

## WHEN ARE MULTIPLICATIVE (GENERALIZED)- $(\sigma, \tau)$ -DERIVATIONS ADDITIVE?

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**ABSTRACT.** Let  $R$  be an associative ring. A multiplicative (generalized)- $(\sigma, \tau)$ -derivation  $F$  is a map on  $R$  satisfying  $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$  for all  $x, y \in R$ , where  $\sigma, \tau$  are homomorphisms on  $R$  and  $g$  is any map on  $R$ . In this article, we have obtained some conditions on  $R$ , which make both  $F$  and  $g$  additive.

### 1. INTRODUCTION

The study of the additivity of mappings on rings as well as operator algebras has been an active area of research. Rickart [10] and Johnson [7] raised questions about when a multiplicative isomorphism becomes additive. Both imposed some sort of minimality conditions on ring  $R$  and answered it. Martindale [8] answered the above questions under some restriction on  $R$  which contains a family of idempotent elements. Daif et al. [1] introduced the definition of multiplicative derivation on  $R$  by choosing a mapping  $d : R \rightarrow R$  such that  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$  and proved that if  $R$  contains nontrivial idempotent elements then any multiplicative derivation is additive. Lu and Xie [3] established a condition on  $R$ , in the case where  $R$  may not contain any non-zero idempotents, that assures that a multiplicative isomorphism is additive, which generalizes Martindale's result. As an application, they showed that under a mild assumption, every multiplicative isomorphism from the radical of a nest algebra onto an arbitrary ring is additive.

Now let us recall the basic definition of Peirce decomposition. Let  $e$  in  $R$  be an idempotent element so that  $e \neq 1, e \neq 0$  ( $R$  need not have an identity). We will

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formally put  $e_1 = e$  and  $e_2 = 1 - e$ . Then, for  $R_{ij} = e_i R e_j$ , where  $i, j = 1, 2$ , one may write  $R$  in its Peirce decomposition as  $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$ , i.e.,  $R = eRe \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)R(1 - e)$ . An element of the subring  $R_{ij}$  will be denoted by  $a_{ij}$ . More transparently,  $e$  induces on  $R$  the generalized matrix ring structure

$$R = \begin{pmatrix} eRe & eR(1 - e) \\ (1 - e)Re & (1 - e)R(1 - e) \end{pmatrix},$$

with the obvious matrix addition and multiplication. Here  $eRe$ ,  $eR(1 - e)$ ,  $(1 - e)Re$  and  $(1 - e)R(1 - e)$  are abelian subgroups of  $R$ .

A map  $F : R \rightarrow R$  is called a multiplicative left centralizer if  $F(xy) = F(x)y$  for all  $x, y \in R$ . In [12], M. S. Tammam El-Sayiad, M. N. Daif, and V. De Filippis proved especially the result for the additivity of the multiplicative left centralizers in prime and semiprime rings which contain an idempotent element. A map  $F$  on  $R$  is called a multiplicative generalized derivation of  $R$  if  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in R$  and some derivation  $d$  of  $R$ . Similarly, a map  $F$  on  $R$  is called a multiplicative semi-derivation of  $R$  if  $F(xy) = F(x)g(y) + xF(y) = F(x)y + g(x)F(y)$  and  $F(g(x)) = g(F(x))$  for all  $x, y \in R$ , where  $g$  is any map on  $R$ . Daif et al. [2, Theorem 2.1] and Siddeeqe et al. [11, Theorem 2.1] proved the additivity of a multiplicative generalized derivation and multiplicative semi-derivation on an arbitrary ring under certain conditions, respectively.

Let  $\mathfrak{R}$  be a ring and  $\sigma, \tau$  be two endomorphisms on  $\mathfrak{R}$ . An additive mapping  $F : \mathfrak{R} \rightarrow \mathfrak{R}$  is called a generalized  $(\sigma, \tau)$ -derivation on  $\mathfrak{R}$  if there exists a  $(\sigma, \tau)$ -derivation  $d : \mathfrak{R} \rightarrow \mathfrak{R}$  such that  $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$  holds for all  $x, y \in \mathfrak{R}$ . A map on a ring  $\mathfrak{R}$  defined as  $x \rightarrow a\sigma(x) + \tau(x)b$ , where  $a, b$  are fixed elements of  $\mathfrak{R}$ , called as generalized  $(\sigma, \tau)$ -inner derivation, is an example of generalized- $(\sigma, \tau)$  derivation. More details about derivation, multiplicative derivation, and generalized derivation can be seen in [4, 5], and [9]. Hou et al. [6] proved that if  $R$  contains nontrivial idempotent elements, then any multiplicative  $(\sigma, \tau)$ -derivation is additive and such map is called  $(\sigma, \tau)$ -derivation. We give the notion of multiplicative (generalized)- $(\sigma, \tau)$  derivation as below.

