is the selection of membership function as more than one membership function can be defined using various operations on fuzzy sets. Atanassov [15] observed that fuzzy sets require more alteration to handle issues in a time domain, and therefore, he introduced the concept of intuitionistic fuzzy sets. After the introduction of intuitionistic fuzzy sets, a progressive development is made in this field. For instance, intuitionistic fuzzy metric spaces were introduced by Park [14], intuitionistic fuzzy topological spaces and intuitionistic fuzzy normed spaces by Saadati and Park [24], etc.

The neutrosophic sets were initially introduced by Smarandache [8] as a generalization of fuzzy sets and intuitionistic fuzzy sets with the help of a membership function, a non-membership function and an indeterminacy function to avoid the complexity arising from uncertainty in settling many practical challenges in real-world activities. For a progressive development on neutrosophic sets, we refer to the reader [7, 25] and [26]. Neutrosophic sets are also used to define a new kind of norm naturally. The credit goes to Kirişci and Şimşek [19] who defined neutrosophic normed space and extended summability theory in these spaces. They defined statistical convergence, statistical Cauchy and established some of their properties in neutrosophic normed space. Some more interesting works on summability in neutrosophic normed spaces can be found in [2–4] and [33].

Many approaches discussed above to minimize the uncertainty have their own drawbacks. The main reason behind this is due to inadequacy of the parametrization. To overcome on these difficulty, Molodtsov [6] introduced the idea of soft sets. These sets find valuable applications in numerous fields, including decision-making ([1,21,23]), medical diagnosis ([30, 34]), data analysis approaches under incomplete information [35], algorithms for COVID-19 outbreak [20], assessment processes [17], etc. Soft sets are further used to define soft norm by Das et al. [29] where they developed soft normed linear spaces from functional point of view.

In 2013, Maji [22] united the concepts of soft sets and neutrosophic sets, which he called neutrosophic soft sets. Quite recently, Bera and Mahapatra [31] used soft sets to define neutrosophic soft normed linear space and introduced the convergence structure in these spaces. In present study, we will continue in this direction and define statistical convergence, statistical Cauchy sequence in neutrosophic soft normed linear space and demonstrate some of their properties.

2. Preliminaries

This section starts with a brief information on soft sets, soft vector spaces and neutrosophic soft normed linear spaces. We begin with the following notations and definitions.

Throughout this work, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ will denote the sets of natural, real and positive real numbers, respectively.

Definition 2.1 ([5])**.** Let $\mathfrak{T} = [0,1]$. A binary operation $\circledast : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ is t-norm if for all $\mathfrak{c}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h} \in \mathfrak{T}$ we have

1) $\circledast$ is continuous, commutative and associative;
2) $\mathfrak{e} = \mathfrak{e} \circledast 1$;
3) $\mathfrak{c} \circledast \mathfrak{e} \leq \mathfrak{g} \circledast \mathfrak{h}$ whenever $\mathfrak{c} \leq \mathfrak{g}$ and $\mathfrak{e} \leq \mathfrak{h}$.

Some examples of t-norm are $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$, $\mathfrak{e} \circledast \mathfrak{g} = \min\{\mathfrak{e}, \mathfrak{g}\}$, $\mathfrak{e} \circledast \mathfrak{g} = \max\{\mathfrak{e} + \mathfrak{g} - 1, 0\}$.

Definition 2.2 ([5])**.** Let $\mathfrak{T} = [0,1]$. A binary operation $\circledcirc : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ is t-conorm if for all $\mathfrak{c}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h} \in \mathfrak{T}$ we have

1) $\circledcirc$ is continuous, commutative and associative;
2) $\mathfrak{e} = \mathfrak{e} \circledcirc 0$;
3) $\mathfrak{c} \circledcirc \mathfrak{e} \leq \mathfrak{g} \circledcirc \mathfrak{h}$ whenever $\mathfrak{c} \leq \mathfrak{g}$ and $\mathfrak{e} \leq \mathfrak{h}$.

Some examples of t-conorm are $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$, $\mathfrak{e} \circledcirc \mathfrak{g} = \max\{\mathfrak{e}, \mathfrak{g}\}$, $\mathfrak{e} \circledcirc \mathfrak{g} = \min\{\mathfrak{e} + \mathfrak{g}, 1\}$.

For any universe set $\mathfrak{U}$ and parameter set $\mathfrak{E}$, the soft set is defined as follows.

Definition 2.3 ([6])**.** A pair $(\mathscr{H}, \mathfrak{E})$ is called a soft set over $\mathfrak{U}$ if and only if $\mathscr{H} : \mathfrak{E} \to \mathfrak{P}(\mathfrak{U})$, where $\mathfrak{P}(\mathfrak{U})$ is the set of all subsets of $\mathfrak{U}$. i.e., the soft set is a parametrized family of subsets of the set $\mathfrak{U}$. Moreover, every set $\mathscr{H}(\mathscr{E}), \mathscr{E} \in \mathfrak{E}$, from this family may be considered as the set of $\mathscr{E}$-elements of the soft set $(\mathscr{H}, \mathfrak{E})$, or as the set of $\mathscr{E}$-approximate elements of the set.

Definition 2.4 ([6])**.** A soft set $(\mathscr{H}, \mathfrak{E})$ over $\mathfrak{U}$ is said to be absolute soft set if for every $\mathscr{E} \in \mathfrak{E}$, $\mathscr{H}(\mathscr{E}) = \mathfrak{U}$. We will denote it by $\widetilde{\mathfrak{U}}$.

Definition 2.5 ([27])**.** Let $\mathbb{R}$ be the set of real numbers, $\mathfrak{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and $\mathfrak{E}$ taken as a set of parameters. Then a mapping $\mathfrak{F} : \mathfrak{E} \to \mathfrak{B}(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, then it is called a soft real number and denoted by $\tilde{\mathfrak{r}}, \tilde{\mathfrak{s}}, \tilde{\mathfrak{t}}$, etc. $\widetilde{0}, \widetilde{1}$ are the soft real numbers where $\widetilde{0}(e) = 0$, $\widetilde{1}(e) = 1$ for all $e \in E$, respectively.

Let $\mathbb{R}(\mathfrak{E})$ and $\mathbb{R}^+(\mathfrak{E})$, respectively, denote the sets of all soft real numbers and all positive soft real numbers.

Definition 2.6 ([28])**.** Let $(\mathscr{H}, \mathfrak{E})$ be a soft set over $\mathfrak{U}$. The set $(\mathscr{H}, \mathfrak{E})$ is said to be a soft point, denoted by $\mathscr{H}_e^{\mathfrak{u}}$ if there is exactly one $e \in \mathfrak{E}$ s.t $\mathscr{H}(e) = \{\mathfrak{u}\}$ for some $\mathfrak{u} \in \mathfrak{U}$ and $\mathscr{H}(e') = \phi$ for all $e' \in \mathfrak{E} - \{e\}$.

Two soft points $\mathscr{H}_e^{\mathfrak{u}}, \mathscr{H}_{e'}^{\mathfrak{w}}$ are said to be equal if $e = e'$ and $\mathfrak{u} = \mathfrak{w}$. Let $\Delta_{\widetilde{\mathfrak{U}}}$ denotes the set of all soft points on $\widetilde{\mathfrak{U}}$.

In case $\mathfrak{U}$ is a vector space over $\mathbb{R}$ and the parameter set $\mathfrak{E} = \mathbb{R}$, the soft point is called a soft vector. Soft vector spaces are used to define soft norm as follows.

Definition 2.7 ([18])**.** Let $\widetilde{\mathfrak{U}}$ be a absolute soft vector space. Then a mapping $\|\cdot\| : \widetilde{\mathfrak{U}} \to \mathbb{R}^+(\mathfrak{E})$ is said to be a soft norm on $\widetilde{\mathfrak{U}}$, if $\|\cdot\|$ satisfies the following conditions:

(i) $\|\mathfrak{u}_e\| \geq \widetilde{0}$ for all $\mathfrak{u}_e \in \widetilde{\mathfrak{U}}$ and

$$\|\mathfrak{u}_e\| = \widetilde{0} \Leftrightarrow \mathfrak{u}_e = \widetilde{\theta}_0,$$

where $\widetilde{\theta}_0$ denotes the zero element of $\widetilde{\mathfrak{U}}$;

(ii) $\|\widetilde{\alpha}\mathfrak{u}_e\| = |\widetilde{\alpha}| \cdot \|\mathfrak{u}_e\|$ for all $\mathfrak{u}_e \in \widetilde{\mathfrak{U}}$ and for every soft scalar$\widetilde{\alpha}$;

(iii) $\|\mathfrak{u}_e + \mathfrak{u}_{e'}\| \leq \|\mathfrak{u}_e\| + \|\mathfrak{u}_{e'}\|$ for all $\mathfrak{u}_e, \mathfrak{u}_{e'} \in \widetilde{\mathfrak{U}}$;

(iv) $\|\mathfrak{u}_e \cdot \mathfrak{u}_{e'}\| = \|\mathfrak{u}_e\| \cdot \|\mathfrak{u}_{e'}\|$ for all $\mathfrak{u}_e, \mathfrak{u}_{e'} \in \widetilde{\mathfrak{U}}$.

The soft vector space $\widetilde{\mathfrak{U}}$ with a soft norm $\|\cdot\|$ on $\widetilde{\mathfrak{U}}$ is said to be a soft normed linear space and is denoted by $(\widetilde{\mathfrak{U}}, \|\cdot\|)$.

We now recall the definition of neutrosophic soft normed linear spaces and the convergence structure in these spaces.

Definition 2.8 ([31])**.** Let $\widetilde{\mathfrak{U}}$ be a soft linear space over the field $\mathfrak{F}$ and $\mathbb{R}(\mathfrak{E}), \Delta_{\widetilde{\mathfrak{U}}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{\mathfrak{U}}$. Then a neutrosophic subset N over $\Delta_{\widetilde{\mathfrak{U}}} \times \mathbb{R}(\mathfrak{E})$ is called a neutrosophic soft norm on $\widetilde{\mathfrak{U}}$ if for $\mathfrak{u}_e, \mathfrak{u}_{e'} \in \widetilde{\mathfrak{U}}$ and $\widetilde{\alpha} \in \mathfrak{F}$ ($\widetilde{\alpha}$ being soft scalar), the following conditions hold:

(i) $0 \leq G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}), B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}), Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \leq 1$ for all $\widetilde{\mathfrak{y}_1} \in \mathbb{R}(\mathfrak{E})$;

(ii) $0 \leq G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) + B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) + Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \leq 3$ for all $\widetilde{\mathfrak{y}_1} \in \mathbb{R}(\mathfrak{E})$;

(iii) $G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} \leq \widetilde{0}$;

(iv) $G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 1,$ with $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ if and only if $\mathfrak{u}_e = \widetilde{\theta_0}$, the null soft vector;

(v) $G_N(\widetilde{\alpha}\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = G_N\left(\mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}_1}}{|\widetilde{\alpha}|}\right)$ for all $\widetilde{\alpha}(\neq \widetilde{0})$, $\widetilde{\mathfrak{y}_1} > \widetilde{0}$;

(vi) $G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \circledast G_N(\mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_2}) \leq G_N(\mathfrak{u}_e \oplus \mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_1} \oplus \widetilde{\mathfrak{y}_2})$ for all $\widetilde{\mathfrak{y}_1}, \widetilde{\mathfrak{y}_2} \in \mathbb{R}(\mathfrak{E})$;

(vii) $G_N(\mathfrak{u}_e, \cdot)$ is continuous non-decreasing function for $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ and $\lim_{\widetilde{\mathfrak{y}_1} \to +\infty} G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 1$;

(viii) $B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 1,$ with $\widetilde{\mathfrak{y}_1} \leq \widetilde{0}$;

(ix) $B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ if and only if $\mathfrak{u}_e = \widetilde{\theta_0}$, the null soft vector;

(x) $B_N(\widetilde{\alpha}\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = B_N\left(\mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}_1}}{|\widetilde{\alpha}|}\right)$ for all $\widetilde{\alpha}(\neq \widetilde{0})$, $\widetilde{\mathfrak{y}_1} > \widetilde{0}$;

(xi) $B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \circledcirc B_N(\mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_2}) \geq B_N(\mathfrak{u}_e \oplus \mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_1} \oplus \widetilde{\mathfrak{y}_2})$ for all $\widetilde{\mathfrak{y}_1}, \widetilde{\mathfrak{y}_2} \in \mathbb{R}(\mathfrak{E})$;

(xii) $B_N(\mathfrak{u}_e, \cdot)$ is continuous non-increasing function for $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ and $\lim_{\widetilde{\mathfrak{y}_1} \to +\infty} B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0$;

(xiii) $Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} \leq \widetilde{0}$;

(xiv)$Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ if and only if $\mathfrak{u}_e = \widetilde{\theta_0}$, the null soft vector;

(xv) $Y_N(\widetilde{\alpha}\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = Y_N\left(\mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}_1}}{|\widetilde{\alpha}|}\right)$ for all $\widetilde{\alpha}(\neq \widetilde{0})$, $\widetilde{\mathfrak{y}_1} > \widetilde{0}$;

(xvi) $Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \circledcirc Y_N(\mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_2}) \geq Y_N(\mathfrak{u}_e \oplus \mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_1} \oplus \widetilde{\mathfrak{y}_2})$ for all $\widetilde{\mathfrak{y}_1}, \widetilde{\mathfrak{y}_2} \in \mathbb{R}(\mathfrak{E})$;

(xvii) $Y_N(\mathfrak{u}_e, \cdot)$ is continuous non-increasing function for $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ and $\lim_{\widetilde{\mathfrak{y}_1} \to +\infty} B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0$.

In this case $N = (G_N, B_N, Y_N)$ is called the neutrosophic soft norm and

$$(\widetilde{\mathfrak{U}}(F), G_N, B_N, Y_N, \circledast, \circledcirc)$$

is an neutrosophic soft normed linear space (briefly $NSNLS$).

Let $(\widetilde{\mathfrak{U}}, \|\cdot\|)$ be a soft normed space. Take the operations $\circledast$ and $\circledcirc$ as $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$. For $\widetilde{\mathfrak{y}} > \widetilde{0}$, define

$$G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \begin{cases} \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} + \|\mathfrak{u}_e\|}, & \text{if } \widetilde{\mathfrak{y}} > \|\mathfrak{u}_e\|, \\ 0, & \text{otherwise,} \end{cases}$$

$$B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \begin{cases} \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}} + \|\mathfrak{u}_e\|}, & \text{if } \widetilde{\mathfrak{y}} > \|\mathfrak{u}_e\|, \\ 1 & \text{otherwise,} \end{cases}$$

$$Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \begin{cases} \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}}}, & \text{if } \widetilde{\mathfrak{y}} > \|\mathfrak{u}_e\|, \\ 1, & \text{otherwise.} \end{cases}$$

Then, $(\widetilde{\mathfrak{U}}(\mathfrak{F}), G_N, B_N, Y_N, \circledast, \circledcirc)$ is the $NSNLS$. From now onwards, unless otherwise stated by $\widetilde{\mathcal{V}}$ we shall denote the $NSNLS$ $(\widetilde{\mathfrak{U}}(\mathfrak{F}), G_N, B_N, Y_N, \circledast, \circledcirc)$.

A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\widetilde{\mathcal{V}}$ is said to be convergent to a soft point $\mathfrak{u}_e \in \widetilde{\mathcal{V}}$ if for $0 < \mathcal{E} < 1$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$ exists $\mathfrak{n}_0 \in \mathbb{N}$ s.t $G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$, $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$. In this case, we write $\lim_{\kappa \to +\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$.

A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\widetilde{\mathcal{V}}$ is said to be Cauchy sequence if for $0 < \mathcal{E} < 1$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$ exists $\mathfrak{n}_0 \in \mathbb{N}$ s.t for all $\kappa, \rho \geq \mathfrak{n}_0$, $G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$, $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) < \mathcal{E}$.

3. Statistical Convergence in NSNLS

In this section, we define statistical convergence in $NSNLS$ and develop some of its properties.

