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ON STATISTICAL SUMMABILITY IN NEUTROSOPHIC SOFT NORMED LINEAR SPACES

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ABSTRACT. In the present paper, we define the notions of statistical convergence and statistical Cauchy sequence in neutrosophic soft normed linear spaces and study some of their properties. We provide examples of a statistical Cauchy sequence that is not statistically convergent and give a useful characterisation of statistical convergence in these spaces.

1. INTRODUCTION

The concept of the statistical convergence was explored by Fast [9] and linked with summability theory by Schoenberg [11].

For any set $\mathscr{K} \subseteq \mathbb{N}$, the natural density of $\mathscr{K}$ is defined by

$$\delta(\mathscr{K}) = \lim_{\mathfrak{n}} \frac{1}{\mathfrak{n}} |\{\kappa \leq \mathfrak{n} : \kappa \in \mathscr{K}\}|$$

provided the limit exists. Further, a number sequence $\mathfrak{u} = (\mathfrak{u}_\kappa)$ is said to be statistical convergent to $\mathfrak{u}_0$ if for each $\mathscr{E} > 0$

$$\lim_{\mathfrak{n}} \frac{1}{\mathfrak{n}} |\{\kappa \leq \mathfrak{n} : |\mathfrak{u}_\kappa - \mathfrak{u}_0| \geq \mathscr{E}\}| = 0,$$

i.e., $\delta(\mathscr{K}_{\mathscr{E}}) = 0$, where $\mathscr{K}_{\mathscr{E}} = \{\kappa \leq \mathfrak{n} : |\mathfrak{u}_\kappa - \mathfrak{u}_0| \geq \mathscr{E}\}$. We write, in this case $\mathcal{S} - \lim_{\kappa} \mathfrak{u}_\kappa = \mathfrak{u}_0$. Subsequently, the idea is developed by several authors including Maddox [10], Fridy [12], Conner [13], Šalát [32] and many others.

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Many problems arising in the areas of science and engineering cannot fit into the framework of classical sets due to complications of uncertainty. As a result, to address these problems, we primarily rely on three approaches: use of probability, interval-based theory, and fuzzy set theory. Among these, fuzzy sets emerge as the most suitable mathematical tool for handling such problems.

The notion of a fuzzy set was initially defined by Zadeh [16] as a generalization of a crisp set with the help of a membership function to deal with those problems that cannot be modeled in the framework of crisp sets. But there are situations which can not be covered by fuzzy sets and therefore we need to extend the idea of fuzzy set. Actually, one drawback of fuzzy sets is the selection of membership function as more than one membership function can be defined using various operations on fuzzy sets. Atanassov [15] observed that fuzzy sets require more alteration to handle issues in a time domain, and therefore, he introduced the concept of intuitionistic fuzzy sets. After the introduction of intuitionistic fuzzy sets, a progressive development is made in this field. For instance, intuitionistic fuzzy metric spaces were introduced by Park [14], intuitionistic fuzzy topological spaces and intuitionistic fuzzy normed spaces by Saadati and Park [24], etc.

The neutrosophic sets were initially introduced by Smarandache [8] as a generalization of fuzzy sets and intuitionistic fuzzy sets with the help of a membership function, a non-membership function and an indeterminacy function to avoid the complexity arising from uncertainty in settling many practical challenges in real-world activities. For a progressive development on neutrosophic sets, we refer to the reader [7, 25] and [26]. Neutrosophic sets are also used to define a new kind of norm naturally. The credit goes to Kirişci and Şimşek [19] who defined neutrosophic normed space and extended summability theory in these spaces. They defined statistical convergence, statistical Cauchy and established some of their properties in neutrosophic normed space. Some more interesting works on summability in neutrosophic normed spaces can be found in [2–4] and [33].

Many approaches discussed above to minimize the uncertainty have their own drawbacks. The main reason behind this is due to inadequacy of the parametrization. To overcome on these difficulty, Molodtsov [6] introduced the idea of soft sets. These sets find valuable applications in numerous fields, including decision-making ([1,21,23]), medical diagnosis ([30, 34]), data analysis approaches under incomplete information [35], algorithms for COVID-19 outbreak [20], assessment processes [17], etc. Soft sets are further used to define soft norm by Das et al. [29] where they developed soft normed linear spaces from functional point of view.

In 2013, Maji [22] united the concepts of soft sets and neutrosophic sets, which he called neutrosophic soft sets. Quite recently, Bera and Mahapatra [31] used soft sets to define neutrosophic soft normed linear space and introduced the convergence structure in these spaces. In present study, we will continue in this direction and define statistical convergence, statistical Cauchy sequence in neutrosophic soft normed linear space and demonstrate some of their properties.

2. Preliminaries

This section starts with a brief information on soft sets, soft vector spaces and neutrosophic soft normed linear spaces. We begin with the following notations and definitions.

Throughout this work, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ will denote the sets of natural, real and positive real numbers, respectively.

Definition 2.1 ([5])**.** Let $\mathfrak{T} = [0,1]$. A binary operation $\circledast : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ is t-norm if for all $\mathfrak{c}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h} \in \mathfrak{T}$ we have

1) $\circledast$ is continuous, commutative and associative;
2) $\mathfrak{e} = \mathfrak{e} \circledast 1$;
3) $\mathfrak{c} \circledast \mathfrak{e} \leq \mathfrak{g} \circledast \mathfrak{h}$ whenever $\mathfrak{c} \leq \mathfrak{g}$ and $\mathfrak{e} \leq \mathfrak{h}$.

Some examples of t-norm are $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$, $\mathfrak{e} \circledast \mathfrak{g} = \min\{\mathfrak{e}, \mathfrak{g}\}$, $\mathfrak{e} \circledast \mathfrak{g} = \max\{\mathfrak{e}+\mathfrak{g}-1, 0\}$.

Definition 2.2 ([5])**.** Let $\mathfrak{T} = [0,1]$. A binary operation $\circledcirc : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ is t-conorm if for all $\mathfrak{c}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h} \in \mathfrak{T}$ we have

1) $\circledcirc$ is continuous, commutative and associative;
2) $\mathfrak{e} = \mathfrak{e} \circledcirc 0$;
3) $\mathfrak{c} \circledcirc \mathfrak{e} \leq \mathfrak{g} \circledcirc \mathfrak{h}$ whenever $\mathfrak{c} \leq \mathfrak{g}$ and $\mathfrak{e} \leq \mathfrak{h}$.

Some examples of t-conorm are $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$, $\mathfrak{e} \circledcirc \mathfrak{g} = \max\{\mathfrak{e}, \mathfrak{g}\}$, $\mathfrak{e} \circledcirc \mathfrak{g} = \min\{\mathfrak{e} + \mathfrak{g}, 1\}$.

