ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 50, Number 7, 2026

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić
. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Belgrade : Donat Graf). - 24 cm

Deset puta godišnje. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

Published By:	Faculty of Science University of Kragujevac Radoja Domanovića 12 34000 Kragujevac Serbia Tel.: +381 (0)34 336223 Fax: +381 (0)34 335040 Email: `krag_j_math@kg.ac.rs` Website: `http://kjm.pmf.kg.ac.rs`
Designed By:	Thomas Lampert
Front Cover:	Željko Mališić
Printed By:	Donat Graf, Belgrade, Serbia In 2026, the journal is published in one volume with ten issues.

Contents

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1035–1047.

GENERALIZED CAPUTO PROPORTIONAL BOUNDARY VALUE LANGEVIN FRACTIONAL DIFFERENTIAL EQUATIONS VIA KURATOWSKI MEASURE OF NONCOMPACTNESS

SAMIRA ZERBIB[1], KHALID HILAL[1], AND AHMED KAJOUNI[1]

Abstract. This manuscript aims to discuss the existence of solutions for nonlinear boundary value Langevin fractional differential equations involving the generalized Caputo proportional fractional derivative via Kuratowski measure of noncompactness in an arbitrary Banach space. Using the measure of noncompactness approach and Mönch's fixed point theorem, we demonstrate the existence result. An illustrative example is provided as an application to illustrate our main results.

1. Introduction

The concept of fractional calculus has enormous potential to change the way we see and model the nature around us. With a particular attention of physicists as well as engineers, remarkable research activities has been devoted to fractional calculus. They claimed that the use of fractional derivative and integration operators are desirable for describing the properties of several materials. Indeed, we discovered that a number of theoretical and experimental investigations demonstrate that differential equations with fractional derivatives govern specific thermal systems (heat diffusion) [1], physical systems (electricity) [2], and reological systems (viscoelasticity) [3]. For additional information regarding fractional differential equations (see [4, 5, 22–25]). Recently, particular attention has been focused on the study of Langevin fractional differential equations. The Langevin equation, which describes the evolution of physical events in fluctuating environments, was first proposed by Langevin in 1908 and is a key theory

Key words and phrases. Differential equations, generalized Caputo proportional fractional derivative, measure of noncompactness, fixed point theorem.
2020 *Mathematics Subject Classification*. Primary: 34A12. Secondary: 34K37.
https://doi.org/10.46793/KgJMat2607.1035Z
Received: November 13, 2023.
Accepted: September 04, 2024.

of Brownian motion [6, 7]. A more flexible fractional model for simulating fractal processes is the Langevin equation's fractional model, which is a generalization of the classical one and yields a fractional Gaussian process parametrized by two indices [8,9]. In [10], the authors established the existence of solutions for nonlinear Langevin equations with multistrip boundary and nonlocal multipoint conditions. The authors of [11], studied an antiperiodic boundary value problem for the Langevin equation with two fractional orders. For more details on Langevin fractional differential equations, we recommend reading [15–17].

Inspired by the previously stated research, we address the existence of solutions for the following nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative

$$\begin{cases} {}^{C}_{\delta}D^{\alpha,g}_{a^+}\left({}^{C}_{\delta}D^{\beta,g}_{a^+}+\chi\right)(x(t)-K(t,x(t))=L(t,x(t)), \quad t\in\Delta=[a,b],\\ (x(t)-K(t,x(t))_{t=a}=(x(t)-K(t,x(t))_{t=b}=0,\\ {}^{C}_{\delta}D^{\beta,g}_{a^+}(x(t)-K(t,x(t))_{t=a}=\mu, \quad {}^{C}_{\delta}D^{\beta,g}_{a^+}(x(t)-K(t,x(t))_{t=b}=\nu, \quad \mu,\nu\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $\Delta=[a,b]$ be a finite interval of $\mathbb{R}$ with $(a>0)$, $1<\alpha<2$, $0<\beta<1$, ${}^{C}_{\delta}D^{\alpha,g}_{0^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order α, $\chi\in\mathbb{R}^*$, and $K,L\in C(\Delta\times E,E)$ are given functions checking certain assumptions that will be defined later and E is a Banach space with the norm $\|\cdot\|$.

To the best of our knowledge, this is the first time that the nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative (1.1) is being studied.

2. Preliminaires

Some definitions and properties of the generalized Caputo proportional fractional derivative and the Kuratowski measure of noncompactness are the focus of this section.

• The Banach space of continuous functions with the norm $\|h\|_\infty=\sup\{\|h(t)\| : t\in\Delta\}$ is denoted as $C(\Delta,E)$.

• We say that the function $f:\Delta\times E\to E$ satisfies the Caratheodory conditions if:
(i) $f(t,x)$ is measurable with respect to t for every $x\in E$;
(ii) $f(t,x)$ is continuous with respect to $x\in E$ for every $t\in\Delta$.

• Throughout this paper we consider that the function $g:\Delta\to\mathbb{R}$ is a strictly positive increasing differentiable function.

Definition 2.1 ([12, 13])**.** Let $0<\delta<1$, $\alpha>0$, $\Phi\in L^1([a,b],E)$. The left-sided generalized proportional fractional integral with respect to g of order α of the function Φ is given by:

$${}_{\delta}I^{\alpha;g}_{a^+}\Phi(t)=\frac{1}{\delta^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\alpha-1}\Phi(s)ds,$$

where $\Gamma(\alpha)=\int_a^{+\infty}e^{-\tau}\tau^{\alpha-1}d\tau$, $\alpha>0$, is the Euler gamma function.

Definition 2.2 ([12, 13]). Let $0 < \delta < 1$, $\zeta, \rho : [0,1] \times \mathbb{R} \to [0, +\infty)$ be continuous functions such that $\lim_{\delta \to 0^+} \zeta(\delta, t) = 0$, $\lim_{\delta \to 1^-} \zeta(\delta, t) = 1$, $\lim_{\delta \to 0^+} \rho(\delta, t) = 1$, $\lim_{\delta \to 1^-} \rho(\delta, t) = 0$, and $\zeta(\delta, t) \neq 0$, $\rho(\delta, t) \neq 0$ for each $\delta \in [0, 1]$, $t \in \mathbb{R}$. Then, the proportional derivative of order δ with respect to g of the function Φ is given by

$$ {}_{\delta}D^{g}\Phi(t) = \rho(\delta, t)\Phi(t) + \zeta(\delta, t)\frac{\Phi'(t)}{g'(t)}. $$

In particular, if $\zeta(\delta, t) = \delta$ and $\rho(\delta, t) = 1 - \delta$, then we have

$$ {}_{\delta}D^{g}\Phi(t) = (1 - \delta)\Phi(t) + \delta\frac{\Phi'(t)}{g'(t)}. $$

Definition 2.3 ([12, 13]). Let $\delta \in (0, 1]$. The left-sided generalized Caputo proportional fractional derivative of order $n - 1 < \alpha < n$ is defined by

$$ {}^{C}_{\delta}D^{\alpha;g}_{a^+}\Phi(t) = \frac{1}{\delta^{n-\alpha}\Gamma(n - \alpha)} \int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))} g'(s)(g(t) - g(s))^{n-\alpha-1}({}_{\delta}D^{n,g}\Phi)(s)ds, $$

where $n = [\alpha] + 1$ and ${}_{\delta}D^{n,g} = \underbrace{{}_{\delta}D^{g}{}_{\delta}D^{g} \cdots {}_{\delta}D^{g}}_{n-\text{times}}$.

Lemma 2.1 ([12, 13]). *Let $t \in \Delta$, $\delta \in (0, 1]$, $\alpha, \theta > 0$, and $\Phi \in L^1(\Delta, E)$ then we have*

$$ {}_{\delta}I^{\alpha;g}_{a^+}({}_{\delta}I^{\theta;g}_{a^+}\Phi(t)) = {}_{\delta}I^{\theta;g}_{a^+}({}_{\delta}I^{\alpha;g}_{a^+}\Phi(t)) = {}_{\delta}I^{\alpha+\theta;g}_{a^+}\Phi(t). $$

Throughout this paper, as a simplification, we set

$$ \Omega^{\theta}_{g}(t, a) = e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a))^{\theta}. \tag{2.1} $$

Lemma 2.2 ([12, 13]). *Let $\alpha > 0$, $\theta > 0$, and $\delta \in (0, 1]$. Then, we have*

(i) $({}_{\delta}I^{\alpha;g}_{a^+}e^{\frac{\delta-1}{\delta}(g(\tau)-g(a))}(g(\tau) - g(a))^{\theta-1})(t) = \frac{\Gamma(\theta)}{\delta^{\alpha}\Gamma(\alpha+\theta)}e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a))^{\alpha+\theta-1}$;

(ii) ${}^{C}_{\delta}D^{\alpha;g}_{a^+}e^{\frac{\delta-1}{\delta}(g(\tau)-g(a))}(g(\tau) - g(a))^{\theta-1})(t) = \frac{\delta^{\alpha}\Gamma(\theta)}{\Gamma(\theta-\alpha)}e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a))^{\theta-\alpha-1}$.

Lemma 2.3 ([12, 13]). *Let $\alpha > 0$, $\delta \in (0, 1]$, and $\Phi \in L^1(\Delta, E)$. Then, we have* $\lim_{t \to a}({}_{\delta}I^{\alpha;g}_{a^+}\Phi(t)) = 0$.

Lemma 2.4 ([14]). *Let $\delta \in (0, 1]$, $n - 1 < \alpha < n$, $n = [\alpha] + 1$. Then, we have*

$$ {}_{\delta}I^{\alpha;g}_{a^+}({}^{C}_{\delta}D^{\alpha,g}_{a^+}\Phi(t)) = \Phi(t) - \sum_{k=0}^{n-1} \frac{({}_{\delta}D^{k,g}\Phi)(0)}{\delta^{k}\Gamma(k + 1)}\Omega^{k}_{g}(t, a). $$

Next, we give some definitions and properties concerning the Kuratowski measure of noncompactness.

Definition 2.4 ([18])**.** Let E be a Banach space and F be a bounded subset of E. We define the Kuratowski measure of noncompactness by the mapping $\Lambda : F \to [0, +\infty)$ defined as follows

$$\Lambda(A) = \inf \left\{ \varrho > 0 / A \subseteq \cup_{i=1}^{n} A_i \text{ and } \operatorname{diam}(A_i) \leq \varrho \right\}.$$

Lemma 2.5 ([18])**.** *Let N and G be two bounded subset of the Banach space E, then we have the following properties:*

(1) $N \subseteq G \Rightarrow \Lambda(N) \leq \Lambda(G)$;
(2) $\Lambda(N + G) \leq \Lambda(N) + \Lambda(G)$;
(3) $\Lambda(\alpha N) = |\alpha| \Lambda(N)$, $\alpha \in \mathbb{R}$;
(4) $\Lambda(N) = 0 \Leftrightarrow N$ *is relatively compact in E;*
(5) $\Lambda(N \cup G) = \max\{\Lambda(N), \Lambda(G)\}$;
(6) $\Lambda(N) = \Lambda(\overline{N}) = \Lambda(\overline{\operatorname{conv}} N)$, *where* $\overline{N}$ *and* $\operatorname{conv} N$ *denote the closure and the convex hull of* N, *respectively.*

Definition 2.5 ([21])**.** Let $\Upsilon : F \to E$ be a continuous bounded operator. Then, Υ is Λ-Lipschitz if there exists a constant $\theta > 0$, such that for all $N \subset F$ we have

$$\Lambda(\Upsilon(N)) \leq \theta \Lambda(N).$$

Lemma 2.6 ([19])**.** *Consider that $N \in C(\Delta, E)$ is a bounded and equicontinuous subset. Then, the function $t \mapsto \Lambda(N(t))$ is continuous on Δ and*

$$\Lambda\left(\int_a^b x(s)ds\right) \leq \int_a^b \Lambda(N(s))ds,$$

where $N(s) = \{x(s) : x \in N\}$, $s \in \Delta$.

Theorem 2.1 (Mönch's fixed point theorem [20])**.** *Let W be a closed, bounded and convex subset in a Banach space E such that $0 \in E$ and let $J : W \to W$ be a continuous mapping satisfying:*

$$U = \overline{\operatorname{conv}} J(U) \quad \text{or} \quad U = J(U) \cup \{0\} \Rightarrow \Lambda(U) = 0, \quad \text{for all } U \subset W. \tag{2.2}$$

Then, the mapping J has a fixed point.

3. Auxiliary Results

In this section, we prove the equivalence between an integral equation and our problem (1.1), and we give an existence theorem of the solution to the nonlinear boundary value Langevin fractional differential equation (1.1).

Lemma 3.1. *Let $\Delta = [a, b]$ and $K, L \in C(\Delta \times E, E)$. Then, x is a solution of problem* (1.1) *if and only if*

$$\begin{aligned} x(t) =& {}_{\delta}I_{a^+}^{\alpha+\beta,g} L(t, x(t)) - \chi {}_{\delta}I_{a^+}^{\beta,g} (x(t) - K(t, x(t)) + \mu({}_{\delta}I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ & + \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))} (g(t) - g(a))^{\beta+1} + K(t, x(t)), \end{aligned} \tag{3.1}$$

where

$$\Pi = \left(\frac{\nu -_\delta I_{a^+}^{\alpha;g} L(b,x(b)) - \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta^\beta \Gamma(\beta+2) e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))} \right). \tag{3.2}$$

Proof. Consider that $x(t)$ is a solution of the nonlinear boundary value Langevin fractional differential equation (1.1). Applying the operator ${}_\delta I_{0^+}^{\alpha;g}(\cdot)$ on both sides of the problem (1.1) and using Lemma 2.4, we get

$$\begin{aligned} {}_\delta^C D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right) =& {}_\delta I_{a^+}^{\alpha;g} L(t,x(t)) - \chi\left(x(t) - K(t,x(t))\right) + C_0 e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &+ C_1 \frac{e^{\frac{\delta-1}{\delta}(g(t)-g(a))}}{\delta}(g(t)-g(a)). \end{aligned} \tag{3.3}$$

Now, applying the operator ${}_\delta I_{0^+}^{\beta;g}(\cdot)$ on both sides of the integral equation (3.3) and using Lemma 2.4, we get

$$\begin{aligned} &x(t) - K(t,x(t)) \\ =& {}_\delta I_{a^+}^{\alpha+\beta;g} L(t,x(t)) - \chi {}_\delta I_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right) + C_0({}_\delta I_{a^+}^{\beta;g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &+ C_1({}_\delta I_{a^+}^{\beta;g}) \left(\frac{e^{\frac{\delta-1}{\delta}(g(t)-g(a))}}{\delta}(g(t)-g(a)) \right) + C_2 e^{\frac{\delta-1}{\delta}(g(t)-g(a))}, \end{aligned} \tag{3.4}$$

with C_0, C_1 and $C_2 \in \mathbb{R}$.

Using Lemma 2.2 (i), the integral equation (3.4) becomes

$$\begin{aligned} &x(t) - K(t,x(t)) \\ =& {}_\delta I_{a^+}^{\alpha+\beta;g} L(t,x(t)) - \chi {}_\delta I_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right) + C_0({}_\delta I_{a^+}^{\beta;g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\ &+ C_1 \frac{e^{\frac{\delta-1}{\delta}(g(t)-g(a))}}{\delta^{\beta+1}\Gamma(\beta+2)}(g(t)-g(a))^{\beta+1} + C_2 e^{\frac{\delta-1}{\delta}(g(t)-g(a))}. \end{aligned} \tag{3.5}$$

Next, let's determine the constants C_0, C_1, and C_2. In the integral equation (3.5), putting $t = a$, using the Lemma 2.3, and the initial condition $\left(x(t) - K(t,x(t))\right)_{t=a} = 0$, then we have

$$C_2 = \left(x(t) - K(t,x(t))\right)_{t=a} = 0.$$

Now we proceed to determine C_0, by applying the operator ${}_\delta^C D_{a^+}^{\beta;g}(\cdot)$ on both sides of the integral equation (3.5) with $C_2 = 0$ and using Lemma 2.2 (ii), we get the integral equation (3.3), putting $t = a$ in this equation, using Lemma 2.3, and the initial condition ${}_\delta^C D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=a} = \mu$, we get

$$C_0 = {}_\delta^C D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=a} = \mu.$$

This time, putting $t = b$ in the integral equation (3.3) with $(C_0 = \mu)$ and using the initial condition ${}_\delta^C D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=b} = \nu$, we get

$${}_\delta^C D_{a^+}^{\beta;g}\left(x(t) - K(t,x(t))\right)_{t=b} = {}_\delta I_{a^+}^{\alpha;g} L(t,x(t))|_{t=b} - \chi\left(x(t) - K(t,x(t))\right)_{t=b}$$

$$+ \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))} + C_1 \frac{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta}(g(b)-g(a)),$$

this implies that

$$\nu = {}_\delta I_{a^+}^{\alpha,g} L(b,x(b)) + \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))} + C_1 \frac{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta}(g(b)-g(a)),$$

therefore

$$C_1 = \delta \left(\frac{\nu - {}_\delta I_{a^+}^{\alpha,g} L(b,x(b)) - \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))} \right).$$

Substituting C_0, C_1, and C_2 in (3.5) we obtain

$$\begin{aligned}
&x(t) - K(t,x(t)) \\
=&{}_\delta I_{a^+}^{\alpha+\beta,g} L(t,x(t)) - \chi {}_\delta I_{a^+}^{\beta,g}\,(x(t) - K(t,x(t)) + \mu({}_\delta I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\
&+ \left(\frac{\nu - {}_\delta I_{a^+}^{\alpha,g} L(b,x(b)) - \mu e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}{\delta^\beta \Gamma(\beta+2) e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))} \right) e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1}.
\end{aligned}$$

This implies that

$$\begin{aligned}
x(t) =&{}_\delta I_{a^+}^{\alpha+\beta,g} L(t,x(t)) - \chi {}_\delta I_{a^+}^{\beta,g}\,(x(t) - K(t,x(t)) + \mu({}_\delta I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\
&+ \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1} + K(t,x(t)),
\end{aligned}$$

where Π is given by (3.2).

The opposite follows by direct computation. $\square$

Next, we introduce the following conditions.

(C_1) The functions $L, K \in C(\Delta \times E, E)$ satisfy the Carathéodory conditions.

(C_2) The function $K : \Delta \times E \to E$ is Λ-Lipschitz, i.e., there exists a constant $\theta > 0$, such that for all $N \subset E$ we have

$$\Lambda(K(t,N)) \leq \theta \Lambda(N).$$

(C_3) There exists a constant ξ such that for each $x \in C(\Delta, E)$ and $t \in \Delta$, we have

$$\|K(t,x)\| \leq \xi \|x\|.$$

(C_4) There exist $\omega \in C(\Delta, \mathbb{R}^+)$ and $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that ψ being nondecreasing. Then, for all $t \in \Delta$ and $x \in E$, we have:

$$\|L(t,x)\| \leq \omega(t)\psi(\|x\|).$$

(C_5) For every bounded subset $N \subseteq E$ and for each $t \in \Delta$, we have:

$$\Lambda(L(t,N)) \leq \omega(t)\Lambda(N).$$

Let $x \in C(\Delta, E)$ and $t \in \Delta$, then we define the operator Σ as follows:

$$\begin{aligned}
(\Sigma x)(t) =&{}_\delta I_{a^+}^{\alpha+\beta,g} L(t,x(t)) - \chi {}_\delta I_{a^+}^{\beta,g}\,(x(t) - K(t,x(t)) + \mu({}_\delta I_{a^+}^{\beta,g}) e^{\frac{\delta-1}{\delta}(g(t)-g(a))} \\
&+ \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1} + K(t,x(t)),
\end{aligned}$$

where Π is given by (3.2).

Note that if the operator Σ has a fixed point, then this implies that the nonlinear Langevin fractional differential equation (1.1) has a solution in $C(\Delta, E)$.

It is now appropriate for us to provide the main result of this paper, which is the existence theorem of problem (1.1).

Theorem 3.1. *Assume that all conditions* (C_1)*–*(C_5) *hold. Then, the nonlinear boundary value Langevin fractional differential equation* (1.1) *has at least a solution on* Δ *provided that the following inequality is satisfied:*

$$\varpi = \frac{\omega^*(g(b)-g(a))^{\alpha+\beta}}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{|\chi|(g(b)-g(a))^{\beta}(1+\theta)}{\delta^{\beta}\Gamma(\beta+1)} + \theta < 1. \tag{3.6}$$

Proof. Let $\Sigma : C(\Delta, E) \to C(\Delta, E)$. Then we consider the set B_ρ defined as:

$$B_\rho = \{x \in C(\Delta, E) : \|x\| \le \rho\},$$

where $\rho \ge \frac{\Theta_1}{1-\Theta_2}$, with

$$\begin{aligned}\Theta_1 =& \frac{\omega^*\psi(\rho)}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(g(b)-g(a))^{\alpha+\beta}\\ &+ \frac{|\mu|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} + \Pi(g(b)-g(a))^{\beta+1},\end{aligned} \tag{3.7}$$

where $\omega^* = \sup_{t\in\Delta}\{\omega(t)\}$, and

$$\Theta_2 = \frac{|\chi|\,(1+\xi)}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta} + \xi. \tag{3.8}$$

It is easy to see that B_ρ is a convex, closed, and bounded subset of the Banach space $C(\Delta, E)$.

The proof is given in the several steps.

Step 1. $\Sigma(B_\rho) \subset B_\rho$.

Let $x \in B_\rho$ and $t \in \Delta$, then by using the fact that $e^{\frac{\delta-1}{\delta}(g(\cdot)-g(\cdot))} < 1$ we have

$$\begin{aligned}&\|(\Sigma x)(t)\|\\ \le& \frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\alpha+\beta-1}\|L(s,x(s))\|ds\\ &+ \frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}\|x(s)\|ds\\ &+ \frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}\|K(s,x(s))\|ds\\ &+ \frac{|\mu|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}e^{\frac{\delta-1}{\delta}(g(s)-g(a))}ds\\ &+ \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1} + \|K(t,x(t))\|\end{aligned}$$

$$\begin{aligned}
\leq&\frac{\omega^*\psi(\rho)}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(g(b)-g(a))^{\alpha+\beta}+\frac{|\chi|\rho}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta}\\
&+\frac{|\chi|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta}\xi\rho\\
&+\frac{|\mu|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta}+\Pi(g(b)-g(a))^{\beta+1}+\xi\rho\\
\leq&\frac{\omega^*\psi(\rho)}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(g(b)-g(a))^{\alpha+\beta}\\
&+\frac{|\mu|}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta}+\Pi(g(b)-g(a))^{\beta+1}\\
&+\rho\left(\frac{|\chi|\,(1+\xi)}{\delta^{\beta}\Gamma(\beta+1)}(g(b)-g(a))^{\beta}+\xi\right)\\
=&\Theta_1+\rho\Theta_2\leq\rho,
\end{aligned}$$

where Θ_1 and Θ_2 are given by (3.7) and (3.8), respectively.

Therefore, Σ maps B_ρ into itself.

Step 2. The operator Σ is continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of B_ρ such that $x_n\to x$ as $n\to+\infty$ in B_ρ. Then, we have

$$\begin{aligned}
&\|(\Sigma x_n)(t)-(\Sigma x)(t)\|\\
\leq&{}_{\delta}I_{a^+}^{\alpha+\beta,g}\|L(t,x_n(t))-L(t,x(t))\|+\chi_{\delta}I_{a^+}^{\beta,g}\|x_n(t)-x(t)\|\\
&+\chi_{\delta}I_{a^+}^{\beta,g}\|K(t,x_n(t)-K(t,x(t))\|+\|K(t,x_n(t)-K(t,x(t))\|,
\end{aligned}$$

by using the fact that the functions L and K satisfy the Carathéodory conditions, and by Lebesgue dominated convergence theorem, from the above inequality, we get:

$$\|(\Sigma x_n)(t)-(\Sigma x)(t)\|\to 0\quad\text{as } n\to+\infty.$$

This implies that the operator Σ is continuous.

Step 3. The operator Σ is equicontinuous.

Let $t_1,t_2\in\Delta$, $t_1<t_2$, and $x\in B_\rho$. Then, we have

$$\begin{aligned}
&\|(\Sigma x)(t_2)-(\Sigma x)(t_1)\|\\
\leq&\frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{t_1}\left(\Omega_g^{\alpha+\beta-1}(t_2,a)-\Omega_g^{\alpha+\beta-1}(t_1,a)\right)g'(s)\|L(s,x(s))\|ds\\
&+\frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\Omega_g^{\alpha+\beta-1}(t_2,a)g'(s)\|L(s,x(s))\|ds\\
&+|\chi|\left[\frac{1}{\delta^{\beta}\Gamma(\beta)}\int_a^{t_1}\left(\Omega_g^{\beta-1}(t_2,a)-\Omega_g^{\beta-1}(t_1,a)\right)g'(s)\|(x(s)-K(s,x(s)))\|ds\right.\\
&\left.+\frac{1}{\delta^{\beta}\Gamma(\beta)}\int_{t_1}^{t_2}\Omega_g^{\beta-1}(t_2,a)g'(s)\|(x(s)-K(s,x(s)))\|ds\right]
\end{aligned}$$

$$
\begin{aligned}
&+|\mu|\left[\frac{1}{\delta^{\beta}\Gamma(\beta)}\int_a^{t_1}\left(\Omega_g^{\beta-1}(t_2,a)-\Omega_g^{\beta-1}(t_1,a)\right)g'(s)e^{\frac{\delta-1}{\delta}(g(s)-g(a))}ds\right.\\
&\left.+\frac{1}{\delta^{\beta}\Gamma(\beta)}\int_{t_1}^{t_2}\Omega_g^{\beta-1}(t_2,a)g'(s)e^{\frac{\delta-1}{\delta}(g(s)-g(a))}ds\right]\\
&+\Pi\left(\Omega_g^{\beta+1}(t_2,a)-\Omega_g^{\beta+1}(t_1,a)\right)+\|K(t_2,x(t_2))-K(t_1,x(t_1))\|,
\end{aligned}
$$

where $\Omega_g^{(\cdot)}(t,a)$ is given by (2.1).