A multiplicative (generalized)- $(\sigma, \tau)$ -derivation is a map satisfying  $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$  for all  $x, y \in \mathfrak{R}$ , where  $g$  is any map on  $\mathfrak{R}$ . Similarly a map  $F : \mathfrak{R} \rightarrow \mathfrak{R}$  is called a reverse multiplicative (generalized)- $(\sigma, \tau)$ -derivation on  $\mathfrak{R}$  if  $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$  holds for all  $x, y \in \mathfrak{R}$ . Here  $\sigma$  and  $\tau$  are endomorphisms and  $g$  is any map on  $\mathfrak{R}$ .

Now, we construct an example to support the definition of multiplicative (generalized) -  $(\sigma, \tau)$ - derivation, which is not additive as follows.

*Example 1.1.* Let  $C[0, 1]$  be the ring of all complex-valued continuous functions defined on  $[0, 1]$ . It can be easily verified that  $\mathfrak{R} = C[0, 1] \times C[0, 1]$ , forms a ring with regard

to component wise operations. Define the maps  $F, g, \sigma$  and  $\tau : \mathfrak{R} \rightarrow \mathfrak{R}$  such that:

$$F(h(x), k(x)) = \begin{cases} (\bar{h}(x) \log |h(x)|, \bar{k}(x) \log |k(x)|), & \text{if } h(x) \neq 0 \text{ and } k(x) \neq 0, \\ (0, \bar{k}(x) \log |k(x)|), & \text{if } h(x) = 0 \text{ and } k(x) \neq 0, \\ (\bar{h}(x) \log |h(x)|, 0), & \text{if } h(x) \neq 0 \text{ and } k(x) = 0, \\ (0, 0), & \text{if } h(x) = 0 \text{ and } k(x) = 0, \end{cases}$$

$g(h(x), k(x)) = F(h(x), k(x))$  and  $\sigma(h(x), k(x)) = \tau(h(x), k(x)) = (\bar{h}(x), \bar{k}(x))$ , where  $\bar{h}$  denotes the conjugate of  $h$ . It can be easily proved that  $\sigma, \tau$  are automorphisms of  $\mathfrak{R}$  and  $F$  is a multiplicative (generalized)- $(\sigma, \tau)$ -derivation of  $\mathfrak{R}$ , i.e.,  $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$  for all  $x, y \in \mathfrak{R}$ . But  $F$  is not additive on  $\mathfrak{R}$ .

Motivated by the above-cited results, we have proved the additivity of multiplicative (generalized)- $(\sigma, \tau)$ -derivation under some conditions on  $\mathfrak{R}$  as follows.

**Theorem 1.1.** *Let  $\mathfrak{R}$  be an associative ring with identity containing an idempotent  $e(e \neq 0, 1)$  which satisfies the following conditions:*

- (i)  $x\mathfrak{R} = (0) \implies x = 0$ ,
- (ii)  $\mathfrak{R}x = (0) \implies x = 0$ .

*If  $F$  is any multiplicative (generalized)- $(\sigma, \tau)$ -derivation on  $\mathfrak{R}$ , i.e.,  $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$  holds for all  $x, y \in \mathfrak{R}$ , where  $\sigma, \tau$  are endomorphisms on  $\mathfrak{R}$  and  $g$  is any map on  $\mathfrak{R}$  which is additive on  $\mathfrak{R}_{11}$  and  $\mathfrak{R}_{22}$ , then  $F$  and  $g$  are additive.*

For proof of Theorem 1.1, first we will prove some auxiliary results as follows.

**Lemma 1.1.**  $F(0) = 0$ .

*Proof.* By the definition of  $F$ , we have  $F(0) = F(00) = F(0)\sigma(0) + \tau(0)g(0) = F(0)0 + 0g(0) = 0$ , which completes the proof.  $\square$

**Lemma 1.2.** *For any  $a_{11} \in \mathfrak{R}_{11}, a_{22} \in \mathfrak{R}_{22}, b_{12} \in \mathfrak{R}_{12}$  and  $b_{21} \in \mathfrak{R}_{21}$ , the following hold*

- (i)  $g(a_{11} + b_{21}) = g(a_{11}) + g(b_{21})$ ,
- (ii)  $g(a_{22} + b_{12}) = g(a_{22}) + g(b_{12})$ .

*Proof.* We prove only (i), and the proof of (ii) goes similarly.