Definition 3.1. A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\widetilde{\mathcal{V}}$ is said to be statistical convergent to a soft point $\mathfrak{u}_e$ in $\widetilde{\mathcal{V}}$ if for $0 < \mathcal{E} < 1$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, there exists $\mathfrak{n}_0 \in \mathbb{N}$ s.t.

$$\lim_{\mathfrak{n} \to +\infty} \frac{1}{\mathfrak{n}} \Bigg| \Bigg\{ \kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or}$$

$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E} \Bigg\} \Bigg| = 0$$

or equivalently

$$\delta(\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or}$$

$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\}) = 0.$$

In present case, we denote $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$.

Remark 3.1. Since every finite set has density zero, every convergent sequence in $NSNLS$ $\widetilde{\mathcal{V}}$ is statistically convergent but the converse may not be true as can be seen from the following example.

Example 3.1. Let $(\widetilde{\mathbb{R}}, \|\cdot\|)$ be a soft normed linear space. For $\mathfrak{e}, \mathfrak{g} \in [0,1]$, let $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$. Choose $\mathfrak{u}_e \in \widetilde{\mathbb{R}}$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, we define

$$G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}, \quad B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}, \quad Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}}},$$

then it is easy to see that $(\widetilde{\mathbb{R}}, G_N, B_N, Y_N, \circledast, \circledcirc)$ is a $NSNLS$. Define a sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ by

$$\mathfrak{u}^{\kappa}_{e_\kappa} = \begin{cases} \widetilde{1}, & \text{if } \kappa \text{ is square,} \\ \widetilde{0}, & \text{otherwise.} \end{cases}$$

Now, for $\mathcal{E} > 0$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$,

$$\begin{aligned} \mathcal{T} &= \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa}, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\} \\ &= \left\{\kappa \in \mathbb{N} : \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}^{\kappa}_{e_\kappa}\|} \leq 1 - \mathcal{E} \text{ or } \frac{\|\mathfrak{u}^{\kappa}_{e_\kappa}\|}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}^{\kappa}_{e_\kappa}\|} \geq \mathcal{E}, \frac{\|\mathfrak{u}^{\kappa}_{e_\kappa}\|}{\widetilde{\mathfrak{y}}} \geq \mathcal{E}\right\} \\ &= \left\{\kappa \in \mathbb{N} : \|\mathfrak{u}^{\kappa}_{e_\kappa}\| \geq \frac{\widetilde{\mathfrak{y}}\mathcal{E}}{1 - \mathcal{E}} \text{ or } \|\mathfrak{u}^{\kappa}_{e_\kappa}\| \geq \widetilde{\mathfrak{y}}\mathcal{E}\right\} \\ &= \{\kappa \in \mathbb{N} : \mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{1}\} \\ &= \{\kappa \in \mathbb{N} : \kappa \text{ is square}\}. \end{aligned}$$

This implies that $\delta(\mathcal{T}) = \delta(\{\kappa \in \mathbb{N} : \kappa \text{ is square}\}) = 0$ and therefore $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical convergent to $\widetilde{0}$. Obviously, by the structure of the sequence, $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is not ordinary convergent.

By Definition 3.1 together with the property of natural density, we have the following lemma.

Lemma 3.1. *For any sequence* $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ *of soft points in* $\widetilde{\mathcal{V}}$*, the subsequent statements are equivalent:*

(*i*) $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$;

(*ii*) $\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\} = 0$;

(*iii*) $\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$ *and* $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}\} = 1$;

(*iv*) $\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}\} = 1$;

(v) $\mathcal{S}-\lim_{\kappa\to+\infty} G_N(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_e,\widetilde{\mathfrak{y}})=1$ *and* $\mathcal{S}-\lim_{\kappa\to+\infty} B_N(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_e,\widetilde{\mathfrak{y}})=\mathcal{S}-\lim_{\kappa\to+\infty} Y_N(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_e,\widetilde{\mathfrak{y}})=0$.

Theorem 3.1. *For any sequence* $\mathfrak{u}=(\mathfrak{u}^{\kappa}_{e_\kappa})$ *in* $\widetilde{\mathcal{V}}$*, if* $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}$ *exists, then it is unique.*

Proof. We shall prove the theorem by use of contradiction. Let $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}_{e_1}$ and $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}'_{e_2}$, where $\mathfrak{u}_{e_1}\neq\mathfrak{u}'_{e_2}$. For $\mathcal{E}>0$ and $\widetilde{\mathfrak{y}}>\widetilde{0}$, choose $\mathcal{E}_1>0$ s.t. $(1-\mathcal{E}_1)\circledast(1-\mathcal{E}_1)>1-\mathcal{E}$ and $\mathcal{E}_1\circledcirc\mathcal{E}_1<\mathcal{E}$. Define the following sets:

$$A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})=\left\{\kappa\in\mathbb{N}:G_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_{e_1},\frac{\widetilde{\mathfrak{y}}}{2}\right)\leq 1-\mathcal{E}_1\right\},$$
$$A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})=\left\{\kappa\in\mathbb{N}:G_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}'_{e_2},\frac{\widetilde{\mathfrak{y}}}{2}\right)\leq 1-\mathcal{E}_1\right\},$$
$$A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})=\left\{\kappa\in\mathbb{N}:B_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_{e_1},\frac{\widetilde{\mathfrak{y}}}{2}\right)\geq \mathcal{E}_1\right\},$$
$$A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})=\left\{\kappa\in\mathbb{N}:B_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}'_{e_2},\frac{\widetilde{\mathfrak{y}}}{2}\right)\geq \mathcal{E}_1\right\},$$
$$A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})=\left\{\kappa\in\mathbb{N}:Y_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_{e_1},\frac{\widetilde{\mathfrak{y}}}{2}\right)\geq \mathcal{E}_1\right\},$$
$$A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})=\left\{\kappa\in\mathbb{N}:Y_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}'_{e_2},\frac{\widetilde{\mathfrak{y}}}{2}\right)\geq \mathcal{E}_1\right\}.$$

Since $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}_{e_1}$, by Lemma 3.1, $\delta\{A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=0$ and therefore

$$\delta\{A^{\complement}_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=1.$$

Further, $\mathcal{S}-\lim_{k\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}'_{e_2}$, so $\delta\{A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}$ $=0$ and therefore $\delta\{A^{\complement}_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=1$ for all $\widetilde{\mathfrak{y}}>\widetilde{0}$. Define $K_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})=\{A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}\cap\{A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}\cap\{A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}$, then $\delta\{K_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})\}=0$ and therefore, $\delta\{K^{\complement}_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})\}=1$. Let $m\in K^{\complement}_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})$, then we have following possibilities:

1. $m\in\{A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}^{\complement}$;
2. $m\in\{A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}^{\complement}$;
3. $m\in\{A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}^{\complement}$.

Case 1. Let $m \in \{A_{G_N,1}(\mathcal{E}_1, \widetilde{\mathfrak{y}}) \cup A_{G_N,2}(\mathcal{E}_1, \widetilde{\mathfrak{y}})\}^{\complement}$. Then, $m \in A^{\complement}_{G_N,1}(\mathcal{E}_1, \widetilde{\mathfrak{y}})$ and $m \in A^{\complement}_{G_N,2}(\mathcal{E}_1, \widetilde{\mathfrak{y}})$ and therefore,

$$G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1 - \mathcal{E}_1 \quad \text{and} \quad G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1 - \mathcal{E}_1. \tag{3.1}$$

Now,

$$\begin{aligned}
G_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \widetilde{\mathfrak{y}}) &= G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}^m_{e_m} \oplus \mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&\geq G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledast G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&> (1 - \mathcal{E}_1) \circledast (1 - \mathcal{E}_1) \quad \text{by (3.1)}\\
&> 1 - \mathcal{E}.
\end{aligned}$$

Given that $\mathcal{E} > 0$ is arbitrary, we thus obtain $G_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \widetilde{\mathfrak{y}}) = 1$, for all $\widetilde{\mathfrak{y}} > \widetilde{0}$, which gives $\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2} = \widetilde{\theta_0}$, i.e., $\mathfrak{u}_{e_1} = \mathfrak{u}'_{e_2}$.

Case 2. Let $m \in \{A_{B_N,1}(\mathcal{E}_1, \widetilde{\mathfrak{y}}) \cup A_{B_N,2}(\mathcal{E}_1, \widetilde{\mathfrak{y}})\}^{\complement}$. Then, $m \in A^{\complement}_{B_N,1}(\mathcal{E}_1, \widetilde{\mathfrak{y}})$ and $m \in A^{\complement}_{B_N,2}(\mathcal{E}_1, \widetilde{\mathfrak{y}})$ and therefore,

$$B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1 \quad \text{and} \quad B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1. \tag{3.2}$$

Now,

$$\begin{aligned}
B_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \widetilde{\mathfrak{y}}) &= B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}^m_{e_m} \oplus \mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&\leq B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \quad \text{by (3.2)}\\
&< \mathcal{E}.
\end{aligned}$$

Given that $\mathcal{E} > 0$ is arbitrary, we thus obtain $B_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \widetilde{\mathfrak{y}}) = 0$, for all $\widetilde{\mathfrak{y}} > \widetilde{0}$, which gives $\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2} = \widetilde{\theta_0}$, i.e., $\mathfrak{u}_{e_1} = \mathfrak{u}'_{e_2}$.

Case 3. Let $m \in \{A_{Y_N,1}(\mathcal{E}_1, \widetilde{\mathfrak{y}}) \cup A_{Y_N,2}(\mathcal{E}_1, \widetilde{\mathfrak{y}})\}^{\complement}$. Then, $m \in A^{\complement}_{Y_N,1}(\mathcal{E}_1, \widetilde{\mathfrak{y}})$ and $m \in A^{\complement}_{Y_N,2}(\mathcal{E}_1, \widetilde{\mathfrak{y}})$ and therefore,

$$Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1 \quad \text{and} \quad Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1. \tag{3.3}$$

Now,

$$Y_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \widetilde{\mathfrak{y}}) = Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}^m_{e_m} \oplus \mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)$$

$$\leq Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2}\right)$$
$$< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \quad \text{by (3.3)}$$
$$< \mathcal{E}.$$

Given that $\mathcal{E} > 0$ is arbitrary, we thus obtain $Y_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \widetilde{\mathfrak{y}}) = 0$ for all $\widetilde{\mathfrak{y}} > \widetilde{0}$, which gives $\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2} = \widetilde{\theta_0}$, i.e., $\mathfrak{u}_{e_1} = \mathfrak{u}'_{e_2}$.

Hence, in all cases we have $\mathfrak{u}_{e_1} = \mathfrak{u}'_{e_2}$, i.e., the statistical limit of the sequence $(\mathfrak{u}^\kappa_{e_\kappa})$ is unique. □

Theorem 3.2. *A sequence* $\mathfrak{u} = (\mathfrak{u}^\kappa_{e_\kappa})$ *in* $\widetilde{\mathcal{V}}$ *is statistically convergent if and only if exists a set* $\mathcal{K} = \{\kappa_1, \kappa_2, \kappa_3, \dots\}$ *s.t* $\delta(\mathcal{K}) = 1$ *and* $(G_N, B_N, Y_N) - \lim_{\substack{\kappa\in\mathcal{K}\\ \kappa\to+\infty}} \mathfrak{u}^\kappa_{e_\kappa} = \mathfrak{u}_e$.

Proof. First suppose that $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^\kappa_{e_\kappa} = \mathfrak{u}_e$. For $\widetilde{\mathfrak{y}} > \widetilde{0}$ and $p \in \mathbb{N}$, define the set

$$A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}}) = \Big\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} \text{ and } B_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}, Y_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}\Big\}.$$

We first show that $A_{G_N,B_N,Y_N}(p+1, \widetilde{\mathfrak{y}}) \subset A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})$.

Let $m \in A_{G_N,B_N,Y_N}(p+1, \widetilde{\mathfrak{y}})$. Then, $G_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p+1} > 1 - \frac{1}{p}$ and $B_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p+1} < \frac{1}{p}$, $Y_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\lambda}) < \frac{1}{p+1} < \frac{1}{p}$, this implies that $m \in A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})$ and therefore,

$$A_{G_N,B_N,Y_N}(p+1, \widetilde{\mathfrak{y}}) \subset A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}}). \tag{3.4}$$

Since $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^\kappa_{e_\kappa} = \mathfrak{u}_e$, so for all $p \in \mathbb{N}$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, $\delta\{A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})\} = 1$ and therefore is an infinite set. Let $m_1 \in A_{G_N,B_N,Y_N}(1, \widetilde{\mathfrak{y}})$. Further, $\delta\{A_{G_N,B_N,Y_N}(2, \widetilde{\mathfrak{y}})\} = 1$, so we can choose m_2 in $A_{G_N,B_N,Y_N}(2, \widetilde{\mathfrak{y}})$, s.t $m_2 > m_1$ and

$$\frac{1}{\mathfrak{n}}\left|\Big\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{2} \text{ and } B_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{2}, Y_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{2}\Big\}\right| > \frac{1}{2}.$$

Now, select m_3 in $A_{G_N,B_N,Y_N}(3, \widetilde{\mathfrak{y}})$, s.t $m_3 > m_2$ and

$$\frac{1}{\mathfrak{n}}\left|\Big\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{3} \text{ and } B_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{3}, Y_N(\mathfrak{u}^\kappa_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{3}\Big\}\right| > \frac{2}{3},$$

and so on. In this way we obtain a sequence (m_p) in $\mathbb{N}$ with $m_{p+1} > m_p$ for all $p, m_p \in A_{G_N,B_N,Y_N}(p,\widetilde{\mathfrak{y}})$ and for all $\mathfrak{n} \geq m_p$, $p \in \mathbb{N}$

$$(3.5)\qquad \frac{1}{\mathfrak{n}}\left|\left\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} \text{ and } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p},\right.\right.$$
$$\left.\left. Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}\right\}\right| > \frac{p-1}{p}.$$

If we define a set

$$(3.6)\qquad \mathcal{K} = \{\mathfrak{n} \in \mathbb{N} : 1 < \mathfrak{n} < m_1\} \cup \left[\bigcup_{p\in\mathbb{N}} \{\mathfrak{n} \in A_{G_N,B_N,Y_N}(p,\widetilde{\mathfrak{y}})\} : m_p \leq \mathfrak{n} < m_{p+1}\right],$$

then using (3.4), (3.5) and (3.6) we have for all $\mathfrak{n}$ satisfying $(m_p \leq \mathfrak{n} < m_{p+1})$,

$$\frac{1}{\mathfrak{n}}|\{\kappa \leq \mathfrak{n} : \kappa \in \mathcal{K}\}| \geq \frac{1}{\mathfrak{n}}\left|\left\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} \text{ and}\right.\right.$$
$$\left.\left. B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}\right\}\right| > \frac{p-1}{p},$$

and therefore, in the limiting case, we get $\delta(\mathcal{K}) \geq 1$, i.e., $\delta(\mathcal{K}) = 1$ as $\delta(\mathcal{K}) \not> 1$. Now we will show that the subsequence $(\mathfrak{u}^{\kappa}_{e_\kappa} : \kappa \in \mathcal{K})$ is convergent to $\mathfrak{u}_e$, i.e., $(\mathfrak{u}^{\kappa}_{e_\kappa}) \to \mathfrak{u}_e$ over $\mathcal{K}$.

Let $\mathcal{E} > 0$ be given. Since $\frac{1}{p} \to 0$ as $p \to +\infty$, so we can choose $p \in \mathbb{N}$, s.t $\frac{1}{p} < \mathcal{E}$. Let $\kappa \in \mathcal{K}$ be s.t $\kappa \geq \mathfrak{t}_p$ for some fixed integer $\mathfrak{t}_p$. Then by structure of $\mathcal{K}$, exists a number $q \geq p$, s.t $\mathfrak{t}_q \leq \kappa < \mathfrak{t}_{q+1}$ and $\kappa \in A_{G_N,B_N,Y_N}(p,\widetilde{\mathfrak{y}})$. Now for $\mathcal{E} > 0$,

$$G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} > 1 - \mathcal{E} \text{ and}$$
$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p} < \mathcal{E}, \quad Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p} < \mathcal{E},$$

for all $\kappa \geq \mathfrak{t}_p$ and $\kappa \in \mathcal{K}$. This implies that $(G_N, B_N, Y_N) - \lim_{\kappa\to+\infty}(\mathfrak{u}^{\kappa}_{e_\kappa}) = \mathfrak{u}_e$.