By using the continuity of the functions $\Omega_g^{(\cdot)}(t,a)$, K, and by Lebesgue dominated convergence theorem, from the above inequality, we get $|(\Sigma x)(t_2)-(\Sigma x)(t_1)|\to 0$ as $t_1\to t_2$.

Therefore, the operator Σ is equicontinuous.

Step 4. Checking for condition (2.2) of Theorem 2.1.

Let $W\subseteq\overline{\text{conv}}(\Sigma(W)\cup\{0\})$ be a bounded and equicontinuous subset. Then, the function $J(t)=\Lambda(W(t))$ is continuous on Δ. Next, by using Lemma 2.6, the conditions (C_2) and (C_5), we obtain:

$$
\begin{aligned}
J(t)=&\Lambda(W(t))\le\Lambda\left(\text{conv}((\Sigma W)(t)\cup\{0\})\right)\le\Lambda((\Sigma W)(t))\\
\le&\Lambda\left\{\frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)\right.\\
&\left.\times(g(t)-g(s))^{\alpha+\beta-1}L(s,x(s))ds:x\in W\right\}\\
&+\Lambda\left\{\frac{\chi}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}x(s)ds:x\in W\right\}\\
&+\Lambda\left\{\frac{\chi}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}K(s,x(s))ds:x\in W\right\}\\
&+\Lambda\left\{\mu({}_{\delta}I_{a^+}^{\beta;g})e^{\frac{\delta-1}{\delta}(g(t)-g(a))}:t\in\Delta\right\}\\
&+\Lambda\left\{\Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\beta+1}:t\in\Delta\right\}\\
&+\Lambda\left\{K(t,x(t)):x\in W\right\}\\
\le&\frac{1}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\alpha+\beta-1}\Lambda\left(L(s,W(s))\right)ds\\
&+\frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}\Lambda\left(W(s)\right)ds\\
&+\frac{|\chi|}{\delta^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\beta-1}\Lambda\left(K(s,W(s))\right)ds\\
&+\Lambda\left(K(t,W(t))\right)\\
\le&\frac{\omega^*\|J\|}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t g'(s)(g(t)-g(s))^{\alpha+\beta-1}ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\chi|\|J\|}{\delta^{\beta}\Gamma(\beta)}\int_a^t g^{'}(s)(g(t)-g(s))^{\beta-1}ds\\
&+\frac{\theta|\chi|\|J\|}{\delta^{\beta}\Gamma(\beta)}\int_a^t g^{'}(s)(g(t)-g(s))^{\beta-1}ds+\theta\|J\|\\
\leq&\|J\|\left\{\frac{\omega^*(g(b)-g(a))^{\alpha+\beta}}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{|\chi|(g(b)-g(a))^{\beta}(1+\theta)}{\delta^{\beta}\Gamma(\beta+1)}+\theta\right\}.
\end{aligned}
$$

This implies that

$$\|J\|_\infty \leq \varpi\|J\|_\infty,$$

where ϖ is given by (3.6).

According to condition (3.6), we conclude that $\|J\| = 0$. Therefore, for all $t \in \Delta$, we have $J(t) = 0$ which implies that $\Lambda(W(t)) = 0$. Then, $W(t)$ is relatively compact in E. Hence, by using Arzelá-Ascoli theorem, we conclude that W is relatively compact in B_ρ. We observe that, all conditions of Theorem 3.1 are satisfied. As a result, there is a fixed point for the operator Σ on B_ρ which solves problem (1.1). The proof is then finished. ☐

4. Example

An exemplary example to demonstrate the applicability of our results will be covered in this section.

Let $\Delta = [0, \pi/2]$ and $E = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \to 0\}$ be the Banach space of real sequences $(x_n)_{n\in\mathbb{N}^*}$, such that $x_n \to 0$, with the norm $\|x\|_\infty = \sup_{n\in\mathbb{N}^*}|x_n|$.

We consider the following nonlinear boundary value Langevin fractional differential equation:

$$
\begin{cases}
{}^{C}_{\frac{1}{2}}D^{\frac{3}{2},t}_{0^+}\left({}^{C}_{\frac{1}{2}}D^{\frac{1}{2},t}_{0^+}+\frac{2}{25}\right)\left(x(t)-\frac{1}{19}x(t)\cos^2(t)\right)\\
=\frac{1}{11+t^2}\left(x^4+\frac{1}{5^n}\sin(|x(t)|)\right),\quad t\in\Delta=[0,\frac{\pi}{2}],\\
x(0)=x\left(\frac{\pi}{2}\right)=0,\\
{}^{C}_{\delta}D^{\beta,g}_{0^+}\left(x(t)\left(1-\frac{1}{19}\cos^2(t)\right)\right)_{t=0}=\mu,\\
{}^{C}_{\delta}D^{\beta,g}_{a^+}\left(x(t)\left(1-\frac{1}{19}\cos^2(t)\right)\right)_{t=\frac{\pi}{2}}=\nu,\quad \mu,\nu\in\mathbb{R},
\end{cases}
\tag{4.1}
$$

Comparing System (4.1) with Problem (1.1). Then, $\alpha = \frac{3}{2}$, $\beta = \delta = \frac{1}{2}$, $g(t) = t$, $\chi = \frac{2}{25}$, and $L, K : \Delta \times E \to E$, such that:

$$L(t,x)=\frac{1}{11+t^2}\left(x_n^4+\frac{1}{5^n}\sin(|x_n|)\right),\quad K(t,x)=\frac{1}{19}x_n\cos^2(t),$$

with $x = \{x_n\} \in E$.

Now we check for conditions (C_1)-(C_5).

It is easy to see that the functions L and K satisfy the Caratheodory conditions. We have:

$$\|K(t,x)\|\leq\frac{1}{19}\|x\| \quad \text{and} \quad \|L(t,x)\|\leq\frac{1}{11+t^2}\left(\|x\|^4+1\right).$$

Then, the conditions (C_3) and (C_4) hold with $\omega(t)=\frac{1}{11+t^2}$, $\omega^*(t)=\frac{1}{11}$, $\psi(x)=x^4+1$, and $\xi=\frac{1}{19}$. For each bounded set $N\subset E$, we have

$$\Lambda\left(L(t,N)\right)\leq\omega(t)\Lambda\left(N\right)\quad\text{and}\quad\Lambda\left(K(t,N)\right)\leq\frac{1}{19}\Lambda\left(N\right).$$

Then, the conditions (C_2) and (C_5) are satisfied with $\theta=\frac{1}{19}$.

Next, checking for the condition (3.6). Then, we have:

$$\begin{aligned}&\frac{\omega^*(g(b)-g(a))^{\alpha+\beta}}{\delta^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{\frac{2}{25}(g(b)-g(a))^{\beta}(1+\theta)}{\delta^{\beta}\Gamma(\beta+1)}+\theta\\=&\frac{\frac{1}{11}\times\left(\frac{\pi}{2}\right)^{\frac{3}{2}+\frac{1}{2}}}{\left(\frac{1}{2}\right)^{\frac{3}{2}+\frac{1}{2}}\Gamma(\frac{3}{2}+\frac{1}{2}+1)}+\frac{\frac{2}{25}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(1+\frac{1}{19})}{\left(\frac{1}{2}\right)^{\frac{1}{2}}\Gamma(\frac{1}{2}+1)}+\frac{1}{19}\simeq 0,6684<1.\end{aligned}$$

We note that all conditions of Theorem 3.1 hold. Then, the nonlinear boundary value Langevin fractional differential equation (4.1) has at least one solution on $[0,\frac{\pi}{2}]$.

5. Conclusion

The theory of nonlinear boundary value Langevin fractional differential equations has been developed in this manuscript. We have established the existence of solutions for a given nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative in arbitrary Banach space. Using the measure of noncompactness approach and Mönch's fixed point theorem, we were able to demonstrate the existence of solutions. Finally, a suitable example effectively illustrates our results.

References

[1] I. S. Jesus and J. A. T. Machado, *Fractional control of heat diffusion systems.* Nonlinear Dyn. **54**(3) (2008), 263–282. `https://doi.org/10.1007/s11071-007-9322-2`

[2] V. H. Schmidt and J. H. Drumheller, *Dielectric properties of lithium hydrazinium sulfate*, Phys. Rev. B **4** (1971), 4582–4597. `https://doi.org/10.1103/PhysRevB.4.4582`

[3] R. L. Bagley and R. A. Calico, *Fractional order state equations for the control of viscoelastically damped structures*, J. Guid. Control Dyn. **14**(2) (1991), 304–311. `https://doi.org/10.2514/3.20641`

[4] S. Zerbib, K. Hilal and A. Kajouni, *Some new existence results on the hybrid fractional differential equation with variable order derivative*, Results Nonlinear Anal. **6**(1) (2023), 34–48. `https://doi.org/10.31838/rna/2023.06.01.004`

[5] H. Khalid, A. Kajouni and S. Zerbib, *Hybrid fractional differential equation with nonlocal and impulsive conditions*, Filomat **37**(10) (2023), 3291–3303. `https://doi.org/10.2298/FIL2310291H`

[6] R. Zwanzig, *Nonequilibrium Statistical Mechanics*, Oxf. Univ. Press, Oxford, UK. 2001.

[7] W. T. Coffey, Y. P. Kalmykov and J. T. Waldron, *The Langevin Equation*, 2nd Ed., World Sci., Singapore, 2004.

[8] F. Mainardi and P. Pironi, *The fractional Langevin equation: Brownian motion revisited*, Extracta Math. **10** (1996), 140–154.

[9] J. W. Bruce, *Fractal physiology and the fractional calculus: a perspective*, Front. Physiol. **1** (2010), 12 pages. `https://doi.org/10.3389/fphys.2010.00012`

[10] B. Ahmad, A. Alsaedi and S. Salem, *On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders*, Adv. Differ. Equ. **2019** (2019), 1–14. https://doi.org/10.1186/s13662-019-2003-x

[11] H. Fazli and J. J. Nieto, *Fractional Langevin equation with anti-periodic boundary conditions*, Chaos Solitons Fractals. **114** (2018), 332–337. https://doi.org/10.1016/j.chaos.2018.07.009

[12] F. Jarad, M. A. Alqudah and T. Abdeljawad, *On more general forms of proportional fractional operators*, Open Math. **18** (2020), 167–176. https://doi.org/10.1515/math-2020-0014

[13] F. Jarad, T. Abdeljawad, S. Rashid and Z. Hammouch, *More properties of the proportional fractional integrals and derivatives of a function with respect to another function*, Adv. Differ. Equ. **2020** (2020), 1–16. https://doi.org/10.1186/s13662-020-02767-x

[14] I. Mallah, I. Ahmed, A. Akgul, F. Jarad and S. Alha, *On ψ-Hilfer generalized proportional fractional operators*, AIMS Math. **7**(1) (2021), 82–103. http://hdl.handle.net/20.500.12416/5850

[15] B. Ahmad, J. J. Nieto, A. Alsaedi and M. EI-Shahed, *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Real World Appl. **13** (2012), 599–606. https://doi.org/10.1016/j.nonrwa.2011.07.052

[16] A. Salem, F. Alzahrani and L. Almaghamsi, *Fractional Langevin equations with nonlocal integral boundary conditions*, Mathematics **7**(5) (2019), Article ID 402. https://doi.org/10.3390/math7050402

[17] H. Baghani and J. Nieto, *On fractional Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Model. Control. **24** (2019), 884–897. https://doi.org/10.1016/j.cnsns.2016.05.023

[18] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Marcel Dekker, New York, NY, USA, 1980.

[19] D. J. Guo, V. Lakshmikantham and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Acad. Publ., Dordrecht, 1996.

[20] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Analysis, Theory, Methods and Applications **4**(5) (1980), 985–999. https://doi.org10.1016/0362-546X(80)90010-3

[21] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[22] S. Zerbib, K. Hilal and A. Kajouni, *On the study of semilinear fractional differential equations involving Atangana-Baleanu-Caputo derivative*, Univ. J. Math. Appl. **6**(3) (2023), 122–129. https://doi.org/10.32323/ujma.1288015

[23] N. C. Eddine, M. A. Ragusa and D. D. Repovs, *Correction to: On the concentrationcompactness principle for anisotropic variable exponent Sobolev spaces and its applications*, Fract. Calc. Appl. Anal. **27**(2) (2024), 725–756. https://doi.org/10.1007/s13540-024-00246-8

[24] K. Hilal, H. Lmou and A. Kajouni, *Existence and stability results for a coupled system of Hilfer fractional Langevin equation with non local integral boundary value conditions*, Filomat **37**(4) (2024), 1241–1259. https://doi.org/10.2298/FIL2304241H

[25] Y. Pandir and Y. Gurefe, *A new version of the generalized F-expansion method for the fractional Biswas-Arshed equation and Boussinesq equation with the beta-derivative*, J. Funct. Spaces **2023**(1) (2023), Article ID 1980382. https://doi.org/10.1155/2023/1980382

[1] Laboratory of applied mathematics and scientific computing,
Sultan Moulay Slimane University,
Beni-Mellal, Morocco
Email address: zerbib.sam123@gmail.com
ORCID iD: https://orcid.org/0000-0002-2782-0055
Email address: hilalkhalid2005@yahoo.fr
ORCID iD: https://orcid.org/0000-0002-0806-2623
Email address: kajjouni@gmail.com
ORCID iD: https://orcid.org/0000-0001-8484-6107

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1049–1055.

GROWTH ESTIMATES FOR CERTAIN CLASS OF MEROMORPHIC FUNCTIONS

ADIL HUSSAIN

Abstract. In this paper, we establish some lower bound estimates for a certain class of rational functions on a disk with prescribed poles and restricted zeros. The results obtained strengthen some known results for rational functions and in turn produce generalizations of some polynomial inequalities as well.

1. Introduction

Let $\mathbb{P}_n$ be the set of all complex polynomials $f(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n. Then it is well known that

$$\max_{|z|=1} |f'(z)| \leq n \max_{|z|=1} |f(z)| \tag{1.1}$$

and

$$\max_{|z|=\rho} |f(z)| \leq \rho^n \max_{|z|=1} |f(z)|, \quad \rho \geq 1. \tag{1.2}$$

The above inequalities are known as Bernstein inequalities, and have been the starting point for a considerable literature in approximation theory. Several papers and research monographs have been written on this subject (see, for example, Marden [7], Rahman and Schmeisser [11], or Milovanović et al. [8]).

If one applies inequality (1.2) to the polynomial $f(z) = z^n f(\frac{1}{z})$ and uses maximum modulus principle, one easily get

$$\max_{|z|=\rho} |f(z)| \geq \rho^n \max_{|z|=1} |f(z)|, \quad 0 < \rho \leq 1. \tag{1.3}$$

Key words and phrases. Rational functions, polynomials, poles, zeros.
2020 *Mathematics Subject Classification.* Primary: 30A10. Secondary: 30C10, 26D10.
https://doi.org/10.46793/KgJMat2607.1049H
Received: August 03, 2022.
Accepted: September 07, 2024.

It is noted that equality holds in (1.1), (1.2) and (1.3) if and only if $f(z)$ has all its zeros at the origin. Inequality (1.3) is due to Varga [14], who attributes it to E. H. Zarantonello. It was shown by Govil, Qazi and Rahman [4] that the inequalities (1.1), (1.2) and (1.3) are all equivalent in the sense that any of these inequalities can be derived from the others. In 1960 Rivlin [13] improved inequality (1.3) for a restricted class of polynomials that do not vanish, in the unit disk $|z| < 1$. In fact, he proved that if $f(z) \neq 0$ in $|z| < 1$, then (1.3) can be replaced by

$$|f(\rho z)| \geq \left(\frac{1+\rho}{2}\right)^n |f(z)|, \quad \text{for } 0 < \rho < 1. \tag{1.4}$$

The inequality is best possible and equality holds for $f(z) = \left(\frac{\lambda+\beta z}{2}\right)^n$, where $|\lambda| = |\beta|$. As a generalization of (1.4), Aziz [2] proved that if $f(z) \neq 0$ in $|z| < 1$, then,

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad \text{for } k \geq 1,\ \rho < 1, \tag{1.5}$$

and

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad \text{for } k \leq 1,\ 0 \leq \rho \leq k^2. \tag{1.6}$$

The result is sharp and equality holds for $f(z) = (z+k)^n$.

Although the literature on polynomial inequalities is vast and growing, interested readers can consult the books of Marden [7] and Milovanović et al. [8]. Rather et al. [12] gave a new perspective to the above inequalities (1.5) and (1.6) and extended them to rational functions with prescribed poles. Essentially, in these inequalities, they replaced the polynomial $f(z)$ by a rational function $r(z)$ with poles $a_1, a_2, \ldots, a_n$ all lying in $|z| > 1$ and z^n by a Blaschke product $B(z)$. Before proceeding towards their results, let us introduce the set of rational functions involved.

For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, let

$$W(z) := \prod_{j=1}^{n} (z - a_j)$$

and

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a}_j z}{z - a_j}\right), \quad \mathbb{R}_n := \mathbb{R}_n(a_1, a_2, \ldots, a_n) = \left\{\frac{f(z)}{W(z)} : f \in \mathbb{P}_n\right\}.$$

Then, $\mathbb{R}_n$ is the set of rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at $+\infty$. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. Throughout this paper, we shall assume that all the poles $a_1, a_2, \ldots, a_n$ lie in $|z| > k$.

Very recently, Rather et al. [12] obtained certain growth estimates for rational functions $r \in \mathbb{R}_n$ having no zero in $|z| < k$. In this direction, they first present an extension of inequality (1.5) to the rational functions. More precisely, they proved the following result.

Theorem 1.1. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < k$, $k \geq 1$, then for every $\rho < 1$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j| - 1}{|a_j| + \rho}\right) |r(z)|. \tag{1.7}$$

Next they proved the following extension of (1.6).

Theorem 1.2. *If $r \in \mathbb{R}_n$ having no zeros in $|z| < k$, $k \leq 1$, then for every $0 \leq \rho \leq k^2$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j| - 1}{|a_j| + \rho}\right) |r(z)|. \tag{1.8}$$

In this paper, we established some lower bound estimates for the maximal modulus of a certain class of rational functions on a disk with prescribed poles and restricted zeros. The results obtained strengthen some known results for rational functions by using some of the extremal coefficients of the underlying polynomial and in turn produce generalizations of some polynomial inequalities as well.

2. Main Results

Theorem 2.1. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < k$, $k \geq 1$, then for every $\rho < 1$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left\{ \left[\left(\frac{k + \rho}{1 + k}\right)^n + \frac{1}{k^{n-1}} \left(\frac{|a_0| - k^n |a_n|}{|a_0| + |a_n|}\right) \left(\frac{1 - \rho}{1 + k}\right)^n \right] \prod_{j=1}^{n} \frac{|a_j| - 1}{|a_j| + \rho} \right\} |r(z)|. \tag{2.1}$$

Remark 2.1. Since $r(z) = \frac{P(z)}{W(z)}$ has all its zeros in $|z| \geq k$, therefore $P(z)$ has all its zeros in $|z| \geq k$, we always have the situation

$$\frac{|a_0| - k^n |a_n|}{|a_0| + |a_n|} \geq 0. \tag{2.2}$$

Therefore, for all rational functions satisfying the hypothesis of Theorem 2.1 excepting those satisfying $|a_0| = k^n |a_n|$, the inequality (2.1) sharpens the inequality (1.7).

Remark 2.2. If we take $k = 1$ in Theorem 2.1, we get the following extension and refinement of (1.4) to the rational functions.

Corollary 2.1. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < 1$, then for every $\rho < 1$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left\{ \left[\left(\frac{1 + \rho}{2}\right)^n + \left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right) \left(\frac{1 - \rho}{2}\right)^n \right] \prod_{j=1}^{n} \frac{|a_j| - 1}{|a_j| + \rho} \right\} |r(z)|.$$

Remark 2.3. Taking $W(z) = (z - \alpha)^n$, $|\alpha| > 1$, in Theorem 2.1, inequality (2.1) then reduces to the following inequality

$$\begin{aligned} &|f(\rho z)| \\ (2.3) \qquad &\geq \left\{ \left[\left(\frac{k+\rho}{1+k} \right)^n + \frac{1}{k^{n-1}} \left(\frac{|a_0| - k^n |a_n|}{|a_0| + |a_n|} \right) \left(\frac{1-\rho}{1+k} \right)^n \right] \left(\frac{|\alpha| - 1}{|\alpha| + \rho} \right)^n \left| \frac{\rho z - \alpha}{z - \alpha} \right|^n \right\} |f(z)|. \end{aligned}$$

Letting $|\alpha| \to +\infty$ in (2.3), we get refinement of (1.5).

Theorem 2.2. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < k$, $k \leq 1$, then for every $0 \leq \rho \leq k^2$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left\{ \left[\left(\frac{k+\rho}{1+k} \right)^n + \left(\frac{|a_0| - k^n |a_n|}{|a_0| + |a_n|} \right) \left(\frac{\lambda}{1+k} \right)^n \right] \prod_{j=1}^{n} \frac{|a_j| - 1}{|a_j| + \rho} \right\} |r(z)|, \tag{2.4}$$

where $\lambda = \min\{1 - \rho, k + \rho\}$.

Remark 2.4. As in remark 2.1, inequality (2.4) refines (1.8).

Remark 2.5. Taking $W(z) = (z - \alpha)^n$, $|\alpha| > 1$, in Theorem 2.2, inequality (2.4) then reduces to the following inequality

$$\begin{aligned} &|f(\rho z)| \\ (2.5) \qquad &\geq \left\{ \left[\left(\frac{k+\rho}{1+k} \right)^n + \frac{1}{k^{n-1}} \left(\frac{|a_0| - k^n |a_n|}{|a_0| + |a_n|} \right) \left(\frac{\lambda}{1+k} \right)^n \right] \left(\frac{|\alpha| - 1}{|\alpha| + \rho} \right)^n \left| \frac{\rho z - \alpha}{z - \alpha} \right|^n \right\} |f(z)|. \end{aligned}$$

Letting $|\alpha| \to +\infty$ in (2.5), we get refinement of (1.6).

3. Lemmas

For the proofs of our results, we shall make use of the following lemmas.

Lemma 3.1. *For any $0 \leq \rho \leq 1$ and $\rho_j \geq k \geq 1$, $1 \leq j \leq n$, we have*

$$\prod_{j=1}^{n} \left(\frac{\rho + \rho_j}{1 + \rho_j} \right) \geq \left(\frac{k+\rho}{k+1} \right)^n + \frac{1}{k^{n-1}} \left[\frac{\rho_1 \rho_2 \cdots \rho_n - k^n}{\rho_1 \rho_2 \cdots \rho_n + 1} \right] \left(\frac{1-\rho}{1+k} \right)^n.$$

Lemma 3.2. *For any $0 \leq \rho \leq 1$ and $\rho_j \geq k \geq 1$, $1 \leq j \leq n$, we have*

$$\prod_{j=1}^{n} \left(\frac{\rho + \rho_j}{1 + \rho_j} \right) \geq \left(\frac{k+\rho}{k+1} \right)^n + \frac{1}{k^{n-1}} \left[\frac{\rho_1 \rho_2 \cdots \rho_n - k^n}{\rho_1 \rho_2 \cdots \rho_n + 1} \right] \left(\frac{\lambda}{1+k} \right)^n,$$

where $\lambda = \min\{1 - \rho, k + \rho\}$.

The above two lemmas are due to Kumar and Milovanović [6].