For any universe set $\mathfrak{U}$ and parameter set $\mathfrak{E}$, the soft set is defined as follows.

Definition 2.3 ([6])**.** A pair $(\mathscr{H}, \mathfrak{E})$ is called a soft set over $\mathfrak{U}$ if and only if $\mathscr{H} : \mathfrak{E} \to \mathfrak{P}(\mathfrak{U})$, where $\mathfrak{P}(\mathfrak{U})$ is the set of all subsets of $\mathfrak{U}$. i.e., the soft set is a parametrized family of subsets of the set $\mathfrak{U}$. Moreover, every set $\mathscr{H}(\mathscr{E}), \mathscr{E} \in \mathfrak{E}$, from this family may be considered as the set of $\mathscr{E}$-elements of the soft set $(\mathscr{H}, \mathfrak{E})$, or as the set of $\mathscr{E}$-approximate elements of the set.

Definition 2.4 ([6])**.** A soft set $(\mathscr{H}, \mathfrak{E})$ over $\mathfrak{U}$ is said to be absolute soft set if for every $\mathscr{E} \in \mathfrak{E}$, $\mathscr{H}(\mathscr{E}) = \mathfrak{U}$. We will denote it by $\widetilde{\mathfrak{U}}$.

Definition 2.5 ([27])**.** Let $\mathbb{R}$ be the set of real numbers, $\mathfrak{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and $\mathfrak{E}$ taken as a set of parameters. Then a mapping $\mathfrak{F} : \mathfrak{E} \to \mathfrak{B}(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, then it is called a soft real number and denoted by $\widetilde{\mathfrak{r}}, \widetilde{\mathfrak{s}}, \widetilde{\mathfrak{t}}$, etc. $\widetilde{0}, \widetilde{1}$ are the soft real numbers where $\widetilde{0}(e) = 0$, $\widetilde{1}(e) = 1$ for all $e \in E$, respectively.

Let $\mathbb{R}(\mathfrak{E})$ and $\mathbb{R}^+(\mathfrak{E})$, respectively, denote the sets of all soft real numbers and all positive soft real numbers.

Definition 2.6 ([28])**.** Let $(\mathscr{H}, \mathfrak{E})$ be a soft set over $\mathfrak{U}$. The set $(\mathscr{H}, \mathfrak{E})$ is said to be a soft point, denoted by $\mathscr{H}_e^{\mathfrak{u}}$ if there is exactly one $e \in \mathfrak{E}$ s.t $\mathscr{H}(e) = \{\mathfrak{u}\}$ for some $\mathfrak{u} \in \mathfrak{U}$ and $\mathscr{H}(e') = \phi$ for all $e' \in \mathfrak{E} - \{e\}$.

Two soft points $\mathscr{H}_e^{\mathfrak{u}}, \mathscr{H}_{e'}^{\mathfrak{w}}$ are said to be equal if $e = e'$ and $\mathfrak{u} = \mathfrak{w}$. Let $\Delta_{\widetilde{\mathfrak{U}}}$ denotes the set of all soft points on $\widetilde{\mathfrak{U}}$.

In case $\mathfrak{U}$ is a vector space over $\mathbb{R}$ and the parameter set $\mathfrak{E} = \mathbb{R}$, the soft point is called a soft vector. Soft vector spaces are used to define soft norm as follows.

Definition 2.7 ([18])**.** Let $\widetilde{\mathfrak{U}}$ be a absolute soft vector space. Then a mapping $\|\cdot\| : \widetilde{\mathfrak{U}} \to \mathbb{R}^+(\mathfrak{E})$ is said to be a soft norm on $\widetilde{\mathfrak{U}}$, if $\|\cdot\|$ satisfies the following conditions:

(i) $\|\mathfrak{u}_e\| \geq \widetilde{0}$ for all $\mathfrak{u}_e \in \widetilde{\mathfrak{U}}$ and

$$\|\mathfrak{u}_e\| = \widetilde{0} \Leftrightarrow \mathfrak{u}_e = \widetilde{\theta}_0,$$

where $\widetilde{\theta}_0$ denotes the zero element of $\widetilde{\mathfrak{U}}$;

(ii) $\|\widetilde{\alpha}\mathfrak{u}_e\| = |\widetilde{\alpha}| \cdot \|\mathfrak{u}_e\|$ for all $\mathfrak{u}_e \in \widetilde{\mathfrak{U}}$ and for every soft scalar $\widetilde{\alpha}$;

(iii) $\|\mathfrak{u}_e + \mathfrak{u}_{e'}\| \leq \|\mathfrak{u}_e\| + \|\mathfrak{u}_{e'}\|$ for all $\mathfrak{u}_e, \mathfrak{u}_{e'} \in \widetilde{\mathfrak{U}}$;

(iv) $\|\mathfrak{u}_e \cdot \mathfrak{u}_{e'}\| = \|\mathfrak{u}_e\| \cdot \|\mathfrak{u}_{e'}\|$ for all $\mathfrak{u}_e, \mathfrak{u}_{e'} \in \widetilde{\mathfrak{U}}$.

The soft vector space $\widetilde{\mathfrak{U}}$ with a soft norm $\|\cdot\|$ on $\widetilde{\mathfrak{U}}$ is said to be a soft normed linear space and is denoted by $(\widetilde{\mathfrak{U}}, \|\cdot\|)$.

We now recall the definition of neutrosophic soft normed linear spaces and the convergence structure in these spaces.

Definition 2.8 ([31])**.** Let $\widetilde{\mathfrak{U}}$ be a soft linear space over the field $\mathfrak{F}$ and $\mathbb{R}(\mathfrak{E}), \Delta_{\widetilde{\mathfrak{U}}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{\mathfrak{U}}$. Then a neutrosophic subset N over $\Delta_{\widetilde{\mathfrak{U}}} \times \mathbb{R}(\mathfrak{E})$ is called a neutrosophic soft norm on $\widetilde{\mathfrak{U}}$ if for $\mathfrak{u}_e, \mathfrak{u}_{e'} \in \widetilde{\mathfrak{U}}$ and $\widetilde{\alpha} \in \mathfrak{F}$ ($\widetilde{\alpha}$ being soft scalar), the following conditions hold:

(i) $0 \leq G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}), B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}), Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \leq 1$ for all $\widetilde{\mathfrak{y}_1} \in \mathbb{R}(\mathfrak{E})$;

(ii) $0 \leq G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) + B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) + Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \leq 3$ for all $\widetilde{\mathfrak{y}_1} \in \mathbb{R}(\mathfrak{E})$;