(i) For any  $t_{n1} \in \mathfrak{R}_{n1}$  where  $n \in \{1, 2\}$ , we have

$$\begin{aligned} \tau(t_{n1})(g(a_{11}) + g(b_{21})) &= \tau(t_{n1})g(a_{11}) + \tau(t_{n1})g(b_{21}) \\ &= F(t_{n1}a_{11}) - F(t_{n1})\sigma(a_{11}) + F(t_{n1}b_{21}) - F(t_{n1})\sigma(b_{21}) \\ &= F(t_{n1}a_{11}) - F(t_{n1})\sigma(a_{11}) - F(t_{n1})\sigma(b_{21}) \\ &= F(t_{n1}(a_{11} + b_{21})) - F(t_{n1})\sigma(a_{11} + b_{21}) \\ &= \tau(t_{n1})g(a_{11} + b_{21}). \end{aligned}$$

This implies that

$$(1.1) \quad \tau(t_{n1})[(g(a_{11}) + g(b_{21})) - g(a_{11} + b_{21})] = 0$$

and

$$\begin{aligned}
 \tau(t_{n2})(g(a_{11}) + g(b_{21})) &= \tau(t_{n2})g(a_{11}) + \tau(t_{n2})g(b_{21}) \\
 &= F(t_{n2}a_{11}) - F(t_{n2})\sigma(a_{11}) + F(t_{n2}b_{21}) - F(t_{n2})\sigma(b_{21}) \\
 &= F(t_{n2}b_{21}) - F(t_{n2})\sigma(a_{11}) - F(t_{n2})\sigma(b_{21}) \\
 &= F(t_{n2}(a_{11} + b_{21})) - F(t_{n2})\sigma(a_{11} + b_{21}) \\
 &= \tau(t_{n2})(a_{11} + b_{21}).
 \end{aligned}$$

Thus, we obtain that

$$(1.2) \quad \tau(t_{n2})[(g(a_{11}) + g(b_{21})) - g(a_{11} + b_{21})] = 0.$$

Using (1.1) and (1.2), we have  $\Re[g(a_{11} + b_{21}) - g(a_{11}) - g(b_{21})] = (0)$ . Using the hypothesis of Theorem 1.1, we get  $g(a_{11} + b_{21}) = g(a_{11}) + g(b_{21})$ .  $\square$

**Lemma 1.3.** *For any  $a_{11} \in \Re_{11}$ ,  $a_{22} \in \Re_{22}$ ,  $b_{12} \in \Re_{12}$ ,  $b_{21} \in \Re_{21}$  and  $b_{22} \in \Re_{22}$ , we have the following:*

- (i)  $F(a_{11} + b_{12}) = F(a_{11}) + F(b_{12})$ ,
- (ii)  $F(a_{22} + b_{21}) = F(a_{22}) + F(b_{21})$ ,
- (iii)  $F(a_{11} + b_{22}) = F(a_{11}) + F(b_{22})$ ,
- (iv)  $F(a_{11} + b_{21}) = F(a_{11}) + F(b_{21})$ ,
- (v)  $F(a_{22} + b_{12}) = F(a_{22}) + F(b_{12})$ .

*Proof.* Proofs of (i), (ii) and (iii) are similar to each other. Similarly the proofs of (iv) and (v) are on the same pattern. Therefore we prove only (i) and (iv).

(i) For any  $t_{1n} \in \Re_{1n}$ , where  $n \in \{1, 2\}$ , we have

$$\begin{aligned}
 (F(a_{11}) + F(b_{12}))\sigma(t_{1n}) &= F(a_{11})\sigma(t_{1n}) + F(b_{12})\sigma(t_{1n}) \\
 &= F(a_{11}t_{1n}) - \tau(a_{11})g(t_{1n}) + F(b_{12}t_{1n}) - \tau(b_{12})g(t_{1n}) \\
 &= F((a_{11} + b_{12})t_{1n}) - \tau(a_{11} + b_{12})g(t_{1n}) \\
 &= F(a_{11} + b_{12})\sigma(t_{1n}).
 \end{aligned}$$

Thus, we obtain that

$$(1.3) \quad [F(a_{11} + b_{12}) - F(a_{11}) - F(b_{12})]\sigma(t_{1n}) = 0.$$

For any  $t_{2n} \in \Re_{2n}$ , where  $n \in \{1, 2\}$ , we have

$$\begin{aligned}
 (F(a_{11}) + F(b_{12}))\sigma(t_{2n}) &= F(a_{11})\sigma(t_{2n}) + F(b_{12})\sigma(t_{2n}) \\
 &= F(a_{11}t_{2n}) - \tau(a_{11})g(t_{2n}) + F(b_{12}t_{2n}) - \tau(b_{12})g(t_{2n}) \\
 &= F((a_{11} + b_{12})t_{2n}) - \tau(a_{11})g(t_{2n}) - \tau(b_{12})g(t_{2n}) \\
 &= F((a_{11} + b_{12})t_{2n}) - \tau(a_{11} + b_{12})g(t_{2n}) \\
 &= F(a_{11} + b_{12})\sigma(t_{2n}).
 \end{aligned}$$

This implies that

$$(1.4) \quad [F(a_{11} + b_{12}) - F(a_{11}) - F(b_{12})]\sigma(t_{2n}) = 0.$$

Using (1.3) and (1.4), we arrive at  $[F(a_{11} + b_{12}) - F(a_{11}) - F(b_{12})]\mathfrak{R} = (0)$ . Using the hypothesis of Theorem 1.1, we get  $F(a_{11} + b_{12}) = F(a_{11}) + F(b_{12})$ .