4. Proof of the Theorems

Proof of Theorem 2.1. By hypothesis, $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$ has all its zeros in $|z| \leq k,\ k \geq 1$. It follows that $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $0 \leq \theta < 2\pi,\ 1 \leq j \leq n$, are zeros of $P(z)$, then we write $P(z) = c\prod_{j=1}^{n}(z - \rho_j e^{i\theta_j})$, where $\rho_j \geq k \geq 1,\ j = 1, 2, \ldots, n$. Hence, for $\rho < 1$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| = \left|\frac{P(\rho e^{i\theta})}{P(e^{i\theta})}\right| \cdot \left|\frac{W(e^{i\theta})}{W(\rho e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \cdot \prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right|. \tag{4.1}$$

Now,

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i(\theta-\theta_j)} - \rho_j}{e^{i(\theta-\theta_j)} - \rho_j}\right| = \prod_{j=1}^{n}\left(\frac{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\theta-\theta_j)}{1 + \rho_j^2 - 2\rho_j\cos(\theta-\theta_j)}\right)^{\frac{1}{2}}$$

$$\geq \prod_{j=1}^{n}\frac{\rho + \rho_j}{1 + \rho_j} \quad \text{as } \rho < 1. \tag{4.2}$$

Using Lemma 3.1 to the right hand side of (4.2), we get

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \geq \left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{\rho_1\rho_2\cdots\rho_n - k^n}{\rho_1\rho_2\cdots\rho_n + 1}\right)\left(\frac{1-\rho}{1+k}\right)^n.$$

Using the fact that $\rho_1\rho_2\cdots\rho_n = \frac{|a_0|}{|a_n|}$,

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \geq \left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right)\left(\frac{1-\rho}{1+k}\right)^n, \tag{4.3}$$

also for $|a_j| > 1,\ j = 1, 2, \ldots, n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right| \geq \prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}. \tag{4.4}$$

Using inequalities (4.3) and (4.4) in (4.1), we get

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| \geq \left[\left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right)\left(\frac{1-\rho}{1+k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}.$$

Thus,

$$|r(\rho z)| \geq \left\{\left[\left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right)\left(\frac{1-\rho}{1+k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}\right\}|r(z)|.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. By hypothesis, $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$ has all its zeros in $|z| \leq k,\ k \geq 1$. It follows that $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore if $z_j = \rho_j e^{i\theta_j}, 0 \leq \theta < 2\pi,\ 1 \leq j \leq n$, are zeros of $P(z)$, then we write $P(z) =$

$c\prod_{j=1}^{n}(z-\rho_j e^{i\theta_j})$, where $\rho_j \geq k \geq 1,\ j=1,2,\dots,n$. Hence, for $\rho < 1$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| = \left|\frac{P(\rho e^{i\theta})}{P(e^{i\theta})}\right| \cdot \left|\frac{W(e^{i\theta})}{W(\rho e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| \cdot \prod_{j=1}^{n}\left|\frac{e^{i\theta}-a_j}{\rho e^{i\theta}-a_j}\right|. \tag{4.5}$$

Now,

$$\begin{aligned}\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{\rho e^{i(\theta-\theta_j)}-\rho_j}{e^{i(\theta-\theta_j)}-\rho_j}\right| = \prod_{j=1}^{n}\left(\frac{\rho^2+\rho_j^2-2\rho\rho_j\cos(\theta-\theta_j)}{1+\rho_j^2-2\rho_j\cos(\theta-\theta_j)}\right)^{\frac{1}{2}}\\ &\geq \prod_{j=1}^{n}\frac{\rho+\rho_j}{1+\rho_j} \quad \text{as } 0\leq\rho\leq k^2. \end{aligned}\tag{4.6}$$

Using Lemma 3.2 to the right hand side of (4.6), we get

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| \geq \left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{\rho_1\rho_2\cdots\rho_n-k^n}{\rho_1\rho_2\cdots\rho_n+1}\right)\left(\frac{1-\rho}{1+k}\right)^n.$$

Using the fact that $\rho_1\rho_2\cdots\rho_n = \frac{|a_0|}{|a_n|}$,

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| \geq \left(\frac{k+\rho}{1+k}\right)^n + \left(\frac{|a_0|-k^n|a_n|}{|a_0|+|a_n|}\right)\left(\frac{\lambda}{1+k}\right)^n, \tag{4.7}$$

also for $|a_j|>1,\ j=1,2,\dots,n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta}-a_j}{\rho e^{i\theta}-a_j}\right| \geq \prod_{j=1}^{n}\frac{|a_j|-1}{|a_j|+\rho}. \tag{4.8}$$

Using inequalities (4.7) and (4.8) in (4.5), we get

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| \geq \left[\left(\frac{k+\rho}{1+k}\right)^n + \left(\frac{|a_0|-k^n|a_n|}{|a_0|+|a_n|}\right)\left(\frac{\lambda}{1+k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j|-1}{|a_j|+\rho}.$$

Thus,

$$|r(\rho z)| \geq \left\{\left[\left(\frac{k+\rho}{1+k}\right)^n + \left(\frac{|a_0|-k^n|a_n|}{|a_0|+|a_n|}\right)\left(\frac{\lambda}{1+k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j|-1}{|a_j|+\rho}\right\}|r(z)|.$$

This completes the proof of Theorem 2.2. □

5. Conclusions

A sequence of publications (see, for example [5, 9] and [10]) on inequalities for rational functions has been published in recent years, and significant progress has been made. Both mathematics and practical fields are interested in inequalities of these types. In this work, we continue our investigation of inequalities of this nature by taking into consideration the location of all the zeros and extremal coefficients of the underlying polynomial.

Acknowledgements. The author is extremely grateful to the anonymous referees for many useful suggestions and comments.

References

[1] N. C. Ankeny and T. J. Rivlin, *On a theorem of S. Bernstein,* Pacific J. Math. **5** (1955), 849–852.

[2] A. Aziz, *Growth of polynomials whose zeros are within or outside a circle,* Bull. Austral. Math. Soc. **35** (1987), 247–256.

[3] S. Bernstein, *Sur l'ordre de la meilleure approximation des functions continues par des polynomes de degré donné*, Mem. Acad. R. Belg. **4** (1912), 1–103.

[4] N. K. Govil, M. A. Qazi and Q. I. Rahman, *Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius*, Math. Inequal. Appl. **6** (2003), 453–467.

[5] A. Hussain and A. Wani, *Turán-type inequalities for certain class of meromorphic functions*, Int. J. Non linear. Anal. **16** (2025), 17–26. https://doi.org/10.22075/ijnaa.2023.29759.4249

[6] P. Kumar and G. V. Milovanović, *On sharpening and generalization of Rivlin's inequality,* Turk. J. Math. **46** (2022), 1436–1445. https://doi.org/10.3906/mat-2203-82

[7] M. Marden, *Geometry of Polynomials,* 2nd Ed. Math. Surveys, No. 3, Amer. Math. Soc., Providence, RI, 1966.

[8] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias., *Topics in Polynomials, Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[9] A. Mir and S. Hans, *Inequalities concerning rational functions in the complex domain,* Sib. Math. J. **63** (2022), 1012–1022. https://doi.org/10.1134/s0037446622050196

[10] A. Mir and T. Fayaz, *Inequalities for a rational function with prescribed poles,* Sib. Math. J. **65** (2024), 899–908. https://doi.org/10.1134/S0037446624040153

[11] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press, 2002.

[12] N. A. Rather, M. Shafi and I. Dar, *Growth estimate for rational functions with prescribed poles and restricted zeros,* Kragujevac J. Math. **49** (2025), 305–311. https://doi.org/10.46793/KgJMat2502.305R

[13] T. J. Rivlin, *On the maximum modulus of polynomials*, Amer. Math. Monthly **67** (1960), 251–253.

[14] R. S. Varga, *A comparision of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomials*, J. Soc. Indust. Appl. Math. **5** (1957), 39–46.

Department of Mathematics,
University of Kashmir,
Hazratbal Srinagar, India
Email address: malikadil6909@gmail.com
ORCID iD: https://orcid.org/0000-0002-9646-9940

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1057–1062.

ON THE DISCRETE HARDY INEQUALITY WITH VARIABLE EXPONENT

RENÉ ERLÍN CASTILLO[1], BABAR SULTAN[2], AND HÉCTOR CAMILO CHAPARRO[3]

Abstract. We obtain the celebrated Hardy's inequality in the context of variable exponent sequence spaces.

1. Introduction

The Hardy inequality is a classical result in analysis and the mathematical inequalities, named after mathematician Godfrey Harold Hardy. It has a long and very rich history, starting in 1915 when G.H. Hardy [1] needed an estimate for arithmetic means, more precisely, an inequality of the form

$$\sum_{n=1}^{+\infty}\left|\frac{1}{n}A_n\right|^2 \leq C\sum_{n=1}^{+\infty}|a_n|^2,$$

with $A_n=\sum_{k=1}^{n}a_k$ and $\{a_k\}_{k\in\mathbb{N}}$ a sequence. Later, this inequality was extended to

$$\sum_{n=1}^{+\infty}\left|\frac{1}{n}A_n\right|^p \leq C_p\sum_{n=1}^{+\infty}|a_n|^p, \quad 1<p<+\infty.$$

In his famous 1925 paper [2], Hardy proved the discrete and the integral version of what is known today as Hardy's inequalities. The discrete version of Hardy's inequality asserts that

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{+\infty}a_n^p, \tag{1.1}$$

where $p>1$ and $\{a_k\}_{k=1}^{+\infty}$ is a sequence of nonnegative real numbers.

Key words and phrases. Hardy's inequality, variable exponent, sequence space.
2020 *Mathematics Subject Classification.* Primary: 42B25. Secondary: 26D15.
https://doi.org/10.46793/KgJMat2607.1057C
Received: February 29, 2024.
Accepted: September 27, 2024.

Since then, this type of inequality has been intensively studied. There are a lot of research articles on the topic, as well as surveys and books. For example, the survey [4] contains several proofs and historical aspects are given. In [3], authors proved improved classical Hardy inequality for sequences of non-negative real numbers. In [6], Lefévre obtained a short direct proof of the discrete Hardy inequality provided that constant p' is optimal. For more on Hardy's inequality, we refer the reader to [5] and [7].

In this note, we define the (variable) Lebesgue sequence spaces, equipped with Luxemburg norm. We will prove Hölder's inequality in this setting. Finally, our main goal is to obtain the Hardy inequality (1.1) for these spaces.

2. $\ell_{p(\cdot)}$ spaces and Hölder's Inequality

We give here some definitions and results that will be useful in the coming section.

Let H be a measurable subset of $\mathbb{R}^n$ and let $p : H \to [1, +\infty)$ be a measurable function. We suppose that

$$1 \le p_-(H) \le p(x) \le p_+(H) < +\infty, \tag{2.1}$$

where $p_- := \operatorname{ess}\inf_{x\in H} p(x)$, $p_+ := \operatorname{ess}\sup_{x\in H} p(x)$.

Let $p(\cdot) > 1$. The conjugate exponent of $p(\cdot)$ is denoted by $p'(\cdot)$ and is defined as

$$p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}.$$

Definition 2.1. $\ell_{p(\cdot)}$ denotes the variable exponent Lebesgue sequence space (sometimes denoted by $\ell_{p(\cdot)}(\mathbb{N})$), which is the set of all sequences $a = \{a_n\}_{n\in\mathbb{N}}$ such that

$$I_{p(\cdot)}(a) = \sum_{n=1}^{+\infty} |a_n|^{p(x)} < +\infty.$$

We equip $\ell_{p(\cdot)}$ with the Luxemburg norm

$$\|a\|_{p(\cdot)} = \inf\left\{\lambda : I_{p(\cdot)}\left(\frac{a}{\lambda}\right) \le 1\right\},$$

where we use the convention that $\inf\emptyset = +\infty$.

Note that

$$I_{p(\cdot)}\left(\frac{a}{\|a\|_{p(\cdot)}}\right) \le 1,$$

where $\|a\|_{p(\cdot)} \neq 0$.

We state and prove Hölder's inequality for $\ell_{p(\cdot)}$.

Theorem 2.1. *Let $a \in I_{p(\cdot)}$ and $b \in I_{q(\cdot)}$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then,*

$$\sum_{n=1}^{+\infty} |a_n b_n| \le k_{p(\cdot)} \|a\|_{p(\cdot)} \|b\|_{q(\cdot)}.$$

Proof. If $\|a\|_{p(\cdot)} = 0$ and $\|b\|_{q(\cdot)} = 0$, then $ab = 0$, so there is nothing to prove. Therefore we may assume that $\|a\|_{p(\cdot)} > 0$ and $\|b\|_{q(\cdot)} > 0$, moreover by homogeneity we may assume that $\|a\|_{p(\cdot)} = \|b\|_{q(\cdot)} = 1$.

By Young's inequality for variable exponent we have

$$\begin{aligned}\sum_{n=1}^{+\infty} |a_n b_n| \leq &\sum_{n=1}^{+\infty} \frac{1}{p(x)} |a_n|^{p(x)} + \sum_{n=1}^{+\infty} \frac{1}{q(x)} |a_n|^{q(x)} \\ \leq &\frac{1}{p_-} I_{p(\cdot)}(a) + \frac{1}{q_-} I_{q(\cdot)}(b) \\ \leq &\frac{1}{p_-} I_{p(\cdot)} \left(\frac{a}{\|a\|_{p(\cdot)}} \right) + \frac{1}{q_-} I_{q(\cdot)} \left(\frac{b}{\|b\|_{q(\cdot)}} \right) \\ \leq &\frac{1}{p_-} + \frac{1}{q_-} \leq \frac{1}{p_-} + 1 - \frac{1}{p_+}.\end{aligned}$$

Finally

$$\sum_{n=1}^{+\infty} |a_n b_n| \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+} \right) \|a\|_{p(\cdot)} \|b\|_{q(\cdot)} = k_{p(\cdot)} \|a\|_{p(\cdot)} \|b\|_{q(\cdot)},$$

where $k_{p(\cdot)} = 1 + \frac{1}{p_-} - \frac{1}{p_+}$. So, the proof is now complete. $\square$

Remark 2.1. For the sake of completeness and the convenience of the reader, we would like to present another proof of Theorem 2.1. In order to do so, let us first check that $q_- = q_+$. Since $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then $\frac{1}{p(x)} = \frac{q(x)-1}{q(x)}$ so $q(x) = p(x)(q(x) - 1)$. Now,

$$\begin{aligned}q_- &= \operatorname{ess\,inf} q(x) = \operatorname{ess\,inf} (p(x)) (q(x) - 1) = \operatorname{ess\,inf} (-p(x)) (1 - q(x)) \\ &= - \operatorname{ess\,sup} (p(x)) (1 - q(x)) = - \operatorname{ess\,sup} (-p(x)) (q(x) - 1) = - \operatorname{ess\,sup} (-q(x)) \\ &= - \operatorname{ess\,sup} (-q_+) \\ &= q_+.\end{aligned}$$

Thus, $\frac{1}{p_-} + \frac{1}{q_-} = \frac{1}{p_-} + \frac{1}{q_+} = \frac{1}{p_-} + 1 - \frac{1}{p_+}$.

Alternative proof of Theorem 2.1. Young's inequality can also be viewed as

$$|a_n b_n| \leq \frac{1}{p(x)} |a_n|^{p(x)} + \frac{1}{q(x)} |a_n|^{q(x)} \leq \frac{1}{p_-} |a_n|^{p_+} + \frac{1}{q_-} |a_n|^{q_+}.$$

Then,

$$\begin{aligned}\sum_{n=1}^{+\infty} \left| \frac{a_n b_n}{\|a\|_{p(\cdot)} \|b\|_{q(\cdot)}} \right| \leq &\frac{1}{p_-} \sum_{n=1}^{+\infty} \left| \frac{a_n}{\|a\|_{p(\cdot)}} \right|^{p_+} + \frac{1}{q_-} \sum_{n=1}^{+\infty} \left| \frac{b_n}{\|b\|_{q(\cdot)}} \right|^{q_+} \\ \leq &\frac{1}{p_-} \frac{\sum_{n=1}^{+\infty} |a_n|^{p_+}}{\|a\|_{p(\cdot)}^{p_+}} + \frac{1}{q_-} \frac{\sum_{n=1}^{+\infty} |b_n|^{q_+}}{\|b\|_{q(\cdot)}^{q_+}} \\ = &\frac{1}{p_-} + \frac{1}{q_-} = \frac{1}{p_-} + 1 - \frac{1}{p_+} = 1 + \frac{1}{p_-} - \frac{1}{p_+}.\end{aligned}$$

So,

$$\sum_{n=1}^{+\infty} |a_n b_n| \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|a\|_{p(\cdot)} \|b\|_{q(\cdot)} = k_{p(\cdot)} \|a\|_{p(\cdot)} \|b\|_{q(\cdot)},$$

where $k_{p(\cdot)} = 1 + \frac{1}{p_-} - \frac{1}{p_+}$. □

3. Hardy's Inequality on $\ell_{p(\cdot)}$ Spaces

We state and prove our main result, which is the $\ell_{p(\cdot)}$ version of (1.1).

Theorem 3.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{+\infty} a_n^{p_+}$ converges, then*

$$\sum_{n=1}^{+\infty} \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{p_+} \leq C(p_-, p_+) \sum_{n=1}^{+\infty} a_n^{p_+},$$

where $C(p_-, p_+) = \left[\frac{p_-}{p_- - 1}\left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\right]^{p_+}$.

Proof. Let $\alpha_n = \frac{A_n}{n}$ where $A_n = a_1 + a_2 + \cdots + a_n$. Then, $A_n = n\alpha_n$, so $a_1 + a_2 + \cdots + a_n = n\alpha_n$, from which we get that

$$a_n = n\alpha_n - (n-1)\alpha_{n-1}.$$

Now, let us consider

$$\begin{aligned}\alpha_n^{p_+} - q_-\alpha_n^{p_+-1} a_n =& \alpha_n^{p_+} - q_-[n\alpha_n - (n-1)\alpha_{n-1}]\alpha_n^{p_+-1} \\ =& \alpha_n^{p_+} - q_- n\alpha_n^{p_+} + q_-(n-1)\alpha_{n-1}\alpha_n^{p_+-1}.\end{aligned}$$

By Young's inequality

$$\begin{aligned}q_-(n-1)\alpha_{n-1}\alpha_n^{p_+-1} \leq& q_-(n-1)\left[\frac{\alpha_{n-1}^{p_+}}{p_-} + \frac{\alpha_n^{q_+(p_+-1)}}{q_-}\right] \\ =& q_-(n-1)\left[\frac{\alpha_{n-1}^{p_+}}{p_-}\right] + (n-1)\alpha_n^{p_+} \\ =& \frac{(n-1)\alpha_{n-1}^{p_+}}{p_- - 1} + (n-1)\alpha_n^{p_+}.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_n^{p_+} - q_-\alpha_n^{p_+-1} a_n \leq& \alpha_n^{p_+} - q_- n\alpha_n^{p_+} + \frac{(n-1)}{p_- - 1}\alpha_{n-1}^{p_+} + (n-1)\alpha_n^{p_+} \\ =& \alpha_n^{p_+} - \frac{p_-}{p_- - 1} n\alpha_n^{p_+} + \frac{(n-1)}{p_- - 1}\alpha_{n-1}^{p_+} + (n-1)\alpha_n^{p_+} \\ =& \frac{1}{p_- - 1}\Big[(n-1)\alpha_{n-1}^{p_+} - n\alpha_n^{p_+}\Big].\end{aligned}$$

From which

$$\begin{aligned}\sum_{n=1}^{N}\alpha_n^{p_+}-\frac{p_-}{p_--1}\sum_{n=1}^{N}\alpha_n^{p_+-1}a_n \leq&\frac{1}{p_--1}\sum_{n=1}^{N}\Big[(n-1)\alpha_{n-1}^{p_+}-n\alpha_n^{p_+}\Big]\\ =&\frac{1}{p_--1}\Big[-\alpha_1^{p_+}+\alpha_1^{p_+}-2\alpha_2^{p_+}+\cdots-N\alpha_N^{p_+}\Big]\\ =&-\frac{N\alpha_N^{p_+}}{p_--1}\\ \leq&0.\end{aligned}$$

Then,

$$\sum_{n=1}^{N}\alpha_n^{p_+}\leq\frac{p_-}{p_--1}\sum_{n=1}^{N}\alpha_n^{p_+-1}a_n.$$

By Hölder's inequality we have

$$\begin{aligned}\sum_{n=1}^{+\infty}\alpha_n^{p_+}\leq&\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\left(\sum_{n=1}^{+\infty}a_n^{p_+}\right)^{\frac{1}{p_+}}\left(\sum_{n=1}^{+\infty}\alpha_n^{q_+(p_+-1)}\right)^{\frac{1}{q_+}}\\ =&\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\left(\sum_{n=1}^{+\infty}a_n^{p_+}\right)^{\frac{1}{p_+}}\left(\sum_{n=1}^{+\infty}\alpha_n^{p_+}\right)^{\frac{1}{q_+}}.\end{aligned}$$

Then,

$$\left(\sum_{n=1}^{+\infty}\alpha_n^{p_+}\right)^{1-\frac{1}{q_+}}\leq\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\left(\sum_{n=1}^{+\infty}a_n^{p_+}\right)^{\frac{1}{p_+}}.$$

This implies

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{p_+}\leq C(p_-,p_+)\sum_{n=1}^{+\infty}a_n^{p_+},$$

where $C(p_-,p_+)=\left[\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\right]^{p_+}$, which completes the proof. $\square$

References

[1] G. H. Hardy, *Notes on some points in the integral calculus, XLI. On the convergence of certain integrals and series*, Messenger Math. **45** (1915), 163–166.

[2] G. H. Hardy, *Notes on some points in the integral calculus. LX. An inequality between integrals*, Messenger Math. **54** (1925), 150–156.

[3] M. Keller, Y. Pinchover and F. Pogorzelski, *An improved discrete Hardy inequality*, Amer. Math. Monthly **125**(4) (2018), 347–350. `http://dx.doi.org/10.1080/00029890.2018.1420995`

[4] A. Kufner, L. Maligranda and L. E. Persson, *The prehistory of the Hardy inequality*, Amer. Math. Monthly **113**(8) (2006), 715–732. `https://doi.org/10.2307/27642033`

[5] A. Kufner, L. Maligranda, L. E. Persson, *The Hardy inequality. About Its History and Some Related Results*, Vydavatelský Servis, Prague, 2007. `https://urn.kb.se/resolve?urn=urn:nbn:se:ltu:diva-16695`

[6] P. Lefévre, *A short direct proof of the discrete Hardy inequality*, Arch. Math. **114**(2) (2019), 195–198. `https://doi.org/10.1007/s00013-019-01395-6`

[7] N. Masmoudi, *About the Hardy Inequality*, in: D. Schleicher, M. Lackmann (Eds.), *An Invitation to Mathematics*, Springer, Berlin, Heidelberg, 2011, 165–180. https://doi.org/10.1007/978-3-642-19533-4_11

[1]Department of Mathematics,
Universidad Nacional de Colombia,
Bogotá, Colombia.
Email address: recastillo@unal.edu.co
ORCID iD: https://orcid.org/0000-0003-1113-5827

[2]Department of Mathematics,
Quaid-I-Azam University,
Islamabad, Pakistan.
Email address: babarsultan40@yahoo.com
ORCID iD: https://orcid.org/0000-0003-2833-4101

[3]Program of Mathematics,
Universidad de Cartagena,
Cartagena de Indias, Colombia.
Email address: hchaparrog@unicartagena.edu.co
ORCID iD: https://orcid.org/0000-0002-0723-8199

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1063–1080.

A NOTE ON THE BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF FRACTIONAL INTEGRALS IN GRAND VARIABLE HERZ-MORREY SPACES

MEHVISH SULTAN[1] AND BABAR SULTAN[2]

Abstract. In this paper we obtain the boundedness of the higher order commutators of the fractional integral operator of variable order on the grand variable Herz-Morrey spaces.

1. Introduction

We consider the Riesz potential operator

$$I^{\lambda(x)}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda(x)}}dy, \quad 0 < \lambda(x) < n.$$

Note that $\lambda(x)$ is the order of the Riesz potential operator which is variable. Researchers in harmonic analysis, functional analysis, and related areas often use these variable exponent function spaces to address challenging problems that cannot be adequately handled by classical function spaces. They offer a powerful framework for studying functions with diverse characteristics, making them a valuable tool in modern mathematical research. This field has seen substantial advancements in recent times. This progress involves the development of new techniques, deeper understanding of the structure and properties of these function spaces, some instances of this work are in [6, 7, 14, 15].

In [16], authors describe a Sobolev-type theorem that concerns the potential operator $I^{\lambda(x)}$ mapping from a Lebesgue space $L^{p(\cdot)}$ into a weighted Lebesgue space $L^{p(\cdot)}_w$

Key words and phrases. Lebesgue spaces, BMO spaces, weighted estimates, fractional integrals, grand variable Herz-Morrey spaces.

2020 *Mathematics Subject Classification.* Primary: 46E30. Secondary: 47B38.

https://doi.org/10.46793/KgJMat2607.1063S

Received: March 20, 2024.

Accepted: September 27, 2024.

in $\mathbb{R}^n$. The theorem is established under specific conditions on the exponent function $p(x)$, which is allowed to vary with the point x.

Grand variable Herz spaces is the generalization of Herz spaces, the boundedeness of some operators can be checked in [19, 25–27]. Moreover the grand variable Herz-Morrey spaces are the generalization of grand variable Herz spaces and they were introduced in [20, 21]. Grand weighted Herz spaces and grand weighted Herz-Morrey spaces are the generalizations of weighted Herz spaces and weighted Herz-Morrey spaces respectively, see for instance [22, 23]. For more results in these spaces see [1–4, 9, 10, 18, 24].

Inspired by the concept, in this article we will obtain the boundedness of higher order commutators of fractional integrals of variable order in grand variable Herz-Morrey spaces. We divided this article into different sections. Apart from introduction, a section is dedicated for providing necessary definitions, lemmas, and preliminary results. Then in the last section we will prove our main results.

2. Preliminaries

This section is dedicated to some necessary definitions and important lemmas to prove our main results.

2.1. **Function spaces with variable exponent.** For this section we refer to [7, 14, 15].

Definition 2.1. Let Q be a measurable set in $\mathbb{R}^n$ and $r(\cdot)\colon\ Q \to [1, +\infty)$ be a measurable function. We suppose that

$$1 \leq r_-(Q) \leq r(q) \leq r_+(Q) < +\infty, \tag{2.1}$$

where $r_- := \operatorname{ess}\inf_{q\in Q} r(q)$, $r_+ := \operatorname{ess}\sup_{q\in Q} r(q)$.

(a) Variable Lebesgue spaces $L^{r(\cdot)}(Q)$ are defined as

$$L^{r(\cdot)}(Q) = \left\{ h \text{ measurable} : \int_Q \left(\frac{|h(x)|}{\gamma} \right)^{r(x)} dx < +\infty, \text{ where } \gamma \text{ is a constant} \right\}.$$

Norm in $L^{r(\cdot)}(Q)$ can be defined as

$$\|h\|_{L^{r(\cdot)}(Q)} = \inf \left\{ \gamma > 0 : \int_Q \left(\frac{|h(x)|}{\gamma} \right)^{r(x)} dx \leq 1 \right\}.$$

(b) The space $L^{r(\cdot)}_{\text{loc}}(Q)$ is defined as

$$L^{r(\cdot)}_{\text{loc}}(Q) := \left\{ h : h \in L^{r(\cdot)}(K) \text{ for all compact subsets } K \subset Q \right\}.$$

The log-condition is stated as follows:

$$|r(x) - r(y)| \leq \frac{C(r)}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2},\ x, y \in Q, \tag{2.2}$$

where $C(r) > 0$.