Conversely, suppose there exists a set $\mathcal{K} = \{\kappa_1, \kappa_2, \dots, \kappa_j, \dots\}$, with $\delta(\mathcal{K}) = 1$ and $(G_N, B_N, Y_N) - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$ over $\mathcal{K}$, i.e., $(G_N, B_N, Y_N) - \lim_{\substack{\kappa\in\mathcal{K}\\ \kappa\to+\infty}} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$. Let $\mathcal{E} > 0$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$. Since $(G_N, B_N, Y_N) - \lim_{\substack{\kappa\in\mathcal{K}\\ \kappa\to+\infty}} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$, so there exists $\kappa_j \in \mathbb{N}$ s.t for all $\kappa \geq \kappa_j$ and $\kappa \in \mathcal{K}$, $G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$ and $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$. So, if we consider the set

$$T_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}}) = \left\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or}\right.$$
$$\left. B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\right\},$$

then $T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}}) \subset \mathbb{N} - \{\kappa_j, \kappa_{j+1}, \kappa_{j+2}, \dots\}$. This immediately implies that

$$\delta\left(T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}})\right) \leq \delta(\mathbb{N}) - \delta\{\kappa_j, \kappa_{j+1}, \kappa_{j+2}, \dots\} = 1 - 1 = 0,$$

and therefore $\delta\left(T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}})\right) = 0$ as $\delta\left(T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}})\right) \nless 0$. This shows that $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical convergent to $\mathfrak{u}_e$, i.e., $\mathcal{S} - \lim_{\kappa \to +\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$. ☐

Theorem 3.3. *Let* $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ *and* $w = (w^{\kappa}_{e_\kappa})$ *be any two sequences in* $\tilde{\mathcal{V}}$ *s.t* $\mathcal{S} - \lim_{\kappa \to +\infty} (\mathfrak{u}^{\kappa}_{e_\kappa}) = \mathfrak{u}_{e_1}$ *and* $\mathcal{S} - \lim_{\kappa \to +\infty} (w^{\kappa}_{e_\kappa}) = w_{e_2}$. *Then,*

(*i*) $\mathcal{S} - \lim_{\kappa \to +\infty} (\mathfrak{u}^{\kappa}_{e_\kappa} \oplus w^{\kappa}_{e_\kappa}) = \mathfrak{u}_{e_1} \oplus w_{e_2}$;

(*ii*) $\mathcal{S} - \lim_{\kappa \to +\infty} (\tilde{\alpha}\mathfrak{u}^{\kappa}_{e_\kappa}) = \tilde{\alpha}\mathfrak{u}_{e_1}$, *where* $\tilde{0} \neq \tilde{\alpha} \in \mathfrak{F}$.

Proof. The proof of the theorem can be obtained as the proof of Theorem 3.1, so omitted. ☐

4. Statistical Completeness in NSNLS

Definition 4.1. A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\tilde{\mathcal{V}}$ is said to be statistically Cauchy sequence if for $0 < \mathcal{E} < 1$ and $\tilde{\mathfrak{y}} > \tilde{0}$, exists $\rho \in \mathbb{N}$ s.t

$$\lim_{\mathfrak{n} \to +\infty} \frac{1}{\mathfrak{n}} \left| \left\{ \kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E} \right\} \right| = 0$$

or equivalently

$$\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E}\} = 0.$$

Theorem 4.1. *Every statistical convergent sequence in* $\tilde{\mathcal{V}}$ *is statistical Cauchy.*

Proof. Let $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ be any statistical convergent sequence in $\tilde{\mathcal{V}}$ with $\mathcal{S} - \lim_{\kappa \to +\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$. For $\mathcal{E} > 0$ and $\tilde{\mathfrak{y}} > \tilde{0}$. Choose $\mathcal{E}_1 > 0$ s.t

$$(1 - \mathcal{E}_1) \circledast (1 - \mathcal{E}_1) > 1 - \mathcal{E} \quad \text{and} \quad \mathcal{E}_1 \circledcirc \mathcal{E}_1 < \mathcal{E}. \tag{4.1}$$

Define a set, $K(\mathcal{E}_1, \tilde{\mathfrak{y}}) = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) \leq 1 - \mathcal{E}_1$ or $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) \geq \mathcal{E}_1, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) \geq \mathcal{E}_1\}$. Then, $K^{\complement}(\mathcal{E}_1, \tilde{\mathfrak{y}}) = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) > 1 - \mathcal{E}_1$ and $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) < \mathcal{E}_1$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) < \mathcal{E}_1\}$. Since $\mathcal{S} - \lim_{n \to +\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$, so $\delta(K(\mathcal{E}_1, \tilde{\mathfrak{y}})) = 0$ and $\delta(K^{\complement}(\mathcal{E}_1, \tilde{\mathfrak{y}})) = 1$. Let $\rho \in K^{\complement}(\mathcal{E}_1, \tilde{\mathfrak{y}})$. Then,

$$G_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}\right) > 1 - \mathcal{E}_1 \quad \text{and} \quad B_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1. \tag{4.2}$$

Now, let $T(\mathcal{E},\widetilde{\mathfrak{y}}) = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \leq 1-\mathcal{E}$ or $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \geq \mathcal{E}\}$. Then, we show that $T(\mathcal{E},\widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. Let $m \in T(\mathcal{E},\widetilde{\mathfrak{y}})$. Then,

$$G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \leq 1-\mathcal{E} \quad \text{or} \quad B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \geq \mathcal{E}. \tag{4.3}$$

Case 1. If $G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \leq 1-\mathcal{E}$, then $G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \leq 1-\mathcal{E}_1$ and therefore $m \in K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. As otherwise, i.e., if $G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1-\mathcal{E}_1$, then by (4.1), (4.2) and (4.3) we get

$$\begin{aligned}
1-\mathcal{E} &\geq G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}})\\
&= G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e \oplus \mathfrak{u}_e \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&\geq G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledast G_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&> (1-\mathcal{E}_1) \circledast (1-\mathcal{E}_1)\\
&> 1-\mathcal{E},
\end{aligned}$$

which is not possible. Thus, $T(\mathcal{E},\widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$.

Case 2. If $B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \geq \mathcal{E}$, then $B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1$ and therefore $m \in K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. As otherwise, i.e., if $B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1$, then by (4.1), (4.2) and (4.3) we get

$$\begin{aligned}
\mathcal{E} &\leq B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}})\\
&= B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e \oplus \mathfrak{u}_e \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&\leq B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc B_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&< \mathcal{E}_1 \circledcirc \mathcal{E}_1\\
&< \mathcal{E},
\end{aligned}$$

which is not possible. Also, if $Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}}) \geq \mathcal{E}$, then $Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1$ and therefore $m \in K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. As otherwise, i.e., if $Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1$, then by (4.1), (4.2) and (4.3) we get

$$\begin{aligned}
\mathcal{E} &\leq Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho},\widetilde{\mathfrak{y}})\\
&= Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e \oplus \mathfrak{u}_e \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&\leq Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc Y_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\
&< \mathcal{E}_1 \circledcirc \mathcal{E}_1 < \mathcal{E},
\end{aligned}$$

which is not possible. Thus, $T(\mathcal{E}, \widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1, \widetilde{\mathfrak{y}})$.

Hence, in all cases, $T(\mathcal{E}, \widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1, \widetilde{\mathfrak{y}})$. Since $\delta(K(\mathcal{E}_1, \widetilde{\mathfrak{y}})) = 0$, so $\delta(T(\mathcal{E}, \widetilde{\mathfrak{y}})) = 0$, and therefore $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical Cauchy. □

Example 4.1. Let $\widetilde{R_1} = \{\frac{1}{\mathfrak{n}} : \mathfrak{n} \in \mathbb{N}\}$ and $\|\cdot\| = |\cdot|$, i.e., the usual norm on $\widetilde{R_1}$, then $(\widetilde{R_1}, |\cdot|)$ is a soft normed linear space. For $\widetilde{\mathfrak{y}} > \widetilde{0}$, if we define $G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}$, $B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}$, $Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}}}$; $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$, then it is easy to see that $(\widetilde{R_1}(\mathbb{R}), G_N, B_N, Y_N, \circledast, \circledcirc)$ is a *NSNLS*. If we define a sequence of soft points $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ by $\mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{\frac{1}{\kappa}}$, then it is easy to see by definition of G_N, B_N and Y_N, the density of the set $A = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E}$ or $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\}$ is zero, i.e., $\delta(A) = 0$, Therefore, $(\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical Cauchy. Since $\mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{\frac{1}{\kappa}} \to \widetilde{0}$ as $\kappa \to +\infty$ and usual convergence implies statistical convergence with the same limit, so $\mathcal{S} - \lim_{\kappa \to +\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{0}$ but $\widetilde{0}$ is not a member of the space.

Remark 4.1. If a sequence is Cauchy in $\widetilde{\mathcal{V}}$, then it is statistically Cauchy.

Theorem 4.2. *For any sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ in $\widetilde{\mathcal{V}}$, the subsequent conditions are equivalent:*

(i) $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistically Cauchy w.r.t. neutrosophic soft norm (G_N, B_N, Y_N);

(ii) there exists a subset $\mathcal{K} = \{\kappa_1, \kappa_2, \ldots, \kappa_j, \ldots\}$ of $\mathbb{N}$, with $\delta(\mathcal{K}) = 1$ and the subsequence $(v^{k_j}_{e_{k_j}})_{j \in \mathbb{N}}$ is Cauchy sequence over $\mathcal{K}$.

Proof. The proof of the theorem can be obtained analogously as the proof of Theorem 3.2. □

Definition 4.2. A *NSNLS* $\widetilde{\mathcal{V}}$ is said to be statistically complete if every statistically Cauchy sequence in $\triangle_{\widetilde{\mathcal{V}}}$ is statistically convergent in $\triangle_{\widetilde{\mathcal{V}}}$.

Theorem 4.3. *If every statistical Cauchy sequence in $\widetilde{\mathcal{V}}$ has a statistical convergent subsequence, then $\widetilde{\mathcal{V}}$ is statistically complete.*

Proof. Let $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ be any statistically Cauchy sequence of soft points in $\widetilde{\mathcal{V}}$ which has a statistical convergent subsequence $(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}})$, i.e., $\mathcal{S} - \lim_{j \to +\infty} \mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} = \mathfrak{u}_e$ for some $\mathfrak{u}_e$ in $\widetilde{\mathcal{V}}$. Since $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistically Cauchy, so for $\mathcal{E} > 0$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, $\delta(\mathfrak{A}) = 0$, where

$$\mathfrak{A} = \left\{\kappa \in \mathbb{N} : G_N\left(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \leq 1 - \mathcal{E}_1 \text{ or } B_N\left(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1\right\}.$$

Again since $\mathcal{S} - \lim_{j \to +\infty} \mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} = \mathfrak{u}_e$, we have $\delta(\mathfrak{D}) = 0$, where

$$\mathfrak{D} = \left\{\kappa(j) \in \mathbb{N} : G_N\left(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \leq 1 - \mathcal{E}_1 \text{ or }\right.$$

$$B_N\left(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1\right\}.$$

Now define

$$\begin{aligned}\mathfrak{R} = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1-\mathcal{E} \quad \text{or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E},\\ Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\}.\end{aligned}$$

Now we claim that $\mathfrak{A}^{\complement} \cap \mathfrak{D}^{\complement} \subseteq \mathfrak{R}^{\complement}$. Let $m \in \mathfrak{A}^{\complement} \cap \mathfrak{D}^{\complement}$. Then, $m \in \mathfrak{A}^{\complement}$ and $m \in \mathfrak{D}^{\complement}$. If $m \in \mathfrak{A}^{\complement}$, then

$$G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1-\mathcal{E}_1 \text{ and } B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, \tag{4.4}$$

and if $m \in \mathfrak{D}^{\complement}$, then $m = \kappa(j_0)$ for $j_0 \in \mathbb{N}$ and

$$G_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1-\mathcal{E}_1 \text{ and } B_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1. \tag{4.5}$$

Now,

$$\begin{aligned} G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) &= G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \oplus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\ &\geq G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledast G_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\ &> (1-\mathcal{E}_1) \circledast (1-\mathcal{E}_1) \quad \text{for } \rho = \kappa(j_0)\\ &> 1-\mathcal{E} \end{aligned}$$

and

$$\begin{aligned} B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) &= B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \oplus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\ &\leq B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc B_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\ &< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \quad \text{for } \rho = \kappa(j_0)\\ &< \mathcal{E},\\ Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) &= Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \oplus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right)\\ &\leq Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc Y_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right)\\ &< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \quad \text{for } \rho = \kappa(j_0)\\ &< \mathcal{E}, \quad \text{by (4.4) and (4.5)}, \end{aligned}$$

which implies that $m \in \mathfrak{R}^{\complement}$, so $\mathfrak{A}^{\complement} \cap \mathfrak{D}^{\complement} \subseteq \mathfrak{R}^{\complement}$ or $\mathfrak{R} \subseteq \mathfrak{A} \cup \mathfrak{D}$. Therefore, $\delta(\mathfrak{R}) \leq \delta(\mathfrak{A} \cup \mathfrak{D}) = 0$. This shows that $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistically convergent and therefore statistically complete. □

5. Conclusion

The neutrosophic soft norm is a very powerful tool due to its parameterized nature to analyze many problems arising in different areas such as **decision-making**, **pattern recognition**, **medical diagnosis**, **data mining**, and **deriving insights from data**, particularly when there is inherent uncertainty in the data. In this paper, we introduce the ideas of statistical convergence, statistical Cauchy sequences, and statistical completeness in a more general setting, i.e., in neutrosophic soft normed linear spaces. The results presented in this paper will be helpful in analyzing many problems where the fuzzy norm is not sufficient to work, and we look forward to a generalized norm like the neutrosophic soft norm.

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WHEN ARE MULTIPLICATIVE (GENERALIZED)-(σ,τ)-DERIVATIONS ADDITIVE?

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Abstract. Let R be an associative ring. A multiplicative (generalized)-(σ,τ)-derivation F is a map on R satisfying $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ for all $x, y \in R$, where σ, τ are homomorphisms on R and g is any map on R. In this article, we have obtained some conditions on R, which make both F and g additive.

1. Introduction

The study of the additivity of mappings on rings as well as operator algebras has been an active area of research. Rickart [10] and Johnson [7] raised questions about when a multiplicative isomorphism becomes additive. Both imposed some sort of minimality conditions on ring R and answered it. Martindale [8] answered the above questions under some restriction on R which contains a family of idempotent elements. Daif et al. [1] introduced the definition of multiplicative derivation on R by choosing a mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ and proved that if R contains nontrivial idempotent elements then any multiplicative derivation is additive. Lu and Xie [3] established a condition on R, in the case where R may not contain any non-zero idempotents, that assures that a multiplicative isomorphism is additive, which generalizes Martindale's result. As an application, they showed that under a mild assumption, every multiplicative isomorphism from the radical of a nest algebra onto an arbitrary ring is additive.

Now let us recall the basic definition of Peirce decomposition. Let e in R be an idempotent element so that $e \neq 1, e \neq 0$ (R need not have an identity). We will

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formally put $e_1 = e$ and $e_2 = 1 - e$. Then, for $R_{ij} = e_i R e_j$, where $i, j = 1, 2$, one may write R in its Peirce decomposition as $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$, i.e., $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. An element of the subring R_{ij} will be denoted by a_{ij}. More transparently, e induces on R the generalized matrix ring structure

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix},$$

with the obvious matrix addition and multiplication. Here eRe, $eR(1-e)$, $(1-e)Re$ and $(1-e)R(1-e)$ are abelian subgroups of R.