(iii) $G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} \leq \widetilde{0}$;

(iv) $G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 1,$ with $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ if and only if $\mathfrak{u}_e = \widetilde{\theta_0}$, the null soft vector;

(v) $G_N(\widetilde{\alpha}\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = G_N\left(\mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}_1}}{|\widetilde{\alpha}|}\right)$ for all $\widetilde{\alpha}(\neq \widetilde{0}),\ \widetilde{\mathfrak{y}_1} > \widetilde{0}$;

(vi) $G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \circledast G_N(\mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_2}) \leq G_N(\mathfrak{u}_e \oplus \mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_1} \oplus \widetilde{\mathfrak{y}_2})$ for all $\widetilde{\mathfrak{y}_1}, \widetilde{\mathfrak{y}_2} \in \mathbb{R}(\mathfrak{E})$;

(vii) $G_N(\mathfrak{u}_e, \cdot)$ is continuous non-decreasing function for $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ and $\lim\limits_{\widetilde{\mathfrak{y}_1} \to +\infty} G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 1$;

(viii) $B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 1,$ with $\widetilde{\mathfrak{y}_1} \leq \widetilde{0}$;

(ix) $B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ if and only if $\mathfrak{u}_e = \widetilde{\theta_0}$, the null soft vector;

(x) $B_N(\widetilde{\alpha}\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = B_N\left(\mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}_1}}{|\widetilde{\alpha}|}\right)$ for all $\widetilde{\alpha}(\neq \widetilde{0}),\ \widetilde{\mathfrak{y}_1} > \widetilde{0}$;

(xi) $B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \circledcirc B_N(\mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_2}) \geq B_N(\mathfrak{u}_e \oplus \mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_1} \oplus \widetilde{\mathfrak{y}_2})$ for all $\widetilde{\mathfrak{y}_1}, \widetilde{\mathfrak{y}_2} \in \mathbb{R}(\mathfrak{E})$;

(xii) $B_N(\mathfrak{u}_e, \cdot)$ is continuous non-increasing function for $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ and $\lim\limits_{\widetilde{\mathfrak{y}_1} \to +\infty} B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0$;

(xiii) $Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} \leq \widetilde{0}$;

(xiv) $Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0,$ with $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ if and only if $\mathfrak{u}_e = \widetilde{\theta_0}$, the null soft vector;

(xv) $Y_N(\widetilde{\alpha}\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = Y_N\left(\mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}_1}}{|\widetilde{\alpha}|}\right)$ for all $\widetilde{\alpha}(\neq \widetilde{0})$, $\widetilde{\mathfrak{y}_1} > \widetilde{0}$;

(xvi) $Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) \circledcirc Y_N(\mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_2}) \geq Y_N(\mathfrak{u}_e \oplus \mathfrak{u}_{e'}, \widetilde{\mathfrak{y}_1} \oplus \widetilde{\mathfrak{y}_2})$ for all $\widetilde{\mathfrak{y}_1}, \widetilde{\mathfrak{y}_2} \in \mathbb{R}(\mathfrak{E})$;

(xvii) $Y_N(\mathfrak{u}_e, \cdot)$ is continuous non-increasing function for $\widetilde{\mathfrak{y}_1} > \widetilde{0}$ and $\lim_{\widetilde{\mathfrak{y}_1} \to +\infty} B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}_1}) = 0$.

In this case $N = (G_N, B_N, Y_N)$ is called the neutrosophic soft norm and

$$(\widetilde{\mathfrak{U}}(F), G_N, B_N, Y_N, \circledast, \circledcirc)$$

is an neutrosophic soft normed linear space (briefly $NSNLS$).

Let $(\widetilde{\mathfrak{U}}, \|\cdot\|)$ be a soft normed space. Take the operations $\circledast$ and $\circledcirc$ as $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$. For $\widetilde{\mathfrak{y}} > \widetilde{0}$, define

$$G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \begin{cases} \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} + \|\mathfrak{u}_e\|}, & \text{if } \widetilde{\mathfrak{y}} > \|\mathfrak{u}_e\|, \\ 0, & \text{otherwise,} \end{cases}$$

$$B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \begin{cases} \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}} + \|\mathfrak{u}_e\|}, & \text{if } \widetilde{\mathfrak{y}} > \|\mathfrak{u}_e\|, \\ 1 & \text{otherwise,} \end{cases}$$

$$Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \begin{cases} \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}}}, & \text{if } \widetilde{\mathfrak{y}} > \|\mathfrak{u}_e\|, \\ 1, & \text{otherwise.} \end{cases}$$

Then, $(\widetilde{\mathfrak{U}}(\mathfrak{F}), G_N, B_N, Y_N, \circledast, \circledcirc)$ is the $NSNLS$. From now onwards, unless otherwise stated by $\widetilde{\mathcal{V}}$ we shall denote the $NSNLS$ $(\widetilde{\mathfrak{U}}(\mathfrak{F}), G_N, B_N, Y_N, \circledast, \circledcirc)$.

A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\widetilde{\mathcal{V}}$ is said to be convergent to a soft point $\mathfrak{u}_e \in \widetilde{\mathcal{V}}$ if for $0 < \mathcal{E} < 1$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$ exists $\mathfrak{n}_0 \in \mathbb{N}$ s.t $G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$, $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$. In this case, we write $\lim_{\kappa \to +\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$.

A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\widetilde{\mathcal{V}}$ is said to be Cauchy sequence if for $0 < \mathcal{E} < 1$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$ exists $\mathfrak{n}_0 \in \mathbb{N}$ s.t for all $\kappa, \rho \geq \mathfrak{n}_0$, $G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$, $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) < \mathcal{E}$.

3. Statistical Convergence in NSNLS

In this section, we define statistical convergence in $NSNLS$ and develop some of its properties.

Definition 3.1. A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\widetilde{\mathcal{V}}$ is said to be statistical convergent to a soft point $\mathfrak{u}_e$ in $\widetilde{\mathcal{V}}$ if for $0 < \mathcal{E} < 1$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, there exists $\mathfrak{n}_0 \in \mathbb{N}$ s.t.

$$\lim_{\mathfrak{n} \to +\infty} \frac{1}{\mathfrak{n}} \Bigg| \Bigg\{ \kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or }$$

$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E} \Bigg\} \Bigg| = 0$$

or equivalently

$$\delta(\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or }$$

$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\}) = 0.$$

In present case, we denote $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$.

Remark 3.1. Since every finite set has density zero, every convergent sequence in $NSNLS$ $\widetilde{\mathcal{V}}$ is statistically convergent but the converse may not be true as can be seen from the following example.