(iv) For any  $t_{2n} \in \mathfrak{R}_{2n}$ , where  $n \in \{1, 2\}$ , we have

$$\begin{aligned} (F(a_{11}) + F(b_{21}))\sigma(t_{2n}) &= F(a_{11})\sigma(t_{2n}) + F(b_{21})\sigma(t_{2n}) \\ &= F(a_{11}t_{2n}) - \tau(a_{11})g(t_{2n}) + F(b_{21}t_{2n}) - \tau(b_{21})g(t_{2n}) \\ &= -\tau(a_{11} + b_{21})g(t_{2n}) \\ &= -F((a_{11} + b_{21})t_{2n}) + F(a_{11} + b_{21})\sigma(t_{2n}) \\ &= F(a_{11} + b_{21})\sigma(t_{2n}). \end{aligned}$$

This implies that

$$(1.5) \quad [F(a_{11} + b_{21}) - F(a_{11}) - F(b_{21})]\sigma(t_{2n}) = 0.$$

For any  $t_{1n} \in \mathfrak{R}_{1n}$ , where  $n \in \{1, 2\}$ , we have

$$\begin{aligned} F(a_{11} + b_{21})\sigma(t_{1n}) &= F((a_{11} + b_{21})t_{1n}) - \tau(a_{11} + b_{21})g(t_{1n}) \\ &= F((e_2 + a_{11})(t_{1n} + b_{21}t_{1n})) - \tau(a_{11} + b_{21})g(t_{1n}) \\ &= F(e_2 + a_{11})\sigma(t_{1n} + b_{21}t_{1n}) + \tau(e_2 + a_{11})g(t_{1n} + b_{21}t_{1n}) - \tau(a_{11} + b_{21})g(t_{1n}). \end{aligned}$$

Using Lemma 1.3 (iii) and Lemma 1.2, we have

$$\begin{aligned} &F(a_{11} + b_{21})\sigma(t_{1n}) \\ &= F(e_2)\sigma(t_{1n}) + F(e_2)\sigma(b_{21}t_{1n}) + F(a_{11})\sigma(t_{1n}) \\ &\quad + F(a_{11})\sigma(b_{21}t_{1n}) + \tau(e_2)g(t_{1n}) + \tau(e_2)g(b_{21}t_{1n}) + \tau(a_{11})g(t_{1n}) \\ &\quad + \tau(a_{11})g(b_{21}t_{1n}) - \tau(a_{11})g(t_{1n}) - \tau(b_{21})g(t_{1n}) \\ &= F(e_2t_{1n}) + F(e_2b_{21}t_{1n}) + F(a_{11}t_{1n}) + F(a_{11}b_{21}t_{1n}) - \tau(a_{11})g(t_{1n}) - \tau(b_{21})g(t_{1n}) \\ &= F(b_{21}t_{1n}) + F(a_{11}t_{1n}) - \tau(a_{11})g(t_{1n}) - \tau(b_{21})g(t_{1n}) \\ &= F(b_{21})\sigma(t_{1n}) + F(a_{11})\sigma(t_{1n}) \\ &= (F(b_{21}) + F(a_{11}))\sigma(t_{1n}). \end{aligned}$$

This shows that

$$(1.6) \quad [F(a_{11} + b_{21}) - F(a_{11}) - F(b_{21})]\sigma(t_{1n}) = 0.$$

Using (1.5) and (1.6), we have  $[F(a_{11} + b_{21}) - F(a_{11}) - F(b_{12})]\mathfrak{R} = (0)$ . With the help of hypothesis of Theorem 1.1, we obtain  $F(a_{11} + b_{21}) = F(a_{11}) + F(b_{21})$ .  $\square$

