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SOME APPLICATIONS RELATED TO ADMISSIBLE FUNCTIONS FOR HIGHER-ORDER DERIVATIVES OF MEROMORPHIC MULTIVALENT FUNCTIONS

ABBAS KAREEM WANAS[1] AND ZAINAB SWAYEH GHALI[2]

Abstract. In the present manuscript, we obtain some differential subordination and superordination results for higher-order derivatives of meromorphic multivalent functions in the punctured unit disk by investigating appropriate families of admissible functions. These results are applied to obtain differential sandwich results.

1. Introduction

We denote by Σ_p the family of all functions f of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and multivalent in the punctured unit disk

$$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

A function $f \in \Sigma_p$ is meromorphic multivalent starlike if $f(z) \neq 0$ and

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in U^*),$$

and $f \in \Sigma_p$ is meromorphic multivalent convex if $f'(z) \neq 0$ and

$$-\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in U^*).$$

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Upon differentiating both sides of (1.1) j-times with respect to z, we obtain

$$f^{(j)}(z) = \frac{(-1)^j (p+j-1)!}{(p-1)!} z^{-p-j} + \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} a_n z^{n-j} \quad (p, j \in \mathbb{N};\ p > j).$$

Let $\mathcal{H}(U)$ be the collection of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ be the sub-collection of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

with $\mathcal{H} = \mathcal{H}[1, 1]$.

Let f and g be members of $\mathcal{H}(U)$. The function f is said to be subordinate to g, or (equivalently) g is said to be superordinate to f, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$. In such a case, we write $f \prec g$ or $f(z) \prec g(z)$, $z \in U$. Further, if the function g is univalent in U, then we have the following equivalence (see [5])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 1.1 ([6])**.** Let $F, h \in \mathcal{H}(U)$ and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. If F is analytic in U and satisfies the following (second-order) differential subordination:

$$\phi\left(F(z), zF'(z), z^2F''(z); z\right) \prec h(z), \tag{1.2}$$

then F is called a solution of the differential subordination (1.2). The univalent function q is called a dominant of the solutions of the differential subordination or more simply a dominant if $F(z) \prec q(z)$ for all F satisfying (1.2). A dominant $\check{q}$ that satisfies $\check{q}(z) \prec q(z)$ for all dominants q of (1.2) is said to be the best dominant.

Definition 1.2 ([7])**.** Let $F, h \in \mathcal{H}(U)$ and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. If F and

$$\phi\left(F(z), zF'(z), z^2F''(z); z\right)$$

are univalent in U for $\zeta \in \bar{U}$ and satisfy the following (second-order) differential superordination:

$$h(z) \prec \phi\left(F(z), zF'(z), z^2F''(z); z\right), \tag{1.3}$$

then F is called a solution of the differential superordination (1.3). An analytic function q is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $q(z) \prec F(z)$ for all F satisfying (1.3). A univalent subordinant $\check{q}$ that satisfies $q(z) \prec \check{q}(z)$ for all subordinants q of (1.3) is said to be the best subordinant.

Definition 1.3 ([6])**.** Denote by Q the set consisting of all functions q that are analytic and injective on $\bar{U} \backslash E(q)$, where

$$E(q) = \left\{\xi \in \partial U : \lim_{z \to \xi} q(z) = \infty\right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial U \backslash E(q)$.

Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition 1.4 ([6])**.** Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}$. The family of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; z) \notin \Omega$, whenever

$$r = q(\xi), \quad s = k\xi q'(\xi) \quad \text{and} \quad \operatorname{Re}\left\{\frac{t}{s} + 1\right\} \geq k \operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$z \in U$, $\xi \in \partial U \backslash E(q)$ and $k \geq n$.

We simply write $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

Definition 1.5 ([6])**.** Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The family of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; \xi) \in \Omega$, whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \operatorname{Re}\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{m} \operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\xi \in \partial U$ and $m \geq n \geq 1$.

In particular, we write $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$.

In our investigations we shall need the following lemmas.

Lemma 1.1 ([6])**.** *Let $\psi \in \Psi_n[\Omega, q]$, with $q(0) = a$. If $F \in \mathcal{H}[a, n]$ satisfies*

$$\psi\left(F(z), zF'(z), z^2F''(z); z\right) \in \Omega,$$

then $F(z) \prec q(z)$.

Lemma 1.2 ([6])**.** *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $F \in Q(a)$ and*

$$\psi\left(F(z), zF'(z), z^2F''(z); z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\psi\left(F(z), zF'(z), z^2F''(z); z\right) : z \in U, \zeta \in \bar{U}\right\}$$

implies $q(z) \prec F(z)$.