Example 3.1. Let $(\widetilde{\mathbb{R}}, \|\cdot\|)$ be a soft normed linear space. For $\mathfrak{e}, \mathfrak{g} \in [0,1]$, let $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$. Choose $\mathfrak{u}_e \in \widetilde{\mathbb{R}}$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, we define

$$G_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}, \quad B_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}, \quad Y_N(\mathfrak{u}_e, \widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}}},$$

then it is easy to see that $(\widetilde{\mathbb{R}}, G_N, B_N, Y_N, \circledast, \circledcirc)$ is a $NSNLS$. Define a sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ by

$$\mathfrak{u}^{\kappa}_{e_\kappa} = \begin{cases} \widetilde{1}, & \text{if } \kappa \text{ is square,} \\ \widetilde{0}, & \text{otherwise.} \end{cases}$$

Now, for $\mathcal{E} > 0$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$,

$$\begin{aligned}
\mathcal{T} &= \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa}, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\} \\
&= \left\{\kappa \in \mathbb{N} : \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}^{\kappa}_{e_\kappa}\|} \leq 1 - \mathcal{E} \text{ or } \frac{\|\mathfrak{u}^{\kappa}_{e_\kappa}\|}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}^{\kappa}_{e_\kappa}\|} \geq \mathcal{E}, \frac{\|\mathfrak{u}^{\kappa}_{e_\kappa}\|}{\widetilde{\mathfrak{y}}} \geq \mathcal{E}\right\} \\
&= \left\{\kappa \in \mathbb{N} : \|\mathfrak{u}^{\kappa}_{e_\kappa}\| \geq \frac{\widetilde{\mathfrak{y}}\mathcal{E}}{1 - \mathcal{E}} \text{ or } \|\mathfrak{u}^{\kappa}_{e_\kappa}\| \geq \widetilde{\mathfrak{y}}\mathcal{E}\right\} \\
&= \{\kappa \in \mathbb{N} : \mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{1}\} \\
&= \{\kappa \in \mathbb{N} : \kappa \text{ is square}\}.
\end{aligned}$$

This implies that $\delta(\mathcal{T}) = \delta(\{\kappa \in \mathbb{N} : \kappa \text{ is square}\}) = 0$ and therefore $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical convergent to $\widetilde{0}$. Obviously, by the structure of the sequence, $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is not ordinary convergent.

By Definition 3.1 together with the property of natural density, we have the following lemma.

Lemma 3.1. *For any sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\widetilde{\mathcal{V}}$, the subsequent statements are equivalent:*

(*i*) $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$;

(*ii*) $\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\} = 0$;

(*iii*) $\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$ *and* $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}\} = 1$;

(*iv*) $\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}\} = \delta\{\kappa \in \mathbb{N} : Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}\} = 1$;

(v) $\mathcal{S}-\lim_{\kappa\to+\infty} G_N(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_e,\widetilde{\mathfrak{y}})=1$ *and* $\mathcal{S}-\lim_{\kappa\to+\infty} B_N(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_e,\widetilde{\mathfrak{y}})=\mathcal{S}-\lim_{\kappa\to+\infty} Y_N(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_e,\widetilde{\mathfrak{y}})=0.$

Theorem 3.1. *For any sequence* $\mathfrak{u}=(\mathfrak{u}^{\kappa}_{e_\kappa})$ *in* $\widetilde{\mathcal{V}}$, *if* $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}$ *exists, then it is unique.*

Proof. We shall prove the theorem by use of contradiction. Let $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}_{e_1}$ and $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}'_{e_2}$, where $\mathfrak{u}_{e_1}\neq\mathfrak{u}'_{e_2}$. For $\mathcal{E}>0$ and $\widetilde{\mathfrak{y}}>\widetilde{0}$, choose $\mathcal{E}_1>0$ s.t. $(1-\mathcal{E}_1)\circledast(1-\mathcal{E}_1)>1-\mathcal{E}$ and $\mathcal{E}_1\circledcirc\mathcal{E}_1<\mathcal{E}$. Define the following sets:

$$
\begin{aligned}
A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})&=\left\{\kappa\in\mathbb{N}:G_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_{e_1},\frac{\widetilde{\mathfrak{y}}}{2}\right)\le 1-\mathcal{E}_1\right\},\\
A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})&=\left\{\kappa\in\mathbb{N}:G_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}'_{e_2},\frac{\widetilde{\mathfrak{y}}}{2}\right)\le 1-\mathcal{E}_1\right\},\\
A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})&=\left\{\kappa\in\mathbb{N}:B_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_{e_1},\frac{\widetilde{\mathfrak{y}}}{2}\right)\ge \mathcal{E}_1\right\},\\
A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})&=\left\{\kappa\in\mathbb{N}:B_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}'_{e_2},\frac{\widetilde{\mathfrak{y}}}{2}\right)\ge \mathcal{E}_1\right\},\\
A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})&=\left\{\kappa\in\mathbb{N}:Y_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}_{e_1},\frac{\widetilde{\mathfrak{y}}}{2}\right)\ge \mathcal{E}_1\right\},\\
A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})&=\left\{\kappa\in\mathbb{N}:Y_N\left(\mathfrak{u}^{\kappa}_{e_\kappa}\ominus\mathfrak{u}'_{e_2},\frac{\widetilde{\mathfrak{y}}}{2}\right)\ge \mathcal{E}_1\right\}.
\end{aligned}
$$

Since $\mathcal{S}-\lim_{\kappa\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}_{e_1}$, by Lemma 3.1, $\delta\{A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=0$ and therefore

$$\delta\{A^{\complement}_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=1.$$

Further, $\mathcal{S}-\lim_{k\to+\infty}\mathfrak{u}^{\kappa}_{e_\kappa}=\mathfrak{u}'_{e_2}$, so $\delta\{A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}$ $=0$ and therefore $\delta\{A^{\complement}_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=\delta\{A^{\complement}_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}=1$ for all $\widetilde{\mathfrak{y}}>\widetilde{0}$. Define $K_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})=\{A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}\cap\{A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}\cap\{A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}$, then $\delta\{K_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})\}=0$ and therefore, $\delta\{K^{\complement}_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})\}=1$. Let $m\in K^{\complement}_{G_N,B_N,Y_N}(\mathcal{E},\widetilde{\mathfrak{y}})$, then we have following possibilities:

1. $m\in\{A_{G_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{G_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}^{\complement}$;
2. $m\in\{A_{B_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{B_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}^{\complement}$;
3. $m\in\{A_{Y_N,1}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\cup A_{Y_N,2}(\mathcal{E}_1,\widetilde{\mathfrak{y}})\}^{\complement}$.