**Lemma 1.4.** *F is additive on  $\mathfrak{R}_{12}$ .*

*Proof.* Let  $a_{12}, b_{12} \in \mathfrak{R}_{12}$  and  $t_{1n} \in \mathfrak{R}_{1n}$ , where  $n \in \{1, 2\}$ , we have

$$\begin{aligned} (F(a_{12}) + F(b_{12}))\sigma(t_{1n}) &= F(a_{12})\sigma(t_{1n}) + F(b_{12})\sigma(t_{1n}) \\ &= F(a_{12}t_{1n}) - \tau(a_{12})g(t_{1n}) + F(b_{12}t_{1n}) - \tau(b_{12})g(t_{1n}) \\ &= -\tau(a_{12} + b_{12})g(t_{1n}) \\ &= -F((a_{12} + b_{12})t_{1n}) + F(a_{12} + b_{12})\sigma(t_{1n}) \end{aligned}$$

$$=F(a_{12} + b_{12})\sigma(t_{1n}).$$

This gives us

$$(1.7) \quad [F(a_{12} + b_{12}) - F(a_{12}) - F(b_{12})]\sigma(t_{1n}) = 0.$$

For any  $t_{2n} \in \mathfrak{R}_{2n}$ , where  $n \in \{1, 2\}$ , we have

$$\begin{aligned} & F(a_{12} + b_{12})\sigma(t_{2n}) \\ &= F((a_{12} + b_{12})t_{2n}) - \tau(a_{12} + b_{12})g(t_{2n}) \\ &= F((e + a_{12})(t_{2n} + b_{12}t_{2n})) - \tau(a_{12} + b_{12})g(t_{2n}) \\ &= F(e + a_{12})\sigma(t_{2n} + b_{12}t_{2n}) + \tau(e + a_{12})g(t_{2n} + b_{12}t_{2n}) - \tau(a_{12} + b_{12})g(t_{2n}). \end{aligned}$$

Using Lemma 1.3 (i) and Lemma 1.2, we obtain

$$\begin{aligned} & F(a_{12} + b_{12})\sigma(t_{2n}) \\ &= F(e)\sigma(t_{2n}) + F(e)\sigma(b_{12}t_{2n}) + F(a_{12})\sigma(t_{2n}) + F(a_{12})\sigma(b_{12}t_{2n}) + \tau(e)g(t_{2n}) \\ & \quad + \tau(e)g(b_{12}t_{2n}) + \tau(a_{12})g(t_{2n}) + \tau(a_{12})g(b_{12}t_{2n}) - \tau(a_{12})g(t_{2n}) - \tau(b_{12})g(t_{2n}) \\ &= F(et_{2n}) + F(eb_{12}t_{2n}) + F(a_{12}t_{2n}) + F(a_{12}b_{12}t_{2n}) - \tau(a_{12})g(t_{2n}) - \tau(b_{12})g(t_{2n}) \\ &= F(b_{12}t_{2n}) + F(a_{12}t_{2n}) - \tau(a_{12})g(t_{2n}) - \tau(b_{12})g(t_{2n}) \\ &= F(b_{12})\sigma(t_{2n}) + F(a_{12})\sigma(t_{2n}) \\ &= (F(b_{12}) + F(a_{12}))\sigma(t_{2n}). \end{aligned}$$

This implies that

$$(1.8) \quad [F(a_{12} + b_{12}) - F(a_{12}) - F(b_{12})]\sigma(t_{2n}) = 0.$$

Using (1.7) and (1.8), we have  $[F(a_{12} + b_{12}) - F(a_{12}) - F(b_{12})]\mathfrak{R} = (0)$ . Using the hypothesis of Theorem 1.1, we get  $F$  is additive on  $\mathfrak{R}_{12}$ . □

**Lemma 1.5.**  $F$  is additive on  $\mathfrak{R}_{21}$ .

*Proof.* Proof is similar to Lemma 1.4. □

**Lemma 1.6.**  $F$  is additive on  $\mathfrak{R}_{11}$ .

*Proof.* Let  $a_{11}, b_{11} \in \mathfrak{R}_{11}$ . We have,

$$F(a_{11} + b_{11}) = F(e(a_{11} + b_{11})) = F(e)\sigma(a_{11} + b_{11}) + \tau(e)g(a_{11} + b_{11}).$$

Since  $g$  is additive on  $\mathfrak{R}_{11}$ , we get  $F(a_{11} + b_{11}) = F(a_{11}) + F(b_{11})$ . □

**Lemma 1.7.**  $F$  is additive on  $\mathfrak{R}_{11} + \mathfrak{R}_{12} = e\mathfrak{R}$ .

*Proof.* Let  $a_{11} + a_{12}, b_{11} + b_{12} \in e\mathfrak{R}$ . We have,

$$F((a_{11} + a_{12}) + (b_{11} + b_{12})) = F((a_{11} + b_{11}) + (a_{12} + b_{12})).$$

Using Lemma 1.3 (i), we have

$$F((a_{11} + a_{12}) + (b_{11} + b_{12})) = F(a_{11} + b_{11}) + F(a_{12} + b_{12}).$$

Lemma 1.4 and Lemma 1.6, provide us

$$F((a_{11} + a_{12}) + (b_{11} + b_{12})) = F(a_{11}) + F(a_{12}) + F(b_{11}) + F(b_{12}).$$