And the decay condition: there exists a number $r_\infty \in (1, +\infty)$, such that

$$|r(x) - r_\infty| \le \frac{C}{\ln(e + |x|)}, \tag{2.3}$$

and also decay condition

$$|r(x) - r_0| \le \frac{C}{\ln|x|}, \quad |x| \le \frac{1}{2}, \tag{2.4}$$

holds for some $r_0 \in (1, +\infty)$.

We use these notations in this article.

(i) $B(x, r)$ is the ball of radius r and center at the point x.
(ii) $B_k := B(0, 2^k) = \{x \in \mathbb{R}^n : |x| < 2^k\}$ for all $k \in \mathbb{Z}$.
(iii) $R_{t,\tau} := B_\tau \setminus B_t = \{x : t < |x| < \tau\}$ is a spherical layer.
(iv) $R_k := B_k \setminus B_{k-1}$.
(v) $\chi_k := \chi_{R_k}$, $\chi_{t,\tau} := \chi_{R_{t,\tau}}$.
(vi) Let $f \in L^1_{\mathrm{loc}}(H)$ be a locally integrable function, then the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined as

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n} \int\limits_{B(x,r)} |f(x)|dx, \quad x \in H,$$

where

$$B(x, r) := \{y \in H : |x - y| < r\}.$$

(vii) The set $\mathcal{P}(H)$ consists of all measurable functions $q(\cdot)$ satisfying $q_- > 1$ and $q_+ < +\infty$.
(viii) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(H)$ consists of all functions $q \in \mathcal{P}(H)$ satisfying (2.1) and (2.2).
(ix) $\mathcal{P}_\infty(H)$ and $\mathcal{P}_{0,+\infty}(H)$ are subsets of $\mathcal{P}(H)$ and the values of these subsets lie in $[1, +\infty)$ which satisfy the condition (2.3) and both conditions (2.3) and (2.4), respectively.
(x) In what follows, we denote $\chi_l = \chi_{R_l}$, $R_l = B_l \setminus B_{l-1}$, $B_l = B(0, 2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}$ for all $l \in \mathbb{Z}$.
(xi) By $p'(x) = p(x)/(p(x) - 1)$, we denote the conjugate exponent of $p(\cdot)$.
(xii) C is a constant, its value varies from line to line and is independent of the main parameters involved.

2.2. **Variable exponent Herz spaces.** In this section we will define variable exponent Herz spaces.

Definition 2.2. Let $p, q \in [1, +\infty)$ and $\alpha \in \mathbb{R}$, the norms of classical versions of non-homgeneous and homgeneous Herz spaces are given below,

$$\|f\|_{K^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(B(0,1))} + \left\{ \sum_{i\in\mathbb{N}} 2^{i\alpha q} \left(\int\limits_{F_{2^{i-1},2^i}} |f(y)|^p dy \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

where

$$\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{i\in\mathbb{Z}} 2^{i\alpha q} \left(\int\limits_{F_{2^{i-1},2^i}} |f(y)|^p dy \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

respectively, where $F_{2^{i-1},2^i} := B(0, 2^i) \backslash B(0, 2^{i-1})$.

Definition 2.3. Let $q \in [1, +\infty)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$. The homgeneous version of variable exponent Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined as

$$\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} = \left(\sum_{i=-\infty}^{+\infty} \|2^{\alpha i} f \chi_i\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}.$$

Definition 2.4. Let $q \in [1, +\infty)$, $\alpha \in \mathbb{R}$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The non-homgeneous version of variable exponent Herz space $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined as

$$K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p(\cdot)}(B(0,1))} + \left(\sum_{i=-\infty}^{+\infty} \|2^{\alpha i} f \chi_i\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}.$$

2.3. **Herz-Morrey spaces.** Now we will define variable Herz-Morrey spaces.

Definition 2.5. Let $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $0 < q < +\infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 \leq \beta < +\infty$. A variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\sum_{i=-\infty}^{k_0} \|2^{\alpha(\cdot)i} f \chi_i\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}.$$

Definition 2.6. To define homogeneous version of GVHM spaces, let $p : \mathbb{R}^n \to [1,+\infty)$, $q \in [1,+\infty)$, $\theta > 0$, $0 \leq \beta < +\infty$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The GVHM spaces are given by

$$M\dot{K}^{\alpha(\cdot),q),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),q),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{i=-\infty}^{k_0} \|2^{i\alpha(\cdot)} f\chi_i\|^{q(1+\epsilon))}_{L^{p(\cdot)}} \right)^{\frac{1}{q(1+\epsilon)}}.$$

The non-homogeneous version of GVHM spaces can be defined in a similar way.

As grand variable Herz-Morrey spaces are the generalization of grand variable Herz spaces, $\beta = 0$, grand variable Herz-Morrey spaces become grand variable Herz spaces.

Definition 2.7 (BMO space)**.** A BMO function is a locally integrable function u whose mean oscillation given by $\frac{1}{|B|}\int_B |u(y) - u_B|dy$ is bounded. Mathematically,

$$\|u\|_{BMO} = \sup_B \frac{1}{|B|} \int\limits_B |u(y) - u_B|dy < +\infty.$$

2.4. **Basic Lemmas.**

Lemma 2.1. ([17]) Let $D > 1$ and $q \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then,

$$\frac{1}{c_0} r^{\frac{n}{q(0)}} \leq \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}} \leq c_0 r^{\frac{n}{q(0)}}, \quad \text{for } 0 < r \leq 1,$$

and

$$\frac{1}{c_\infty} r^{\frac{n}{q_\infty}} \leq \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}} \leq c_\infty r^{\frac{n}{q_\infty}}, \quad \text{for } r \geq 1,$$

respectively, where $c_0 \geq 1$ and $c_\infty \geq 1$ depend on D but do not depend on r.

Hölder's inequality in the variable exponent case has the form:

$$\int\limits_\Omega |f(x)g(x)|\, dx \leq \kappa \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $\kappa = \frac{1}{p_-} + \frac{1}{(p')_-}$ and Ω is a measurable subset of $\mathbb{R}^n$.

The next statement is the generalized Hölder inequality for variable exponent Lebesgue spaces (see [7, 14]).

Lemma 2.2. *Let Q be a measurable subset of $\mathbb{R}^n$ and $p(\cdot)$ be an exponent such that $1 \leq p_-(Q) \leq p_+(Q) < +\infty$ and $q_-, r_- > 1$. Then,*

$$\|fg\|_{L^{r(\cdot)}(Q)} \leq 2^{1/r_-} \|f\|_{L^{p(\cdot)}(Q)} \|g\|_{L^{q(\cdot)}(Q)}$$

holds, where $f \in L^{p(\cdot)}(Q)$, $g \in L^{q(\cdot)}(Q)$ and $\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$.

Lemma 2.3 ([13]). *Let $b \in BMO(\mathbb{R}^n)$, $m \in \mathbb{N}$, $r_0, t_0 \in \mathbb{Z}$, with $r_0 < t_0$. Then, we have*

$$C^{-1}\|b\|^m_{BMO(\mathbb{R}^n)} \leq \sup_{B:ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}} \|(b-b_B)^m \chi_B\|_{L^{q(\cdot)}} \leq C\|b\|^m_{BMO(\mathbb{R}^n)},$$

$$\|(b-b_{B_{r_0}})^m \chi_{B_{t_0}}\|_{L^{q(\cdot)}} \leq C(t_0-r_0)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_{B_{t_0}}\|_{L^{q(\cdot)}}.$$

Assuming that the order of the Riesz potential operator $\lambda(x)$ is not continuous but rather is a measurable function in $\mathbb{R}^n$ adds an interesting twist to the study of potential operators. We mentioned that the measurable function $\lambda(x)$ satisfies certain conditions:

(a) $\lambda_0 := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} \lambda(x) > 0$;
(b) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} r(x)\lambda(x) < n$;
(c) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} r(\infty)\lambda(x) < n$.

Proposition 2.1 ([16]). *Let $r(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}_{0,\infty}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, and assume*

$$1 < r(\infty) \leq r(x) \leq r_+ < +\infty.$$

Let $\lambda(x)$ be a measurable function in $\mathbb{R}^n$ satisfying the conditions (1), (2) *and* (3). *Under these conditions, we establish the following weighted Sobolev-type estimate for the fractional operator $I^{\lambda(\cdot)}$:*

$$\|(1+|x|)^{-\gamma(x)} I^{\lambda(\cdot)}(f)\|_{L^{r(\cdot)}} \leq C \, \|f\|_{L^{r'(\cdot)}},$$

where

$$\frac{1}{r(x)} = \frac{1}{r'(x)} - \frac{\lambda(x)}{n}$$

is the Sobolev exponent

$$\gamma(x) = C\lambda(x)\left(1 - \frac{\lambda(x)}{n}\right) \leq \frac{n}{4}C,$$

where C is the Dini-Lipschitz constant from the inequality (2.3) *in which $\alpha(\cdot)$ replaced by $r(\cdot)$.*

Remark 2.1. (i) Under the given condition (2.3): $|\lambda(x) - \lambda_\infty| \leq \frac{C_\infty}{\ln(e+|x|)}$ for $x \in \mathbb{R}^n$. Then, we can check that $(1+|z|)^{-\gamma(z)}$ is equivalent to the weight $(1+|z|)^{-\gamma_\infty}$.

(ii) The behavior of the Riesz potential operator over a bounded domain and how this behavior doesn't change significantly when replacing the variable order $\lambda(x)$ with $\lambda(y)$. The difference in potentials is unessential if the function $\lambda(x)$ satisfies the smoothness logarithmic condition as given in (2.2)

$$C_1 \, |x-y|^{n-\lambda(y)} \,| \leq |x-y|^{n-\lambda(x)} \leq C_2 \, |x-y|^{n-\lambda(y)}.$$

Let b be a locally integrable function, $0 < \lambda(x) < n$, and $m \in \mathbb{N}$; the higher order commutators of fractional integrable operator of variable order $I_m^{\lambda(\cdot),b}$ are defined by

$$I_m^{\lambda(\cdot),b} f(x) = \int_{\mathbb{R}^n} \frac{[b(x)-b(y)]^m}{|x-y|^{n-\lambda(x)}} f(y)dy. \tag{2.5}$$

When $m=0$, then $I_0^{\lambda(\cdot),b}=I^{\lambda(\cdot)}$ and for $m=1$, $I_1^{\lambda,b}=[b,I^{\lambda}]$. The Hardy-Littlewood-Sobolev theorem provides an important result regarding the boundedness of the fractional integral operator, denoted as I^{λ}, from one Lebesgue space to another under certain conditions. Specifically, the theorem asserts that this boundedness holds when the exponents of the Lebesgue spaces satisfy the equation:

$$\frac{1}{r_1}-\frac{1}{r_2}=\frac{\lambda}{n}, \tag{2.6}$$

where $0<r_1<r_2<+\infty$.

The next proposition is a generalization of variable exponents Herz spaces in [5]. We omit the proof of Proposition 2.2 since it is essentially similar to the proof given in [5] and with slight modification, we can obtain following result in grand variable Herz-Morrey spaces.

Proposition 2.2. *Let α,p,q are as defined in Definition 2.6, then*

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} &= \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{i=-\infty}^{k_0}\|2^{i\alpha(\cdot)}f\chi_i\|_{L^{p(\cdot)}}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\ &\approx \max\Bigg\{\sup_{\epsilon>0}\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\|f\chi_i\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},\\ &\quad \sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\|f\chi_i\|_{L^{p(\cdot)}}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\ &\quad +\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{i=0}^{k_0}2^{i\alpha_\infty q(1+\epsilon)}\|f\chi_i\|_{L^{p(\cdot)}}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\Bigg\}.\end{aligned}$$

3. Boundedness Results for Grand Variable Herz-Morrey Spaces

Theorem 3.1. *Let $1<q<+\infty$, $b\in BMO(\mathbb{R}^n)$, $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\lambda(\cdot)}{n}$ and $0<\beta<+\infty$. If $0<\lambda(\cdot)<n$ and $\alpha,p_2\in\mathcal{P}_{0,\infty}(\mathbb{R}^n)$ such that*

$$\frac{-n}{p_{1\infty}}<a_\infty<\frac{n}{p'_{1\infty}},\quad \frac{-n}{p_1(0)}<\alpha(0)<\frac{n}{p'_1(0)},$$

then

$$\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f)\right\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)}\leq C\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f\in M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)$ and $f(x)=\sum_{t=-\infty}^{+\infty}f(x)\chi_t(x)=\sum_{t=-\infty}^{+\infty}f_t(x)$, we have

$$\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f)\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&= \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{i=-\infty}^{k_0} \left\| 2^{i\alpha(\cdot)} (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f)\chi_i \right\|_{p_2(\cdot)}^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
&\le \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{i=-\infty}^{k_0} \left(\sum_{t=-\infty}^{\infty} \left\| 2^{i\alpha(\cdot)} (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
&\le \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{i=-\infty}^{k_0} \left(\sum_{t=-\infty}^{i} \left\| 2^{i\alpha(\cdot)} (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
&+ \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{i=-\infty}^{k_0} \left(\sum_{t=i+1}^{\infty} \left\| 2^{i\alpha(\cdot)} (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
&=: E_1 + E_2.
\end{aligned}
$$

Now we will find the estimate for E_1, for each $i \in \mathbb{Z}$ with $t \le i$ a.e. Let $x \in R_i$, $y \in R_t$. Then, $|x-y| \sim |x| \sim 2^i$, and we get

$$
\begin{aligned}
\left| I_m^{\lambda(\cdot),b}(f\chi_t)(x) \right| &\le C \int_{R_t} |x-y|^{\lambda(x)-n} |b(x)-b(y)|^m |f(y)| dy \\
&\le C 2^{-in} \int_{R_t} |x|^{\lambda(x)} |b(x)-b(y)|^m |f(y)| dy \\
&\le C 2^{-in} |x|^{\lambda(x)} \left(|b(x)-b_{B_t}|^m \int\limits_{R_t} |f_t(y)| dy + \int\limits_{R_t} |f_t(y)| |b(y)-b_{B_t}|^m dy \right) \\
&\le C 2^{-in} |x|^{\lambda(x)} \|f_t\|_{p_1(\cdot)} \left(|b(x)-b_{B_t}|^m \|\chi_t\|_{p_1'(\cdot)} + \|((b-b_{B_t})^m \chi_t\|_{p_1'(\cdot)} \right).
\end{aligned}
$$

It is known, see e.g. [30] that

$$
\begin{aligned}
I^{\lambda(\cdot)}((b(x)-b_{B_t})^m \chi_i)(x) &\ge I^{\lambda(\cdot)}(\chi_{B_i})(x) \cdot (\chi_i)(x) \\
&= \int_{B_i} \frac{|b(x)-b_{B_t}|^m}{|x-y|^{\lambda(x)-n}} dy \cdot \chi_i(x) \\
&\ge C |b(x)-b_{B_t}|^m |x|^{\lambda(x)} \cdot \chi_i(x) \\
&\ge C |b(x)-b_{B_t}|^m |x|^{\lambda(x)} \cdot \chi_i(x).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\left\| \chi_i (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t) \right\|_{p_2(\cdot)} \\
\le & C 2^{-in} \|f_t\|_{p_1(\cdot)} \left(\left\| (1+|x|)^{-\gamma(x)} |x|^{\lambda(x)} (b-b_{B_t})^m \chi_i \right\|_{p_2(\cdot)} \|\chi_t\|_{p_1'(\cdot)} \right. \\
& \left. + \|(b-b_{B_t})^m \chi_t\|_{p_1'(\cdot)} \left\| (1+|x|)^{-\gamma(x)} I^{\lambda(\cdot)}(\chi_{B_i}) \right\|_{p_2(\cdot)} \right) \\
\le & C 2^{-in} \|f_t\|_{p_1(\cdot)} \left(\left\| (1+|x|)^{-\gamma(x)} I^{\lambda(\cdot)}((b-b_{B_t})^m \chi_i) \right\|_{p_2(\cdot)} \|\chi_t\|_{p_1'(\cdot)} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\|(b-b_{B_t})^m\chi_t\|_{p_1'(\cdot)}\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}(\chi_{B_i})\right\|_{p_2(\cdot)}\Big)\\
\leq &C2^{-in}\|f_t\|_{p_1(\cdot)}\Big((i-t)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_i\|_{p_1(\cdot)}\|\chi_t\|_{p_1'(\cdot)}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_t\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}\Big)\\
\leq &C2^{-in}(i-t)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f_t\|_{p_1(\cdot)}\|\chi_t\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}.
\end{aligned}
$$

By Lemma 2.8,

$$
2^{-in}\|\chi_t\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}\leq C2^{-in}2^{\frac{in}{p_1(0)}}2^{\frac{tn}{p_1'(0)}}\leq C2^{\frac{(t-i)n}{p_1'(0)}}. \tag{3.1}
$$

Splitting E_1 by using Minkowski's inequality we have

$$
\begin{aligned}
E_1\leq&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{i=-\infty}^{k_0}\left(\sum_{t=-\infty}^{i}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\approx&\max\Bigg\{\sup_{\epsilon>0}\sup_{k_0<0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},\\
&\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{i=-0}^{k_0}2^{i\alpha_\infty q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\Bigg\}\\
:=&\max\left\{E_1',E_{11}+E_{12}\right\}.
\end{aligned}
$$

We will find the estimate for E_{11} and E_{12}, estimate for E_1' is obtained similarly. For E_{11} we get

$$
\begin{aligned}
E_{11}\leq&\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\leq&C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}\Bigg(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\sum_{t=-\infty}^{i}(i-t)^m 2^{-in}\|f\chi_t\|_{p_1(\cdot)}\|\chi_t\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\Bigg)^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}(i-t)^m 2^{\frac{(t-i)n}{p_1'(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}.
\end{aligned}
$$

Let $b_1 := \frac{n}{p_1'(0)} - \alpha(0)$, applying the fact $2^{-q(1+\epsilon)} < 2^{-q}$, Hölder's inequality and Fubini's theorem we get

$$
\begin{aligned}
E_{11}\leq& C\|b\|_{BMO)(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}\left(\sum_{t=-\infty}^{i}2^{\alpha(0)t}\|f\chi_t\|_{p_1(\cdot)}(i-t)^m 2^{b_1(t-i)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\\
&\times\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^\theta\sum_{i=-\infty}^{-1}\left(\sum_{t=-\infty}^{i}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{b_1 q(1+\epsilon)(t-i)/2}\right)\right.\\
&\times\left.\left(\sum_{t=-\infty}^{i}(i-t)^{m(q(1+\epsilon))'/2}2^{b_1(q(1+\epsilon))'(t-i)/2}\right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\\
&\times\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^\theta\sum_{i=-\infty}^{-1}\sum_{t=-\infty}^{i}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{b_1 q(1+\epsilon)(t-i)/2}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\\
&\times\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^\theta\sum_{t=-\infty}^{-1}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\sum_{i=t}^{-1}2^{b_1 q(1+\epsilon)(t-i)/2}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=-\infty}^{-1}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}\\
=& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=-\infty}^{k_0}\left\|2^{a(\cdot)t}f\chi_t\right\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now for E_{12}, by using Minkowski's inequality we have

$$\begin{aligned}
E_{12} \leq & \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
& \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{i} \left\| \chi_i (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t) \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
\leq & \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
& \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{-1} \left\| \chi_i (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t) \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
& + \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
& \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=0}^{i} \left\| \chi_i (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t) \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
:= & A_1 + A_2.
\end{aligned}$$

The estimate for A_2 follows in a similar manner to E_{11} by replacing $p_1'(0)$ with $p_{1\infty}'$. For A_1 we have

$$2^{-in} \|\chi_i\|_{p_1(\cdot)} \|\chi_t\|_{p_1'(\cdot)(\mathbb{R}^n)} \leq C 2^{-in} 2^{\frac{in}{p_{1\infty}}} 2^{\frac{tn}{p_1'(0)}} \leq C 2^{\frac{-in}{p_{1\infty}}} 2^{\frac{tn}{p_1'(0)}}. \tag{3.2}$$

As $\alpha_\infty - \frac{n}{p'_{1_\infty}} < 0$, we have

$$\begin{aligned}
A_1 \leq & C \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
& \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{-1} \left\| \chi_i (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t) \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
\leq & \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \Bigg(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \\
& \times \left(\sum_{j=-\infty}^{-1} 2^{-in} \|f\chi_t\|_{p_1(\cdot)} \|b\|_{BMO(\mathbb{R}^n)}^m (i-t)^m \|\chi_t\|_{p_1'(\cdot)} \|\chi_i\|_{p_1(\cdot)} \right)^{q(1+\epsilon)} \Bigg)^{\frac{1}{q(1+\epsilon)}} \\
\leq & C \|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
& \times \left[\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{-1} (i-t)^m 2^{-in} 2^{\frac{in}{p_{1\infty}}} 2^{\frac{tn}{p_1'(0)}} \|f\chi_t\|_{p_1(\cdot)} \right)^{q(1+\epsilon)} \right]^{\frac{1}{q(1+\epsilon)}} \\
\leq & C \|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta}
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\epsilon^{\theta}\sum_{i=0}^{k_0}2^{i\alpha_\infty q(1+\epsilon)}\left(\sum_{t=-\infty}^{-1}(i-t)^m 2^{\frac{-in}{p'_{1\infty}}}2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left[\epsilon^{\theta}\sum_{i=0}^{k_0}2^{k\left(\alpha_\infty-\frac{n}{p'_{1\infty}}\right)q(1+\epsilon)}\left(\sum_{t=-\infty}^{-1}(i-t)^m 2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}(i-t)^m 2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}2^{t\alpha(0)}(i-t)^m 2^{-t\alpha(0)}2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}2^{t\alpha(0)q(1+\epsilon)}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{t\left(\frac{n}{p'_1(0)}-\alpha(0)\right)q(1+\epsilon)/2}\right)\right.\\
&\times\left.\left(\sum_{t=-\infty}^{-1}(i-t)^{m(q(1+\epsilon))'/2}2^{j\left(\frac{n}{p'_1(0)}-\alpha(0)\right)q(1+\epsilon)'/2}\right)^{q(1+\epsilon)/(q(1+\epsilon))'}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}2^{t\alpha(0)q(1+\epsilon)}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{t\left(\frac{n}{p'_1(0)}-\alpha(0)\right)q(1+\epsilon)/2}\right)\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{t=-\infty}^{-1}2^{a(0)q(1+\epsilon)t}\,\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}\\
=& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{t=-\infty}^{k_0}\left\|2^{a(\cdot)t}f\chi_t\right\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\,\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now we will find the estimate for E_2, for every $i\in\mathbb{Z}$ with $t\geq i+1$ a.e. Let $x\in R_i$ and $y\in R_t$, we known that $|x-y|\approx|y|\approx 2^t$, we consider

$$
\left|I_m^{\lambda(\cdot),b}(f\chi_t)(x)\right|\leq C\int_{R_t}|x-y|^{\lambda(x)-n}\,|b(x)-b(y)|^m|f(y)|dy
$$

$$
\begin{aligned}
&\leq C2^{-tn}\int_{R_t}|x|^{\lambda(x)}|b(x)-b(y)|^m|f(y)|dy\\
&\leq C2^{-tn}|x|^{\lambda(x)}\left(|b(x)-b_{B_t}|^m\int\limits_{R_t}|f_t(y)|dy+\int\limits_{R_t}|f_t(y)||b(y)-b_{B_t}|^m dy\right)\\
&\leq C2^{-tn}|x|^{\lambda(x)}\left\|f_t\right\|_{p_1(\cdot)}\left(|b(x)-b_{B_t}|^m\left\|\chi_t\right\|_{p_1'(\cdot)}+\|((b-b_{B_t})^m\chi_t\|_{p_1'(\cdot)}\right).
\end{aligned}
$$

Similarly to E_1, for E_2, we have

$$
\begin{aligned}
&\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\|_{p_2(\cdot)}\\
\leq &C2^{-tn}\left\|f_t\right\|_{p_1(\cdot)}\Big(\left\|(1+|x|)^{-\gamma(x)}|x|^{\lambda(x)}(b-b_{B_t})^m\chi_i\right\|_{p_2(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\\
&+\left\|(b-b_{B_t})^m\chi_t\right\|_{p_1'(\cdot)}\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}(\chi_{B_i})\right\|_{p_2(\cdot)}\Big)\\
\leq &C2^{-tn}\left\|f_t\right\|_{p_1(\cdot)}\Big(\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}((b-b_{B_t})^m\chi_i)\right\|_{p_2(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\\
&+\left\|(b-b_{B_t})^m\chi_t\right\|_{p_1'(\cdot)}\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}(\chi_{B_i})\right\|_{p_2(\cdot)}\Big)\\
\leq &C2^{-tn}\left\|f_t\right\|_{p_1(\cdot)}\Big((i-t)^m\left\|b\right\|^m_{BMO(\mathbb{R}^n)}\left\|\chi_i\right\|_{p_1(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\\
&+\left\|b\right\|^m_{BMO(\mathbb{R}^n)}\left\|\chi_t\right\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}\Big)\\
\leq &C2^{-tn}(i-t)^m\left\|b\right\|^m_{BMO(\mathbb{R}^n)}\left\|f_t\right\|_{p_1(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}.
\end{aligned}
$$

Next we find estimate of E_2

$$
\begin{aligned}
E_2\leq&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{i=-\infty}^{k_0}\left(\sum_{t=i+1}^{\infty}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\approx&\max\Bigg\{\sup_{\epsilon>0}\sup_{k_0<0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},\\
&\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}
\end{aligned}
$$

$$\times\left(\epsilon^{\theta}\sum_{i=-0}^{k_0}2^{i\alpha_\infty q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\Bigg\}$$
$$:=\max\left\{E_2',E_{21}+E_{22}\right\}.$$