In recent years, several authors obtained many interesting results in differential subordination and superordination, such as Seoudy [12], Wanas and Srivastava [19], Lupas and Catas [4] and others (see, for example, [1–3, 8–11, 13–18, 20]). In this investigation, we consider certain suitable families of admissible functions and derive some differential subordination and superordination properties for higher-order derivatives of meromorphic multivalent functions.

2. Subordination Results

Definition 2.1. Let Ω be a set in C and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_j[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; z) \notin \Omega$, whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi)}{q(\xi)}, \ q(\xi) \neq 0 \quad \text{and} \quad \mathrm{Re}\left\{\frac{w + v^2}{v}\right\} \geq k\mathrm{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

where $z \in U$, $\xi \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem 2.1. *Let $\phi \in \Phi_j[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

$$\begin{aligned}\left\{\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1,\right.\right.\\ \left.\left.\frac{z^2 f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) : z \in U\right\} \subset \Omega,\end{aligned} \tag{2.1}$$

then

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Proof. Define the function F by

$$F(z) = \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}. \tag{2.2}$$

Then, the function F is analytic in U. After some calculation, we have

$$\frac{zF'(z)}{F(z)} = \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1. \tag{2.3}$$

Further computations show that

$$\begin{aligned}\frac{z^2F''(z)}{F(z)} + \frac{zF'(z)}{F(z)} - \left(\frac{zF'(z)}{F(z)}\right)^2 &= z\left[\frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1\right]' \\ &= \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2.\end{aligned} \tag{2.4}$$

Now, we define the transforms from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s}{r}, \quad w = \frac{r(t+s) - s^2}{r^2}.$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi\left(r, \frac{s}{r}, \frac{r(t+s) - s^2}{r^2}; z\right). \tag{2.5}$$

The proof will make use of Lemma 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.5) that

(2.6)
$$\psi\Big(F(z), zF'(z), z^2F''(z); z\Big) = \phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right).$$

Therefore, (2.1) becomes $\psi\left(F(z), zF'(z), z^2F''(z); z\right) \in \Omega$.

To complete the proof, we next show that the admissibility condition for $\phi \in \Phi_j[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.4.

Note that
$$\frac{t}{s} + 1 = \frac{w + v^2}{v},$$
and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, $F(z) \prec q(z)$ or equivalently
$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z). \qquad \square$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi_j[h(U), q]$ is written as $\Phi_j[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_j[h, q]$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \prec h(z), \tag{2.7}$$

then
$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

By taking $\phi(u, v, w; z) = u + \frac{v}{\beta u + \gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 2.2, we state the following corollary.

Corollary 2.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U, with $h(0) = 1$, and*
$$\operatorname{Re}\{\beta h(z) + \gamma\} > 0.$$

If $f \in \Sigma_p$ satisfies

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} + \frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z)+(p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2+\gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)} \prec h(z),$$

then

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of q on ∂U is not known.

Corollary 2.2. *Let $\Omega \in \mathbb{C}$ and q be univalent in U with $q(0)=1$. Let $\phi \in \Phi_j[h, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \in \Omega,$$

then

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Proof. Theorem 2.1 yields

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_\rho(z).$$

The result is now deduced from the fact that $q_\rho(z) \prec q(z)$. □

Theorem 2.3. *Let h and q be univalent in U with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(1) *$\phi \in \Phi_j[h, q_\rho]$ for some $\rho \in (0,1)$;*

(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_j[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.7), *then*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Proof. Case (1). By applying Theorem 2.1, we obtain

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_\rho(z).$$

Since $q_\rho(z) \prec q(z)$, we deduce

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z).$$

Case (2). Let

$$F(z) = \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \quad \text{and} \quad F_\rho(z) = F(\rho z).$$

Then,

$$\phi\left(F_\rho(z), zF'_\rho(z), z^2F''_\rho(z); \rho z\right) = \phi\left(F(\rho z), zF'(\rho z), z^2F''(\rho z); \rho z,\right) \in h_\rho(U).$$

By using Theorem 2.1 and the comment associated with

$$\phi\left(F(z), zF'(z), z^2F''(z); w(z)\right) \in \Omega,$$

where w is any function mapping U into U, with $w(z) = \rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $F(z) \prec q(z)$. Therefore,

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z). \qquad \square$$

The next result gives the best dominant of the differential subordination (2.7).

Theorem 2.4. *Let h be univalent in U and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)}{q(z)}, \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left(\frac{zq'(z)}{q(z)}\right)^2; z\right) = h(z) \tag{2.8}$$

has a solution q, with $q(0) = 1$, and satisfies one of the following conditions:

(1) *$q \in Q_1$ and $\phi \in \Phi_j[h, q]$;*

(2) *q is univalent in U and $\phi \in \Phi_j[h, q_\rho]$ for some $\rho \in (0, 1)$;*

(3) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_j[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.7), then

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q(z)$$

and q is the best dominant.