A map $F : R \to R$ is called a multiplicative left centralizer if $F(xy) = F(x)y$ for all $x, y \in R$. In [12], M. S. Tammam El-Sayiad, M. N. Daif, and V. De Filippis proved especially the result for the additivity of the multiplicative left centralizers in prime and semiprime rings which contain an idempotent element. A map F on R is called a multiplicative generalized derivation of R if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ and some derivation d of R. Similarly, a map F on R is called a multiplicative semi-derivation of R if $F(xy) = F(x)g(y) + xF(y) = F(x)y + g(x)F(y)$ and $F(g(x)) = g(F(x))$ for all $x, y \in R$, where g is any map on R. Daif et al. [2, Theorem 2.1] and Siddeeque et al. [11, Theorem 2.1] proved the additivity of a multiplicative generalized derivation and multiplicative semi-derivation on an arbitrary ring under certain conditions, respectively.

Let $\Re$ be a ring and σ, τ be two endomorphisms on $\Re$. An additive mapping $F : \Re \to \Re$ is called a generalized (σ, τ)- derivation on $\Re$ if there exists a (σ, τ)-derivation $d : \Re \to \Re$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \Re$. A map on a ring $\Re$ defined as $x \to a\sigma(x) + \tau(x)b$, where a, b are fixed elements of $\Re$, called as generalized (σ, τ)-inner derivation, is an example of generalized-(σ, τ) derivation. More details about derivation, multiplicative derivation, and generalized derivation can seen in [4,5], and [9]. Hou et al. [6] proved that if R contains nontrivial idempotent elements, then any multiplicative (σ, τ)-derivation is additive and such map is called (σ, τ)-derivation. We give the notion of multiplicative (generalized)-(σ, τ) derivation as below.

A multiplicative (generalized)-(σ, τ)-derivation is a map satisfying $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ for all $x, y \in \Re$, where g is any map on $\Re$. Similarly a map $F : \Re \to \Re$ is called a reverse multiplicative (generalized)-(σ, τ)-derivation on $\Re$ if $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$ holds for all $x, y \in \Re$. Here σ and τ are endomorphisms and g is any map on $\Re$.

Now, we construct an example to support the definition of multiplicative (generalized) - (σ, τ)- derivation, which is not additive as follows.

Example 1.1. Let $C[0, 1]$ be the ring of all complex-valued continuous functions defined on $[0, 1]$. It can be easily verified that $\Re = C[0, 1] \times C[0, 1]$, forms a ring with regard to component wise operations. Define the maps F, g, σ and $\tau : \Re \to \Re$ such that:

$$F\left(h(x), k(x)\right) = \begin{cases} \left(\bar{h}(x)\log|h(x)|, \bar{k}(x)\log|k(x)|\right), & \text{if } h(x) \neq 0 \text{ and } k(x) \neq 0, \\ \left(0, \bar{k}(x)\log|k(x)|\right), & \text{if } h(x) = 0 \text{ and } k(x) \neq 0, \\ \left(\bar{h}(x)\log|h(x)|, 0\right), & \text{if } h(x) \neq 0 \text{ and } k(x) = 0, \\ (0,0), & \text{if } h(x) = 0 \text{ and } k(x) = 0, \end{cases}$$

$g(h(x), k(x)) = F(h(x), k(x))$ and $\sigma\left(h(x), k(x)\right) = \tau\left(h(x), k(x)\right) = \left(\bar{h}(x), \bar{k}(x)\right)$, where $\bar{h}$ denotes the conjugate of h. It can be easily proved that σ, τ are automorphisms of $\Re$ and F is a multiplicative (generalized)-(σ, τ)-derivation of $\Re$, i.e., $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ for all $x, y \in \Re$. But F is not additive on $\Re$.

Motivated by the above-cited results, we have proved the additivity of multiplicative (generalized)-(σ, τ)-derivation under some conditions on $\Re$ as follows.

Theorem 1.1. *Let $\Re$ be an associative ring with identity containing an idempotent $e(e \neq 0, 1)$ which satisfies the following conditions:*

(*i*) $x\Re = (0) \implies x = 0$,
(*ii*) $\Re x = (0) \implies x = 0$.

If F is any multiplicative (generalized)-(σ, τ)-derivation on $\Re$, i.e., $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ holds for all $x, y \in \Re$, where σ, τ are endomorphisms on $\Re$ and g is any map on $\Re$ which is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.

For proof of Theorem 1.1, first we will prove some auxiliary results as follows.

Lemma 1.1. $F(0) = 0$.

Proof. By the definition of F, we have $F(0) = F(00) = F(0)\sigma(0) + \tau(0)g(0) = F(0)0 + 0g(0) = 0$, which completes the proof. □

Lemma 1.2. *For any $a_{11} \in \Re_{11}$, $a_{22} \in \Re_{22}$, $b_{12} \in \Re_{12}$ and $b_{21} \in \Re_{21}$, the following hold*

(*i*) $g(a_{11} + b_{21}) = g(a_{11}) + g(b_{21})$,
(*ii*) $g(a_{22} + b_{12}) = g(a_{22}) + g(b_{12})$.

Proof. We prove only (*i*), and the proof of (*ii*) goes similarly.

(*i*) For any $t_{n1} \in \Re_{n1}$ where $n \in \{1, 2\}$, we have

$$\begin{aligned} \tau(t_{n1})(g(a_{11}) + g(b_{21})) &= \tau(t_{n1})g(a_{11}) + \tau(t_{n1})g(b_{21}) \\ &= F(t_{n1}a_{11}) - F(t_{n1})\sigma(a_{11}) + F(t_{n1}b_{21}) - F(t_{n1})\sigma(b_{21}) \\ &= F(t_{n1}a_{11}) - F(t_{n1})\sigma(a_{11}) - F(t_{n1})\sigma(b_{21}) \\ &= F(t_{n1}(a_{11} + b_{21})) - F(t_{n1})\sigma(a_{11} + b_{21}) \\ &= \tau(t_{n1})g(a_{11} + b_{21}). \end{aligned}$$

This implies that

$$\tau(t_{n1})[(g(a_{11}) + g(b_{21})) - g(a_{11} + b_{21})] = 0 \tag{1.1}$$

and

$$\begin{aligned}\tau(t_{n2})(g(a_{11})+g(b_{21})) &= \tau(t_{n2})g(a_{11})+\tau(t_{n2})g(b_{21})\\ &= F(t_{n2}a_{11})-F(t_{n2})\sigma(a_{11})+F(t_{n2}b_{21})-F(t_{n2})\sigma(b_{21})\\ &= F(t_{n2}b_{21})-F(t_{n2})\sigma(a_{11})-F(t_{n2})\sigma(b_{21})\\ &= F(t_{n2}(a_{11}+b_{21}))-F(t_{n2})\sigma(a_{11}+b_{21})\\ &= \tau(t_{n2})(a_{11}+b_{21}).\end{aligned}$$

Thus, we obtain that

$$\tau(t_{n2})[(g(a_{11})+g(b_{21}))-g(a_{11}+b_{21})]=0. \tag{1.2}$$

Using (1.1) and (1.2), we have $\Re[g(a_{11}+b_{21})-g(a_{11})-g(b_{21})]=(0)$. Using the hypothesis of Theorem 1.1, we get $g(a_{11}+b_{21})=g(a_{11})+g(b_{21})$. □

Lemma 1.3. *For any* $a_{11}\in\Re_{11},\ a_{22}\in\Re_{22},\ b_{12}\in\Re_{12},\ b_{21}\in\Re_{21}$ *and* $b_{22}\in\Re_{22}$, *we have the following:*

(i) $F(a_{11}+b_{12})=F(a_{11})+F(b_{12})$,
(ii) $F(a_{22}+b_{21})=F(a_{22})+F(b_{21})$,
(iii) $F(a_{11}+b_{22})=F(a_{11})+F(b_{22})$,
(iv) $F(a_{11}+b_{21})=F(a_{11})+F(b_{21})$,
(v) $F(a_{22}+b_{12})=F(a_{22})+F(b_{12})$.

Proof. Proofs of (i), (ii) and (iii) are similar to each other. Similarly the proofs of (iv) and (v) are on the same pattern. Therefore we prove only (i) and (iv).

(i) For any $t_{1n}\in\Re_{1n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{11})+F(b_{12}))\sigma(t_{1n}) &= F(a_{11})\sigma(t_{1n})+F(b_{12})\sigma(t_{1n})\\ &= F(a_{11}t_{1n})-\tau(a_{11})g(t_{1n})+F(b_{12}t_{1n})-\sigma(b_{12})g(t_{1n})\\ &= F((a_{11}+b_{12})t_{1n})-\tau(a_{11}+b_{12})g(t_{1n})\\ &= F(a_{11}+b_{12})\sigma(t_{1n}).\end{aligned}$$

Thus, we obtain that

$$[F(a_{11}+b_{12})-F(a_{11})-F(b_{12})]\sigma(t_{1n})=0. \tag{1.3}$$

For any $t_{2n}\in\Re_{2n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{11})+F(b_{12}))\sigma(t_{2n}) &= F(a_{11})\sigma(t_{2n})+F(b_{12})\sigma(t_{2n})\\ &= F(a_{11}t_{2n})-\tau(a_{11})g(t_{2n})+F(b_{12}t_{2n})-\tau(b_{12})g(t_{2n})\\ &= F((a_{11}+b_{12})t_{2n})-\tau(a_{11})g(t_{2n})-\tau(b_{12})g(t_{2n})\\ &= F((a_{11}+b_{12})t_{2n})-\tau(a_{11}+b_{12})g(t_{2n})\\ &= F(a_{11}+b_{12})\sigma(t_{2n}).\end{aligned}$$

This implies that

$$[F(a_{11}+b_{12})-F(a_{11})-F(b_{12})]\sigma(t_{2n})=0. \tag{1.4}$$

Using (1.3) and (1.4), we arrive at $[F(a_{11}+b_{12})-F(a_{11})-F(b_{12})]\Re=(0)$. Using the hypothesis of Theorem 1.1, we get $F(a_{11}+b_{12})=F(a_{11})+F(b_{12})$.

(*iv*) For any $t_{2n}\in\Re_{2n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{11})+F(b_{21}))\sigma(t_{2n}) &= F(a_{11})\sigma(t_{2n})+F(b_{21})\sigma(t_{2n})\\ &= F(a_{11}t_{2n})-\tau(a_{11})g(t_{2n})+F(b_{21}t_{2n})-\tau(b_{21})g(t_{2n})\\ &= -\tau(a_{11}+b_{21})g(t_{2n})\\ &= -F((a_{11}+b_{21})t_{2n})+F(a_{11}+b_{21})\sigma(t_{2n})\\ &= F(a_{11}+b_{21})\sigma(t_{2n}).\end{aligned}$$

This implies that

$$[F(a_{11}+b_{21})-F(a_{11})-F(b_{21})]\sigma(t_{2n})=0. \tag{1.5}$$

For any $t_{1n}\in\Re_{1n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}&F(a_{11}+b_{21})\sigma(t_{1n}) = F((a_{11}+b_{21})t_{1n})-\tau(a_{11}+b_{21})g(t_{1n})\\ &=F((e_2+a_{11})(t_{1n}+b_{21}t_{1n}))-\tau(a_{11}+b_{21})g(t_{1n})\\ &=F(e_2+a_{11})\sigma(t_{1n}+b_{21}t_{1n})+\tau(e_2+a_{11})g(t_{1n}+b_{21}t_{1n})-\tau(a_{11}+b_{21})g(t_{1n}).\end{aligned}$$

Using Lemma 1.3 (*iii*) and Lemma 1.2, we have

$$\begin{aligned}&F(a_{11}+b_{21})\sigma(t_{1n})\\ &=F(e_2)\sigma(t_{1n})+F(e_2)\sigma(b_{21}t_{1n})+F(a_{11})\sigma(t_{1n})\\ &\quad+F(a_{11})\sigma(b_{21}t_{1n})+\tau(e_2)g(t_{1n})+\tau(e_2)g(b_{21}t_{1n})+\tau(a_{11})g(t_{1n})\\ &\quad+\tau(a_{11})g(b_{21}t_{1n})-\tau(a_{11})g(t_{1n})-\tau(b_{21})g(t_{1n})\\ &=F(e_2t_{1n})+F(e_2b_{21}t_{1n})+F(a_{11}t_{1n})+F(a_{11}b_{21}t_{1n})-\tau(a_{11})g(t_{1n})-\tau(b_{21})g(t_{1n})\\ &=F(b_{21}t_{1n})+F(a_{11}t_{1n})-\tau(a_{11})g(t_{1n})-\tau(b_{21})g(t_{1n})\\ &=F(b_{21})\sigma(t_{1n})+F(a_{11})\sigma(t_{1n})\\ &=(F(b_{21})+F(a_{11}))\sigma(t_{1n}).\end{aligned}$$

This shows that

$$[F(a_{11}+b_{21})-F(a_{11})-F(b_{21})]\sigma(t_{1n})=0. \tag{1.6}$$

Using (1.5) and (1.6), we have $[F(a_{11}+b_{21})-F(a_{11})-F(b_{12})]\Re=(0)$. With the help of hypothesis of Theorem 1.1, we obtain $F(a_{11}+b_{21})=F(a_{11})+F(b_{21})$. □

Lemma 1.4. *F is additive on $\Re_{12}$.*

Proof. Let $a_{12},b_{12}\in\Re_{12}$ and $t_{1n}\in\Re_{1n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{12})+F(b_{12}))\sigma(t_{1n}) &=F(a_{12})\sigma(t_{1n})+F(b_{12})\sigma(t_{1n})\\ &=F(a_{12}t_{1n})-\tau(a_{12})g(t_{1n})+F(b_{12}t_{1n})-\tau(b_{12})g(t_{1n})\\ &=-\tau(a_{12}+b_{12})g(t_{1n})\\ &=-F((a_{12}+b_{12})t_{1n})+F(a_{12}+b_{12})\sigma(t_{1n})\end{aligned}$$

$$=F(a_{12}+b_{12})\sigma(t_{1n}).$$

This gives us

$$[F(a_{12}+b_{12})-F(a_{12})-F(b_{12})]\sigma(t_{1n})=0. \tag{1.7}$$

For any $t_{2n}\in\Re_{2n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}
&F(a_{12}+b_{12})\sigma(t_{2n})\\
=&F((a_{12}+b_{12})t_{2n})-\tau(a_{12}+b_{12})g(t_{2n})\\
=&F((e+a_{12})(t_{2n}+b_{12}t_{2n}))-\tau(a_{12}+b_{12})g(t_{2n})\\
=&F(e+a_{12})\sigma(t_{2n}+b_{12}t_{2n})+\tau(e+a_{12})g(t_{2n}+b_{12}t_{2n})-\tau(a_{12}+b_{12})g(t_{2n}).
\end{aligned}$$

Using Lemma 1.3 (i) and Lemma 1.2, we obtain

$$\begin{aligned}
&F(a_{12}+b_{12})\sigma(t_{2n})\\
=&F(e)\sigma(t_{2n})+F(e)\sigma(b_{12}t_{2n})+F(a_{12})\sigma(t_{2n})+F(a_{12})\sigma(b_{12}t_{2n})+\tau(e)g(t_{2n})\\
&+\tau(e)g(b_{12}t_{2n})+\tau(a_{12})g(t_{2n})+\tau(a_{12})g(b_{12}t_{2n})-\tau(a_{12})g(t_{2n})-\tau(b_{12})g(t_{2n})\\
=&F(et_{2n})+F(eb_{12}t_{2n})+F(a_{12}t_{2n})+F(a_{12}b_{12}t_{2n})-\tau(a_{12})g(t_{2n})-\tau(b_{12})g(t_{2n})\\
=&F(b_{12}t_{2n})+F(a_{12}t_{2n})-\tau(a_{12})g(t_{2n})-\tau(b_{12})g(t_{2n})\\
=&F(b_{12})\sigma(t_{2n})+F(a_{12})\sigma(t_{2n})\\
=&(F(b_{12})+F(a_{12}))\sigma(t_{2n}).
\end{aligned}$$

This implies that

$$[F(a_{12}+b_{12})-F(a_{12})-F(b_{12})]\sigma(t_{2n})=0. \tag{1.8}$$

Using (1.7) and (1.8), we have $[F(a_{12}+b_{12})-F(a_{12})-F(b_{12})]\Re=(0)$. Using the hypothesis of Theorem 1.1, we get F is additive on $\Re_{12}$. ☐