We will find the estimates for E_{21} and E_{22}, the estimate for E_2' is obtained similarly. First, we will find an estimate for E_{22},

$$\begin{aligned}E_{22}\leq&C\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left(\epsilon^{\theta}\sum_{i=o}^{k_0}2^{i\alpha_\infty q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\\leq&C\|b\|_{BMO)(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left(\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=i+1}^{\infty}2^{\alpha_\infty t}\left\|f\chi_t\right\|_{p_1(\cdot)}(i-t)^m2^{d(i-t)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},\end{aligned}$$

where $d=\frac{n}{p_{1\infty}}+\alpha_\infty>0$. Then, we use Hölder's theorem for series and $2^{-q(1+\epsilon)}<2^{-q}$ to obtain

$$\begin{aligned}E_{22}\leq&C\|b\|_{BMO)(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=i+1}^{\infty}2^{\alpha_\infty q(1+\epsilon)t}\left\|f\chi_t\right\|_{p_1(\cdot)}^{q(1+\epsilon)}2^{dq(1+\epsilon)(i-t)/2}\right)\right.\\&\left.\times\left(\sum_{t=i+1}^{\infty}(i-t)^{m(q(1+\epsilon))'/2}2^{d(q(1+\epsilon))'(i-t)/2}\right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}}\right]^{\frac{1}{q(1+\epsilon)}}\\\leq&C\left\|b\right\|_{BMO(\mathbb{R}^n)}^m\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left[\epsilon^{\theta}\sum_{i=0}^{k_0}\sum_{t=i+1}^{\infty}2^{\alpha_\infty q(1+\epsilon)t}\left\|f\chi_t\right\|_{p_1(\cdot)}^{q(1+\epsilon)}2^{dq(1+\epsilon)(i-t)/2}\right]^{\frac{1}{q(1+\epsilon)}}\\\leq&C\left\|b\right\|_{BMO(\mathbb{R}^n)}^m\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left(\epsilon^{\theta}\sum_{t=0}^{\infty}2^{\alpha_\infty q(1+\epsilon)t}\left\|f\chi_t\right\|_{p_1(\cdot)}^{q(1+\epsilon)}\sum_{i=0}^{t-1}2^{dq(1+\epsilon)(i-t)/2}\right)^{\frac{1}{q(1+\epsilon)}}\\\leq&C\left\|b\right\|_{BMO(\mathbb{R}^n)}^m\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\end{aligned}$$

$$\times\left(\epsilon^{\theta}\sum_{t=0}^{\infty}\sum_{j=-\infty}^{t}2^{\alpha_{\infty}q(1+\epsilon)j}\|f\chi_t\|_{p_1(\cdot)}^{q(1+\epsilon)}\sum_{i=0}^{t-1}2^{dq(1+\epsilon)(i-t)/2}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^{m}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{t=0}^{\infty}\sum_{i=0}^{t-2}2^{dq(1+\epsilon)(i-t)/2}\right)^{\frac{1}{q(1+\epsilon)}}\|f\|_{M\dot{K}_{\beta,p_1(\cdot)}^{a(\cdot),q),\theta}(\mathbb{R}^n)}$$
$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^{m}\|f\|_{M\dot{K}_{\beta,p_1(\cdot)}^{a(\cdot),q),\theta}(\mathbb{R}^n)}.$$

For E_{21}, by using Minkowski's inequality,

$$E_{21}\leq\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},$$
$$E_{21}\leq\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=k+2}^{-1}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$+\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=0}^{\infty}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$:=B_1+B_2.$$

The estimate for B_1 can be obtained in a similar way to that of E_{22} by replacing $p_{1\infty}$ with $p_1(0)$ and using the inequality $\frac{n}{p_1(0)}+a(0)>0$, $\frac{n}{p_{1\infty}}+\alpha_{\infty}>0$. For B_2 we have

$$2^{-tn}\|\chi_i\|_{p_1(\cdot)}\|\chi_t\|_{p_1'(\cdot)}\leq C2^{-tn}2^{\frac{in}{p_1(0)}}2^{\frac{tn}{q'1\infty}}\leq C2^{\frac{in}{p_1(0)}}2^{\frac{-tn}{p_{1\infty}}}, \tag{3.3}$$

and

$$B_2\leq C\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=0}^{\infty}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^{m}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=0}^{\infty}(i-t)^m2^{-tn}2^{\frac{in}{p_1(0)}}2^{\frac{tn}{p'_{1\infty}}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$

$$
\begin{aligned}
\leq & C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
& \times \left(\epsilon^\theta \sum_{i=-\infty}^{-1} \left(\sum_{t=0}^{\infty} 2^{t\alpha(0)} (i-t)^m 2^{\frac{in}{p_1(0)}+ia(0)} 2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)} \|f\chi_t\|_{p_1(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
\leq & C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
& \times \left(\epsilon^\theta \sum_{i=-\infty}^{-1} \left(\sum_{t=0}^{\infty} 2^{t\alpha(0)(q(1+\epsilon))} \left(2^{\frac{in}{p_1(0)}+ia(0)} 2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)} \right)^{q(1+\epsilon)/2} \right) \right. \\
& \left. \times \|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)} \left(\sum_{t=0}^{k_0} \left((i-t)^m 2^{\frac{in}{p_1(0)}+ia(0)} 2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)} \right)^{(q(1+\epsilon))'/2} \right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}} \right)^{\frac{1}{q(1+\epsilon)}} \\
\leq & C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{i=-\infty}^{-1} \sum_{t=0}^{\infty} 2^{t\alpha(0)(q(1+\epsilon))} \|f\chi_t\|^{q(1+\epsilon)/2}_{p_1(\cdot)} \right. \\
& \left. \times \left(2^{\frac{in}{p_1(0)}+ia(0)} 2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\
\leq & C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=0}^{\infty} 2^{t\alpha(0)(q(1+\epsilon))} \|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)} \right. \\
& \left. \times \sum_{i=-\infty}^{t} \left(2^{\frac{in}{p_1(0)}+ia(0)} 2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)} \right)^{q(1+\epsilon)/2} \right)^{\frac{1}{q(1+\epsilon)}} \\
\leq & C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=0}^{\infty} \sum_{j=-\infty}^{t} 2^{ja(0)(q(1+\epsilon))} \|f\chi_j\|^{q(1+\epsilon)}_{p_1(\cdot)} \right. \\
& \left. \times \sum_{i=-\infty}^{t} 2^{\frac{in}{p_1(0)}+ia(0)} 2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)} \right)^{\frac{1}{q(1+\epsilon)}} \\
\leq & C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=0}^{\infty} \sum_{i=-\infty}^{t} 2^{\frac{in}{p_1(0)}+ia(0)} 2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)} \right)^{\frac{1}{q(1+\epsilon)}} \\
& \times \|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)} \\
\leq & C \|b\|^m_{BMO(\mathbb{R}^n)} \|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)} .
\end{aligned}
$$

Combining the estimates for E_1 and E_2 yields

$$
\left\| (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f) \right\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)} \leq C \|b\|^m_{BMO(\mathbb{R}^n)} \|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)} ,
$$

which completes the proof. □

References

[1] A. Abdalmonem and A. Scapellato, *Fractional operators with homogeneous kernels in weighted Herz spaces with variable exponent*, Appl. Anal. **101**(6) (2022), 1953–1962.

[2] A. Abdalmonem, A. Scapellato, *Intrinsic square functions and commutators on Morrey-Herz spaces with variable exponents*, Math. Methods Appl. Sci. **44**(2021), 12408–12425.

[3] A. Abdalmonem, O. Abdalrhman and S. Tao, *Boundedness of fractional integral with variable kernel and their commutators on variable exponent Herz spaces*, Applied Math. **7**(10) (2016).

[4] A. Ajaib and A. Hussain, *Weighted $CBMO$ estimates for commutators of matrix Hausdorff operator on the Heisenberg group*, Open Math. **18**(1) (2020), 496–511.

[5] A. Almeida and D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl. **394**(2) (2012), 781–795.

[6] R. E. Castillo and H. Rafeiro, *An Introductory Course in Lebesgue Spaces*, CMS Books, Mathematics/Ouvrages de Mathematiques de ta SMC, Springer, Cham, 2016.

[7] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*, Birkhäuser, Heidelberg, 2013.

[8] L.Grafakos, X.Li and D.Yang, *Bilinear operator on Herz-type Hardy spaces*, Trans. Amer. Math. Soc. **350**(3) (1998), 1249–1275.

[9] A. Hussain, I. Khan and A. Mohamed, *Variable Herz-Morrey estimates for rough fractional Hausdorff operator*, J. Inequal. Appl. **33** (2024).

[10] A. Hussain and A. Ajaib, *Some results for the commutators of generalized Hausdorff operator*, J. Math. Inequal. **13**(4) (2019), 1129–1146.

[11] M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math. **36**(1) (2010), 33–50.

[12] M. Izuki, *Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponents*, Math. Sci. J. **13**(10) (2009), 243–253.

[13] M. Izuki, *Boundedness of commutators on Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo. **59** (2010), 199–213.

[14] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Vol. 1, Birkhuser/Springer, Cham, 2016.

[15] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Vol. 2, Birkhuser/Springer, Cham, 2016.

[16] V. Kokilashvili and S. Samko, *On Sobolev theorem for Riesz-type potentials in the Lebesgue spaces with variable exponent*, Z. Anal. Anwend. **22** (2003), 899–910. `https://www.doi.org/10.4171/ZAA/1178`

[17] S. Samko, *Variable exponent Herz spaces*, Mediterr. J. Math. **10**(4) (2013), 2007–2025.

[18] M. Sultan, B. Sultan and A. Hussain, *Grand Herz-Morrey spaces with variable exponent*, Math. Notes **114**(5) (2023), 957–977.

[19] M. Sultan, B. Sultan, A. Aloqaily, N. Mlaiki, *Boundedness of some operators on grand Herz spaces with variable exponent*, AIMS Math. **8**(6) (2023), 12964–12985.

[20] B. Sultan, F. Azmi, M. Sultan, M. Mehmood and N. Mlaiki, *Boundedness of Riesz potential operator on grand Herz-Morrey spaces*, Axioms **11**(11) (2022), Article ID 583.

[21] M. Sultan, B. Sultan, A. Khan and T. Abdeljawad, *Boundedness of Marcinkiewicz integral operator of variable order in grand Herz-Morrey spaces*, AIMS Math. **8**(9) (2023), 22338–22353.

[22] B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M. A. Alghafli and N. Mlaik, *Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent*, AIMS Math. **8**(1) (2023), 752–764.

[23] B Sultan, F. M. Azmi, M. Sultan, T. Mahmood, N. Mlaiki and N. Souayah, *Boundedness of fractional integrals on grand weighted Herz-Morrey spaces with variable exponent*, Fractal fract. **6**(11) (2022), Article ID 660.

[24] B. Sultan and M. Sultan, *Boundedness of higher order commutators of Hardy operators on grand Herz-Morrey spaces*, Bull. Sci. Math. **190** (2024).

[25] B. Sultan, M. Sultan, A. Khan and T. Abdeljawad, *Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces*, J. Inequal. Appl. **2024** (2024), Article ID 93.

[26] B. Sultan and M. Sultan, *Boundedness of commutators of rough Hardy operators on grand variable Herz spaces*, Forum Math. (2023). https://doi.org/10.1515/forum-2023-0152

[27] B. Sultan, M. Sultan and I. Khan, *On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces*, Commun. Nonlinear Sci. Numer. Simul. **126** (2023). https://doi.org/10.1016/j.cnsns.2023.107464

[28] J. L. Wu, *Boundedness for fractional Hardy-type operator on Herz-Morrey spaces with variable exponent*, Bull. Korean Math. Soc. **51** (2014), 423–435.

[29] H. Wang, *Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent*, Czech. Math. J. **66** (2016), 251–269.

[30] J. L. Wu and W. J. Zhao, *Boundedness for fractional Hardy-type operator on variable-exponent Herz-Morrey spaces*, Kyoto J. Math. **56** (2016), 831–845.

[1]Department of Mathematics,
Capital University of Science and Technology,
Islamabad, Pakistan
Email address: mehvishsultanbaz@gmail.com
ORCID iD: https://orcid.org/0009-0005-2379-0476

[2]Department of Mathematics,
Quaid-I-Azam University,
Islamabad, Pakistan
Email address: babarsultan40@yahoo.com
ORCID iD: https://orcid.org/0000-0003-2833-4101

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1081–1086.

A NOTE ON A NEW SHARP RESULT IN SPACES OF PLURIHARMONIC FUNCTIONS AND RELATED PROBLEMS

ROMI SHAMOYAN[1]

Abstract. We provide a new sharp result in Bergman spaces of pluriharmonic functions related to the trace operator, extending previously known assertions. Related new estimates for other pluriharmonic spaces in product domains will be also discussed.

1. Introduction

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, \dots, z_n)\colon z_k \in \mathbb{C}, 1 \leq k \leq n\}$ be the n-dimensional space of complex coordinates. We denote the unit polydisk by

$$U^n = \{z \in \mathbb{C}^n\colon\ |z_k| < 1, 1 \leq k \leq n\}$$

and the distinguished boundary of U^n by

$$T^n = \{z \in \mathbb{C}^n\colon\ |z_k| = 1, 1 \leq k \leq n\}.$$

We define Lusin cone in a usual manner as follows

$$\Gamma_\alpha(\xi) = \{z \in U\colon\ |1 - z\xi| < \alpha(1 - |z|)\}, \quad \text{where } \alpha > 1, \xi \in T.$$

We use m_{2_n} to denote the volume measure on U^n and m_n to denote the normalized Lebesgue measure on T^n. Let $H(U^n)$ be the space of all holomorphic functions on U^n. When $n = 1$, we simply denote U^1 by U, T^1 by T, m_{2_n} by m_2, m_n by m. We refer to [4, 5, 13] for further details.

Key words and phrases. Trace operator, pluriharmonic functions, analytic functions, product domains, polydisk, unit disk.

2020 *Mathematics Subject Classification.* Primary: 30J99. Secondary: 46A18.

https://doi.org/10.46793/KgJMat2607.1081S

Received: February 12, 2024.

Accepted: September 27, 2024.

The Hardy spaces, denoted by $H^p(U^n)$, $0<p<+\infty$, are defined by

$$H^p(U^n)=\left\{f\in H(U^n)\colon \sup_{0<r<1} M_p(f,r)<+\infty\right\},$$

where

$$M_p^p(f,r)=\int_{T^n}|f(r\xi)|^p\,dm_n(\xi),\quad M_\infty(f,r)=\max_{\xi\in T^n}|f(r\xi)|,$$

$r\in(0,1)$, $f\in H(U^n)$.

As usual, we denote by $\vec{\alpha}$ the vector $(\alpha_1,\dots,\alpha_n)$.

For $\alpha_j>-1$, $j=1,\dots,n$, $0<p<+\infty$, recall that the weighted Bergman space $A^p_{\vec{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk satisfying the condition (see [4, 5, 13])

$$\|f\|^p_{A^p_{\vec{\alpha}}}=\int_{U^n}|f(z)|^p\prod_{i=1}^n(1-|z_i|^2)^{\alpha_i}dm_{2_n}(z)<+\infty.$$

Let

$$\mathbb{Z}^n_+=\{(k_1,\dots,k_n)\colon k_j\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}\},$$
$$\mathbb{Z}^n_-=\{(k_1,\dots,k_n)\colon k_j\in\mathbb{Z}_-, j=1,\dots,n\}.$$

If u is n-harmonic (harmonic by each variable), then as usual

$$u(r_1e^{i\varphi_1},\dots,r_ne^{i\varphi_n})=\sum_{k_1,\dots,k_n=-\infty}^{+\infty}C_{k_1,\dots,k_n}\prod_{j=1}^n r_j^{|k_j|}e^{ik_j\varphi_j}.$$

We define a fractional derivative of order α of an n-harmonic function in a usual way as follows

$$\mathcal{D}^\alpha u(\vec{r}e^{i\vec{\varphi}})=\sum_{k_1,\dots,k_n=-\infty}^{+\infty}C_{k_1,\dots,k_n}\frac{\Gamma(\alpha+|k|+1)}{\Gamma(\alpha+1)\Gamma(|k|+1)}\prod_{j=1}^n r_j^{|k_j|}e^{ik_j\varphi_j},$$

where

$$\frac{\Gamma(\alpha+|k|+1)}{\Gamma(\alpha+1)\Gamma(|k|+1)}=\prod_{j=1}^n\frac{\Gamma(\alpha_j+|k_j|+1)}{\Gamma(\alpha_j+1)\Gamma(|k_j|+1)},\quad \alpha_j>-1.$$

Let further

$$h^p(\vec{\alpha})=\Big\{u \text{ is } n\text{-harmonic:} \int_{U^n}\prod_{k=1}^n(1-|z_k|)^{\alpha_k}|u(z_1,\dots,z_n)|^p\,dm_{2_n}(z)<+\infty,$$
$$0<p<+\infty,\alpha_k>-1,k=1,\dots,n\Big\}.$$

Note that it is easy to check that $\mathcal{D}^\alpha u$ is n-harmonic if u is n-harmonic.

We will call a function u pluriharmonic if $u=\mathrm{Re}(f)$, $f\in H(U^n)$, and we denote

$$\tilde{h}^p(\vec{\alpha})=\{u \text{ is pluriharmonic:} \|u\|_{h^p(\vec{\alpha})}<+\infty\}.$$

If $1\le p<+\infty$, $j=1,\dots,n$, both $h^p(\alpha)$ and $\tilde{h}^p(\vec{\alpha})$ are Banach spaces, for $0<p\le1$, $j=1,\dots,n$, $h^p(\alpha)$ and $\tilde{h}^p(\vec{\alpha})$ classes are quasinormed spaces.

Some interesting results for pluriharmonic functions can be found in [8–13]. Some results on related plurisubharmonic functions can be seen in [6, 7].

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The notation $A \asymp B$ means that there is a positive constant C, such that $\frac{B}{C} \leq A \leq CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

2. Main Results

Let us remind the main definition.

Definition 2.1 (see [1–5]). Let $\mathcal{X} \subset H(U), \mathcal{Y} \subset H(U^n)$ be subspaces of $H(U)$ and $H(U^n)$. We say that the diagonal of $\mathcal{Y}$ coincides with $\mathcal{X}$ if for any function f, $f \in \mathcal{Y}$, $f(z, \dots, z) \in \mathcal{X}$, and the reverse is also true: for every function g from $\mathcal{X}$ there exists an expansion $f(z_1, \dots, z_n)$, $f \in \mathcal{Y}$ such that $f(z, \dots, z) = g(z)$.

Note that when $\operatorname{diag}(\mathcal{Y}) = \mathcal{X}$, then $\|f\|_{\mathcal{X}} \asymp \inf_{\Phi} \|\Phi(f)\|_{\mathcal{Y}}$, where $\Phi(f)$ is an arbitrary analytic extension of f from the diagonal of the polydisk to the polydisk. The study of the diagonal map and its applications was first suggested by W. Rudin in [1–5]. Later several papers appeared in which the complete solutions were given for classical holomorphic spaces such as the Hardy, Bergman classes see [1–5] and references there. Recently, a complete answer was given for so-called mixed norm spaces in [1–5]. For many other classes, the answer is unknown. The goal of this note is to add various new results in this research area.

Theorems on the diagonal map have numerous applications in the theory of holomorphic functions. Analogues of the diagonal map problem so-called trace problems in various functional spaces in $\mathbb{R}^n$ are well-known (see, for example, [1–5]).

Note that, very similarly, the definition of the diagonal map (Definition 2.1) can be extended to spaces of pluriharmonic functions in the unit polydisk. In this case, the restriction of such a function to the diagonal $(z, \dots, z)$ is an n-harmonic function. Moreover to the following sharp result is valid.

Theorem 2.1. *Let* $0 < p < +\infty$, $\alpha_j > -1$, $j = 1, \dots, n$. *Then,*

$$\operatorname{diag}(\widetilde{h}^p(\vec{\alpha})) = h^p\Big(\sum_{j=1}^{n} \alpha_j + 2n - 2\Big), \quad n \geq 1.$$

Remark 2.1. Since $\widetilde{h}^p(\vec{\alpha})$ classes contain holomorphic Bergman spaces in polydisk, Theorem 1 can be considered as an extension of the theorems on the diagonal map in $A^p_{\vec{\alpha}}(U^n)$ - Bergman classes in the polydisk (see [1–5]).

Proof. The proof of Theorem 2.1 almost repeats the corresponding proof for the classical Bergman classes in the polydisk (see, for example, [1–5]). We add few remarks that are needed. For every harmonic function v such that

$$\int_U |v(z)|^p (1-|z|)^\alpha dm_2(z) < +\infty, \quad \alpha > 1, 0 < p < +\infty,$$

where

$$v(z) = C(\beta) \int_U v(w)(1-|w|^2)^{n(\beta+2)-2} \mathrm{Re}\Big(\frac{1}{1-\langle \bar{w}, z\rangle}\Big)^{n(\beta+2)} dm_2(w),$$

we put

$$u(z_1,\dots,z_n) = C(\beta) \int_U v(w)(1-|w|^2)^{n(\beta+2)-2} \mathrm{Re}\left(\frac{1}{\prod_{k=1}^n (1-\langle z_k, \bar{w}\rangle)^{\beta+2}}\right) dm_2(w).$$

Note that $u(z,\dots,z) = v(z)$, $z \in U^n$, and u is a pluriharmonic function. Indeed to prove the last assertion we have

$$\begin{aligned}
v(w) &= \sum_{k=-\infty}^{+\infty} C_k \rho^{|k|} e^{ik\theta}, \quad w = \rho e^{i\theta}, z = re^{i\varphi},\\
u(z_1,\dots,z_n) &= C(\beta) \int_0^1 \int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} C_k \rho^{|k|} e^{ik\theta} (1-\rho^2)^{n(\beta+2)-2}\\
&\quad\times \sum_{(k_1,\dots,k_n)\in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n} \frac{\Gamma(\beta+|k|+2)}{\Gamma(\beta+2)\Gamma(|k|+1)} r_1^{|k_1|}\cdots r_n^{|k_n|}\\
&\quad\times \prod_{j=1}^n e^{ik_j\varphi_j} \rho^{|k_j|} e^{-ik_j\theta} \rho\, d\rho\, d\theta\\
&= C(\beta) \sum_{(k_1,\dots,k_n)\in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n} C_k \frac{\Gamma(\beta+|k|+2)}{\Gamma(\beta+2)\Gamma(|k|+1)} \prod_{j=1}^n r_j^{|k_j|} e^{ik_j\varphi_j}\\
&\quad\times \int_0^1 (1-\rho^2)^{n(\beta+2)-2} \rho^{2\left(\sum_{j=1}^n |k_j|\right)+1} d\rho\\
&= \sum_{(k_1,\dots,k_n)\in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n} C_{k_1,\dots,k_n} r_1^{|k_1|}\cdots r_n^{|k_n|} e^{ik_1\varphi_1}\cdots e^{ik_n\varphi_n},
\end{aligned}$$

hence u is pluriharmonic (see, for example, [1–5]). □

The rest is a repetition of the arguments in the proof of the well-known theorem on traces in classical analytic Bergman spaces in the unit polydisk. We refer the reader to [1–5] for this

$$\int_T \sup_{z_1\in\Gamma_t(\xi)} \cdots \sup_{z_n\in\Gamma_t(\xi)} |f(z_1,\dots,z_n)|^p\, d\xi,$$

$$\int_T \int_{\Gamma_r(\xi)} \cdots \int_{\Gamma_r(\xi)} |f(w)|^q \prod_{k=1}^{n} (1-|w_k|)^{\alpha_k} dm_2(w_1)\dots dm_2(w_2) dm(\xi),$$

$$\int_0^1 \left(\int_{|z_1|<r} \cdots \int_{|z_n|<r} |f(z)|^p \cdot \prod_{j=1}^{n} (1-|z_j|)^{\alpha_j} dm_{2_n}(z) \right)^{\frac{q}{p}} (1-r)^\beta dr$$

(we consider only $q=p$ case for this quazinorm below).

The proofs of these assertions, as in the case of Bergman spaces which were considered above, are the similar to the proofs for analytic function spaces cases provided earlier in [3].

We denote last two spaces indicated in the polydisk by $\widetilde{M}^{p,\vec{\alpha}}$ and $\widetilde{N}^p_{\vec{\alpha},\beta}$ of pluriharmonic functions.

Theorem 2.2. 1) *We have*

$$h^p_{|\alpha|+2n-1}(U) \subset \operatorname{diag}(\widetilde{M}^{p,\vec{\alpha}})(U^n),$$

where $|\alpha| = \sum_{k=1}^n \alpha_k$, $0<p<+\infty$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>-1$, $j=1,\dots,n$.

2) *We have*

$$h^p_{\beta+|\alpha|+2n-1}(U) \subset \operatorname{diag}(\widetilde{N}^p_{\vec{\alpha},\beta})(U^n),$$

where $0<p<+\infty$, $\alpha_j>-1$, $j=1,\dots,n$, $\beta>-1$.

Related problems on the diagonal can be considered for Bergman-Sobolev type function classes of n-harmonic functions, that is, for spaces of n-harmonic functions with finite quazinorms $\left\|\mathcal{D}^\beta f\right\|_{A^p_\alpha}$ with some restrictions on the parameters p, α and β.

References

[1] R. F. Shamoyan and O. Mihić, *On traces of Q_p type spaces and mixed norm analytic function spaces on polyballs*, Šiauliai Math. Semin. **5**(13) (2010), 101–119.

[2] R. F. Shamoyan and O. Mihić, *On a sharp trace theorem in BMOA type spaces in pseudoconvex domains*, C. R. Acad. Bulgare Sci. **70**(2) (2017), 161–166.

[3] M. Pavlović, M. Jevtić and R. F. Shamoyan, *A note on diagonal mapping in some spaces of analytic functions*, Publ. Math. Debrecen **74**(1–2) (2009), 45–58.