With the help of Lemma 1.3 (i), we get

$$F((a_{11} + a_{12}) + (b_{11} + b_{12})) = F(a_{11} + a_{12}) + F(b_{11} + b_{12}).$$

That is,  $F$  is additive on  $e\mathfrak{R}$ . □

**Lemma 1.8.**  $F$  is additive on  $\mathfrak{R}_{22}$ .

*Proof.* Proof is similar as Lemma 1.6. □

**Lemma 1.9.**  $F$  is additive on  $\mathfrak{R}_{21} + \mathfrak{R}_{22} = e_2\mathfrak{R} = (1 - e)\mathfrak{R}$ .

*Proof.* Let  $a_{21} + a_{22}, b_{21} + b_{22} \in (1 - e)\mathfrak{R}$ . We have

$$F((a_{21} + a_{22}) + (b_{21} + b_{22})) = F((a_{21} + b_{21}) + (a_{22} + b_{22})).$$

Using Lemma 1.3 (ii), we get

$$F((a_{21} + a_{22}) + (b_{21} + b_{22})) = F(a_{21} + b_{21}) + F(a_{22} + b_{22}).$$

Lemma 1.5 and Lemma 1.8 provide us

$$F((a_{21} + a_{22}) + (b_{21} + b_{22})) = F(a_{21}) + F(a_{22}) + F(b_{21}) + F(b_{22}).$$

Lemma 1.3 (ii) provides us

$$F((a_{21} + a_{22}) + (b_{21} + b_{22})) = F(a_{21} + a_{22}) + F(b_{21} + b_{22}).$$

That is,  $F$  is additive on  $(1 - e)\mathfrak{R}$ . □

**Lemma 1.10.**  $F$  is additive on  $\mathfrak{R}_{22} + \mathfrak{R}_{12} = \mathfrak{R}(1 - e)$ .

*Proof.* Let  $a_{22} + a_{12}, b_{22} + b_{12} \in \mathfrak{R}(1 - e)$ . We have

$$F((a_{22} + a_{12}) + (b_{22} + b_{12})) = F((a_{22} + b_{22}) + (a_{12} + b_{12})).$$

Using Lemma 1.3 (v), we get

$$F((a_{22} + a_{12}) + (b_{22} + b_{12})) = F(a_{22} + b_{22}) + F(a_{12} + b_{12}).$$

Lemma 1.4 and Lemma 1.8, provide us

$$F((a_{22} + a_{12}) + (b_{22} + b_{12})) = F(a_{22}) + F(a_{12}) + F(b_{22}) + F(b_{12}).$$

With the help of Lemma 1.3 (v), we get

$$F((a_{22} + a_{12}) + (b_{22} + b_{12})) = F(a_{22} + a_{12}) + F(b_{22} + b_{12}).$$

Hence,  $F$  is additive on  $\mathfrak{R}(1 - e)$ . □

**Lemma 1.11.**  $F$  is additive on  $\mathfrak{R}_{11} + \mathfrak{R}_{21} = \mathfrak{R}e$ .

*Proof.* Let  $a_{11} + a_{21}, b_{11} + b_{21} \in \mathfrak{R}e$ . We have

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F((a_{11} + b_{11}) + (a_{21} + b_{21})).$$

By Lemma 1.3 (iv), we get

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F(a_{11} + b_{11}) + F(a_{21} + b_{21}).$$

Lemma 1.5 and Lemma 1.6, provide us

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F(a_{11}) + F(a_{21}) + F(b_{11}) + F(b_{21}).$$

Finally, we conclude by Lemma 1.3 (iv)

$$F((a_{11} + a_{21}) + (b_{11} + b_{21})) = F(a_{11} + a_{21}) + F(b_{11} + b_{21}).$$

That is,  $F$  is additive on  $\mathfrak{R}e$ . □

Now, we prove Theorem 1.1.

*Proof of Theorem 1.1.* (i) First we prove that  $g$  is additive.

Let  $t \in e\mathfrak{R} = \mathfrak{R}_{11} + \mathfrak{R}_{12}$ ,  $x, y \in \mathfrak{R}$ . We have  $tx, ty \in e\mathfrak{R}$ .

$$\begin{aligned} \tau(t)(g(x) + g(y)) &= \tau(t)g(x) + \tau(t)g(y) \\ &= F(tx) - F(t)\sigma(x) + F(ty) - F(t)\sigma(y) \\ &= F(tx) + F(ty) - F(t)\sigma(x) - F(t)\sigma(y). \end{aligned}$$