Lemma 1.5. *F is additive on $\Re_{21}$.*

Proof. Proof is similar to Lemma 1.4. ☐

Lemma 1.6. *F is additive on $\Re_{11}$.*

Proof. Let $a_{11},b_{11}\in\Re_{11}$. We have,

$$F(a_{11}+b_{11})=F(e(a_{11}+b_{11}))=F(e)\sigma(a_{11}+b_{11})+\tau(e)g(a_{11}+b_{11}).$$

Since g is additive on $\Re_{11}$, we get $F(a_{11}+b_{11})=F(a_{11})+F(b_{11})$. ☐

Lemma 1.7. *F is additive on $\Re_{11}+\Re_{12}=e\Re$.*

Proof. Let $a_{11}+a_{12},b_{11}+b_{12}\in e\Re$. We have,

$$F((a_{11}+a_{12})+(b_{11}+b_{12}))=F((a_{11}+b_{11})+(a_{12}+b_{12})).$$

Using Lemma 1.3 (i), we have

$$F((a_{11}+a_{12})+(b_{11}+b_{12}))=F(a_{11}+b_{11})+F(a_{12}+b_{12}).$$

Lemma 1.4 and Lemma 1.6, provide us

$$F((a_{11}+a_{12})+(b_{11}+b_{12})) = F(a_{11})+F(a_{12})+F(b_{11})+F(b_{12}).$$

With the help of Lemma 1.3 (i), we get

$$F((a_{11}+a_{12})+(b_{11}+b_{12})) = F(a_{11}+a_{12})+F(b_{11}+b_{12}).$$

That is, F is additive on $e\Re$. □

Lemma 1.8. *F is additive on $\Re_{22}$.*

Proof. Proof is similar as Lemma 1.6. □

Lemma 1.9. *F is additive on $\Re_{21}+\Re_{22} = e_2\Re = (1-e)\Re$.*

Proof. Let $a_{21}+a_{22}, b_{21}+b_{22} \in (1-e)\Re$. We have

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F((a_{21}+b_{21})+(a_{22}+b_{22})).$$

Using Lemma 1.3 (ii), we get

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F(a_{21}+b_{21})+F(a_{22}+b_{22}).$$

Lemma 1.5 and Lemma 1.8 provide us

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F(a_{21})+F(a_{22})+F(b_{21})+F(b_{22}).$$

Lemma 1.3 (ii) provides us

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F(a_{21}+a_{22})+F(b_{21}+b_{22}).$$

That is, F is additive on $(1-e)\Re$. □

Lemma 1.10. *F is additive on $\Re_{22}+\Re_{12} = \Re(1-e)$.*

Proof. Let $a_{22}+a_{12}, b_{22}+b_{12} \in \Re(1-e)$. We have

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F((a_{22}+b_{22})+(a_{12}+b_{12})).$$

Using Lemma 1.3 (v), we get

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F(a_{22}+b_{22})+F(a_{12}+b_{12}).$$

Lemma 1.4 and Lemma 1.8, provide us

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F(a_{22})+F(a_{12})+F(b_{22})+F(b_{12}).$$

With the help of Lemma 1.3 (v), we get

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F(a_{22}+a_{12})+F(b_{22}+b_{12}).$$

Hence, F is additive on $\Re(1-e)$. □

Lemma 1.11. *F is additive on $\Re_{11}+\Re_{21} = \Re e$.*

Proof. Let $a_{11} + a_{21}, b_{11} + b_{21} \in \Re e$. We have

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F((a_{11} + b_{11}) + (a_{21} + b_{21})).$$

By Lemma 1.3 (iv), we get

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F(a_{11} + b_{11}) + F(a_{21} + b_{21}).$$

Lemma 1.5 and Lemma 1.6, provide us

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F(a_{11}) + F(a_{21}) + F(b_{11}) + F(b_{21}).$$

Finally, we conclude by Lemma 1.3 (iv)

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F(a_{11} + a_{21}) + F(b_{11} + b_{21}).$$

That is, F is additive on $\Re e$. □

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. (i) First we prove that g is additive.
Let $t \in e\Re = \Re_{11} + \Re_{12},\ x, y \in \Re$. We have $tx, ty \in e\Re$.

$$\begin{aligned} \tau(t)(g(x) + g(y)) &= \tau(t)g(x) + \tau(t)g(y) \\ &= F(tx) - F(t)\sigma(x) + F(ty) - F(t)\sigma(y) \\ &= F(tx) + F(ty) - F(t)\sigma(x) - F(t)\sigma(y). \end{aligned}$$

Lemma 1.7 provides us

$$\tau(t)(g(x) + g(y)) = F(t(x + y)) - F(t)\sigma(x + y) = \tau(t)g(x + y).$$

This implies that

$$\tau(t)[(g(x) + g(y)) - g(x + y)] = 0. \tag{1.9}$$

Also, let $m \in (1 - e)\Re = \Re_{21} + \Re_{22}$. This shows that $mx, my \in (1 - e)\Re$,

$$\begin{aligned} \tau(m)(g(x) + g(y)) &= \tau(m)g(x) + \tau(m)g(y) \\ &= F(mx) - F(m)\sigma(x) + F(my) - F(m)\sigma(y) \\ &= F(mx) + F(my) - F(m)\sigma(x) - F(m)\sigma(y). \end{aligned}$$

Using Lemma 1.9, we get

$$\tau(m)(g(x) + g(y)) = F(m(x + y)) - F(m)\sigma(x + y) = \tau(m)(x + y).$$

This implies that

$$\tau(m)[(g(x) + g(y)) - g(x + y)] = 0. \tag{1.10}$$

On adding (1.9) and (1.10), we have

$$\tau(t + m)[g(x + y) - g(x) - g(y)] = 0.$$

Since τ is onto on $\Re$, we get

$$\Re[g(x + y) - g(x) - g(y)] = (0).$$

Using hypothesis, we conclude that

$$g(x+y) = g(x) + g(y).$$

(ii) Now, we prove that F is additive.

Let $t \in \Re e = \Re_{11} + \Re_{21},\ m \in \Re(1-e) = \Re_{22} + \Re_{12}$, and $a, b \in \Re$. Then $at, bt \in \Re e$ and $am, bm \in \Re(1-e)$,

$$\begin{aligned}(F(a) + F(b))\sigma(t) &= F(a)\sigma(t) + F(b)\sigma(t) = F(at) - \tau(a)g(t) + F(bt) - \tau(b)g(t)\\ &= F(at) + F(bt) - (\tau(a) + \tau(b))g(t).\end{aligned}$$

Using Lemma 1.11, we get

$$(F(a) + F(b))\sigma(t) = F((a+b)t) - \tau(a+b)g(t) = F(a+b)\sigma(t).$$

This implies that

$$[F(a+b) - F(a) - F(b)]\sigma(t) = 0. \tag{1.11}$$

Also,

$$\begin{aligned}(F(a) + F(b))\sigma(m) &= F(a)\sigma(m) + F(b)\sigma(m)\\ &= F(am) - \tau(a)g(m) + F(bm) - \tau(b)g(m)\\ &= F(am) + F(bm) - (\tau(a) + \tau(b))g(m).\end{aligned}$$

Lemma 1.10 provides us

$$(F(a) + F(b))\sigma(m) = F((a+b)m) - \tau(a+b)g(m) = F(a+b)\sigma(m).$$

This implies that

$$[F(a+b) - F(a) - F(b)]\sigma(m) = 0. \tag{1.12}$$

On adding (1.11)and (1.12), we have

$$[F(a+b) - F(a) - F(b)]\sigma(t+m) = 0.$$

Since σ is onto on $\Re$, we conclude that

$$[F(a+b) - F(a) - F(b)]\Re = (0).$$

Using hypothesis, we get $F(a+b) = F(a) + F(b)$, i.e., F is additive. □

Corollary 1.1. *Let $\Re$ be a semi-prime ring with identity containing an idempotent $e \neq 0, 1$. If F is any multiplicative (generalized)-(σ, τ)-derivation on $\Re$, i.e., $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ holds for all $x, y \in \Re$, where σ, τ are endomorphisms on $\Re$ and g is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.*

Now we construct an example to support the necessity of the condition that "g is additive on both $\Re_{11}$ and $\Re_{22}$" in Theorem 1.1 for additivity of F on $\Re$.

Example 1.2. Let S be an integral domain ring with the unity and

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in S \right\}$$

be the ring of upper triangular matrices over S. Let $C[0,1]$ be the ring of all complex-valued continuous functions defined on $[0,1]$. It can be easily shown that $\Re = C[0,1] \times M$ forms a ring concerning component-wise operations. Define the maps F, g, σ and $\tau : \Re \to \Re$ as follows

$$F\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{cases} \left(\bar{f}(x)\log|\bar{f}(x)|, \begin{bmatrix} 0 & a+c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) \neq 0, \\ \left(0, \begin{bmatrix} 0 & a+c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) = 0, \end{cases}$$

$$\sigma\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \left(\bar{f}(x), \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix}\right), \tau\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \left(\bar{f}(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right),$$

$$g\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{cases} \left(\bar{f}(x)\log|\bar{f}(x)|, \begin{bmatrix} 0 & a-c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) \neq 0, \\ \left(0, \begin{bmatrix} 0 & a-c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) = 0, \end{cases}$$

where $\bar{f}$ denotes the conjugate of f. One can verify that $\Re$ satisfies both the conditions (i) and (ii) of Theorem 1.1 and also both σ, τ are automorphisms. F is a multiplicative (generalized)-(σ, τ) derivation, *i.e.*, $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ for all $x, y \in \Re$. Obviously a non trivial idempotent element of the ring $\Re$ is $e = \left(\mathbf{1}(x), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$ and then $1 - e = \left(\mathbf{0}(x), \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$, where $\mathbf{1}(x)$ and $\mathbf{0}(x)$ are the constant functions on $[0,1]$ defined as; $\mathbf{1}(x) = 1$ for all $x \in [0,1]$ and $\mathbf{0}(x) = 0$ for each $x \in [0,1]$. Then,

$$\Re_{11} = \left\{ \left(f(x), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \mid f(x) \in C[0,1], a \in S \right\}$$

and

$$\Re_{22} = \left\{ \left(\mathbf{0}(x), \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\right) \mid \mathbf{0}(x) \in C[0,1], a \in S \right\}.$$

Clearly, it can be proved that g is additive on $\Re_{22}$ but not additive on $\Re_{11}$. But F is not additive on $\Re$.

Similarly, if we choose another non-trivial idempotent element as $e_1 = \left(\mathbf{0}(x), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$, then $(1 - e_1) = \left(\mathbf{1}(x), \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$. In this case,

$$\Re_{11} = \left\{ \left(\mathbf{0}(x), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \mid a \in S \right\}$$

and

$$\Re_{22} = \left\{ \left(f(x), \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\right) \mid f(x) \in C[0,1], a \in S \right\}.$$

Here, one can observe that g is additive on $\Re_{11}$ but not additive on $\Re_{22}$. However, F is not additive on $\Re$.

Theorem 1.2. *Let $\Re$ be an associative ring with identity containing an idempotent $e(e \neq 0, 1)$ which satisfies the conditions (i) and (ii) of Theorem 1.1. If F is any reverse-multiplicative (generalized)-(σ,τ)-derivation on $\Re$, i.e., $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$ holds for all $x, y \in \Re$, where σ, τ are endomorphisms on $\Re$ and g is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.*

Proof. The proof is in the same pattern as done for multiplicative (generalized)-(σ,τ)-derivation in Theorem 1.1. ☐

Corollary 1.2. *Let $\Re$ be a semi-prime ring with identity containing an idempotent $e \neq 0, 1$. If F is any reverse-multiplicative (generalized)-(σ,τ)-derivation on $\Re$, i.e., $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$ holds for all $x, y \in \Re$, where σ, τ are endomorphisms on $\Re$ and g is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.*

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ON THE ZEROS OF POLYNOMIALS WITH REAL COEFFICIENTS

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Abstract. The Eneström-Kakeya theorem provides essential bounds on the location of the zeros of a polynomial with positive coefficients. Lot of research work has been done regarding the classical theorem known as Eneström-Kakeya theorem concerning the regions containing zeros of a polynomial. This theorem states that if $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ is a polynomial with degree n with real coefficients satisfying $0 \leq f_0 \leq f_1 \leq f_2 \leq \cdots \leq f_n$, then all the zeros of $F(z)$ lie in $|z| \leq 1$. In this article, we prove several extensions of this theorem which impose restrictions only on the coefficients $f_0, f_1, \ldots, f_{n-1}$ and leaves the coefficient f_n to vary freely over the whole complex plane.

1. Introduction

The classical Eneström-Kakeya Theorem gives us information about the position of the zeros of a polynomial whose coefficients are nonnegative and satisfy a monotonicity condition. It was independently proved by G. Eneström in 1893 [6] and Kakeya in 1912 [12].

Theorem 1.1 (Eneström-Kakeya Theorem). *If $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ is a polynomial of degree n with real coefficients satisfying $0 \leq f_0 \leq f_1 \leq \cdots \leq f_n$, then $F(z)$ has all its zeros in the region $|z| \leq 1$.*

In literature, there exist several extentions and generalizations of Theorem 1.1 (see [1, 2], [5]-[10], [13]). Joyal, Labelle, and Rahman [11] extended Theorem 1.1 to the

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polynomials whose coefficients satisfy a monotonicity condition but need not be non negative. In fact, they proved the following result.

Theorem 1.2. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n with real coefficients satisfying the condition $f_0 \leq f_1 \leq \cdots \leq f_n$. Then, $F(z)$ has all its zeros lying in the region*

$$|z| \leq \frac{1}{|f_n|}(f_n - f_0 + |f_0|).$$

In 1996, Aziz and Zargar [3] proved the following generalisation of Theorem 1.1.

Theorem 1.3. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n. If for some positive number k with $k \geq 1$, $0 \leq f_0 \leq f_1 \leq \cdots \leq f_{n-1} \leq kf_n$. Then, all the zeros of $F(z)$ lie in the disc*

$$|z + k - 1| \leq k.$$

In 2012, Aziz and Zargar [4] also proved the following generalization of Theorem 1.1.

Theorem 1.4. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n. If for some positive numbers k and s with $k \geq 1$, $0 < s \leq 1$, $0 \leq sf_0 \leq f_1 \leq \cdots \leq f_{n-1} \leq kf_n$, then all the zeros of $F(z)$ lie in the closed disc*

$$|z + k - 1| \leq k + \frac{2f_0}{f_n}(1 - s).$$

In 2015, E. R. Nwaeze [**?**] proved the following result concerning the zeros of polynomials, which is a generalization of Theorem 1.3.

Theorem 1.5. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n. If for some real numbers γ and δ, $f_0 - \delta \leq f_1 \leq \cdots \leq f_{n-1} \leq f_n + \gamma$, then all the zeros of $F(z)$ lie in the disc*

$$\left| z + \frac{\gamma}{f_n} \right| \leq \frac{1}{|f_n|}\left(f_n + \gamma - f_0 + \delta + |\delta| + |f_0| \right).$$

Aziz and Zargar [3] also relaxed the hypothesis of Theorem 1.2 in several ways and among other things they proved the following result.

Theorem 1.6. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial of degree n with real coefficients such that for some k with $k \geq 1$, $kf_n \geq f_{n-1} \geq \cdots \geq f_1 \geq f_0$. Then, all the zeros of $F(z)$ lie in the disc*

$$|z + k - 1| \leq \frac{kf_n - f_0 + |f_0|}{|f_n|}.$$

Shah and Liman [19] extended Theorem 1.6 to the polynomials with complex coefficients by proving the following result.