[4] R. F. Shamoyan and S. Kurilenko, *On traces of analytic Herz and Bloch type spaces in bounded strongly pseudoconvex domains in* $\mathbb{C}^n$, Probl. Anal. Issues Anal. **4(22)**(1) (2015), 73–94. `https://doi.org/10.15393/j3.art.2015.2669`

[5] A. E. Djrbashian and F. A. Shamoian, *Topics in the Theory of A^p_α Spaces*, Leipzig, Teubner, 1988.

[6] A. Zeriahi, *A minimum principle for plurisubharmonic functions*, Indiana Univ. Math. J. **56**(6) (2007), 2671–2518. `https://doi.org/10.1512/iumj.2007.56.3209`

[7] H. Song Do, *A class of maximal plurisubharmonic function*, C. R. Math. Acad. Sci. Paris **357**(11–12) (2019), 858–862. `https://doi.org/10.1016/j.crma.2019.11.003`

[8] P. Bartolomeo and G. Tomassini, *Traces of pluriharmonic functions*, Compos. Math. **44**(1–3) (1981), 29–39.
[9] V. Beloshapka, *Function pluriharmonic on manifolds*, Mathematics of the USSR-Izvestiya **12**(3) (1978), 439–447. https://doi.org/10.1070/IM1978v012n03ABEH001989
[10] W. Rudin, *Pluriharmonic functions in the ball*, Proc. Amer. Math. Soc. **62**(1) (1977), 44–46.
[11] P. Bartolomeis and G. Tomassini, *Traces of Pluriharmonic Functions*, Springer, Berlin, 1980, 10–17.
[12] E. Bedford and P. Feredbush, *Pluriharmonic boundary values*, Tohoku Math. J. **2** (1974), 505–511.
[13] W. Rudin, *Function Theory in the Polydisk*, W. A. Benjamin Inc., New York, 1969.

[1]Department of Mathematics,
Bryansk State Technical University,
Brynsk, Russia 241050
Email address: rsham@mail.ru
ORCID iD: https://orcid.org/0000-0002-8415-9822

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1087–1103.

STUDY OF DOUBLE PHASE-CHOQUARD PROBLEM IN GENERALIZED ψ-HILFER FRACTIONAL DERIVATIVE SPACES WITH p-LAPLACIAN OPERATOR

ELHOUSSAIN ARHRRABI[1] AND HAMZA EL-HOUARI[2,*]

Abstract. In this paper, our focus is on a specific class of non-linear ψ-Hilfer fractional generalized double phase-Choquard differential equations involving the p-Laplacian operator with Dirichlet boundary conditions. The equation is given by:

$$\begin{cases} \mathcal{L}^{\gamma,\beta;\psi} u = \left(\displaystyle\int_\Omega \frac{G\big(u(x)\big)}{|x-y|^\lambda} dx \right) g\big(u(y)\big), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\mathcal{L}^{\gamma,\beta;\psi}$ is defined as:

$$\mathcal{L}^{\gamma,\beta;\psi} u := \ \mathbb{D}_T^{\gamma,\beta;\psi} \Big(|\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{p-2} \ \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u + \mathbf{a}(x) |\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{q-2} \ \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u \Big),$$

where $\mathbb{D}_T^{\gamma,\beta;\psi}$ and $\mathbb{D}_{0^+}^{\gamma,\beta;\psi}$ are ψ-Hilfer fractional derivatives of order $\frac{1}{p} < \gamma < 1$ and type $0 \leq \beta \leq 1$ and $\mathbf{a}(\cdot)$ is non-negative weight function, and $G(\cdot)$ represents Choquard nonlinearities satisfying a certain growth conditions. By employing the mountain pass theorem without the Palais-Smale condition, along with the Hardy-Littlewood-Sobolev inequality, we establish the existence of a weak solution to the aforementioned problem. Our main results are novel and contribute to the literature on problems involving ψ-Hilfer derivatives with the p-Laplacian operator. This investigation enhances the scope of understanding in this specific class of problems.

Key words and phrases. Generalized ψ-Hilfer derivative, double phase-Choquard equation, mountain pass theorem, Hardy-Littlewood-Sobolev inequality.

2020 *Mathematics Subject Classification.* Primary: 26A33. Secondary: 35R15, 35R11, 35A15.

`https://doi.org/10.46793/KgJMat2607.1087A`

Received: March 31, 2024.

Accepted: September 27, 2024.

1. Introduction

The equation known as the Choquard equation, given by

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy\right) u, \quad u \in H^1(\mathbb{R}^3), \tag{1.1}$$

was initially introduced by Choquard in 1976 and has since captured considerable attention in the realms of physics and mathematical analysis [35]. This equation serves as an approximation to the Hartree-Fock theory of a one-component plasma, providing insights into intricate interactions between particles. Lion in [33] studied the normalized solutions of the following problem

$$-\Delta u + \lambda u = \left(\int_{\mathbb{R}^3} u^2(y) V(|x-y|) dy\right) u(x), \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

where V is some given positive function. In the special case where $V = 1/|x|$, equation (1.2) returns to equation (1.1). Furthermore, Penrose proposed it as a model for elucidating the self-gravitational collapse of a quantum mechanical wave function, underscoring its significance in comprehending essential quantum phenomena [36].

In the context of Choquard equations driven by a p-Laplacian operator, P. Le in [32], established the existence of weak solutions to the following semilinear Choquard equation, which appears as a model in quantum mechanics,

$$-\Delta_p u = \left(\frac{1}{|x|^{n-\alpha}} * |u|^q\right) |u|^{q-2} u, \quad u \in \mathbb{R}^n,$$

where $2 \leq p < q \leq n$ and $\max\{0, n-2p\} < \alpha < n$. In [1], the authors studied the existence of semiclassical ground state solutions to the following generalized Choquard equation

$$-\Delta_p u + |u|^{p-2} = \left(\int_{\mathbb{R}^3} \frac{F(u(y))}{|x-y|} dy\right) f(u(x)), \quad \text{in } \mathbb{R}^n.$$

Alternatively, the fractional diffusion integrodifferential equation problems are as follows

$$\begin{cases} a^2 \dfrac{\partial^2}{\partial x^2} T(x,t) = \displaystyle\int_0^t \frac{T(x,\tau)(t-\tau)^{-(\lambda+1)}}{\Gamma(-\lambda)} d\tau, & x \in \mathbb{R}^+, a > 0, 0 < \lambda \leq 1, \\ T(x,0) = \theta(x), \quad T(0,t) = 0, \quad t > 0. \end{cases} \tag{1.3}$$

The authors in [46] discuss the exactly solution and the asymptotic behavior of the problem (1.3) for different values of λ, i.e., for $\lambda = 1$ and $0 < \lambda < 1$. The authors in [31] solved exactly the following fractional diffusion equation based on Riemann-Liouville fractional derivatives

$$\mathbb{D}_{0+}^{\alpha} f(r,t) = C_\alpha \Delta f(r,t),$$

where $f(r,t)$ denotes the unknown field and C_α denotes the fractional diffusion constant with dimensions $[cm/s^\alpha]$ and D_{0+}^{α} is the Riemann-Liouville derivative of order α.

Numerous researchers have suggested employing fractional time derivatives to address problems involving linear or non-linear differential equations. The essential question is whether there exists a connection between fractional derivatives and gradient terms. The answer is provided in [45], where the authors extend gradient elasticity models to characterize materials exhibiting fractional non-locality and fractality. They derive a generalization of three-dimensional continuum gradient elasticity theory, starting from integral relations and assuming a weak non-locality of power-law (fractional) type. This results in constitutive relations featuring fractional Laplacian terms, achieved through the application of fractional Taylor series in wave-vector space. Subsequently, the authors explore non-linear field equations with fractional derivatives of non-integer order to describe nonlinear elastic effects in gradient materials with power-law long-range interactions within the framework of weak non-locality approximation. The specific constitutive relationship detailed in this study could serve as the foundation for developing a fractional extension of deformation theory in gradient plasticity. On the other hand, related to the double phase problem, the stationary general reaction diffusion double-phase is given by the form

$$u_t = \operatorname{div}[A(x)\nabla u] + b(x,u), \quad \text{with } A(x) = |\nabla u|^{p-2} + |\nabla u|^{q-2}, \tag{1.4}$$

where the function u represents a concentration, and $\operatorname{div}[A(x)\nabla u]$ relates to diffusion with diffusion coefficient $A(x)$. The term $b(x,u)$ corresponds to sources and loss processes, and this type of problem has applications in physics and allied fields such as biophysics, plasma physics, solid state physics, and chemical reaction design. For more information, refer to [4,7]. One example of this type of problem is the following equation:

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(x,u) \quad \text{in } \mathbb{R}^n, \quad 1 < p \leq q < +\infty,$$

which is connected with the general reaction-diffusion system (1.4). The equation involves two distinct materials with power-hardening exponents p and q. Many authors have established existence, multiplicity, and regularity results for this type of problem in bounded or unbounded domains, as discussed in [23, 24, 30] and the references therein.

In the context of fractional differential equation boundary-value problems with p-Laplacian operator, for the existence and non-existence of weak solutions to the nonlinear examination of solution existence and stability can be found in [39], the equation as the form

$$\begin{cases} \mathbb{D}_T^{\alpha,\beta;\psi}\left(|\mathbb{D}_{0+}^{\alpha,\beta;\psi}u(x)|^{p-2}\ \mathbb{D}_{0+}^{\alpha,\beta;\psi}u(x)\right) = \lambda|u(x)|^{p-2}u(x) + b(x)|u(x)|^{q-1}u(x), \\ \mathbf{I}_{0+}^{\beta(\beta-1);\psi}u(0) = \mathbf{I}_T^{\beta(\beta-1);\psi}u(T) = 0, \end{cases}$$

where $\mathbb{D}_{0+}^{\alpha,\beta;\psi}$, $\mathbb{D}_T^{\alpha,\beta;\psi}$ are ψ-Hilfer fractional derivatives left-sided and right-sided of order $\frac{1}{p} < \alpha < 1$, type $0 \leq \beta \leq 1$, $1 < q < p-1 < +\infty$, $b \in L^\infty(\Omega)$ and $\mathbf{I}_{0+}^{\beta(\beta-1);\psi}$,

$\mathbf{I}_T^{\beta(\beta-1);\psi}$ are ψ-Riemann-Liouville fractional integrals left-sided and right-sided, for all $x \in \Omega = [0, T]$.

In 2023, Sousa et al. [44], discussed the existence and regularity of weak solutions for ψ-Hilfer fractional boundary value problem by using an extension of the Lax-Milgram theorem to the following nonlinear boundary value problem

$$\begin{cases} \mathbb{D}_T^{\alpha,\beta;\psi}\Big(|\mathbb{D}_{0^+}^{\alpha,\beta;\psi}u(x)|^{p-2}\ \mathbb{D}_{0^+}^{\alpha,\beta;\psi}u(x)\Big) + u(x) = \lambda\Phi(t, u(x)), \qquad t \in (0, T),\\ \mathbf{I}_{0^+}^{\beta(\beta-1);\psi}u(0) = \mathbf{I}_T^{\beta(\beta-1);\psi}u(T) = 0, \end{cases}$$

where $\mathbb{D}_T^{\alpha,\beta;\psi}$, $\mathbb{D}_{0^+}^{\alpha,\beta;\psi}$ are ψ-Hilfer fractional derivatives left-sided and right-sided of order $\frac{1}{2} < \alpha < 1$, type $0 \le \beta \le 1$, respectively, $\mathbf{I}_{0^+}^{\beta(\beta-1);\psi}$, $\mathbf{I}_T^{\beta(\beta-1);\psi}$ are ψ-Riemann-Liouville fractional integrals left-sided and right-sided of order $\beta(\beta-1)$, respectively, λ is a parameter and $\Phi : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [38], the authors established the existence of solutions to the following new class of singular double phase p-Laplacian equation problems with a ψ-Hilfer fractional operator combined from a parametric term, namely:

$$\begin{cases} \mathbb{D}_T^{\alpha,\beta;\psi}\Big(|\mathbb{D}_{0^+}^{\alpha,\beta;\psi}u|^{p-2}\ \mathbb{D}_{0^+}^{\alpha;\beta;\psi}u + \mu(x)|\mathbb{D}_{0^+}^{\alpha,\beta;\psi}u|^{q-2}\ \mathbb{D}_{0^+}^{\alpha,\beta;\psi}u\Big)\\ = \xi(x)u^{-\sigma} + \lambda u^{r-1}, & \text{in} \quad \Omega = [0, T] \times [0, T],\\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We can not quote all reference in the existence of solution for fractional equation, for that we refer for interesting reader to [2, 3, 5, 8–22, 25–29, 40–45].

Motivated by these results, we turn our attention to the exploration of the existence solution in a suitable fractional ψ-Hilfer derivative space for the double phase-Choquard problem with p-Laplacian operator presented in this paper, namely

$$\begin{cases} \mathcal{L}^{\gamma,\beta;\psi}u = \left(\displaystyle\int_\Omega \frac{G(u(x))}{|x-y|^\lambda}dx\right) g(u(y)), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

with

$$\mathcal{L}^{\gamma,\beta;\psi}u := \ \mathbb{D}_T^{\gamma,\beta;\psi}\Big(|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{p-2}\ \mathbb{D}_{0^+}^{\gamma,\beta;\psi}u + \mathbf{a}(x)|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{q-2}\ \mathbb{D}_{0^+}^{\gamma,\beta;\psi}u\Big),$$

where $\mathbb{D}_T^{\gamma,\beta;\psi}$ and $\mathbb{D}_{0^+}^{\gamma,\beta;\psi}$ are ψ-Hilfer fractional derivatives of order $\frac{1}{p} < \gamma < 1$ and type $0 \le \beta \le 1$ and $\mathbf{a}(x)$ is non-negative weight function with compact support in Ω and $g : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying:

- (g_1) $N \ge 2$, $1 < p < N$, $p < q < p + \frac{\alpha p}{N}$ and $\mathbf{a}(\cdot) \in C_0^\infty(\Omega)$ with $0 < \alpha \le 1$;
- (g_2) the growth condition, i.e.,

$$|g(\xi)| \le c_1\Big(|\xi|^{r_1-1} + |\xi|^{r_2-1}\Big), \quad \text{for all } \xi \in \mathbb{R}^N \text{ and } c_1 > 0,$$

where

$$p < \tau r_1 \le \tau r_2 < p^*, \tag{1.6}$$

and $\tau = 2N/(2N-\lambda) > 1$ and p^* being the critical Sobolev exponent to p;

(g_3) there is $\alpha > q$ such that

$$0 < \alpha G(\xi) \leq 2g(\xi)\xi, \quad \text{where } G(\xi) := \int_0^\xi g(\tau)d\tau.$$

To our surprise, these results represent the first contributions available in the literature for the ψ-Hilfer fractional generalized double phase-Choquard differential equations involving the p-Laplacian operator with Dirichlet boundary conditions within the framework of ψ-fractional derivative space $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. Our approach to establishing existence results for problem (1.5) hinges on utilizing the mountain pass theorem without the Palais-Smale condition [6]. Initially, we demonstrate that the energy functional $\mathfrak{E}$ connected to the problem (1.5) adheres to the mountain pass geometry. We establish the boundedness of a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ that does not satisfy the Palais-Smale condition. Ultimately, we leverage the properties of the transformed sequence along with technical skills to achieve the existence result for problem (1.5). One of the key challenges in this approach lies in utilizing the Hardy-Littlewood-Sobolev inequality for nonlinearities involving ψ-Hilfer fractional derivative.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of Musielak spaces and ψ-fractional derivative spaces. Moving on to Section 3, we present the existing solutions to problems (1.5), along with their corresponding proofs.

2. Preliminary

In this section we refer to [38]. Consider the nonlinear function $\mathcal{H} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\mathcal{H}(x,u) = u^p + \mathbf{a}(x)u^q.$$

Let $\mathbf{M}(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. Then, Musielak space $L^{\mathcal{H}}\big(\Omega\big)$ is given by

$$L^{\mathcal{H}}(\Omega) = \left\{ u \in \mathbf{M}(\Omega) : \varrho^{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)dx < +\infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{ \delta > 0 : \varrho^{\mathcal{H}}\left(\frac{u}{\delta}\right) \leq 1 \right\}.$$

Moreover, we define the weighted space

$$L^q_{\mathbf{a}}(\Omega) = \left\{ \mathbf{f} \in \mathbf{M}\big(\Omega\big) : \int_\Omega \mathbf{a}(x)|\mathbf{f}|^q dx < +\infty \right\},$$

with the seminorm

$$\|\mathbf{f}\|_{\mathbf{a},q} = \left(\int_\Omega \mathbf{a}(x)|\mathbf{f}|^q \right)^{\frac{1}{q}}.$$

ψ-fractional derivative space. Let $A := [c,d]$, $-\infty \le c < d \le +\infty$, $n-1<\gamma<n$, $n \in \mathbb{N}$, $\mathbf{f}$, $\psi \in C^n(I,\mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $u \in A$.

• The left-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_c^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)}\int_0^x \psi'(y)(\psi(x)-\psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{dv}. \tag{2.1}$$

• The right-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_d^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)}\int_x^d \psi'(y)(\psi(y)-\psi(x))^{\gamma-1}\mathbf{f}(y)\mathrm{dv}. \tag{2.2}$$

• The left-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \le \beta \le 1$ is defined by

$$\mathbb{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\beta(n-\gamma);\psi}\Big(\frac{1}{\psi'(x)}\cdot\frac{d}{dx}\Big)^n \mathbf{I}_c^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

• The right-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \le \beta \le 1$ is defined by

$$\mathbb{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\beta(n-\gamma);\psi}\Big(-\frac{1}{\psi'(x)}\cdot\frac{d}{dx}\Big)^n \mathbf{I}_d^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

Choosing $\beta \to 1$, we obtain ψ-Caputo fractional derivatives left-sided and right-sided, given by

$$\mathbb{D}_c^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_c^{(n-\gamma);\psi}\Big(\frac{1}{\psi'(x)}\cdot\frac{d}{dx}\Big)^n \mathbf{f}(x), \tag{2.3}$$

$$\mathbb{D}_d^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_d^{(n-\gamma);\psi}\Big(-\frac{1}{\psi'(x)}\cdot\frac{d}{dx}\Big)^n \mathbf{f}(x). \tag{2.4}$$

Remark 2.1. The ψ-Hilfer fractional derivatives defined as above can be written in the following form

$$\mathbb{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\mu-\gamma;\psi}\mathbb{D}_c^{\gamma;\psi}\mathbf{f}(x)$$

and

$$\mathbb{D}_d^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\mu-\gamma;\psi}\mathbb{D}_d^{\gamma;\psi}\mathbf{f}(x),$$

with $\mu = \gamma + \beta(n-\gamma)$ and $\mathbf{I}_c^{\mu-\gamma;\psi}$, $\mathbf{I}_d^{\mu-\gamma;\psi}$, $\mathbb{D}_c^{\gamma;\psi}$ and $\mathbb{D}_d^{\gamma;\psi}$ as defined in (2.1), (2.2), (2.3) and (2.4).

In this paper, we take $\Omega = A_1 \times \cdots \times A_N = [c_1,d_1] \times \cdots \times [c_N,d_N]$ where $0 < c_i < d_i$ for all $i \in \mathbb{N}$, $0 < \gamma_1, \dots, \gamma_N < 1$. Consider also $\psi(\cdot)$ to be an increasing and positive monotone function on $(c_1,d_1), \dots, (c_N,d_N)$, having a continuous derivative $\psi'(\cdot)$ on $(c_1,d_1], \dots, (c_N,d_N]$.

• The ψ-Riemann-Liouville fractional partial integral of order γ of N-variables $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_N)$ is defined by

$$\mathbf{I}_{c,x}^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)}\int_{A_1}\int_{A_2}\cdots\int_{A_N}\psi'(y)(\psi(x)-\psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}v,$$

with $\psi'(y)(\psi(x)-\psi(y))^{\gamma-1} = \psi'(v_1)(\psi(x_1)-\psi(v_1))^{\gamma_1-1}\cdots\psi'(v_N)(\psi(x_N)-\psi(v_N))^{\gamma_N-1}$ and $\Gamma(\gamma) = \Gamma(\gamma_1)\Gamma(\gamma_2)\cdots\Gamma(\gamma_N)$, $u_i = u_1u_2\cdots u_N$ and $\mathrm{d}v_i = \mathrm{d}v_1\mathrm{d}v_2\cdots\mathrm{d}v_N$, for all $i \in \{1, 2, \ldots, N\}$.

• The ψ-Hilfer fractional partial derivative of N-variables of order γ and type β $(0 \leq \beta \leq 1)$ is defined by

$$\mathbb{D}^{\gamma,\beta;\psi}_{c,x_i}\mathbf{f}(x_i) = \mathbf{I}^{\beta(n-\gamma);\psi}_{c,x_i}\Big(\frac{1}{\psi'(x_i)}\cdot\frac{\partial^N}{\partial x_i}\Big)\mathbf{I}^{(1-\beta)(n-\gamma);\psi}_{c,x_i}\mathbf{f}(x_i),$$

with $\partial x_i = \partial x_1, \partial x_2, \ldots, \partial x_N$ and $\psi'(x_i) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$ for all $i \in \{1, 2, \ldots, N\}$. Analogously, it is defined $\mathbb{D}^{\gamma,\beta;\psi}_{d,x_i}(\cdot)$.

• The left-sided ψ-fractional derivative space $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ is defined by

$$\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) = \Big\{u \in \mathbf{L}^{\mathcal{H}}(\Omega):\ |\mathbb{D}^{\gamma,\beta;\psi}_{0+}u| \in \mathbf{L}^{\mathcal{H}}(\Omega); u = 0 \text{ a.e } \Omega\setminus 0\Big\},$$

equipped with the norm

$$\|u\|_{0,\mathcal{H}} = \|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}} = \||\mathbb{D}^{\gamma,\beta;\psi}_{0+}u|\|_{\mathcal{H}}$.

Remark 2.2. Note that $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) := \overline{C_0^\infty(\Omega)}^{\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)}$, and the equivalent norm on $\mathbb{H}^{\gamma,\beta;\psi}_{0,\mathcal{H}}(\Omega)$ is given by $\|u\|_{0,\mathcal{H}} = \|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}}$.

The results below will be needed for our purposes.

Proposition 2.1 ([39])**.** *Let (g_1) be satisfied. Then, the following embeddings hold:*
(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ are continuous for all $r \in [1, p]$;
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [1, p^*]$;
(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in [1, p^*)$;
(iy) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_{\mathbf{a}}(\Omega)$ is continuous;
(vi) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous.

Proposition 2.2 ([39])**.** *Let (g_1) be satisfied, $v \in L^{\mathcal{H}}(\Omega)$, $c > 0$ and $\varrho^{\mathcal{H}}$ previously defined. Then, the following hold:*
(i) if $v \neq 0$, then $\|v\|_{\mathcal{H}} = c$ if and only if $\varrho^{\mathcal{H}}\left(\frac{v}{c}\right) = 1$;
(ii) $\|v\|_{\mathcal{H}} < 1$ (resp. $> 1, = 1$) if and only if $\varrho^{\mathcal{H}}(v) < 1$ (resp. $> 1, = 1$);
(iii) if $\|v\|_{\mathcal{H}} < 1$, then $\|v\|^q_{\mathcal{H}} \leq \varrho^{\mathcal{H}}(v) \leq \|v\|^p_{\mathcal{H}}$;
(iy) if $\|v\|_{\mathcal{H}} > 1$, then $\|v\|^p_{\mathcal{H}} \leq \varrho^{\mathcal{H}}(v) \leq \|\boldsymbol{y}\|^q_{\mathcal{H}}$;
(y) $\|v\|_{\mathcal{H}} \to 0$ if and only if $\varrho^{\mathcal{H}}(v) \to 0$;
(vi) $\|v\|_{\mathcal{H}} \to +\infty$ if and only if $\varrho^{\mathcal{H}}(v) \to +\infty$.

Proposition 2.3 ([39])**.** *Let (g_1) be satisfied. Then, the embedding $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [p, p^*]$.*

Remark 2.3. From (1.6) and Proposition 2.3, we have

$$\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) \hookrightarrow L^{\tau r_i}(\Omega), \quad \text{with } r_i \in [p, p^*], i = 1, 2. \tag{2.5}$$

Proposition 2.4 (Hardy-Littlewood-Sobolev inequality)**.** *Let $p, q > 1$ and $0 < \lambda < N$ with $1/p + \lambda/N + 1/q = 2$, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then, there exists a sharp constant $C(p, N, \lambda, q)$, independent of f, g, such that*

$$\left| \int_{\Omega\times\Omega} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \right| \leq C(p, N, \lambda, q)||f||_{L^p(\Omega)}||g||_{L^q(\Omega)}. \tag{2.6}$$

3. Main Result

Our first main result is the following.

Theorem 3.1. *The problem* (1.5) *has a nontrivial solution under the conditions* (g_1)*-*(g_3)*.*

In the proof of Theorem 3.1 we will use variational methods. The energy functional $\mathfrak{E} : \mathbb{H}^{\gamma,\beta;\psi}_{0,\mathcal{H}}(\Omega) \to \mathbb{R}$ associated with (1.5) is given by

$$\mathfrak{E}(u) = \frac{1}{p}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|^p_p + \frac{1}{q}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|^q_{\mathbf{a},q} - \Psi(u),$$

where

$$\Psi(u) = \frac{1}{2}\int_\Omega\int_\Omega \frac{G(u(x))G(u(y))}{|x-y|^\lambda} dxdy.$$

Theorem 3.1 is proved in several steps.

Step 1. The energy functional $\mathfrak{E}$ satisfies the mountain pass geometry, i.e., satisfies the following lemma.