Case 1. Let $m \in \{A_{G_N,1}(\mathscr{E}_1, \tilde{\mathfrak{y}}) \cup A_{G_N,2}(\mathscr{E}_1, \tilde{\mathfrak{y}})\}^{\complement}$. Then, $m \in A^{\complement}_{G_N,1}(\mathscr{E}_1, \tilde{\mathfrak{y}})$ and $m \in A^{\complement}_{G_N,2}(\mathscr{E}_1, \tilde{\mathfrak{y}})$ and therefore,

$$G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\tilde{\mathfrak{y}}}{2}\right) > 1 - \mathscr{E}_1 \quad \text{and} \quad G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2}\right) > 1 - \mathscr{E}_1. \tag{3.1}$$

Now,

$$\begin{aligned} G_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \tilde{\mathfrak{y}}) &= G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}^m_{e_m} \oplus \mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2} \oplus \frac{\tilde{\mathfrak{y}}}{2}\right) \\ &\geq G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\tilde{\mathfrak{y}}}{2}\right) \circledast G_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2}\right) \\ &> (1 - \mathscr{E}_1) \circledast (1 - \mathscr{E}_1) \quad \text{by (3.1)} \\ &> 1 - \mathscr{E}. \end{aligned}$$

Given that $\mathscr{E} > 0$ is arbitrary, we thus obtain $G_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \tilde{\mathfrak{y}}) = 1$, for all $\tilde{\mathfrak{y}} > \tilde{0}$, which gives $\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2} = \widetilde{\theta_0}$, i.e., $\mathfrak{u}_{e_1} = \mathfrak{u}'_{e_2}$.

Case 2. Let $m \in \{A_{B_N,1}(\mathscr{E}_1, \tilde{\mathfrak{y}}) \cup A_{B_N,2}(\mathscr{E}_1, \tilde{\mathfrak{y}})\}^{\complement}$. Then, $m \in A^{\complement}_{B_N,1}(\mathscr{E}_1, \tilde{\mathfrak{y}})$ and $m \in A^{\complement}_{B_N,2}(\mathscr{E}_1, \tilde{\mathfrak{y}})$ and therefore,

$$B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathscr{E}_1 \quad \text{and} \quad B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathscr{E}_1. \tag{3.2}$$

Now,

$$\begin{aligned} B_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \tilde{\mathfrak{y}}) &= B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}^m_{e_m} \oplus \mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2} \oplus \frac{\tilde{\mathfrak{y}}}{2}\right) \\ &\leq B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\tilde{\mathfrak{y}}}{2}\right) \circledcirc B_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2}\right) \\ &< \mathscr{E}_1 \circledcirc \mathscr{E}_1 \quad \text{by (3.2)} \\ &< \mathscr{E}. \end{aligned}$$

Given that $\mathscr{E} > 0$ is arbitrary, we thus obtain $B_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \tilde{\mathfrak{y}}) = 0$, for all $\tilde{\mathfrak{y}} > \tilde{0}$, which gives $\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2} = \widetilde{\theta_0}$, i.e., $\mathfrak{u}_{e_1} = \mathfrak{u}'_{e_2}$.

Case 3. Let $m \in \{A_{Y_N,1}(\mathscr{E}_1, \tilde{\mathfrak{y}}) \cup A_{Y_N,2}(\mathscr{E}_1, \tilde{\mathfrak{y}})\}^{\complement}$. Then, $m \in A^{\complement}_{Y_N,1}(\mathscr{E}_1, \tilde{\mathfrak{y}})$ and $m \in A^{\complement}_{Y_N,2}(\mathscr{E}_1, \tilde{\mathfrak{y}})$ and therefore,

$$Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathscr{E}_1 \quad \text{and} \quad Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathscr{E}_1. \tag{3.3}$$

Now,

$$Y_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \tilde{\mathfrak{y}}) = Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}^m_{e_m} \oplus \mathfrak{u}_{e_1} \ominus \mathfrak{u}'_{e_2}, \frac{\tilde{\mathfrak{y}}}{2} \oplus \frac{\tilde{\mathfrak{y}}}{2}\right)$$

$$\leq Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}_{e_1}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc Y_N\left(\mathfrak{u}^m_{e_m} \ominus \mathfrak{u}^{'}_{e_2}, \frac{\widetilde{\mathfrak{y}}}{2}\right)$$
$$< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \quad \text{by (3.3)}$$
$$< \mathcal{E}.$$

Given that $\mathcal{E} > 0$ is arbitrary, we thus obtain $Y_N(\mathfrak{u}_{e_1} \ominus \mathfrak{u}^{'}_{e_2}, \widetilde{\mathfrak{y}}) = 0$ for all $\widetilde{\mathfrak{y}} > \widetilde{0}$, which gives $\mathfrak{u}_{e_1} \ominus \mathfrak{u}^{'}_{e_2} = \widetilde{\theta_0}$, i.e., $\mathfrak{u}_{e_1} = \mathfrak{u}^{'}_{e_2}$.

Hence, in all cases we have $\mathfrak{u}_{e_1} = \mathfrak{u}^{'}_{e_2}$, i.e., the statistical limit of the sequence $(\mathfrak{u}^{\kappa}_{e_\kappa})$ is unique. ☐

Theorem 3.2. *A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ in $\widetilde{\mathcal{V}}$ is statistically convergent if and only if exists a set $\mathcal{K} = \{\kappa_1, \kappa_2, \kappa_3, \dots\}$ s.t $\delta(\mathcal{K}) = 1$ and $(G_N, B_N, Y_N) - \lim_{\substack{\kappa\in\mathcal{K}\\ \kappa\to+\infty}} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$.*

Proof. First suppose that $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$. For $\widetilde{\mathfrak{y}} > \widetilde{0}$ and $p \in \mathbb{N}$, define the set

$$A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}}) = \Bigg\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} \text{ and}$$
$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}\Bigg\}.$$

We first show that $A_{G_N,B_N,Y_N}(p+1, \widetilde{\mathfrak{y}}) \subset A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})$.