Theorem 1.7. *Let* $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ *be a polynomial of degree* n *with complex coefficients such that for some real* β, $|arg f_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \dots, n$, *and* $k \geq 1$

$$k|f_n| \geq |f_{n-1}| \geq \cdots \geq |f_1| \geq |f_0|.$$

Then, all the zeros of $F(z)$ *lie in*

$$|z + k - 1| \leq \frac{1}{|f_n|}\left\{(k|f_n| - |f_0)(\cos\alpha + \sin\alpha) + |f_0| + 2\sin\alpha \sum_{j=0}^{n-1}|f_j|\right\}.$$

Rather et al. [15] relaxed the hypothesis of Theorem 1.4 and they proved the following result.

Theorem 1.8. *Let* $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ *be a polynomial of degree* n *with real coefficients such that for some* $k_j \geq 1$, $f_{n-j+1} > 0$, $j = 1, 2, \dots, r$ *where* $1 \leq r \leq n$

$$k_1 f_n \geq k_2 f_{n-1} \geq k_3 f_{n-2} \geq \cdots \geq k_r f_{n-r+1} \geq f_{n-r} \geq f_1 \geq f_0.$$

Then, all the zeros of $F(z)$ *lie in*

$$\begin{aligned}&\left|z + k_1 - 1 - (k_2 - 1)\frac{f_{n-1}}{f_n}\right| \\ \leq &\frac{1}{|f_n|}\left\{k_1 f_n - (k_2 - 1)|f_{n-1}| + 2\sum_{j=2}^{r}(k_j - 1)|f_{n-j+1}| - f_0 + |f_0|\right\}.\end{aligned}$$

Rather et al. [17, 18], extended Theorem 1.7 to the polynomials with complex coefficients and proved following two results.

Theorem 1.9. *Let* $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ *be a polynomial of degree* n *with complex coefficients such that for some real* β, $|arg f_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \dots n$, *and* $k_j \geq 1$, $f_{n-j} \neq 0$, $j = 0, 1, \dots, r$, *where* $1 \leq r \leq n - 1$

$$k_0|f_n| \geq k_1|f_{n-1}| \geq k_2|f_{n-2}| \geq \cdots \geq k_r|f_{n-r}| \geq |f_{n-r-1}| \geq \cdots \geq |f_1| \geq |f_0|.$$

Then, all the zeros of $F(z)$ *lie in*

$$\begin{aligned}&\left|z + k_0 - 1 - (k_1 - 1)\frac{f_{n-1}}{f_n}\right| \\ \leq &\frac{1}{|f_n|}\left\{(k_0|f_n| - |f_0|)(\cos\alpha + \sin\alpha) + 2\sin\alpha\left(\sum_{j=1}^{r} k_j|f_{n-j}| + \sum_{j=r+1}^{n}|f_{n-j}|\right)\right. \\ &\left. - (k_1 - 1)|f_{n-1}| + 2\sum_{j=1}^{r}(k_j - 1)|f_{n-j}| + |f_0|\right\}.\end{aligned}$$

Theorem 1.10. *Let* $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$, $f_j = \alpha_j + i\gamma_j$ *be a polynomial of degree* n *with complex coefficients such that for some* $k_j \geq 1$, $\alpha_{n-j+1} > 0$, $j = 1, 2, \dots, r$, *where* $1 \leq r \leq n$

$$k_1\alpha_n \geq k_2\alpha_{n-1} \geq k_3\alpha_{n-2} \geq \cdots \geq k_r\alpha_{n-r+1} \geq \alpha_{n-r} \geq \cdots \geq \alpha_1 \geq \alpha_0.$$

Then, all the zeros of $F(z)$ lie in

$$\left|z+(k_1-1)\frac{\alpha_n}{f_n}-(k_2-1)\frac{\alpha_{n-1}}{f_n}\right|$$
$$\leq\frac{1}{|f_n|}\left[|k_1\alpha_n-(k_2-1)|\alpha_n-1|+2\left(\sum_{j=2}^{r}(k_j-1)|\alpha_{n-j+1}|+\sum_{j=0}^{n-1}|\gamma_j|\right)-\alpha_0+|\alpha_0|+|\gamma_n|\right].$$

Recently, Rather et al. [16], proved the following results.

Theorem 1.11. *Let $F(z)=\sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial of degree n with complex coefficients such that for some real β, $|arg(k_j+a_{n-j})-\beta|\leq\alpha\leq\frac{\pi}{2}$, $j=0,1,2,\dots n$, and for some number k_j, $j=0,1,\dots,r$, where $1\leq r\leq n-1$*

$$|k_0+f_n|\geq|k_1+f_{n-1}|\geq|k_2+f_{n-2}|\geq\cdots\geq|k_r+f_{n-r}|\geq|f_{n-r-1}|\geq\cdots\geq|f_1|\geq|f_0|.$$

Then, all the zeros of $F(z)$ lie in

$$\left|z+\frac{k_0-k_1}{f_n}\right|$$
$$\leq\frac{1}{|f_n|}\Bigg[(|k_0-f_n|-|f_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\left\{\sum_{j=1}^{r}(|k_j+f_{n-j}|)+\sum_{j=r+1}^{n}|f_{n-j}|\right\}$$
$$+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+|k_r|+|f_0|\Bigg].$$

Theorem 1.12. *Let $F(z)=\sum_{\lambda=0}^{n} f_\lambda z^\lambda$, where $f_j=\alpha_j+\iota\gamma_j$ be a polynomial of degree n with complex coefficients such that for some $k_j\geq 0$, $j=0,1,2,\dots r$, where $1\leq r\leq n-1$*

$$k_0+\alpha_n\geq k_1+\alpha_{n-1}\geq k_2+\alpha_{n-2}\geq\cdots\geq k_r+\alpha_{n-r}\geq\alpha_{n-r-1}\geq\cdots\geq\alpha_1\geq\alpha_0.$$

Then, all the zeros of $F(z)$ lie in

$$\left|z+\frac{k_0-k_1}{f_n}\right|\leq\frac{1}{|f_n|}\left\{\alpha_n+|\alpha_0|-\alpha_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|\sum_{j=1}^{r-1}|\gamma_{j+1}-\gamma_j|+|\gamma_0|+k_r+k_0\right\}.$$

2. Main Results and Proofs

In this article, we first give a result which is an extension of Theorem 1.1. In this result, only the coefficients $f_0,f_1,\dots,f_{n-1}$ satisfy a monotonicity condition, and the coefficient f_n moves freely in the complex plane. Our theorem provides a stronger result than the classic Eneström-Kakeya theorem, which is applicable to a broader class of polynomials. Infact, we first prove the following results.

Theorem 2.1. *Let $F(z)=\sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n such that the coefficients $f_0,f_1,\dots,f_{n-1}$ are real and satisfying the monotonicity condition* $0\leq f_0\leq$

$f_1 \leq \cdots \leq f_{n-1}$. *Then, all the zeros of* $F(z)$ *lie in the union of the regions*

$$\{z \in \mathbb{C} : |z| \leq 1\} \bigcup \left\{ z \in \mathbb{C} : \left| z - \left(1 - \frac{f_{n-1}}{f_n}\right) \right| \leq \frac{f_{n-1}}{|f_n|} \right\}.$$

Proof. Consider the polynomial

$$\begin{aligned} Q(z) &= (1-z)F(z) \\ &= -f_n z^{n+1} + (f_n - f_{n-1})z^n + (f_{n-1} - f_{n-2})z^{n-1} + \cdots + (f_1 - f_0)z + f_0 \\ &= -f_n z^{n+1} + (f_n - f_{n-1})z^n + \phi(z), \end{aligned} \tag{2.1}$$

where $\phi(z) = (f_{n-1} - f_{n-2})z^{n-1} + \cdots + (f_1 - f_0)z + f_0$. For $|z| = 1$, we have that

$$|\phi(z)| \leq |f_{n-1} - f_{n-2}| + |f_{n-2} - f_{n-3}| + \cdots + |f_1 - f_0| + |f_0| = f_{n-1}$$

implies $|z^n \phi(1/z)| \leq f_{n-1}$ for $|z| = 1$. Hence, By Maximum Modulus Theorem

$$|z^n \phi(1/z)| \leq f_{n-1}, \quad \text{for } |z| \leq 1.$$

Replacing z by $1/z$, we get

$$|\phi(z)| \leq f_{n-1}|z^n|, \quad \text{for } |z| \geq 1.$$

Therefore, for $|z| \geq 1$, from equation (2.1), we obtain

$$\begin{aligned} |Q(z)| &= |-f_n z^{n+1} + (f_n - f_{n-1})z^n + \phi(z)| \\ &\geq |z^n||f_n z - (f_n - f_{n-1})| - |\phi(z)| \\ &\geq |z^n||f_n z - (f_n - f_{n-1})| - f_{n-1}|z^n| \\ &= |z^n| \left[|f_n z - (f_n + f_{n-1})| - f_{n-1} \right] \\ &> 0, \end{aligned}$$

which is true if and only if $|f_n z - (f_n + f_{n-1})| > f_{n-1}$, i.e.,

$$\left| z - \left(1 - \frac{f_{n-1}}{f_n}\right) \right| > \frac{f_{n-1}}{|f_n|}.$$

Thus, all the zeros of $g(z)$ whose modulus is greater than or equal to one lie in

$$\left| z - \left(1 - \frac{f_{n-1}}{f_n}\right) \right| \leq \frac{f_{n-1}}{|f_n|}.$$

Hence, all the zeros of $F(z)$ lie in the union of disc

$$|z| \leq 1 \bigcup \left| z - \left(1 - \frac{f_{n-1}}{f_n}\right) \right| \leq \frac{f_{n-1}}{|f_n|}.$$

□

Example 2.1. Consider the polynomial

$$F(z) = 3z^4 + 5z^3 + 2z^2 + z + 1.$$

Here, $f_0 = 1,\ f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 3$, which satisfy $0 \le 1 \le 1 \le 2 \le 5$. We cannot apply Eneström-Kakeya theorem, because $0 \le f_0 \le f_1 \le f_2 \le f_3 \not\le f_4$. By Theorem 2.1, the polynomial $F(z)$ has all its zeros in the union of the regions

$$\{z \in \mathbb{C} : |z| \le 1\} \cup \left\{z \in \mathbb{C} : \left|z - \left(1 - \frac{f_2}{f_3}\right)\right| \le \frac{f_2}{|f_3|}\right\},$$

which is given by

$$\{z \in \mathbb{C} : |z| \le 1\} \cup \left\{z \in \mathbb{C} : \left|z + \left(\frac{2}{3}\right)\right| \le \frac{5}{3}\right\}.$$

This example illustrates the applications of Theorem 2.1 to a broader class of polynomials as against to the Eneström-Kakeya theorem.

Remark 2.1. If the polynomial $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ satisfies the conditions of Theorem 1.1, then the disc

$$\left|z - \left(1 - \frac{f_{n-1}}{f_n}\right)\right| \le \frac{f_{n-1}}{|f_n|}$$

is contained in the disc $|z| \le 1$. Hence, Theorem 2.1 reduces to Theorem 1.1.

Next, we prove the following result in which we impose the monotonicity condition on the coefficients $f_0, f_1, \ldots, f_{n-1}$ and leave the coefficient f_n to vary freely over the whole complex plane and thus broaden the scope of Theorem 1.5.

Theorem 2.2. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n. If for some real numbers γ and δ, $f_0 - \delta \le f_1 \le \cdots \le f_{n-2} \le f_{n-1} + \gamma$, then all the zeros of $F(z)$ lie in the union of regions*

$$\{z \in \mathbb{C} : |z| \le 1\} \bigcup \{z \in \mathbb{C} : |z - \zeta||z - \eta| \le \gamma^*\},$$

where ζ and η are the roots of the quadratic $f_n z^2 + (f_n - f_{n-1})z - \gamma$ and

$$\gamma^* = \frac{f_{n-1} + \gamma + \delta + |\delta| + |f_0| - f_0}{|f_n|}.$$

Proof. Consider the polynomial

$$\begin{aligned} L(z) =&(1 - z)F(z) \\ =&f_n z^{n+1} + (f_n - f_{n-1})\, z^n - \gamma z^{n-1} + \gamma z^{n-1} + (f_{n-1} - f_{n-2})z^{n-1} \\ &+ \cdots + ((f_1 - f_0) + \delta)\, z - \delta z + f_0 \\ =&z^{n-1}[f_n z^2 + ((f_n - f_{n-1})z - \gamma)] + \phi(z), \end{aligned} \tag{2.2}$$

where $\phi(z) = \gamma z^{n-1} + (f_{n-1} - f_{n-2})z^{n-1} + \cdots + (f_1 - f_0 + \delta)z - \delta z + f_0$.
For $|z| = 1$, we have

$$\begin{aligned} |\phi(z)| &\le |\gamma + f_{n-1} - f_{n-2}| + |f_{n-2} - f_{n-3}| + \cdots + |f_1 - f_0 + \delta| + |\delta| - |f_0| \\ &= \gamma + f_{n-1} + \delta + |\delta| + |f_0| - f_0. \end{aligned}$$

Hence, for $|z| = 1$, we have

$$|z^n\phi(1/z)| \leq \gamma + f_{n-1} + |\delta| + |\delta| - |f_0| - f_0.$$

Therefore, by Maximum Modulus Theorem

$$|z^n\phi(1/z)| \leq \gamma + f_{n-1} + \delta + |\delta| - |f_0| - f_0, \quad \text{for } |z| \leq 1.$$

Replacing z by $1/z$, we get for $|z| \geq 1$

$$|\phi(z)| \leq (\gamma + f_{n-1} + |\delta| + |\delta| - |f_0| - f_0)\,|z^{n-1}|, \quad \text{for } |z| \geq 1. \tag{2.3}$$

Therefore, for $|z| \geq 1$, from equation (2.2), we obtain

$$\begin{aligned}|L(z)| &\geq |z^{n-1}|\Big[|f_nz^2 + (f_n - f_{n-1})z - \gamma|\Big] - |\phi(z)| \\ &\geq |z^{n-1}|\Big[|f_nz^2 + (f_n - f_{n-1})z - \gamma|\Big] - \Big[(\gamma + f_{n-1} + \delta + |\delta| - |f_0| - f_0)\Big]|z^{n-1}| \\ &= |z^{n-1}|\Big[|f_nz^2 + (f_n - f_{n-1})z - \gamma| - (\gamma + f_{n-1} + \delta + |\delta| - |f_0| - f_0)\Big] \\ &> 0,\end{aligned}$$

which is true if and only if

$$|f_nz^2 + (f_n - f_{n-1})z - \gamma| > \gamma + f_{n-1} + \delta + |\delta| + |f_0| - f_0,$$

i.e.,

$$|(z-\eta)(z-\zeta)| > \frac{\gamma + f_{n-1} + \delta + |\delta| + |f_0| - f_0}{|f_n|}$$

or

$$|(z-\eta)(z-\zeta)| > \gamma^*.$$

Hence, all the zeros of $F(z)$ lie in the union of disc

$$|z| \leq 1 \bigcup \{z \in \mathbb{C} : |z-\zeta| \cdot |z-\eta| \leq \gamma^*\},$$

where ζ and η are the roots of the quadratic $f_nz^2 + (f_n - f_{n-1})z - \gamma$ and

$$\gamma^* = \frac{f_{n-1} + \gamma + \delta + |\delta| + |f_0| - f_0}{|f_n|}.$$

□

Example 2.2. Consider the polynomial

$$F(z) = z^4 + 5z^3 + 7z^2 + 6z + 10.$$

Here, we can't apply Theorem 1.5 because $f_2 \nleq f_3$. Now if we choose $\delta = 5$, $\gamma = 2$. Then, $f_2 = 7 \leq f_3 + \gamma$.

Therefore, by Theorem 2.2, all the zeros of $F(z)$ lie in the union of the regions

$$|z| \leq 1 \bigcup \{z \in \mathbb{C} : |z-\zeta||z-\eta| \leq \gamma^*\},$$

where ζ and η are roots of the quadratic equation

$$f_4z^2 + (f_4 - f_3)z - \gamma = 0.$$

That is $z^2 - 4z - 2 = 0$. This gives $\zeta = 8 + \sqrt{6}$, $\eta = 8 - \sqrt{6}$. Also,

$$\gamma^* = \frac{f_2 + \gamma + \delta + |\delta| + |f_0| - f_0}{|f_3|}$$

implies

$$\gamma^* = \frac{19}{5} = 3.8.$$

Therefore, all the zeros of $F(z)$ lie in the union of the regions

$$\{z \in \mathbb{C} : |z| \leq 1\} \bigcup \{z \in \mathbb{C} : |z - (8 + \sqrt{6})| \cdot |z - (8 - \sqrt{6})| \leq 3.8\}.$$

This specific example demonstrates the application of Theorem 2.2 to a broader class of polynomials as against Theorem 1.5.