Lemma 3.1. *The functional $\mathfrak{E}$ exhibits the following characteristics.*

(i) For sufficiently small $\rho > 0$, $\mathfrak{E}(u) \geq \eta$ holds for $u \in \mathbb{H}^{\gamma,\beta;\psi}_{0,\mathcal{H}}(\Omega)$ with $\|u\|_{0,\mathcal{H}} = \rho$, where $\eta > 0$.

(ii) There exists an element $e \in \mathbb{H}^{\gamma,\beta;\psi}_{0,\mathcal{H}}(\Omega)$ such that $\|e\|_{0,\mathcal{H}} > \rho$ and $\mathfrak{E}(e) < 0$.

First, we need to demonstrate the following useful property.

Proposition 3.1. *For each $v \in \mathbb{H}^{\gamma,\beta;\psi}_{0,\mathcal{H}}(\Omega)$, we have the following property: G and gv are belong to $L^\tau(\Omega)$.*

Proof. Due to (g_2), if $u \in \mathbb{R}$ and $u(x) \neq 0$, then

$$|g(u(x))| \leq c_1\Big(|u(x)|^{r_1-1} + |u(x)|^{r_2-1}\Big). \tag{3.1}$$

Also, from the last inequality, we deduce

$$|G(v)| = \left|\int_0^v g(x)dx\right| \leq c_1\int_0^v \Big(|u|^{r_1-1} + |u|^{r_2-1}\Big)dx \leq c_1\Big(|v|^{r_1} + |v|^{r_2}\Big).$$

Hence,

$$|G(v)|^\tau \leq c_2\Big(|v|^{\tau r_1} + |v|^{\tau r_2}\Big), \tag{3.2}$$

where $c_2 = c_1^\tau$. Now, utilizing (2.5), we deduce that $G \in L^\tau(\Omega)$. By a similar argument as above, which applies to $g(u)v$, we deduce that $gv \in L^\tau(\Omega)$. □

Lemma 3.2. *For each* $u \in \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. *We have the following properties:*

(*i*)

$$\int_{\Omega\times\Omega} \frac{|G(u(x))g(u(y))v(y)|}{|x-y|^\lambda} dxdy < +\infty, \quad \textit{for all } v \in C_0^\infty(\Omega);$$

(*ii*)

$$\int_{\Omega\times\Omega} \frac{|G(u(x))g(u(y))v(y)|}{|x-y|^\lambda} dxdy \le C\|G(u)\|_{L^\tau}\|g(u)v\|_{L^\tau}, \quad \textit{for all } v \in C_0^\infty(\Omega).$$

Proof. Taking into account Proposition 3.1, equations (2.5), (3.1), and the fact that $(a+b)^p \le 2^{p-1}(a^p+b^p)$, for every $a, b \ge 0$ and $1 \le p < +\infty$, as well as Proposition 2.4, we arrive at the proof of Lemma 3.2. □

Corollary 3.1. *For each* $v \in \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$, *with* $||v||_{0,\mathcal{H}} \le 1$, *the sequence* $\{||g(u_k)v||_{L^\tau}\}$ $k \in \mathbb{N}$ *is bounded.*

Proof. Due to (3.1) and (2.5), we obtain

$$\begin{aligned}
\int_\Omega |g(u_k(y))v(y)|^\tau dy \le & c_2 \int_\Omega \Big(|u_k(y)|^{\tau(r_1-1)} + |u_k(y)|^{\tau(r_2-1)}\Big)|v(y)|^\tau dy \\
\le & c_2 \Bigg[\Big(\int_\Omega |u_k|^{\tau r_1} dy\Big)^{\frac{r_1-1}{r_1}} \Big(\int_\Omega |v(y)|^{\tau r_1} dy\Big)^{\frac{1}{r_1}} + \Big(\int_\Omega |u_k|^{\tau r_2} dy\Big)^{\frac{r_2-1}{r_2}} \\
& \times \Big(\int_\Omega |v(y)|^{\tau r_2} dy\Big)^{\frac{1}{r_2}}\Bigg] \\
= & c_2 \Big(||u_k(y)||^{\tau(r_1-1)}_{L^{\tau r_1}} ||v(y)||^{\tau}_{L^{\tau r_1}} + ||u_k(y)||^{\tau(r_2-1)}_{L^{\tau r_2}} ||v(y)||^{\tau}_{L^{\tau r_2}}\Big) \\
\le & c_2 \Big(||u_k(y)||^{\tau(r_1-1)}_{L^{\tau r_1}} + ||u_k(y)||^{\tau(r_2-1)}_{L^{\tau r_2}}\Big) < +\infty. \qquad (3.3)
\end{aligned}$$

□

Corollary 3.2. *The function* $\mathfrak{E}$ *belongs to* $C^1\left(\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega), \mathbb{R}\right)$, *and we can express it as follows:*

$$\begin{aligned}
\mathfrak{E}'(u)v = & \int_\Omega \Big(|\mathbb{D}^{\gamma,\beta;\psi}_{0^+} u|^{p-2}\, \mathbb{D}^{\gamma,\beta;\psi}_{0^+} u + \mathbf{a}(x)|\mathbb{D}^{\gamma,\beta;\psi}_{0^+} u|^{q-2}\, \mathbb{D}^{\gamma,\beta;\psi}_{0^+} u\Big)\, \mathbb{D}^{\gamma,\beta;\psi}_{0^+} v dx \\
& - \int_\Omega\int_\Omega \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda} dxdy, \qquad (3.4)
\end{aligned}$$

for all $u, v \in \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$.

Proof. Using the analysis presented earlier, along with a similar approach as in Lemma 3.2 of [1], we confirm the validity of this corollary. □

Proof of the main Lemma 3.1. Let prove (i). Applying Propositions 3.1, Lemma 3.4 and Proposition 2.4, we have

$$\Big|\int_{\Omega\times\Omega}\frac{G(u(x))G(u(y))}{|x-y|^{\lambda}}dxdy\Big|\leq c_3\|G(u)\|^2_{L^{\tau}(\Omega)},$$

for all $u\in\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. Due to (3.1) and (3.2), we have

$$\|G(u)\|_{L^{\tau}(\Omega)}\leq c_4\Big(\|u\|^{r_1}_{L^{\tau r_1}}+\|u\|^{r_2}_{L^{\tau r_2}}\Big)\leq c_5\Big(\|u\|^{r_1}_{0,\mathcal{H}}+\|u\|^{r_2}_{0,\mathcal{H}}\Big), \tag{3.5}$$

where c_5 is a constant that does not depend on $u\in\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ with $\|u\|_{0,\mathcal{H}}=\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}}+\|u\|_{\mathcal{H}}<1$ we get that

$$\begin{aligned}\mathfrak{E}(u)&\geq\frac{1}{p}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|^p_p+\frac{1}{q}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|^q_{\mathbf{a},q}-c_6\Big(\|u\|^{r_1}_{0,\mathcal{H}}+\|u\|^{r_2}_{0,\mathcal{H}}\Big)\\&\geq c_6\|u\|^p_{0,\mathcal{H}}-c_7\Big(\|u\|^{r_1}_{0,\mathcal{H}}+\|u\|^{r_2}_{0,\mathcal{H}}\Big),\end{aligned}$$

where c_6 and c_7 are constants that do not depend on u. The fact that $r_2>p/2$, then the result follows by fixing $\|u\|_{0,\mathcal{H}}=\rho$ with $\rho>0$ small enough.

Let prove (ii). Let us fix $u_0\in\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)\backslash\{0\}$ with $u_0>0$ and define

$$J(t)=G\Big(x,\frac{tu_0}{\|u_0\|_{0,\mathcal{H}}}\Big),\quad\text{for } t>0,u\in\Omega,$$

The condition (g_2) implies that

$$\frac{J'(t)}{J(t)}\geq\frac{\alpha}{2t},\quad\text{for } t>0,$$

Integrating this over $[1,s\|u_0\|_{0,\mathcal{H}}]$ with $s>\frac{1}{\|u_0\|_{0,\mathcal{H}}}$, we get

$$G(x,su_0)\geq G\Big(x,\frac{u_0}{\|u_0\|_{0,\mathcal{H}}}\Big)\Big(s\|u_0\|_{0,\mathcal{H}}\Big)^{\alpha/2}.$$

With this, we are able to compose

$$\begin{aligned}\mathfrak{E}(su_0)\leq&\frac{s^p}{p}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_0\|^p_p+\frac{s^q}{q}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_0\|^q_{\mathbf{a},q}\\&-\frac{s^{\alpha}\|u_0\|^{\alpha}_{0,\mathcal{H}}}{2}\int_{\Omega}\Bigg(\int_{\Omega}\frac{G\Big(v,\frac{u_0}{\|u_0\|}\Big)}{|x-y|^{\lambda}}dy\Bigg)G\Big(x,\frac{u_0}{\|u_0\|}\Big)dx.\end{aligned}$$

Since $\alpha>q>p$ we can choose $s>\frac{1}{\|u_0\|_{0,\mathcal{H}}}$ large enough such that $e=su_0$ with $\|e\|_{0,\mathcal{H}}>\rho$ and $\mathfrak{E}(e)<0$. This finishes the proof of the main Lemma 3.1. □

Step 2. The sequence $\{u_k\}_{k\in\mathbb{N}}$, which does not satisfy the (PS)-condition, is a bounded sequence in $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. Recalling that the mountain pass theorem without the

Palais-Smale condition (refer to [6], Theorem 5.4.1) states the existence of a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ such that:

$$\mathfrak{E}(u_k) \to \theta \tag{3.6}$$

and

$$\mathfrak{E}'(u_k) \to 0, \tag{3.7}$$

where $\theta > 0$ is the mountain pass level defined by

$$\theta := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathfrak{E}(\gamma(t)),$$

with

$$\Gamma := \Big\{\gamma \in C\Big(\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega), \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)\Big) : \gamma(0) = 0, \gamma(1) = e\Big\}.$$

Concerning the sequence mentioned earlier, we observe the following auxiliary characteristic.

Lemma 3.3. *The sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$.*

Proof. Note that

$$\mathfrak{E}(u_k) - \frac{\mathfrak{E}'(u_k)u_k}{\alpha} \le \theta + 1 + \|u_k\|_{0,\mathcal{H}}, \tag{3.8}$$

for k large enough. Moreover, from Proposition 2.2 and (g_3) we have for $\|u_k\|_{0,\mathcal{H}} \ge 1$ that

$$\begin{aligned}\mathfrak{E}(u_k) - \frac{\mathfrak{E}'(u_k)u_k}{\alpha} =& \Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_k\|^p_p + \Big(\frac{1}{q} - \frac{1}{\alpha}\Big)\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_k\|^q_{\mathbf{a},q}\\ &+ \int_\Omega\int_\Omega \frac{G(x,u_k(x))}{|x-y|^\lambda}\Big(\frac{g(v,u_k(y))u_k(y)}{\alpha} - \frac{G(v,u_k(y))}{2}\Big)dxdy\\ \ge& \Big(\frac{1}{q} - \frac{1}{\alpha}\Big)\|u_k\|_{0,\mathcal{H}}. \end{aligned}\tag{3.9}$$

Hence, (3.8) and (3.9) owing to the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. ☐

Step 3. Existence of solution, i.e., critical point of $\mathfrak{E}$.

First, we need the following two lemmas.

Lemma 3.4. *The following limits hold for a subsequence:*

(i)

$$\int_{\Omega\times\Omega} \frac{G(x,u_k(x))g(v,u(y))v(y)}{|x-y|^\lambda}dxdv \to \int_{\Omega\times\Omega} \frac{G(x,u(x))g(v,u(y))v(y)}{|x-y|^\lambda}dxdv,$$

for all $v \in C^\infty_c(\Omega)$;

(ii)

$$\int_\Omega\int_\Omega \frac{G(x,u_k(x))\Big(g(v,u_k(y))v(y) - g(v,u(y))v(y)\Big)}{|x-y|^\lambda}dxdy \to 0,$$

for all $v \in C_c^\infty(\Omega)$;

(iii)

$$\int_\Omega \int_\Omega \frac{G(u_k(x))g(u_k(y))v(y)}{|x-y|^\lambda} dxdy \to \int_\Omega \int_\Omega \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda} dxdv,$$

for all $v \in C_c^\infty(\Omega)$.

Proof. (i) Lemma 3.3, (2.5), and Proposition 3.1 collectively establish that $\{G(u_k)\}_{k\in\mathbb{N}}$ forms a bounded sequence in $L^\tau(\mathbb{R}^n)$. Leveraging the continuity of G, along with the previously mentioned information and the pointwise convergence $G(u_k(x)) \to G(u(x))$ almost everywhere in $\mathbb{R}^n$, we deduce that $G(u_k) \rightharpoonup G(u)$ in $L^\tau(\mathbb{R}^n)$. By virtue of Proposition 2.4, it follows that the function

$$H(w) := \int_\Omega \int_\Omega \frac{w(x)g(u(y))v(y)}{|x-y|^\lambda}, \quad w \in L^\tau(\Omega),$$

defines a continuous linear functional. Since $G(u_k) \rightharpoonup G(u)$ in $L^\tau(\Omega)$, it follows that

$$\int_\Omega \int_\Omega \frac{G(u_k(x))g(u_k(y))v(y)}{|x-y|^\lambda} dxdy \to \int_\Omega \int_\Omega \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda} dxdy,$$

which proves (i).

For (ii), since $\{G(u_k)\}_{k\in\mathbb{N}}$ is bounded in $L^\tau(\Omega)$, we have

$$\begin{aligned}&\left| \int_{\Omega\times\Omega} \frac{G(u_k(x))(g(u_k(y))v(y) - g(u(y))v(y))}{|x-y|^\lambda} dxdy \right| \\ \le & c_8 \left\| G(u_k) \right\|_{L^\tau(\Omega)} \left\| g(u_k)v - g(u)v \right\|_{L^\tau(\Omega)} \\ \le & c_9 \left\| g(u_k)v - g(u)v \right\|_{L^\tau(\Omega)}.\end{aligned}$$

Let $v \in C_c^\infty(\Omega)$. Since Ω is bounded the compactness of the embeddings $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega) \hookrightarrow L^{\tau r_2}(\Omega)$, implies that there exist $h \in L^{\tau r_1}(\Omega)$ and $w \in L^{\tau r_2}(\Omega)$ such that

$$u_k(x) \to u(x) \text{ a.e in } \Omega, \quad |u_k(x)| \le h(x) \quad \text{and} \quad |u_k(x)| \le w(x) \text{ a.e in } \Omega. \tag{3.10}$$

Combining (3.10) with Lebesgue's dominated convergence theorem, we infer that

$$\|g(u_k)v - g(u)v\|_{L^\tau(\Omega)} = \|g(u_k)v - g(u)v\|_{L^\tau(\Omega)} \to 0.$$

The proof for (ii) is now complete, and (iii) directly follows from both (i) and (ii). □

Proposition 3.2. *Let hypotheses* (g_1)*-*(g_3) *be satisfied. For a subsequence of* $\{u_k\}_{k\in\mathbb{N}}$, *we have*

$$\mathbb{D}_{0^+}^{\gamma,\beta,\psi} u_k \to \mathbb{D}_{0^+}^{\gamma,\beta,\psi} u, \quad \textit{pointwise a.e. in } \Omega.$$

Consequently, it holds

$$\left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi} u_k\right|^{p-2} \mathbb{D}_{0^+}^{\gamma,\beta,\psi} u_k \rightharpoonup \left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi} u\right|^{p-2} \mathbb{D}_{0^+}^{\gamma,\beta,\psi} u, \quad \textit{in } \left[L^{\frac{p}{p-1}}(\Omega)\right]^N, \tag{3.11}$$

$$\left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi} u_k\right|^{q-2} \mathbb{D}_{0^+}^{\gamma,\beta,\psi} u_k \rightharpoonup \left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi} u\right|^{q-2} \mathbb{D}_{0^+}^{\gamma,\beta,\psi} u, \quad \textit{in } \left[L^{\frac{q}{q-1}}(\Omega)\right]^N. \tag{3.12}$$

Proof. We refer to the proof of Lemma 13 in [37]. □

Lemma 3.5. *The function u is a critical point of $\mathfrak{E}$.*

Proof. First of all, we claim that

$$\mathfrak{E}'(u_k)v \to \mathfrak{E}'(u)v, \quad \text{for all } v \in C_c^\infty(\Omega).$$

To verify such limit, note that

$$\begin{aligned}\mathfrak{E}'(u)v = &\int_\Omega \left(|\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{p-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u + \mathbf{a}(x)|\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{q-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u\right) \cdot \mathbb{D}_{0^+}^{\gamma,\beta;\psi} v dx\\ &- \int_\Omega\int_\Omega \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda} dxdy,\end{aligned}$$

Lemma 3.4, Proposition 3.2 owing to

$$\int_{\Omega\times\Omega} \frac{G(u_k(x))g(u_k(y))v(y)}{|x-y|^\lambda} dxdy \to \int_{\Omega\times\Omega} \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda} dxdy \tag{3.13}$$

and

$$\begin{aligned}&|\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u_k|^{p-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u_k + \mathbf{a}(x)\, |\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u_k|^{q-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u_k\\ \to &|\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{p-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u + \mathbf{a}(x)|\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{q-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u.\end{aligned} \tag{3.14}$$

From relations (3.13) and (3.14), the claim follows. As $\mathfrak{E}'(u_k)v \to 0$, this claim implies that $\mathfrak{E}'(u)v = 0$ for all $v \in C_c^\infty(\mathbb{R}^n)$. With the knowledge that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$, the lemma follows. □

Proof of Theorem 3.1. If $u \neq 0$, then u serves as a nontrivial solution, concluding the theorem. However, if $u = 0$, the task is to locate another solution $v \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)\backslash\{0\}$ for equation (1.5). In pursuit of this objective, the assertion presented below plays a pivotal role in our reasoning.

Claim. There exist $s > 0$, $\vartheta > 0$ and a sequence $(v_n)_n \subset \Omega$ such that

$$\liminf_{n\to+\infty} \int_{B_s(v_n)} |u_k(x)|^p dx \geq \vartheta > 0. \tag{3.15}$$

Proof. In fact, if the above claim does not hold, by Lions's lemma ([34], Lemmma I.1), one has

$$u_k \to 0, \quad \text{in } L^\tau(\Omega). \tag{3.16}$$

Moreover, Proposition 2.4 owing to,

$$\left|\int_\Omega\int_\Omega \frac{G(u_k(x))g(u_k(y))u_k(y)}{|x-y|^\lambda} dxdy\right| \leq C\|G(u_k(x))\|_{L^r(\Omega)}\|g(u_k(y))u_k(y)\|_{L^r(\Omega)}.$$

By (3.3), (3.5), and 3.16, we obtain that

$$\int_\Omega |G(u_k(x))|^\tau dx \to 0 \quad \text{and} \quad \int_\Omega |g(u_k(y))u_k(y)|^\tau dy \to 0.$$

Therefore,

$$\int_\Omega \frac{G(u_k(x))g(u_k(y))u_k(y)}{|x-y|^\lambda} dxdy \to 0. \tag{3.17}$$

Using (3.7) together with (3.17) give

$$\int_\Omega \left(|\mathbb{D}_{0+}^{\gamma,\beta;\psi} u|^{p-2}\, \mathbb{D}_{0+}^{\gamma,\beta;\psi} u + \mathbf{a}(x)\, |\mathbb{D}_{0+}^{\gamma,\beta;\psi} u|^{q-2}\, \mathbb{D}_{0+}^{\gamma,\beta;\psi} u \right) \mathbb{D}_{0+}^{\gamma,\beta;\psi} v dx \to 0.$$

This limit leads to $\mathfrak{E}(u_k) \to 0$, which contradicts (3.6). □

Due to the next lemma, we finish the prove of Theorem 3.1.

Lemma 3.6. *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ be such that $\mathfrak{E} \to \theta$. Then, there exists $\{\tilde{y}_k\}_{k\in\mathbb{N}} \subset \Omega$ such that the translated sequence*

$$\tilde{v} := u_k(x + \tilde{y}_k)$$

has a subsequence which converges in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$.

Proof. By utilizing the fact that $\mathfrak{E}'(u_k)u_k \to 0$ and $\mathfrak{E}(u_k) \to \theta$, we can employ the same reasoning as in the proof of Lemma 3.3 to demonstrate that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. Then, considering $\tilde{u}_k(x) = u_k(x + \tilde{y}_k)$ and a subsequence, we can find $\tilde{u} \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ and $\tilde{u} \neq 0$ according to (3.15). Furthermore, for $(t_k)_{k\in\mathbb{N}} > 0$, we can construct $\tilde{v}_k = t_k\tilde{u}_k \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. Then,

$$\mathfrak{E}(\tilde{v}_k) \le \max_{t\ge 0} \mathfrak{E}(tu_k) = \mathfrak{E}(u_k),$$

and so

$$\mathfrak{E}(\tilde{v}_k) \to \theta. \tag{3.18}$$

Since (3.18) holds, we have that $\{\tilde{v}_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$, which implies that we can assume $\tilde{v}_k \rightharpoonup \tilde{v}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. Moreover, $(t_k)_{k\in\mathbb{N}}$ is bounded and converges to $t_0 > 0$. Suppose for contradiction that $t_0 = 0$. Then, by the boundedness of $\{\tilde{u}_k\}_{k\in\mathbb{N}}$, we have $\|\tilde{v}_k\|_{0,\mathcal{H}} = t_k\|\tilde{u}_k\|_{0,\mathcal{H}} \to 0$, which contradicts $\mathfrak{E}(\tilde{v}_k) \to \theta > 0$. Hence, $t_0 > 0$. Since the weak limit is unique, we have $\tilde{v} = t_0\tilde{u}$ and $\tilde{u} \neq 0$. Thus, $\tilde{v}_k \to \tilde{v}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$, and consequently $\tilde{u}_k \to \tilde{u}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. □

References

[1] C. O. Alves and M. Yang, *Existence of semi classical ground state solutions for a generalized Choquard equation*, J. Differential Equations **257**(11) (2014), 4133–4164. https://doi.org/10.1016/j.jde.2014.08.004

[2] E. Arhrrabi and H. El-Houari, *On a class of generalized capillarity system involving fractional ψ-Hilfer derivative with $p(\cdot)$-Laplacian operator*, Math. Math. Methods Appl. Sci. (2024), 1–23, https://doi.org/10.1002/mma.10495.

[3] E. Arhrrabi and H. El-Houari, *On a class of generalized capillarity phenomena involving fractional ψ-Hilfer derivative with $p(\cdot)$-Laplacian operator*, Kragujevac J. Math. (2026) (to appear).

[4] G. M. Bisci, V. D. Radulescu, and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Cambridge University Press, 2016.

[5] L. S. Chadli, H. El-Houari and H. Moussa, *Multiplicity of solutions for nonlocal parametric elliptic systems in fractional Orlicz-Sobolev spaces*, J. Elliptic Parabol. Equ. **9**(2) (2023), 1131–1164. https://doi.org/10.1007/s41808-023-00238-4

[6] J. H. Chabrowski, *Variational Methods for Potential Operator Equations: with Applications to Nonlinear Elliptic Equations*, Walter de Gruyter, 2011.

[7] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136**(5) (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004

[8] H. El-Houari, H. Moussa and H. Sabiki, *Multiplicity and concentration properties of solutions for double-phase problem in fractional modular spaces*, J. Elliptic Parabol. Equ. (2024), 1–47. https://doi.org/10.1007/s41808-024-00278-4

[9] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of a solution to a nonlocal Schrödinger system problem in fractional modular spaces*, Advances in Operator Theory **7**(1) (2022), 1–30. https://doi.org/10.1007/s43036-021-00166-x

[10] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of solution to M-Kirchhoff system type*, 7th International Conference on Optimization and Applications (ICOA), IEEE, (2022), 1–6. https://doi.org/10.1109/ICOA51614.2021.9442669

[11] H. El-houari, H. Moussa and L. S. Chadli, *A class of elliptic inclusion in fractional Orlicz-Sobolev spaces*, Complex Var. Elliptic Equ. **69**(5) (2024), 755–772. https://doi.org/10.1080/17476933.2022.2159955

[12] H. El-Houari, L. S. Chadli and H. Moussa, *A class of non-local elliptic system in non-reflexive fractional Orlicz-Sobolev spaces*, Asian-Eur. J. Math. **16**(07) (2023), Article ID 2350114. https://doi.org/10.1142/S1793557123501140

[13] H. El-Houari, H. Sabiki and H. Moussa, *On topological degree for pseudomonotone operators in fractional Orlicz-Sobolev spaces: study of positive solutions of non-local elliptic problems*, Adv. Oper. Theory **9**(2) (2024), 16 pages. https://doi.org/10.1007/s43036-023-00313-6

[14] H. El-Houari, L. S. Chadli and H. Moussa, *A weak solution to a non-local problem in fractional Orlicz-Sobolev spaces*, Asia Pac. J. Math. **10**(2) (2023), 17 pages. https://doi.org/10.28924/APJM/10-2

[15] H. El-Houari, L. S. Chadli and H. Moussa, *Ground state solutions for a nonlocal system in fractional Orlicz-Sobolev spaces*, Int. J. Differ. Equ. **2022**(1) (2022), Article ID 3849217.

[16] H. El-Houari, H. Moussa, S. Kassimi and H. Sabiki, *Fractional Musielak spaces: a class of non-local problem involving concave-convex nonlinearity*, J. Elliptic Parabol. Equ. (2023), 1–39. https://doi.org/10.1007/s41808-023-00252-6

[17] H. El-Houari, L. S. Chadli and H. Moussa, *Nehari manifold and fibering map approach for fractional $p(\cdot)$-Laplacian Schrödinger system*, SeMA J (2023), 1–23. https://doi.org/10.1007/s40324-023-00343-3

[18] H. El-Houari and E. Arhrrabi, *Fractional ψ-Hilfer derivative spaces: study of Kirchhoff Problem with $p(\cdot)$-Laplacian Operator*, Bull. Transilv. Univ. Brabësov Ser. III. Math. Comput. Sci. (2024) (to appear).