Proof. It follows from Theorems 2.2 and 2.3, that q is a dominant of (2.7). Since q satisfies (2.8), it is also a solution of (2.7), then q will be dominated by all dominants. Thus, q is the best dominant of (2.7). $\square$

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.1, the family of admissible functions $\Phi_j[\Omega, q]$ denoted by $\Phi_j[\Omega, M]$ can be expressed in the following form.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M>0$. The family of admissible functions $\Phi_j[\Omega, M]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ such that

$$\phi\left(1+Me^{i\theta}, \frac{kM}{M+e^{-i\theta}}, \frac{kM+Le^{-i\theta}}{M+e^{-i\theta}}-\left(\frac{kM}{M+e^{-i\theta}}\right)^2; z\right)\notin\Omega, \tag{2.9}$$

whenever $z\in U$, $\theta\in\mathbb{R}$, $\operatorname{Re}\{Le^{-i\theta}\}\geq k(k-1)M$ for all θ and $k\geq 1$.

Corollary 2.3. *Let $\phi\in\Phi_j[\Omega, M]$. If $f\in\Sigma_p$ satisfies*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)\in\Omega,$$

then

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

When $\Omega=q(U)=\{w:|w-1|<M\}$, the family $\Phi_j[\Omega, M]$ is simply denoted by $\Phi_j[M]$, then Corollary 2.3 takes the following form.

Corollary 2.4. *Let $\phi\in\Phi_j[M]$. If $f\in\Sigma_p$ satisfies*

$$\left|\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)-1\right|<M.$$

Then,

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

Example 2.1. If $M>0$ and $f\in\Sigma_p$ satisfies

$$\left|\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2-p-j+1\right|<M,$$

then

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

This implication follows from Corollary 2.4 by taking $\phi(u,v,w;z)=w-v+1$.

Example 2.2. If $M>0$ and $f\in\Sigma_p$ satisfies

$$\left|\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-2\right|<\frac{M}{M+1},$$

then

$$\left|\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-2}(p+j-2)!}+1\right|<M.$$

This implication follows from Corollary 2.3 by taking $\phi(u,v,w;z)=v$ and $\Omega=h(U)$ where $h(z)=\frac{M}{M+1}z$, $M>0$. To apply Corollary 2.3, we need to show that $\phi\in\Phi_j[\Omega,M]$, that is the admissibility condition (2.9) is satisfied follows from

$$\left|\phi\left(1+Me^{i\theta},\frac{kM}{M+e^{-i\theta}},\frac{kM+Le^{-i\theta}}{M+e^{-i\theta}}-\left(\frac{kM}{M+e^{-i\theta}}\right)^2;z\right)\right|=\frac{kM}{M+1}\geq\frac{M}{M+1},$$

for $z\in U$, $\theta\in\mathbb{R}$ and $k\geq 1$.

3. Superordination Results

In this section, we derive differential superordination. For this purpose the family of admissible functions given in the following definition will be required.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{H}$. The class of admissible functions $\Phi_j'[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ that satisfy the admissibility condition: $\phi(u,v,w;\xi)\in\Omega$, whenever

$$u=q(z),\quad v=\frac{zq'(z)}{mq(z)},\ q(z)\neq 0\quad\text{and}\quad\operatorname{Re}\left\{\frac{w+v^2}{v}\right\}\leq\frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

where $z\in U$, $\xi\in\partial U$ and $m\geq 1$.

Theorem 3.1. *Let $\phi\in\Phi_j'[\Omega,q]$. If $f\in\Sigma_p$,*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}\in Q_1$$

and

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\right.$$
$$\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right)$$

is univalent in U, then

$$\Omega\subset\left\{\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\right.\right.$$
$$\left.\left.\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right):z\in U\right\}\tag{3.1}$$

implies

$$q(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}.$$

Proof. Let F defined by (2.2) and $\psi\left(F(z), zF'(z), z^2F''(z); z\right)$ defined by (2.6). Since $\phi \in \Phi'_j[\Omega, q]$, from (2.6) and (3.1), we have

$$\Omega \subset \left\{\psi\left(F(z), zF'(z), z^2F''(z); z\right) : z \in U\right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_j[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.5. Hence, $\psi \in \Psi'[\Omega, q]$ and by Lemma 1.2, $q(z) \prec F(z)$ or equivalently

$$q(z) \prec \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}. \qquad \square$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi'_j[h(U), q]$ is written as $\Phi'_j[h, q]$. The following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let $\phi \in \Phi'_j[h, q]$, $q \in \mathcal{H}$ and h be analytic in U. If $f \in \Sigma_p$, $\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in Q_1$ and*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)$$

is univalent in U, then

$$h(z) \prec \phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \tag{3.2}$$

implies

$$q(z) \prec \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}.$$

By taking $\phi(u, v, w; z) = u + \frac{v}{\beta u + \gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 3.2, we state the following corollary.