We now prove a result which is a generalization of Theorem 1.3. In this result, the bond of the disc containing all the zeros of a polynomial under certain restricted conditions on the coefficients, involves also the coefficients of the polynomial. In fact, we prove the following result.

Theorem 2.3. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n with real coefficients such that for some real numbers k and λ*

$$kf_n \geq f_{n-1} \quad and \quad \lambda f_j \geq f_{j-1}, \quad for\ j = 0, 1, \ldots, n-1,\ f_{-1} = 0.$$

Then, all the zeros of $f(z)$ lie in the union of the discs

$$\{z \in \mathbb{C}\ /\ |z| \leq 1\} \bigcup \left\{z \in \mathbb{C}\ /\ |z - \lambda + k| \leq \frac{1}{|f_n|}\left(kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i\right)\right\}.$$

Proof. Consider the polynomial

$$\begin{aligned} G(z) =& (\lambda - z)F(z) \\ =& - f_n z^{n+1} + (\lambda f_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2})z^{n-1} + \cdots + (\lambda f_1 - f_0)z + \lambda f_0 \\ =& - f_n z^{n+1} + \lambda f_n z^n - kf_n z^n + (kf_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2})z^{n-1} \\ & + \cdots + (\lambda f_1 - f_0)z + \lambda f_0. \end{aligned}$$

Thus, we can write

$$G(z) = -z^n \left[f_n z - \lambda f_n + kfn\right] + \phi(z), \tag{2.4}$$

where $\phi(z) = (kf_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2})z^{n-1} + \cdots + (\lambda f_1 - f_0)z + \lambda f_0$.

For $|z| = 1$, we have

$$\begin{aligned} |\phi(z)| &\leq |kf_n - f_{n-1}| + |\lambda f_{n-1} - f_{n-2}| + \cdots + |\lambda f_1 - f_0| + |\lambda f_0| \\ &= (kf_n - f_{n-1}) + (\lambda f_{n-1} - f_{n-2} + \lambda f_{n-2} - f_{n-3} + \cdots + \lambda f_1 - f_0 + \lambda f_0) \\ &= kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i, \end{aligned}$$

If $|z| = 1$, then $|\frac{1}{z}| = 1$. Therefore, we can write

$$|z^n \phi(1/z)| \le kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i.$$

Hence, by Maximum Modulus theorem for $|z| \le 1$, we observe the following

$$|z^n \phi(1/z)| \le kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i.$$

Replacing z by $1/z$, we obtain for $|z| \ge 1$

$$|\phi(z)| \le \left(kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i\right)|z|^n.$$

Therefore, for $|z| \ge 1$, from (2.3), we have

$$\begin{aligned} |G(z)| &\ge |z|^n|f_n z - \lambda f_n + kf_n)| - |\phi(z)| \\ &\ge |z^n|\left\{|f_n z - \lambda f_n + kf_n| - \left(kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i\right)\right\} \\ &> 0 \end{aligned}$$

if and only if

$$|z - \lambda + k| > \frac{1}{|f_n|}\left(kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i\right).$$

Thus, all the zeros of $G(z)$ whose modulus is greater than or equal to one lie in the disc

$$|z - \lambda + k| \le \frac{1}{|f_n|}\left(kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i\right).$$

Therefore, all the zeros of $F(z)$ lie in the union of the discs

$$\{z \in \mathbb{C} : |z| \le 1\} \bigcup \left\{z \in \mathbb{C} : |z - \lambda + k| \le \frac{1}{|f_n|}\left(kf_n + (\lambda - 1)\sum_{i=0}^{n-1} f_i\right)\right\}.$$

This completes the proof. □

For $\lambda = 1$, we obtain the following generalization of Theorem 1.3, which is true for any real number k.

Corollary 2.1. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n with real coefficients such that for some real number k,*

$$0 \le f_0 \le f_1 \cdots \le f_{n-1} \le kf_n.$$

Then all the zeros of $F(z)$ lie in the union of the discs

$$\{z \in \mathbb{C} \ : \ |z| \le 1\} \bigcup \left\{z \in \mathbb{C} \ : \ |z - 1 + k| \le \frac{kf_n}{|f_n|}\right\}.$$

Finally, we prove the following result which is an extention of both Theorem 1.2 and Theorem 1.4. In this result, the bond of the disc containing all the zeros of a polynomial under certain restricted conditions on the coefficients, involves also the coefficients of the polynomial. In fact, we prove the following result.

Theorem 2.4. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n with real coefficients such that for some real numbers k, λ and $0 \leq \rho \leq 1$*

$$kf_n \geq f_{n-1},\ \lambda f_j \geq f_{j-1}, \quad \text{for } j = 2, 3, \dots, , n-1 \text{ and } \lambda f_1 \geq \rho f_0.$$

Then all the zeros of $F(z)$ lie in the union of the discs

$$\{z \in \mathbb{C} \ : \ |z| \leq 1\} \bigcup \left\{ z \in \mathbb{C} \ : \ |z - \lambda + k| \leq \frac{1}{|f_n|} \left(kf_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i + (1 - 2\rho) f_0 + |\lambda f_0| \right) \right\}.$$

Proof. Consider the polynomial

$$\begin{aligned} H(z) =&(\lambda - z)F(z) \\ =& - f_n z^{n+1} + (\lambda f_n - f_{n-1}) z^n + (\lambda f_{n-1} - f_{n-2}) z^{n-1} + \cdots + (\lambda f_1 - f_0) z + \lambda f_0 \\ =& - f_n z^{n+1} + \lambda f_n z^n - k f_n z^n + (k f_n - f_{n-1}) z^n + (\lambda f_{n-1} - f_{n-2}) z^{n-1} \\ &+ \cdots + (\lambda f_2 - f_1) z^2 \\ &+ (\lambda f_1 - \rho f_0) z - (1 - \rho) f_0 z + \lambda f_0. \end{aligned}$$

Thus, we can write

$$H(z) = -z^n \left[f_n z - \lambda f_n + k f_n \right] + \phi(z), \tag{2.5}$$

where $\phi(z) = (kf_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2})z^{n-1} + \cdots + (\lambda f_2 - f_1)z^2 + (\lambda f_1 - \rho f_0)z - (1-\rho) f_0 z + \lambda f_0$. For $|z| = 1$, we have

$$\begin{aligned} |\phi(z)| \leq& |kf_n - f_{n-1}| + |\lambda f_{n-1} - f_{n-2}| + \cdots + |\lambda f_2 - f_1| + |\lambda f_1 - \rho f_0| \\ &+ (1-\rho)|f_0| + |\lambda f_0| \\ =&(kf_n - f_{n-1}) + (\lambda f_{n-1} - f_{n-2}) + \cdots + (\lambda f_2 - f_1) + (\lambda f_1 - \rho f_0) \\ &+ (1-\rho)|f_0| + |\lambda f_0| \\ =&kf_n + (\lambda - 1)\left(f_{n-1} + f_{n-2} + f_{n-3} + \cdots + f_1\right) - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0| \\ =&kf_n + (\lambda - 1)\sum_{i=1}^{n-1} a_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|. \end{aligned}$$

If $|z| = 1$, then $|\frac{1}{z}| = 1$. Therefore, we can write

$$|z^n \phi(1/z)| \leq kf_n + (\lambda - 1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|$$

Hence, by Maximum Modulus theorem for $|z| \leq 1$, we observe the following

$$|z^n \phi(1/z)| \leq kf_n + (\lambda - 1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|. \tag{2.6}$$

Replacing z by $1/z$, we obtain for $|z| \geq 1$

$$|\phi(z)| \leq \left(kf_n + (\lambda-1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right)|z|^n.$$

Therefore, for $|z| \geq 1$, from equation (2.6), we get

$$\begin{aligned}|H(z)| &\geq |z|^n|f_n z - \lambda f_n + kf_n)| - |\phi(z)|\\ &\geq |z^n|\left\{|f_n z - \lambda f_n + kf_n| - \left(kf_n + (\lambda-1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right)\right\}\\ &> 0,\end{aligned}$$

if and only if

$$|z - \lambda + k| > \frac{1}{|f_n|}\left(kf_n + (\lambda-1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right).$$

Thus, all the zeros of $H(z)$ whose modulus is greater than or equal to one lie in the disc

$$|z - \lambda + k| \leq \frac{1}{|f_n|}\left(kf_n + (\lambda-1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right).$$

Therefore, all the zeros of $F(z)$ lie in the union of discs

$$\{z \in \mathbb{C}: \ |z| \leq 1\} \bigcup \left\{z \in \mathbb{C}: \ |z - \lambda + k| \leq \frac{1}{|f_n|}\left(kf_n + (\lambda-1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right)\right\}.$$

□

For $\lambda = 1$, we obtain the following generalization of Theorem 1.4, which is true for any positive real number k.

Corollary 2.2. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ is a polynomial of degree n. If for some positive numbers k and ρ such that $\rho \leq 1$, $0 \leq \rho f_0 \leq f_1 \leq \cdots \leq f_{n-1} \leq kf_n$, then all the zeros of $F(z)$ lie in the union of the discs*

$$\{z \in \mathbb{C} \ : \ |z| \leq 1\} \bigcup \left\{z \in \mathbb{C} \ : \ |z + k - 1| \leq \left(k + (1-\rho)\frac{2f_0}{f_n}\right)\right\}.$$

For $\lambda = 1$ and $\rho = 1$, we obtain the following generalization of Theorem 1.2.

Corollary 2.3. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ is a polynomial of degree n such that for some real number k, $f_0 \leq f_1 \leq \cdots \leq f_{n-1} \leq kf_n$. Then all the zeros of $F(z)$ lie in the union of the discs*

$$\{z \in \mathbb{C} \ / \ |z| \leq 1\} \bigcup \left\{z \in \mathbb{C} \ : \ |z| \leq \frac{1}{|f_n|}(kf_n - f_0 + |f_0|)\right\}.$$

3. Conclusion

In this paper, we have presented a significant generalization of the classical Eneström-Kakeya theorem, broadening its scope and applicability to a wider class of polynomials. By relaxing the traditional conditions on polynomial coefficients, we have established new results that constrain the zeros of these generalized polynomials within specific geometric regions in the complex plane.

Our work began with a review of the original Eneström-Kakeya theorem, highlighting its historical importance and the foundational role it plays in understanding polynomial zero distributions. Building upon this foundation, we introduced our generalized conditions and provided rigorous proofs to demonstrate the validity of our results. Through illustrative examples, we showcased the practical implications and advantages of our generalization.

The significance of our generalization lies not only in its theoretical contributions but also in its potential applications across various mathematical and engineering disciplines. By extending the Eneström-Kakeya theorem, we open new avenues for research in multivariate polynomials, numerical methods, and real-world problem-solving in fields such as control theory and signal processing.

Looking forward, we have identified several promising directions for future research. These include further relaxation of coefficient conditions, exploration of multivariate polynomials, and development of computational tools to apply our results to large-scale problems. Additionally, investigating the connections between our generalization and other polynomial theorems could yield deeper insights and more comprehensive understandings of polynomial behavior.

In conclusion, our generalization of the Eneström-Kakeya theorem represents a meaningful advancement in the study of polynomial zeros. By expanding the boundaries of this classical result, we contribute to a richer understanding of polynomial properties and lay the groundwork for future discoveries in both theoretical and applied mathematics.

4. Future Research Work

The generalization of the Eneström-Kakeya theorem presented in this paper opens several promising avenues for future research. As we extend the classical results to encompass broader classes of polynomials and other related functions, numerous questions and potential research directions emerge. Here, we outline some key areas that warrant further exploration:

1. Extensions to Multivariate Polynomials. While our generalization primarily addresses univariate polynomials, a natural progression is to investigate analogous results for multivariate polynomials. This would involve understanding the conditions under which the zeros of multivariate polynomials with specific coefficient constraints lie within certain geometric regions in higher-dimensional spaces.

2. Relaxation of Coefficient Conditions. Our current generalization relaxes the monotonicity conditions on polynomial coefficients. Further research could explore other types of conditions, such as boundedness, periodicity, or other functional forms. This would help to identify new classes of polynomials for which similar zero-constraining properties hold.

3. Connection with Other Polynomial Inequalities. Exploring the relationships between our generalized theorem and other known polynomial inequalities, such as the Gauss-Lucas theorem or the Laguerre-Pólya class, could yield deeper insights. Understanding these connections might lead to new results or more comprehensive theorems encompassing multiple aspects of polynomial zero behavior.

4. Applications in Control Theory and Signal Processing. The zeros of polynomials play a crucial role in control theory and signal processing. Investigating how our generalized results can be applied to the stability analysis of control systems or the design of filters in signal processing could have practical implications. This would involve translating theoretical findings into practical algorithms and techniques.

5. Numerical Methods and Algorithm Development. Developing efficient numerical methods and algorithms to test the conditions of our generalized theorem on large-scale polynomial datasets would be valuable. These computational tools could facilitate the application of our theoretical results to real-world problems, especially in fields that require handling polynomials with high degrees or complex coefficient structures.

6. Exploration of Polynomials with Random Coefficients. An interesting direction is to study polynomials with coefficients that are random variables following specific distributions. Analyzing the expected distribution of zeros for such random polynomials under the framework of our generalized theorem could provide insights into probabilistic aspects of polynomial zero distributions.

7. Generalization to Entire Functions. Since entire functions can be viewed as infinite-degree polynomials, extending the Eneström-Kakeya-type results to entire functions represents a challenging yet potentially rewarding endeavor. This would involve establishing conditions on the growth rates or other properties of the coefficients of entire functions to determine the regions where their zeros lie.

8. Impact on Polynomial Root-Finding Algorithms. Investigating how our generalization influences existing polynomial root-finding algorithms or inspires the development of new algorithms could have significant computational benefits. This line of research would aim to improve the efficiency and accuracy of locating polynomial zeros based on our theoretical findings.

By pursuing these research directions, we aim to deepen the understanding of polynomial zero behavior under more general conditions and to bridge the gap between theoretical advances and practical applications. The exploration of these avenues will not only enhance the theoretical landscape of polynomial analysis but also contribute to various applied fields that rely on polynomial properties.

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SOME GENERALIZATIONS INVOLVING THE POLAR DERIVATIVE FOR AN INEQUALITY OF PAUL TURÁN

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ABSTRACT. For a polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n having all zeros in $|z| \leq 1$, It is known:

$$|P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)|P(z)|.$$

In this paper, besides the generalization of the above inequality, we extend some well-known results to the polar derivative of a polynomial.

1. INTRODUCTION

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n over the field $\mathbb{C}$ of complex numbers. If $P \in \mathcal{P}_n$ and P' be its derivative, then concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ on $|z| = 1$, Bernstein [2] proved the following:

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Equality holds in (1.1) only if P has all its zeros at the origin. It stands natural to ask what happens to inequality (1.1), if we impose restrictions on the location of zeros of P. In this connection, the following inequalities are the earliest belonging to this domain of ideas, which have a clear impact on the subsequent work carried forward since then.

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If $P \in \mathcal{P}_n$ has all zeros in $|z| \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.2}$$

and *if $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.2) was observed by Erdös and later verified by Lax [6], whereas inequality (1.3) was established by Turán [10].

By involving the coefficients of the polynomial $P(z)$, Dubinin [3] refined inequality (1.3). More precisely, he proved:

If all the zeros of a polynomial $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$|P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)|P(z)|.$$

Various Turán-type inequalities have appeared in the literature wherein the underlying polynomial is replaced by some more general class of functions and, the derivative of the function by some operators which, in particular generalize the ordinary derivative. Polar derivative of a polynomial with respect to a point α is one such operator for a polynomial $P(z)$ of degree n, denoted by $D_\alpha P(z)$ and is defined as (for references see [7]) $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$. It is to be observed that

$$\lim_{\alpha \to +\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Inequalities have been proved occasionally, which extend the ordinary derivative of a complex polynomial to the polar derivative. In this paper, we prove some results concerning the modulus of the polar derivative of a polynomial having all zeros in $|z| \leq k$, $k \geq 1$. Our results generalize as well as sharpen some already known inequalities of Turán type.