Lemma 1.7 provides us

$$\tau(t)(g(x) + g(y)) = F(t(x + y)) - F(t)\sigma(x + y) = \tau(t)g(x + y).$$

This implies that

$$(1.9) \quad \tau(t)[(g(x) + g(y)) - g(x + y)] = 0.$$

Also, let  $m \in (1 - e)\mathfrak{R} = \mathfrak{R}_{21} + \mathfrak{R}_{22}$ . This shows that  $mx, my \in (1 - e)\mathfrak{R}$ ,

$$\begin{aligned} \tau(m)(g(x) + g(y)) &= \tau(m)g(x) + \tau(m)g(y) \\ &= F(mx) - F(m)\sigma(x) + F(my) - F(m)\sigma(y) \\ &= F(mx) + F(my) - F(m)\sigma(x) - F(m)\sigma(y). \end{aligned}$$

Using Lemma 1.9, we get

$$\tau(m)(g(x) + g(y)) = F(m(x + y)) - F(m)\sigma(x + y) = \tau(m)g(x + y).$$

This implies that

$$(1.10) \quad \tau(m)[(g(x) + g(y)) - g(x + y)] = 0.$$

On adding (1.9) and (1.10), we have

$$\tau(t + m)[g(x + y) - g(x) - g(y)] = 0.$$

Since  $\tau$  is onto on  $\mathfrak{R}$ , we get

$$\mathfrak{R}[g(x + y) - g(x) - g(y)] = (0).$$

Using hypothesis, we conclude that

$$g(x + y) = g(x) + g(y).$$

(ii) Now, we prove that  $F$  is additive.

Let  $t \in \mathfrak{R}e = \mathfrak{R}_{11} + \mathfrak{R}_{21}$ ,  $m \in \mathfrak{R}(1 - e) = \mathfrak{R}_{22} + \mathfrak{R}_{12}$ , and  $a, b \in \mathfrak{R}$ . Then  $at, bt \in \mathfrak{R}e$  and  $am, bm \in \mathfrak{R}(1 - e)$ ,

$$\begin{aligned} (F(a) + F(b))\sigma(t) &= F(a)\sigma(t) + F(b)\sigma(t) = F(at) - \tau(a)g(t) + F(bt) - \tau(b)g(t) \\ &= F(at) + F(bt) - (\tau(a) + \tau(b))g(t). \end{aligned}$$

Using Lemma 1.11, we get

$$(F(a) + F(b))\sigma(t) = F((a + b)t) - \tau(a + b)g(t) = F(a + b)\sigma(t).$$

This implies that

$$(1.11) \quad [F(a + b) - F(a) - F(b)]\sigma(t) = 0.$$

Also,

$$\begin{aligned} (F(a) + F(b))\sigma(m) &= F(a)\sigma(m) + F(b)\sigma(m) \\ &= F(am) - \tau(a)g(m) + F(bm) - \tau(b)g(m) \\ &= F(am) + F(bm) - (\tau(a) + \tau(b))g(m). \end{aligned}$$

Lemma 1.10 provides us

$$(F(a) + F(b))\sigma(m) = F((a + b)m) - \tau(a + b)g(m) = F(a + b)\sigma(m).$$

This implies that

$$(1.12) \quad [F(a + b) - F(a) - F(b)]\sigma(m) = 0.$$

On adding (1.11) and (1.12), we have

$$[F(a + b) - F(a) - F(b)]\sigma(t + m) = 0.$$

Since  $\sigma$  is onto on  $\mathfrak{R}$ , we conclude that

$$[F(a + b) - F(a) - F(b)]\mathfrak{R} = (0).$$

Using hypothesis, we get  $F(a + b) = F(a) + F(b)$ , i.e.,  $F$  is additive. □

**Corollary 1.1.** *Let  $\mathfrak{R}$  be a semi-prime ring with identity containing an idempotent  $e \neq 0, 1$ . If  $F$  is any multiplicative (generalized)- $(\sigma, \tau)$ -derivation on  $\mathfrak{R}$ , i.e.,  $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$  holds for all  $x, y \in \mathfrak{R}$ , where  $\sigma, \tau$  are endomorphisms on  $\mathfrak{R}$  and  $g$  is additive on  $\mathfrak{R}_{11}$  and  $\mathfrak{R}_{22}$ , then  $F$  and  $g$  are additive.*

Now we construct an example to support the necessity of the condition that “ $g$  is additive on both  $\mathfrak{R}_{11}$  and  $\mathfrak{R}_{22}$ ” in Theorem 1.1 for additivity of  $F$  on  $\mathfrak{R}$ .