Let $m \in A_{G_N,B_N,Y_N}(p+1, \widetilde{\mathfrak{y}})$. Then, $G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p+1} > 1 - \frac{1}{p}$ and $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p+1} < \frac{1}{p}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\lambda}) < \frac{1}{p+1} < \frac{1}{p}$, this implies that $m \in A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})$ and therefore,

$$A_{G_N,B_N,Y_N}(p+1, \widetilde{\mathfrak{y}}) \subset A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}}). \tag{3.4}$$

Since $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$, so for all $p \in \mathbb{N}$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, $\delta\{A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})\} = 1$ and therefore is an infinite set. Let $m_1 \in A_{G_N,B_N,Y_N}(1, \widetilde{\mathfrak{y}})$. Further, $\delta\{A_{G_N,B_N,Y_N}(2, \widetilde{\mathfrak{y}})\} = 1$, so we can choose m_2 in $A_{G_N,B_N,Y_N}(2, \widetilde{\mathfrak{y}})$, s.t $m_2 > m_1$ and

$$\frac{1}{\mathfrak{n}}\Bigg|\Bigg\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{2} \text{ and}$$
$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{2}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{2}\Bigg\}\Bigg| > \frac{1}{2}.$$

Now, select m_3 in $A_{G_N,B_N,Y_N}(3, \widetilde{\mathfrak{y}})$, s.t $m_3 > m_2$ and

$$\frac{1}{\mathfrak{n}}\Bigg|\Bigg\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{3} \text{ and}$$
$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{3}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{3}\Bigg\}\Bigg| > \frac{2}{3},$$

and so on. In this way we obtain a sequence (m_p) in $\mathbb{N}$ with $m_{p+1} > m_p$ for all $p, m_p \in A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})$ and for all $\mathfrak{n} \geq m_p$, $p \in \mathbb{N}$

$$\frac{1}{\mathfrak{n}}\left|\left\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} \text{ and } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}\right\}\right| > \frac{p-1}{p}. \tag{3.5}$$

If we define a set

$$\mathcal{K} = \{\mathfrak{n} \in \mathbb{N} : 1 < \mathfrak{n} < m_1\} \cup \left[\underset{p\in\mathbb{N}}{\cup}\{\mathfrak{n} \in A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})\} : m_p \leq \mathfrak{n} < m_{p+1}\right], \tag{3.6}$$

then using (3.4), (3.5) and (3.6) we have for all $\mathfrak{n}$ satisfying $(m_p \leq \mathfrak{n} < m_{p+1})$,

$$\frac{1}{\mathfrak{n}}|\{\kappa \leq \mathfrak{n} : \kappa \in \mathcal{K}\}| \geq \frac{1}{\mathfrak{n}}\left|\left\{\kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} \text{ and } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p}\right\}\right| > \frac{p-1}{p},$$

and therefore, in the limiting case, we get $\delta(\mathcal{K}) \geq 1$, i.e., $\delta(\mathcal{K}) = 1$ as $\delta(\mathcal{K}) \ngtr 1$. Now we will show that the subsequence $(\mathfrak{u}^{\kappa}_{e_\kappa} : \kappa \in \mathcal{K})$ is convergent to $\mathfrak{u}_e$, i.e., $(\mathfrak{u}^{\kappa}_{e_\kappa}) \to \mathfrak{u}_e$ over $\mathcal{K}$.

Let $\mathcal{E} > 0$ be given. Since $\frac{1}{p} \to 0$ as $p \to +\infty$, so we can choose $p \in \mathbb{N}$, s.t $\frac{1}{p} < \mathcal{E}$. Let $\kappa \in \mathcal{K}$ be s.t $\kappa \geq \mathfrak{t}_p$ for some fixed integer $\mathfrak{t}_p$. Then by structure of $\mathcal{K}$, exists a number $q \geq p$, s.t $\mathfrak{t}_q \leq \kappa < \mathfrak{t}_{q+1}$ and $\kappa \in A_{G_N,B_N,Y_N}(p, \widetilde{\mathfrak{y}})$. Now for $\mathcal{E} > 0$,

$$G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \frac{1}{p} > 1 - \mathcal{E} \text{ and}$$
$$B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p} < \mathcal{E}, \quad Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \frac{1}{p} < \mathcal{E},$$

for all $\kappa \geq \mathfrak{t}_p$ and $\kappa \in \mathcal{K}$. This implies that $(G_N, B_N, Y_N) - \lim_{\kappa\to+\infty}(\mathfrak{u}^{\kappa}_{e_\kappa}) = \mathfrak{u}_e$.

Conversely, suppose there exists a set $\mathcal{K} = \{\kappa_1, \kappa_2, \dots, \kappa_j, \dots\}$, with $\delta(\mathcal{K}) = 1$ and $(G_N, B_N, Y_N) - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$ over $\mathcal{K}$, i.e., $(G_N, B_N, Y_N) - \lim_{\substack{\kappa\in\mathcal{K} \\ \kappa\to+\infty}} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$. Let $\mathcal{E} > 0$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$. Since $(G_N, B_N, Y_N) - \lim_{\substack{\kappa\in\mathcal{K} \\ \kappa\to+\infty}} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$, so there exists $\kappa_j \in \mathbb{N}$ s.t for all $\kappa \geq \kappa_j$ and $\kappa \in \mathcal{K}$, $G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) > 1 - \mathcal{E}$ and $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) < \mathcal{E}$. So, if we consider the set

$$T_{G_N,B_N,Y_N}(\mathcal{E}, \widetilde{\mathfrak{y}}) = \left\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\right\},$$

then $T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}}) \subset \mathbb{N} - \{\kappa_j, \kappa_{j+1}, \kappa_{j+2}, \dots\}$. This immediately implies that

$$\delta\left(T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}})\right) \leq \delta(\mathbb{N}) - \delta\{\kappa_j, \kappa_{j+1}, \kappa_{j+2}, \dots\} = 1 - 1 = 0,$$

and therefore $\delta\left(T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}})\right) = 0$ as $\delta\left(T_{G_N,B_N,Y_N}(\mathcal{E},\tilde{\mathfrak{y}})\right) \nless 0$. This shows that $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical convergent to $\mathfrak{u}_e$, i.e., $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$. ☐

Theorem 3.3. *Let $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ and $w = (w^{\kappa}_{e_\kappa})$ be any two sequences in $\tilde{\mathcal{V}}$ s.t $\mathcal{S} - \lim_{\kappa\to+\infty}(\mathfrak{u}^{\kappa}_{e_\kappa}) = \mathfrak{u}_{e_1}$ and $\mathcal{S} - \lim_{\kappa\to+\infty}(w^{\kappa}_{e_\kappa}) = w_{e_2}$. Then,*
(i) $\mathcal{S} - \lim_{\kappa\to+\infty}(\mathfrak{u}^{\kappa}_{e_\kappa} \oplus w^{\kappa}_{e_\kappa}) = \mathfrak{u}_{e_1} \oplus \ w_{e_2}$;
(ii) $\mathcal{S} - \lim_{\kappa\to+\infty}(\tilde{\alpha}\mathfrak{u}^{\kappa}_{e_\kappa}) = \ \tilde{\alpha}\mathfrak{u}_{e_1}$, where $\tilde{0} \neq \tilde{\alpha} \in \mathfrak{F}$.