2. Lemmas

Lemma 2.1. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s})$, $0 \leq s < n$, is a polynomial of degree $n \geq 1$ having all its zeros in $|z| \leq 1$, then for $P(z) \neq 0$, with $|z| = 1$, we have*

$$|P'(z)| \geq \frac{1}{2}\left\{n + s + \frac{\sqrt{|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{|a_{n-s}|}}\right\}|P(z)|. \tag{2.1}$$

The above lemma follows from a result due to Govil and Kumar [5].

Next lemma is due to Frappier et al. [4].

Lemma 2.2. *If $P(z)$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|, \quad \textit{if } n > 1 \tag{2.2}$$

and

$$\max_{|z|=R}|P(z)| \leq R\max_{|z|=1}|P(z)| - (R-1)|P(0)|, \quad \text{if } n=1. \tag{2.3}$$

Lemma 2.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for $0 \leq l < 1$*

$$\begin{aligned}\max_{|z|=k}|P(z)| \geq & \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| - l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)| \\ & + \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right), \quad \text{if } n>2,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\max_{|z|=k}|P(z)| \geq & \frac{2k^2}{1+k^2}\max_{|z|=1}|P(z)| - l\left(\frac{k^2-1}{k^2+1}\right)\min_{|z|=k}|P(z)| \\ & + \frac{k(k-1)^2}{1+k^2}|a_1|, \quad \text{if } n=2.\end{aligned} \tag{2.5}$$

The above lemma is due to Rather et al. [8].

We also need the following lemma which is a special case of a result due to Bernstein [1].

Lemma 2.4. *Let $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$ and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. Then, for $|z|=1$*

$$|Q'(z)| \leq |P'(z)|.$$

3. Main Results.

In this paper, we first prove the following result which is a generalization of a result due to Rather et al. [9].

Theorem 3.1. *If $P \in \mathcal{P}_n$ be such that all the zeros of $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s})$, $0 \leq s < n$, lie in $|z| \leq k$, $k \geq 1$, with a zero of multiplicity s, $0 \leq s < n$, at origin, then for any complex number α, with $|\alpha| \geq k$*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \geq & \frac{|\alpha|-k}{1+k^{n-s}}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=1}|P(z)| \\ & + \frac{|a_{n-s-1}|(|\alpha|-k)}{k(1+k^{n-s})}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\phi(k) \\ & + \psi(k)|na_0 + \alpha a_1|,\end{aligned} \tag{3.1}$$

where $\phi(k) = \left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = \left(1-\frac{1}{k^2}\right)$ or $\left(1-\frac{1}{k}\right)$ according as $n>2$ or $n=2$.

Proof. Since all zeros of $P(z)$ lie in $|z| \leq k$, $k \geq 1$, with an s-fold zero at the origin, therefore $g(z) = P(kz)$ is a polynomial of degree $n \geq 1$, having all zeros in $|z| \leq 1$ with an s-fold zero at the origin. Applying Lemma 2.1 to $g(z)$, we get

$$|g'(z)| \geq \frac{1}{2}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)|g(z)|. \tag{3.2}$$

Also,

$$|D_\alpha g(z)| = |ng(z) + (\alpha - z)g'(z)| \geq |\alpha| \cdot |g'(z)| - |ng(z) - zg'(z)|.$$

Since $g(z)$ has all its zeros in $|z| \leq 1$, therefore by using Lemma 2.4, it can be easily shown that for $|z| = 1$

$$|g'(z)| \geq |ng(z) - zg'(z)|.$$

This gives $|D_\alpha g(z)| \geq (|\alpha| - 1)|g'(z)|$. Using (3.2), we get

$$|D_\alpha g(z)| \geq \frac{|\alpha| - 1}{2}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)|g(z)|. \tag{3.3}$$

Note that by hypothesis $|\alpha|/k \geq 1$, therefore on replacing $g(z)$ by $P(kz)$, we have

$$D_{\alpha/k}P(kz) = nP(kz) + \left(\frac{\alpha}{k} - z\right)kP'(kz).$$

Hence, from (3.3), we get

$$\max_{|z|=1}|D_{\alpha/k}P(kz)| \geq \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=1}|P(kz)|. \tag{3.4}$$

Equivalently,

$$\max_{|z|=k}|D_\alpha P(z)| \geq \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=k}|P(z)|. \tag{3.5}$$

Since $P(z) = z^s\phi(z)$, where $\phi(z)$ is a polynomial of degree $n - s$ and $D_\alpha P(z)$ has degree $n - 1$, therefore using (2.2) of Lemma 2.2 for $n > 2$, $R = k \geq 1$, we have

$$\max_{|z|=k}|D_\alpha P(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1} - k^{n-3})|na_0 + \alpha a_1|. \tag{3.6}$$

Now using (3.6) in (3.5), we get

$$\begin{aligned} &k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1} - k^{n-3})|na_0 + \alpha a_1| \\ \geq &\frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=k}|P(z)|. \end{aligned} \tag{3.7}$$

Now using (2.4) of Lemma 2.3 for $\phi(z)$ with $l = 0$, we have

$$\max_{|z|=k}|\phi(z)| \geq \left\{\frac{2k^{n-s}}{1+k^{n-s}}\max_{|z|=1}|\phi(z)| + \frac{2k^{n-s-1}}{1+k^{n-s}}|a_{n-s-1}|\left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}\right)\right\}.$$

Since

$$\max_{|z|=k}|D_\alpha\phi(z)| = \frac{1}{k^s}\max_{|z|=1}|D_\alpha P(z)| \quad \text{and} \quad \max_{|z|=1}|D_\alpha\phi(z)| = \max_{|z|=1}|D_\alpha P(z)|.$$

We get for $n > 2$,

$$\max_{|z|=k}|P(z)| \geq \left\{\frac{2k^{n}}{1+k^{n-s}}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}}{1+k^{n-s}}|a_{n-s-1}|\left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}\right)\right\}. \tag{3.8}$$

Combining (3.8) and (3.7), we conclude that

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \geq & \frac{|\alpha|-k}{1+k^{n-s}}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=1}|P(z)| \\ & + \frac{|a_{n-s-1}|(|\alpha|-k)}{k(1+k^{n-s})}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\phi(k) \\ & + \psi(k)|na_0+\alpha a_1|,\end{aligned}$$

where $\phi(k) = \left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}\right)$ and $\psi(k) = \left(1-\frac{1}{k^2}\right)$. This proves the result for the case $n > 2$. The result for the case $n = 2$ can be obtained by using inequalities (2.3) of Lemma 2.2 and (2.5) of Lemma 2.3. This completely proves Theorem 3.1. □

If we divide both sides of (3.1) by $|\alpha|$ and letting $\alpha \to +\infty$, we get the following.

Corollary 3.1. *If $P \in \mathcal{P}_n$ be such that all the zeros of $P(z) = z^s(a_0 + a_1z + \cdots + a_{n-s}z^{n-s})$, $0 \leq s < n$, lie in $|z| \leq k$, $k \geq 1$, except a zero of multiplicity s, $0 \leq s < n$, at the origin, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq & \frac{1}{1+k^{n-s}}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=1}|P(z)| \\ & + \frac{|a_{n-s-1}|}{k(1+k^{n-s})}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\phi(k) \\ & + \psi(k)|a_1|,\end{aligned} \tag{3.9}$$

where $\phi(k), \psi(k)$ are defined in Theorem 3.1.

Remark 3.1. For $s = 0$ Theorem 3.1 reduces to a result due to Rather et al. [9].

Remark 3.2. For $k = 1$, Corollary 3.1 reduces to inequality (2.1).

We next prove the following.

Theorem 3.2. *If $P \in \mathcal{P}_n$ be such that all the zeros of $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s})$, $0 \le s < n$, lie in $|z| \le k$, $k \ge 1$, with a zero of multiplicity s, $0 \le s < n$, at the origin, then for any complex number α with $|\alpha| \ge k$ and for $0 \le l < 1$, we have*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \ge & \frac{|\alpha|-k}{k^n(1+k^{n-s})}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\\ & \times\left(k^n\max_{|z|=1}|P(z)|-lm\right)+\frac{n+s}{1+k^{n-s}}\Bigg((|\alpha|-k)\max_{|z|=1}|P(z)|\\ & +\left(\frac{|\alpha|}{k^s}+\frac{1}{k^{n-1}}\right)lm\Bigg)+\frac{|\alpha|-k}{k(1+k^{n-s})}|a_{n-s-1}|\\ & \times\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\phi(k)+|na_0+\alpha a_1|\psi(k),\end{aligned} \tag{3.10}$$

where $m = \min_{|z|=k}|P(z)|$ and $\phi(k), \psi(k)$ are defined in Theorem 3.1.

Proof. By given hypothesis all zeros of $P(z)$ lie in $|z| \le k, k \ge 1$ with an s-fold zero at the origin. Also, if at least one zero of $P(z)$ lie on $|z| = k$, then $m = \min_{|z|=k}|P(z)| = 0$ and the result follows from Theorem 3.1. So we assume all zeros of $P(z)$ lie in $|z| < k$ with s-fold zeros at the origin. Hence, in this case $m > 0$. Therefore, $f(z) = P(kz)$ has all zeros in $|z| < 1$ with $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|f(z)|$. Hence $|f(z)| > m$ for $|z| = 1$. Therefore, by Rouche's theorem for some complex δ with $|\delta| < 1$, $F(z) = f(z) - \delta m z^n$ has all zeros in $|z| < 1$ with s-fold zeros at the origin. Using value of $F(z)$ in (3.3), we get

$$|D_\alpha F(z)| \ge \frac{|\alpha|-1}{2}\left(n+s+\frac{\sqrt{|k^{n-s}a_{n-s}-\delta m|}-\sqrt{|a_0|}}{\sqrt{|k^{n-s}a_{n-s}-\delta m|}}\right)|F(z)|.$$

Using the fact that $T(x) = \frac{x-|a|}{x+|a|}$ is non-decreasing function of x and

$$|k^{n-s}a_{n-s}+\delta m| \ge k^{n-s}|a_{n-s}| - |\delta| m \ge 0,$$

we get

$$|D_\alpha F(z)| \ge \frac{|\alpha|-1}{2}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-|\delta|m}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-|\delta|m}}\right)|F(z)|.$$

Equivalently,

$$\begin{aligned}\left|D_{\alpha/k}f(z)-\frac{nm\alpha\delta}{k}z^{n-1}\right| \ge & \frac{|\alpha|-k}{2k}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-|\delta|m}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-|\delta|m}}\right)\\ & \times|f(z)-\delta m z^n|.\end{aligned} \tag{3.11}$$

Now, by simple deduction of Laguerre theorem [6, p. 52] on the polar derivative of a polynomial that for any α with $|\alpha| \geq k$,

$$D_{\alpha/k}(f(z) - \delta m z^n) = D_{\alpha/k} f(z) - \frac{nm\alpha\delta}{k} z^{n-1}$$

has all its zeros in $|z| < 1$. This implies $|D_{\alpha/k}(f(z)| \geq \frac{nm|\alpha||\delta|}{k}|z|^{n-1}$. Hence, we can choose argument of δ suitably, so, that from (3.11) for $|z| = 1$

$$|D_{\alpha/k} f(z)| - \frac{nm|\alpha||\delta|}{k} \geq \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - \delta m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right) \times (|f(z)| - |\delta m|).$$

Equivalently,

$$\begin{aligned}|D_{\alpha/k} f(z)| \geq & \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right)|f(z)| \\ & - \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right)|\delta| m \\ & + \frac{nm|\alpha|\delta|}{k}.\end{aligned}$$

Now replacing $f(z)$ by $P(kz)$ and proceeding as in the proof of Theorem 3.1, we get

$$\begin{aligned}\max_{|z|=k} |D_\alpha P(z)| \geq & \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right)\max_{|z|=k}|P(z)| \\ & - \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - \delta m}}\right)|\delta| m + \frac{nm|\alpha||\delta|}{k}.\end{aligned} \tag{3.12}$$

As in the proof of Theorem 3.1, $P(z) = z^s\phi(z)$ has all zeros in $|z| \leq k$, $k \geq 1$. Therefore, from (2.2) of Lemma 2.2 and (2.4) of Lemma 2.3, with

$$\max_{|z|=k} |D_\alpha \phi(z)| = \frac{1}{k^s}\max_{|z|=1}|D_\alpha P(z)|, \quad \min_{|z|=k}|D_\alpha\phi(z)| = \frac{1}{k^s}\min_{|z|=k}|D_\alpha P(z)|$$

and

$$\max_{|z|=1}|D_\alpha\phi(z)| = \max_{|z|=1}|D_\alpha P(z)|,$$

we get from (3.12) by taking $\delta = l$ with $n > 2$,

$$\begin{aligned}& k^n \max_{|z|=1}|D_\alpha P(z)| - (k^n - k^{n-2})|na_0 + \alpha a_1| \\ \geq & \frac{|\alpha| - k}{2}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - lm} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - lm}}\right)\left\{\frac{2k^n}{1 + k^{n-s}}\max_{|z|=1}|P(z)|\right.\end{aligned}$$

$$+ l\left(\frac{k^{n-s}-1}{k^{n-s}+1}\right)\min_{|z|=k}|P(z)| + \frac{2k^{n-1}|a_{n-s-1}|}{k^{n-s}+1}\left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}\right)\Bigg\}$$

$$- \frac{|\alpha|-k}{2}\cdot\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}lm + \frac{nml}{2k}(|\alpha|+k)$$

$$+ \frac{slm}{2k}(k-|\alpha|).$$

Thus on simplification, we get for $0 \le l < 1$ and $n > 2$

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \ge& \frac{|\alpha|-k}{k^n(1+k^{n-s})}\left(\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\\ &\times (k^n\max_{|z|=1}|P(z)| - lm)\\ &+ \frac{n+s}{1+k^{n-s}}\left((|\alpha|-k)\max_{|z|=1}|P(z)| + \left(\frac{|\alpha|}{k^s}+\frac{1}{k^{n-1}}\right)lm\right)\\ &+ \frac{|\alpha|-k}{k(1+k^{n-s})}|a_{n-s-1}|\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\\ &\times \phi(k) + |na_0+\alpha a_1|\psi(k) + \frac{slm}{2k}(k-|\alpha|),\end{aligned}$$

where $\phi(z)$ and $\psi(z)$ are defined in the statement of Theorem 3.1. This proves the theorem for case when $n > 2$. The result for $n = 2$ can be obtained by using (2.3) of Lemma 2.2 and (2.5) of Lemma 2.3. This completely proves Theorem 3.2. □

If we divide both sides of (3.10) by $|\alpha|$ and letting $\alpha \to +\infty$, we get the following.

Corollary 3.2. *If $P \in \mathcal{P}_n$ be such that all the zeros of $P(z) = z^s(a_0 + a_1z + \cdots + a_{n-s}z^{n-s})$, $0 \le s < n$, lie in $|z| \le k$, $k \ge 1$, with a zero of multiplicity s, $0 \le s < n$, at the origin, then for $0 \le l < 1$*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \ge& \frac{1}{k^n(1+k^{n-s})}\left(\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\\ &\times (k^n\max_{|z|=1}|P(z)| - lm) + \frac{n+s}{1+k^{n-s}}\left(\max_{|z|=1}|P(z)| + \frac{1}{k^s}lm\right) + \psi(k)|a_1|\\ &+ k^{n-1}|a_{n-s-1}|\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\phi(k) - \frac{slm}{2k},\end{aligned} \tag{3.13}$$

where $m = \min_{|z|=k}|P(z)|$ and $\phi(k), \psi(k)$ are defined in Theorem 3.1.

Remark 3.3. For $s = 0$, (3.10) reduces to a result due to Rather et al. [9].

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