Corollary 3.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U with $h(0) = 1$. Suppose that the differential equation $q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$ has univalent solution q that satisfies $q(0) = 1$ and $q(z) \prec h(z)$. If $f \in \Sigma_p$,*

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in \mathcal{H} \cap Q_1$$

and

$$\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}+\frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z)+(p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2+\gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)}$$

is univalent in U*, then*

$$h(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}+\frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z)+(p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2+\gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)}$$

implies

$$q(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}.$$

The next result gives the best subordinant of the differential superordination (3.2).

Theorem 3.3. *Let* h *be analytic in* U *and* $\phi:\mathbb{C}^3\times U\to\mathbb{C}$*. Suppose that the differential equation*

$$\phi\left(q(z),\frac{zq'(z)}{q(z)},\frac{z^2q''(z)}{q(z)}+\frac{zq'(z)}{q(z)}-\left(\frac{zq'(z)}{q(z)}\right)^2;z\right)=h(z)$$

has a solution $q\in Q_1$*. If* $\phi\in\Phi_j'[h,q]$*,* $f\in\Sigma_p$*,* $\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}\in Q_1$ *and*

$$\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right)$$

is univalent in U*, then*

$$h(z)\prec\phi\left(\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!},\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}+p+j-1,\frac{z^2f^{(j+1)}(z)}{f^{(j-1)}(z)}+\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}-\left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)^2;z\right)$$

implies

$$q(z)\prec\frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}$$

and q *is the best subordinant.*

Proof. The proof is similar to that of Theorem 2.4 and is omitted. □

4. Sandwich Results

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich result.

Theorem 4.1. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent in U, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_j [h_2, q_2] \cap \Phi'_j [h_1, q_1]$. If $f \in \Sigma_p$,*

$$\frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in \mathcal{H} \cap Q_1$$

and

$$\phi\left(\frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{z f^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2 f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{z f^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{z f^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!}, \frac{z f^{(j)}(z)}{f^{(j-1)}(z)} + p + j - 1, \frac{z^2 f^{(j+1)}(z)}{f^{(j-1)}(z)} + \frac{z f^{(j)}(z)}{f^{(j-1)}(z)} - \left(\frac{z f^{(j)}(z)}{f^{(j-1)}(z)}\right)^2; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_2(z).$$

By combining Corollary 2.1 and Corollary 3.1, we obtain the following sandwich result.

Corollary 4.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h_1, h_2 be convex in U with $h_1(0) = h_2(0) = 1$. Suppose that the differential equations $q_1(z) + \frac{z q_1'(z)}{\beta q_1(z) + \gamma} = h_1(z)$, $q_2(z) + \frac{z q_2'(z)}{\beta q_2(z) + \gamma} = h_2(z)$ have a univalent solutions q_1 and q_2, respectively, that satisfies $q_1(0) = q_2(0) = 1$ and $q_1(z) \prec h_1(z)$, $q_2(z) \prec h_2(z)$. If $f \in \Sigma_p$,*

$$\frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \in \mathcal{H} \cap Q_1$$

and

$$\frac{(p-1)! z^{p+j-1} f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} + \frac{(-1)^{j-1}(p+j-2)! \left[z f^{(j)}(z) + (p+j-1) f^{(j-1)}(z)\right]}{\beta (p-1)! z^{p+j-1} \left(f^{(j-1)}(z)\right)^2 + \gamma (-1)^{j-1}(p+j-2)! f^{(j-1)}(z)}$$

is univalent in U, then

$$
\begin{aligned}
h_1(z) \prec & \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \\
& + \frac{(-1)^{j-1}(p+j-2)!\left[zf^{(j)}(z)+(p+j-1)f^{(j-1)}(z)\right]}{\beta(p-1)!z^{p+j-1}\left(f^{(j-1)}(z)\right)^2+\gamma(-1)^{j-1}(p+j-2)!f^{(j-1)}(z)} \\
\prec & h_2(z)
\end{aligned}
$$

implies

$$q_1(z) \prec \frac{(p-1)!z^{p+j-1}f^{(j-1)}(z)}{(-1)^{j-1}(p+j-2)!} \prec q_2(z).$$

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[1]Department of Mathematics,
College of Science, University of Al-Qadisiyah,
Al Diwaniyah 58001, Al-Qadisiyah, Iraq
Email address: abbas.kareem.w@qu.edu.iq
ORCID iD: https://orcid.org/0000-0001-5838-7365

[2]Department of Mathematics,
College of Science, University of Al-Qadisiyah,
Al Diwaniyah, Al-Qadisiyah, Iraq
Email address: sci.math.mas.22.3@qu.edu.iq