Proof. The proof of the theorem can be obtained as the proof of Theorem 3.1, so omitted. ☐

4. Statistical Completeness in NSNLS

Definition 4.1. A sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ of soft points in $\tilde{\mathcal{V}}$ is said to be statistically Cauchy sequence if for $0 < \mathcal{E} < 1$ and $\tilde{\mathfrak{y}} > \tilde{0}$, exists $\rho \in \mathbb{N}$ s.t

$$\lim_{\mathfrak{n}\to+\infty} \frac{1}{\mathfrak{n}} \Bigg| \Bigg\{ \kappa \leq \mathfrak{n} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E} \Bigg\} \Bigg| = 0$$

or equivalently

$$\delta\{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \text{ or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \tilde{\mathfrak{y}}) \geq \mathcal{E}\} = 0.$$

Theorem 4.1. *Every statistical convergent sequence in $\tilde{\mathcal{V}}$ is statistical Cauchy.*

Proof. Let $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ be any statistical convergent sequence in $\tilde{\mathcal{V}}$ with $\mathcal{S} - \lim_{\kappa\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$. For $\mathcal{E} > 0$ and $\tilde{\mathfrak{y}} > \tilde{0}$. Choose $\mathcal{E}_1 > 0$ s.t

$$(1 - \mathcal{E}_1) \circledast (1 - \mathcal{E}_1) > 1 - \mathcal{E} \quad \text{and} \quad \mathcal{E}_1 \circledcirc \mathcal{E}_1 < \mathcal{E}. \tag{4.1}$$

Define a set, $K(\mathcal{E}_1, \tilde{\mathfrak{y}}) = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) \leq 1 - \mathcal{E}_1$ or $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) \geq \mathcal{E}_1, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) \geq \mathcal{E}_1\}$. Then, $K^{\complement}(\mathcal{E}_1, \tilde{\mathfrak{y}}) = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) > 1 - \mathcal{E}_1$ and $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) < \mathcal{E}_1$, $Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}) < \mathcal{E}_1\}$. Since $\mathcal{S} - \lim_{n\to+\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \mathfrak{u}_e$, so $\delta(K(\mathcal{E}_1, \tilde{\mathfrak{y}})) = 0$ and $\delta(K^{\complement}(\mathcal{E}_1, \tilde{\mathfrak{y}})) = 1$. Let $\rho \in K^{\complement}(\mathcal{E}_1, \tilde{\mathfrak{y}})$. Then,

$$G_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}\right) > 1 - \mathcal{E}_1 \quad \text{and} \quad B_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\tilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1. \tag{4.2}$$

Now, let $T(\mathcal{E},\widetilde{\mathfrak{y}}) = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \leq 1-\mathcal{E}$ or $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\}$. Then, we show that $T(\mathcal{E},\widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. Let $m \in T(\mathcal{E},\widetilde{\mathfrak{y}})$. Then,

$$G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \leq 1-\mathcal{E} \quad \text{or} \quad B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}. \tag{4.3}$$

Case 1. If $G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \leq 1-\mathcal{E}$, then $G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \leq 1-\mathcal{E}_1$ and therefore $m \in K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. As otherwise, i.e., if $G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1-\mathcal{E}_1$, then by (4.1), (4.2) and (4.3) we get

$$\begin{aligned}
1-\mathcal{E} &\geq G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \\
&= G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e \oplus \mathfrak{u}_e \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right) \\
&\geq G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledast G_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \\
&> (1-\mathcal{E}_1) \circledast (1-\mathcal{E}_1) \\
&> 1-\mathcal{E},
\end{aligned}$$

which is not possible. Thus, $T(\mathcal{E},\widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$.

Case 2. If $B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}$, then $B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1$ and therefore $m \in K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. As otherwise, i.e., if $B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1$, then by (4.1), (4.2) and (4.3) we get

$$\begin{aligned}
\mathcal{E} &\leq B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \\
&= B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e \oplus \mathfrak{u}_e \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right) \\
&\leq B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc B_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \\
&< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \\
&< \mathcal{E},
\end{aligned}$$

which is not possible. Also, if $Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}$, then $Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1$ and therefore $m \in K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. As otherwise, i.e., if $Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1$, then by (4.1), (4.2) and (4.3) we get

$$\begin{aligned}
\mathcal{E} &\leq Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \\
&= Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e \oplus \mathfrak{u}_e \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right) \\
&\leq Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc Y_N\left(\mathfrak{u}^{\rho}_{e_\rho} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \\
&< \mathcal{E}_1 \circledcirc \mathcal{E}_1 < \mathcal{E},
\end{aligned}$$

which is not possible. Thus, $T(\mathcal{E},\widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$.

Hence, in all cases, $T(\mathcal{E},\widetilde{\mathfrak{y}}) \subseteq K(\mathcal{E}_1,\widetilde{\mathfrak{y}})$. Since $\delta(K(\mathcal{E}_1,\widetilde{\mathfrak{y}})) = 0$, so $\delta(T(\mathcal{E},\widetilde{\mathfrak{y}})) = 0$, and therefore $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical Cauchy. □

Example 4.1. Let $\widetilde{R_1} = \{\frac{1}{\mathfrak{n}} : \mathfrak{n} \in \mathbb{N}\}$ and $\|\cdot\| = |\cdot|$, i.e., the usual norm on $\widetilde{R_1}$, then $(\widetilde{R_1}, |\cdot|)$ is a soft normed linear space. For $\widetilde{\mathfrak{y}} > \widetilde{0}$, if we define $G_N(\mathfrak{u}_e,\widetilde{\mathfrak{y}}) = \frac{\widetilde{\mathfrak{y}}}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}$, $B_N(\mathfrak{u}_e,\widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}} \oplus \|\mathfrak{u}_e\|}$, $Y_N(\mathfrak{u}_e,\widetilde{\mathfrak{y}}) = \frac{\|\mathfrak{u}_e\|}{\widetilde{\mathfrak{y}}}$; $\mathfrak{e} \circledast \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \circledcirc \mathfrak{g} = \mathfrak{e} + \mathfrak{g} - \mathfrak{e}\mathfrak{g}$, then it is easy to see that $(\widetilde{R_1}(\mathbb{R}), G_N, B_N, Y_N, \circledast, \circledcirc)$ is a $NSNLS$. If we define a sequence of soft points $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ by $\mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{\frac{1}{\kappa}}$, then it is easy to see by definition of G_N, B_N and Y_N, the density of the set $A = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E}$ or $B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\}$ is zero, i.e., $\delta(A) = 0$, Therefore, $(\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistical Cauchy. Since $\mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{\frac{1}{\kappa}} \to \widetilde{0}$ as $\kappa \to +\infty$ and usual convergence implies statistical convergence with the same limit, so $\mathcal{S} - \lim_{\kappa \to +\infty} \mathfrak{u}^{\kappa}_{e_\kappa} = \widetilde{0}$ but $\widetilde{0}$ is not a member of the space.