*Example 1.2.* Let  $S$  be an integral domain ring with the unity and

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in S \right\}$$

be the ring of upper triangular matrices over  $S$ . Let  $C[0, 1]$  be the ring of all complex-valued continuous functions defined on  $[0, 1]$ . It can be easily shown that  $\mathfrak{R} = C[0, 1] \times M$  forms a ring concerning component-wise operations. Define the maps  $F, g, \sigma$  and  $\tau : \mathfrak{R} \rightarrow \mathfrak{R}$  as follows

$$F \left( f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \begin{cases} \left( \bar{f}(x) \log |\bar{f}(x)|, \begin{bmatrix} 0 & a+c \\ 0 & 0 \end{bmatrix} \right), & \text{if } f(x) \neq 0, \\ \left( 0, \begin{bmatrix} 0 & a+c \\ 0 & 0 \end{bmatrix} \right), & \text{if } f(x) = 0, \end{cases}$$

$$\sigma \left( f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \left( \bar{f}(x), \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix} \right), \tau \left( f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \left( \bar{f}(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right),$$

$$g \left( f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \begin{cases} \left( \bar{f}(x) \log |\bar{f}(x)|, \begin{bmatrix} 0 & a-c \\ 0 & 0 \end{bmatrix} \right), & \text{if } f(x) \neq 0, \\ \left( 0, \begin{bmatrix} 0 & a-c \\ 0 & 0 \end{bmatrix} \right), & \text{if } f(x) = 0, \end{cases}$$

where  $\bar{f}$  denotes the conjugate of  $f$ . One can verify that  $\mathfrak{R}$  satisfies both the conditions (i) and (ii) of Theorem 1.1 and also both  $\sigma, \tau$  are automorphisms.  $F$  is a multiplicative (generalized)- $(\sigma, \tau)$  derivation, i.e.,  $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$  for all  $x, y \in \mathfrak{R}$ .

Obviously a non trivial idempotent element of the ring  $\mathfrak{R}$  is  $e = \left( \mathbf{1}(x), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$  and

then  $1 - e = \left( \mathbf{0}(x), \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ , where  $\mathbf{1}(x)$  and  $\mathbf{0}(x)$  are the constant functions on  $[0, 1]$  defined as;  $\mathbf{1}(x) = 1$  for all  $x \in [0, 1]$  and  $\mathbf{0}(x) = 0$  for each  $x \in [0, 1]$ . Then,

$$\mathfrak{R}_{11} = \left\{ \left( f(x), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) \mid f(x) \in C[0, 1], a \in S \right\}$$

and

$$\mathfrak{R}_{22} = \left\{ \left( \mathbf{0}(x), \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right) \mid \mathbf{0}(x) \in C[0, 1], a \in S \right\}.$$

Clearly, it can be proved that  $g$  is additive on  $\mathfrak{R}_{22}$  but not additive on  $\mathfrak{R}_{11}$ . But  $F$  is not additive on  $\mathfrak{R}$ .

Similarly, if we choose another non-trivial idempotent element as  $e_1 = \left( \mathbf{0}(x), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$ , then  $(1 - e_1) = \left( \mathbf{1}(x), \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ . In this case,

$$\mathfrak{R}_{11} = \left\{ \left( \mathbf{0}(x), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) \mid a \in S \right\}$$

and

$$\mathfrak{R}_{22} = \left\{ \left( f(x), \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right) \mid f(x) \in C[0, 1], a \in S \right\}.$$

Here, one can observe that  $g$  is additive on  $\mathfrak{R}_{11}$  but not additive on  $\mathfrak{R}_{22}$ . However,  $F$  is not additive on  $\mathfrak{R}$ .

**Theorem 1.2.** *Let  $\mathfrak{R}$  be an associative ring with identity containing an idempotent  $e (e \neq 0, 1)$  which satisfies the conditions (i) and (ii) of Theorem 1.1. If  $F$  is any reverse-multiplicative (generalized)- $(\sigma, \tau)$ -derivation on  $\mathfrak{R}$ , i.e.,  $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$  holds for all  $x, y \in \mathfrak{R}$ , where  $\sigma, \tau$  are endomorphisms on  $\mathfrak{R}$  and  $g$  is additive on  $\mathfrak{R}_{11}$  and  $\mathfrak{R}_{22}$ , then  $F$  and  $g$  are additive.*

*Proof.* The proof is in the same pattern as done for multiplicative (generalized)- $(\sigma, \tau)$ -derivation in Theorem 1.1.  $\square$

**Corollary 1.2.** *Let  $\mathfrak{R}$  be a semi-prime ring with identity containing an idempotent  $e \neq 0, 1$ . If  $F$  is any reverse-multiplicative (generalized)- $(\sigma, \tau)$ -derivation on  $\mathfrak{R}$ , i.e.,  $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$  holds for all  $x, y \in \mathfrak{R}$ , where  $\sigma, \tau$  are endomorphisms on  $\mathfrak{R}$  and  $g$  is additive on  $\mathfrak{R}_{11}$  and  $\mathfrak{R}_{22}$ , then  $F$  and  $g$  are additive.*

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