Remark 4.1. If a sequence is Cauchy in $\widetilde{\mathcal{V}}$, then it is statistically Cauchy.

Theorem 4.2. *For any sequence $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ in $\widetilde{\mathcal{V}}$, the subsequent conditions are equivalent:*

(i) $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistically Cauchy w.r.t. neutrosophic soft norm (G_N, B_N, Y_N);

(ii) there exists a subset $\mathcal{K} = \{\kappa_1, \kappa_2, \ldots, \kappa_j, \ldots\}$ of $\mathbb{N}$, with $\delta(\mathcal{K}) = 1$ and the subsequence $(v^{k_j}_{e_{k_j}})_{j \in \mathbb{N}}$ is Cauchy sequence over $\mathcal{K}$.

Proof. The proof of the theorem can be obtained analogously as the proof of Theorem 3.2. □

Definition 4.2. A $NSNLS$ $\widetilde{\mathcal{V}}$ is said to be statistically complete if every statistically Cauchy sequence in $\triangle_{\widetilde{\mathcal{V}}}$ is statistically convergent in $\triangle_{\widetilde{\mathcal{V}}}$.

Theorem 4.3. *If every statistical Cauchy sequence in $\widetilde{\mathcal{V}}$ has a statistical convergent subsequence, then $\widetilde{\mathcal{V}}$ is statistically complete.*

Proof. Let $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ be any statistically Cauchy sequence of soft points in $\widetilde{\mathcal{V}}$ which has a statistical convergent subsequence $(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}})$, i.e., $\mathcal{S} - \lim_{j \to +\infty} \mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} = \mathfrak{u}_e$ for some $\mathfrak{u}_e$ in $\widetilde{\mathcal{V}}$. Since $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistically Cauchy, so for $\mathcal{E} > 0$ and $\widetilde{\mathfrak{y}} > \widetilde{0}$, $\delta(\mathfrak{A}) = 0$, where

$$\mathfrak{A} = \Bigg\{\kappa \in \mathbb{N} : G_N\left(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \leq 1 - \mathcal{E}_1 \text{ or } B_N\left(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1\Bigg\}.$$

Again since $\mathcal{S} - \lim_{j \to +\infty} \mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} = \mathfrak{u}_e$, we have $\delta(\mathfrak{D}) = 0$, where

$$\mathfrak{D} = \Bigg\{\kappa(j) \in \mathbb{N} : G_N\left(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \leq 1 - \mathcal{E}_1 \text{ or}$$

$$B_N\left(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\kappa(j)}_{e_{\kappa(j)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \geq \mathcal{E}_1\right\}.$$

Now define

$$\mathfrak{R} = \{\kappa \in \mathbb{N} : G_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \leq 1 - \mathcal{E} \quad \text{or } B_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}, \\ Y_N(\mathfrak{u}^{\kappa}_{e_\kappa} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) \geq \mathcal{E}\}.$$

Now we claim that $\mathfrak{A}^\complement \cap \mathfrak{D}^\complement \subseteq \mathfrak{R}^\complement$. Let $m \in \mathfrak{A}^\complement \cap \mathfrak{D}^\complement$. Then, $m \in \mathfrak{A}^\complement$ and $m \in \mathfrak{D}^\complement$. If $m \in \mathfrak{A}^\complement$, then

$$G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1-\mathcal{E}_1 \text{ and } B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\rho}_{e_\rho}, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, \tag{4.4}$$

and if $m \in \mathfrak{D}^\complement$, then $m = \kappa(j_0)$ for $j_0 \in \mathbb{N}$ and

$$G_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) > 1-\mathcal{E}_1 \text{ and } B_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1, Y_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) < \mathcal{E}_1. \tag{4.5}$$

Now,

$$\begin{aligned} G_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) &= G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \oplus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right) \\ &\geq G_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledast G_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \\ &> (1-\mathcal{E}_1) \circledast (1-\mathcal{E}_1) \quad \text{for } \rho = \kappa(j_0) \\ &> 1-\mathcal{E} \end{aligned}$$

and

$$\begin{aligned} B_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) &= B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \oplus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right) \\ &\leq B_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc B_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \\ &< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \quad \text{for } \rho = \kappa(j_0) \\ &< \mathcal{E}, \\ Y_N(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}_e, \widetilde{\mathfrak{y}}) &= Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \oplus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2} \oplus \frac{\widetilde{\mathfrak{y}}}{2}\right) \\ &\leq Y_N\left(\mathfrak{u}^{m}_{e_m} \ominus \mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}}, \frac{\widetilde{\mathfrak{y}}}{2}\right) \circledcirc Y_N\left(\mathfrak{u}^{\kappa(j_0)}_{e_{\kappa(j_0)}} \ominus \mathfrak{u}_e, \frac{\widetilde{\mathfrak{y}}}{2}\right) \\ &< \mathcal{E}_1 \circledcirc \mathcal{E}_1 \quad \text{for } \rho = \kappa(j_0) \\ &< \mathcal{E}, \quad \text{by (4.4) and (4.5)}, \end{aligned}$$

which implies that $m \in \mathfrak{R}^{\complement}$, so $\mathfrak{A}^{\complement} \cap \mathfrak{D}^{\complement} \subseteq \mathfrak{R}^{\complement}$ or $\mathfrak{R} \subseteq \mathfrak{A} \cup \mathfrak{D}$. Therefore, $\delta(\mathfrak{R}) \leq \delta(\mathfrak{A} \cup \mathfrak{D}) = 0$. This shows that $\mathfrak{u} = (\mathfrak{u}^{\kappa}_{e_\kappa})$ is statistically convergent and therefore statistically complete. $\square$

5. Conclusion

The neutrosophic soft norm is a very powerful tool due to its parameterized nature to analyze many problems arising in different areas such as **decision-making**, **pattern recognition**, **medical diagnosis**, **data mining**, and **deriving insights from data**, particularly when there is inherent uncertainty in the data. In this paper, we introduce the ideas of statistical convergence, statistical Cauchy sequences, and statistical completeness in a more general setting, i.e., in neutrosophic soft normed linear spaces. The results presented in this paper will be helpful in analyzing many problems where the fuzzy norm is not sufficient to work, and we look forward to a generalized norm like the neutrosophic soft norm.

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