[19] E. H. Hamza, A. Elhoussain and J. V. D. D. C. Sousa, *On a class of Kirchhoff problems with nonlocal terms and logarithmic nonlinearity*, J. Pseudo-Differ. Oper. Appl. **15**(3) (2024), 52 pages. https://link.springer.com/article/10.1007/s11868-024-00624-z

[20] A. Elhoussain, E. H. Hamza and J. V. da C. Sousa, *On a class of capillarity phenomenon with logarithmic nonlinearity involving $\theta(\cdot)$-Laplacian operator*, Comput. Appl. Math. **43**(6) (2024), Article ID 344. https://link.springer.com/article/10.1007/s40314-024-02863-8

[21] H. El-Houari, L. S. Chadli and H. Moussa, *Multiple solutions in fractional Orlicz-Sobolev spaces for a class of nonlocal Kirchhoff systems*, Filomat **38**(8) (2024), 2857–2875. https://doi.org/10.2298/FIL2408857E

[22] H. El-Houari, H. Sabiki and H. Moussa, *Multivalued elliptic inclusion in fractional Orlicz-Sobolev spaces*, Complex Anal. Oper. Theory **18**(4) (2024), 94. https://doi.org/10.1007/s11785-024-01541-1

[23] G. M. Figueiredo, *Existence of positive solutions for a class of p&q elliptic problems with critical growth on* $\mathbb{R}^N$, J. Math. Anal. Appl. **378**(2) (2011), 507–518. https://doi.org/10.1016/j.jmaa.2011.02.017

[24] G. M. Figueiredo, *Existence and multiplicity of solutions for a class of p&q elliptic problems with critical exponent*, Math. Nachr. **286**(11–12) (2013), 1129–1141. https://doi.org/10.1002/mana.201100237

[25] A. Ghanmi, H. Mguagli, S. Turki and N. Zeddini, *Existence of positive bounded solutions for some nonlinear elliptic systems*, J. Math. Anal. Appl. **352**(1) (2009), 440–448. https://doi.org/10.1016/j.jmaa.2008.04.029

[26] A. Ghanmi, *Nontrivial solution for Kirchhoff-type problem involving the* $p(x)$*-Laplace operator*, J. Math. **48**(4) (2018), 1145–1158. https://doi.org/10.1216/RMJ-2018-48-4-1145

[27] A. Ghanmi and S. Horrigue, *Existence of positive solutions for a coupled system of nonlinear fractional differential equations*, Ukr. Math. J. **71**(2) (2019), 39–49. https://doi.org/10.1007/s11253-019-01623-w

[28] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of ground state solutions of elliptic system in fractional Orlicz-Sobolev spaces*, Res. Nonlinear Analysis **5**(2) (2022), 112–130. https://doi.org/10.53006/rna.1021871

[29] H. El-Houari, L. S. Chadli and H. Moussa, *On a class of fractional* $\Gamma(\cdot)$*-Kirchhoff-Schrödinger system type*, Cubo **26**(1) (2024), 53–73. http://dx.doi.org/10.56754/0719-0646.2601.053

[30] C. He and G. Li, *The regularity of weak solutions to nonlinear scalar field elliptic equations containing p&q-Laplacians*, Ann. Acad. Sci. Fenn. Math. **33**(2) (2008), 337–371.

[31] R. Hilfer, *Fractional diffusion based on Riemann-Liouville fractional derivatives*, J. Phy. Chem. B **104**(16) (2000), 3914–3917. https://doi.org/10.1021/jp9936289

[32] P. Le, *Liouville theorems for a p-Laplace equation with Hartree type nonlinearity*, Vietnam J. Math. (2023) 1–14. https://doi.org/10.1007/s10013-021-00508-5

[33] P. L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4**(6) (1980), 1063–1072. https://doi.org/10.1016/0362-546X(80)90016-4

[34] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part* 2, Ann. Inst. H. Poincaré C Anal. Non Linéaire **4**(1) (1984), 223–283. https://doi.org/10.1016/S0294-1449(16)30428-0

[35] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57**(2) (1977), 93–105. https://doi.org/10.1002/sapm197757293

[36] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Relativity Gravitation **28** (1996), 581–600. https://doi.org/10.1007/BF02105068

[37] H. M. Srivastava and J. V. da Costa Sousa, *Multiplicity of solutions for fractional-order differential equations via the* $\kappa(x)$*-Laplacian operator and the Genus theory*, Fract. Fract. **6**(9) (2022), Article ID 481. https://doi.org/10.3390/fractalfract6090481

[38] J. Sousa, K. B. Lima and L. S. Tavares, *Existence of solutions for a singular double phase problem involving a* ψ*-Hilfer fractional operator via Nehari manifold*, Qual. Theory Dyn. Syst. **22**(3) (2023), 1–26. https://doi.org/10.1007/s12346-023-00794-z

[39] J. Sousa, C. Vanterler da, J. Zuo and D. O'Regan, *The Nehari manifold for a* ψ*-Hilfer fractional p-Laplacian*, Applicable Anal. **101**(14) (2022), 5076–5106. https://doi.org/10.1080/00036811.2021.1880569

[40] C. E. T. Ledesma and J. V. D. C. Sousa, *Fractional integration by parts and Sobolev type inequalities for ψ-fractional operators*, Math. Meth. Appl. Sci. **45**(16) (2022), 9945–9966. https://doi.org/10.1002/mma.8348

[41] J. V. D. C. Sousa, D. S. Oliveira and S. L. Tavares, *Solutions of the mean curvature equation with the Nehari manifold*, Comput. Appl. Math. **43**(1) (2024), 24 pages. https://doi.org/10.1007/s40314-023-02534-0

[42] J. V. D. C. Sousa, K. D. Kucche and J. J. Nieto, *Existence and multiplicity of solutions for ractional $\kappa(\xi)$-Kirchhoff-type equation*, Qual. Theory Dyn. Sys. **23**(1) (2024), 27 pages. https://doi.org/10.1007/s12346-023-00877-x

[43] J. V. D. C. Sousa, D. S. Oliveira and R. P. Agarwal, *Existence and multiplicity for fractional Dirichlet problem with $\gamma(\xi)$-Laplacian equation and Nehari manifold*, Appl. Anal. Disc. Math. **17**(2) (2023), 480–495. https://www.jstor.org/stable/27281422

[44] J. Sousa, C. Vanterler da, M. A. P. Pulido and E. C. D. Oliveira, *Existence and regularity of weak solutions for ψ-Hilfer fractional boundary value problem*, Mediter. J. Math. **18**(4) (2021), Article ID 147. https://doi.org/10.1007/s00009-021-01789-3

[45] V. E. Tarasov and E. C. Aifantis, *On fractional and fractal formulations of gradient linear and nonlinear elasticity*, Acta Mech. **230** (2019), 2043–2070. https://doi.org/10.1007/s00707-019-2373-x

[46] W. Wyss, *The fractional diffusion equation*, J. Math. Phy. **27**(11) (1986), 2782–2785.

[1]ENSI: School of New Sciences and Engineering, Tangier, Morocco.

[2]AMNEA Group, Department of Mathematics,
Faculty of Sciences and Techniques Errachidia,
Moulay Ismail University of Meknés,
Boutalamine, PO. BOX 509, 52000 Errachidia, Morocco

*Corresponding Author
Email address: arhrrabi.elhoussain@gmail.com
ORCID iD: https://orcid.org/0000-0002-9239-4603
Email address: h.elhouari94@gmail.com
ORCID iD: https://orcid.org/0000-0003-4240-1286

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1105–1117.

ON THE JACOBSON SEMISIMPLE SEMIRINGS

A. K. BHUNIYA AND PUJA SARKAR

Abstract. Based on the minimal and simple representations, we introduce two types of Jacobson semisimplicity, m-semisimplicity and s-semisimplicity, of a semiring S. Every $m(s)$-semisimple semiring is a subdirect product of $m(s)$-primitive semirings. It is shown that a commutative s-primitive semiring is either a two element Boolean algebra or a field. Every s-primitive semiring is isomorphic to a 1-fold transitive subsemiring of the semiring of all endomorphisms of a semimodule over a division semiring.

1. Introduction

A semiring is an algebraic structure satisfying all the axioms of a ring, but one that every element has an additive inverse. The absence of additive inverses forces a semiring to deviate radically from behaving like a ring. For example, ideals are not in bijection with the congruences on a semiring. Further, the presence of additively idempotent semirings makes the class of the semirings abundant. Now semirings have become a part of mainstream mathematics for their importance in theoretical computer science [30], graph theory [16] and automata theory [9, 11, 13]; and for the surprising 'characteristic one analogy' of the usual algebra over fields [10, 12, 25, 26]; and for the role of additively idempotent semirings in tropical mathematics [1, 7, 14, 19, 20, 27].

In a recent paper [24], Katsov and Nam considered two Jacobson type semisimple semirings - J-semisimple semirings and J_s-semisimple semirings. They characterized J-semisimple semirings within the class of all additively cancellative semirings and additively idempotent J_s-semisimple semirings. Besides characterizing the semirings S such that $J(S) = J_s(S)$, a problem stated in [24], Mai and Tuyen [28] extended the

Key words and phrases. Semiring, Jacobson semisimple, faithful, primitive, transitive semiring
2020 *Mathematics Subject Classification.* Primary: 16Y60. Secondary: 16N99, 16D99.
https://doi.org/10.46793/KgJMat2607.1105B
Received: March 07, 2024.
Accepted: September 27, 2024.

study of the J_s-semisimple semirings to the class of all zerosumfree semirings. They proved that every J_s-semisimple zerosumfree commutative semiring is semiisomorphic to a subdirect product of its maximal entire quotients.

This article is a continuation of [6] under the project to develop a radical theory of semirings that can be used to study semiring in general. Based on the minimal and simple representations of a semiring, the present authors introduced and studied two Jacobson type radicals - Jacobson m-radical and s-radical of a semiring in [6]. Here, we define and study two types of Jacobson semisimple semirings - Jacobson m-semisimple and s-semisimple semirings.

Since ideals are not in bijection with the congruences on a semiring, it is not surprising that replacing ideals with the more general notion of congruences on semirings exhibits many excellent properties and several analogies with classical results on the rings [3, 22, 23]. In our approach, the annihilator $ann_S(M)$ of an S-semimodule M is considered as a congruence on S. Similarly to the semirings, subsemimodules are not in bijection with the congruences on a semimodule; which produces three variants of 'irreducibility' of semimodules – minimal semimodules, elementary semimodules, and simple semimodules [8, 21]. In [6], the authors of the present article introduced and characterized two Jacobson type Hoehnke radicals, namely, m-radical and s-radical of a semiring S as two congruences on S. Here we introduce m-semisimple, s-semisimple, m-primitive and s-primitive semirings in an obvious way. Considering radical as a congruence makes it easy to represent these semisimple semirings as a subdirect product of suitable class of primitive semirings. The two element Boolean algebra and the fields are the only commutative s-primitive semirings. Hence the study of commutative s-primitive semirings characterizes the subdirect products of the copies of the two element Boolean algebra and fields.

This paper is organized as follows. Section 2 briefly recaps the necessary definitions and associated facts on semirings and semimodules. Section 3 introduces m-primitive and s-primitive semirings and characterizes Jacobson semisimple semirings as subdirect products of primitive semirings. Every commutative $(s)m$-primitive semiring is a (congruence simple) semifield. Since every congruence simple semifield S with $|S| > 2$ is a field, every commutative s-semisimple semiring is a subdirect product of a family of semirings, each of which is either the 2-element Boolean algebra or a field. Finally, every s-primitive semiring is represented as a 1-fold transitive subsemiring of the semiring of all endomorphisms of a semimodule over a division semiring.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is a nonempty set S with two binary operations '+' and '·' satisfying:

- $(S, +)$ is a commutative monoid with identity element 0;
- $(S, \cdot)$ is a semigroup;
- $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

Moreover, we assume that the additive identity element 0 is absorbing, i.e., $0s = s0 = 0$ for all $s \in S$. There is no consensus whether every semiring contains a multiplicative unity 1. In this article, we follow the convention of Hebisch and Weinert [17], that a semiring does not contain 1, in general. Also, there are many articles on semirings where the existence of unity is not assumed [4, 5, 18, 21, 31, 32]. On the other hand, in Golan [15], it is assumed that every semiring contains 1. If a semiring S with multiplicative identity is such that every nonzero element has a multiplicative inverse, then S is called a *division semiring*. A commutative division semiring is called a *semifield*. A semiring S is said to be an *additively idempotent semiring* if $a + a = a$ for all $a \in S$. Both the two element Boolean algebra $\mathbb{B}$ and the max-plus algebra $\mathbb{R}_{max}$ are additively idempotent semifields.

Definition of the ideals, congruences and homomorphisms of semirings are as usual. Here it is assumed that every semiring homomorphism $\phi : S_1 \to S_2$ satisfies $\phi(0_1) = 0_2$. The kernel of a semiring homomorphism $\phi : S_1 \to S_2$ is defined by $\ker \phi = \{(a, b) \in S_1 \times S_2 \mid \phi(a) = \phi(b)\}$. Then $\ker \phi$ is a congruence on S_1 and $S_1/\ker \phi \simeq \phi(S_1)$.

A semiring S is said to be *congruence-simple* if it has no congruences other than the equality congruence $\Delta_S = \{(s, s) \mid s \in S\}$ and the universal congruence $\nabla_S = S \times S$.

Let I be a (left, right) ideal of a semiring S and μ be a (left, right) congruence relation on S. Then, I is said to be a *μ-saturated (left, right) ideal* of S if for every $s \in S$ and $i \in I$, $(s, i) \in \mu$ implies that $s \in I$. Thus, an ideal I is μ-saturated if and only if $I = \cup_{a \in I}[a]_\mu$.

A *right S-semimodule* is a commutative monoid $(M, +, 0_M)$ equipped with a right action $M \times S \to M$ that satisfies for all $m, m_1, m_2 \in M$ and $s, s_1, s_2 \in S$:

- $(m_1 + m_2)s = m_1 s + m_2 s$;
- $m(s_1 + s_2) = ms_1 + ms_2$;
- $m(s_1 s_2) = (ms_1)s_2$;
- $m0 = 0_M = 0_M s$.

Unless stated otherwise, by an *S-semimodule M*, we mean a right S-semimodule.

The *annihilator* of an S-semimodule M is defined by

$$ann_S(M) = \{(s_1, s_2) \in S \times S \mid ms_1 = ms_2 \text{ for all } m \in M\}.$$

Then $ann_S(M)$ is a congruence on S. If $\psi : S \to End_S(M)$ is the representation of S induced by the right action of S on the semimodule M, then $\ker \psi = ann_S(M)$.

If M is a right S-semimodule then for every congruence ρ on S with $\rho \subseteq ann_S(M)$, the scalar multiplication $m[s]_\rho = ms$ makes M an S/ρ-semimodule.

The following result can be proved easily, and so we omit the proof.

Lemma 2.1. *Let S be a semiring and ρ be a congruence on S.*

(a) *If M is an S/ρ-semimodule, then M becomes an S-semimodule under the scalar multiplication $ms = m[s]$. Moreover, $\rho \subseteq ann_S(M)$.*

(b) *Let M be an S-semimodule and $\rho \subseteq ann_S(M)$. Then, $ann_{S/\rho}(M) = ann_S(M)/\rho$.*

An S-semimodule M is said to be *faithful* if $ann_S(M) = \Delta_S$.

Following Chen et al. [8] we define the following.

Definition 2.1. Let M be an S-semimodule such that $MS \neq 0$. Then M is called

(i) minimal if M has no subsemimodules other than (0) and M;
(ii) simple if it is minimal and the only congruences on M are Δ_M and ∇_M where Δ_M is the equality relation on M and $\nabla_M = M \times M$.

In [21], simple semimodules have been termed as irreducible semimodules. We denote the class of all minimal and simple S-semimodules by $\mathcal{M}(S)$ and $\mathcal{S}(S)$, respectively. If R is a ring then $\mathcal{M}(S) = \mathcal{S}(S)$ [6].

In this paper, we will have many occasions to use the following characterization [6, 21] of the minimal semimodules.

Lemma 2.2. *A nonzero S-semimodule M is minimal if and only if $M = mS$ for all $m(\neq 0) \in M$.*

The classes $\mathcal{M}(S)$ and $\mathcal{S}(S)$ of minimal and simple representations of a semiring S induces the following two notions of Jacobson type radicals of a semiring [6].

Definition 2.2. Let S be a semiring. We define

(a) m-radical of S by $rad_m(S) = \cap_{M\in\mathcal{M}(S)} ann_S(M)$;
(b) s-radical of S by $rad_s(S) = \cap_{M\in\mathcal{S}(S)} ann_S(M)$.

If there are no minimal semimodules over S, then we define $rad_m(S) = \nabla_S$. Similarly, we define $rad_s(S) = \nabla_S$ if there is no simple S-semimodules.

Both the assignments $S \mapsto rad_m(S)$ and $S \mapsto rad_s(S)$ are Hoehnke radicals on S [6].

A right congruence μ on S is said to be a *regular right congruence* if there exists $e \in S$ such that $(es, s) \in \mu$ for every $s \in S$. If μ is a regular right congruence on S, then $M = S/\mu$ is a right S-semimodule such that $MS \neq 0$.

The subsequent two results characterize the regular congruences μ on S such that the quotient semimodule S/μ is a minimal or a simple semimodule over S.

Lemma 2.3 ([6]). *Let M be an S-semimodule. Then, M is minimal if and only if there exists a regular right congruence μ on S such that $S/\mu \simeq M$ and $[0]_\mu$ is a maximal μ-saturated right ideal in S.*

Every simple semimodule is congruence-simple. Therefore, for every right congruence μ on S, if S/μ is simple, then μ is maximal. Hence, the following result follows.

Lemma 2.4 ([6]). *Let M be a S-semimodule. Then M is simple if and only if there exists a maximal regular right congruence μ on S such that $S/\mu \simeq M$ and $[0]_\mu$ is a maximal μ-saturated right ideal in S.*

A regular right congruence μ on S is said to be *m-regular* if $[0]_\mu$ is a maximal μ-saturated right ideal in S; and *s-regular* if it is a maximal regular right congruence such that $[0]_\mu$ is a maximal μ-saturated right ideal in S. We denote the set of all m-regular right congruences on S by $\mathcal{RC}_m(S)$ and the set of all s-regular right congruences on S by $\mathcal{RC}_s(S)$.

The following internal characterization of the m-radical and the s-radical of a semiring was proved in [6].

Lemma 2.5 ([6])**.** *Let S be a semiring. Then,*

(i) $rad_m(S) = \cap_{\mu\in\mathcal{RC}_m(S)}\mu$;
(ii) $rad_s(S) = \cap_{\mu\in\mathcal{RC}_s(S)}\mu$.

The following result has useful applications.

Lemma 2.6 ([6])**.** *Let R and S be two semirings. Then, $rad_m(R\times S) = rad_m(R)\times rad_m(S)$ and $rad_s(R\times S) = rad_s(R)\times rad_s(S)$.*

The reader is referred to [6] for more details on the m-radical and the s-radical of a semiring and to [17] for the undefined terms and notions concerning semirings and semimodules over semirings.

3. Jacobson Semisimple Semirings

In this section, we introduce and study the Jacobson m-semisimple and s-semisimple semirings.

Definition 3.1. A semiring S is said to be *m-semisimple* if $rad_m(S) = \Delta_S$; and *s-semisimple* if $rad_s(S) = \Delta_S$.

Since $\mathcal{RC}_s(S) \subseteq \mathcal{RC}_m(S)$, it follows that $rad_m(S) \subseteq rad_s(S)$. Therefore, every s-semisimple semiring is m-semisimple. The following example shows that the converse does not hold in general.

Example 3.1. Let $\mathbb{H}$ be the ring of all real quaternions. Since $\mathbb{H}$ is a division ring, $rad_m(\mathbb{H}) = rad_s(\mathbb{H}) = \Delta_{\mathbb{H}}$. Also, $rad_m(\mathbb{R}_{max}) = \Delta_{\mathbb{R}_{max}}$ and $\rho = \{(-\infty,-\infty)\}\cup\{(r,s)\mid r,s\in\mathbb{R}\}$ is the only s-regular congruence on $\mathbb{R}_{max}$ implies that $rad_s(\mathbb{R}_{max}) = \rho$ [6, Example 3.9]. Consider the semiring $S = \mathbb{H}\times\mathbb{R}_{max}$. Then, by the Lemma 2.6, we have

$$rad_m(S) = rad_m(\mathbb{H})\times rad_m(\mathbb{R}_{max}) = \Delta_{\mathbb{H}}\times\Delta_{\mathbb{R}_{max}} = \Delta_S,$$
$$rad_s(S) = rad_s(\mathbb{H})\times rad_s(\mathbb{R}_{max}) = \Delta_{\mathbb{H}}\times\rho.$$

Therefore, S is m-semisimple but not s-semisimple.

Now we give an example of an s-semisimple semiring.

Example 3.2. Both the ring $\mathbb{H}$ of all real quaternions and the semiring $\mathbb{N}^0$ of all nonnegative integers are s-semisimple. Hence, it follows from the Lemma 2.6 that the semiring $S = \mathbb{H}\times\mathbb{N}^0$ is both s-semisimple and m-semisimple.

If S is a Jacobson m-semisimple semiring, then $\cap_{M\in\mathcal{M}(S)}ann_S(M) = \Delta_S$ implies that every m-semisimple semiring is a subdirect product of the family of semirings $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$. Also, for every $M \in \mathcal{M}(S)$, by Lemma 2.1, implies that M is a minimal and faithful $S/ann_S(M)$-semimodule. Similarly, every s-semisimple semiring S is a subdirect product of the family of semirings $\{S/ann_S(M) \mid M \in \mathcal{S}(S)\}$ where each quotient semiring $S/ann_S(M)$ has a faithful and simple semimodule M. Intending to characterize the structure of semisimple semirings, we introduce the following two notions.

Definition 3.2. Let S be a semiring. Then, S is called

(i) m-primitive if there is a faithful minimal S-semimodule M;
(ii) s-primitive if there is a faithful simple S-semimodule M.

If S is an m-primitive semiring, then there is a minimal S-semimodule M such that $ann_S(M) = \Delta_S$. Hence $rad_m(S) = \cap_{M\in\mathcal{M}(S)}ann_S(M) = \Delta_S$ and so S is m-semisimple. Similarly, every s-primitive semiring is s-semisimple.

A congruence σ on S is said to be an *m-primitive* (*s-primitive*) *congruence* if the quotient semiring S/σ is an m-primitive (s-primitive) semiring. Thus, σ is m-primitive(s-primitive) if and only if there exists a faithful minimal (simple) S/σ-semimodule M.

If ρ is a right congruence on S, we define

$$(\rho : \nabla_S) = \{(x, y) \in \nabla_S \mid (sx, sy) \in \rho \text{ for all } s \in S\}.$$

Then, $(\rho : \nabla_S)$ is a congruence on S.

Lemma 3.1. *Let σ be a congruence on S. Then, the following conditions are equivalent:*

(a) *σ is m-primitive (s-primitive);*
(b) *$\sigma = ann_S(M)$ for some minimal (simple) S-semimodule M;*
(c) *$\sigma = (\rho : \nabla_S)$ for some $\rho \in \mathcal{RC}_m(S)$ ($\rho \in \mathcal{RC}_s(S)$).*

Proof. We prove the result for m-primitive congruences. The other cases are similar.

(a) ⇒ (b) Let σ be an m-primitive congruence on S. Then there exists a faithful minimal S/σ-semimodule M. Hence, by Lemma 2.1, M is also a minimal S-semimodule such that $\sigma \subseteq ann_S(M)$ and $\Delta_{S/\sigma} = ann_{S/\sigma}(M) = ann_S(M)/\sigma$, i.e., $\sigma = ann_S(M)$.

(b) ⇒ (c) Let M be a minimal S-semimodule and $\sigma = ann_S(M)$. Then M is a minimal and faithful right S/σ-semimodule. Theorem 2.3 implies that there exists $\rho \in \mathcal{RC}_m(S)$ such that $M \simeq S/\rho$; and hence $ann_S(M) = (\rho : \nabla_S)$. Then $ann_{(S/\sigma)}(M) = ann_S(M)/\sigma = \Delta_{(S/\sigma)}$ implies that $ann_S(M) = \sigma$. Hence, $(\rho : \nabla_S) = ann_S(M) = \sigma$.

(c) ⇒ (a) Let $\rho \in \mathcal{RC}_m(S)$ and $\sigma = (\rho : \nabla_S)$. Then S/ρ is a minimal right S-semimodule and $ann_S(S/\rho) = (\rho : \nabla_S) = \sigma$. Since σ is a semiring congruence on S and $\sigma = (\rho : \nabla_S)$, it follows that S/ρ is a minimal right S/σ-semimodule. Also, by Lemma 2.1, we have $ann_{S/\sigma}(S/\rho) = ann_S(S/\rho)/\sigma = \Delta_{S/\sigma}$. Hence, S/ρ is a minimal and faithful right S/σ-semimodule, so σ is a m-primitive congruence on S. □

From the definition, it follows that a semiring S is an m-primitive (s-primitive) semiring if and only if Δ_S is an m-primitive (s-primitive) congruence on S. Thus, we have the following.

Corollary 3.1. *Let S be a semiring. Then, S is*

(a) *m-primitive if and only if there exists $\rho \in \mathcal{RC}_m(S)$ such that $(\rho : \nabla_S) = \Delta_S$;*
(b) *s-primitive if and only if there exists $\rho \in \mathcal{RC}_s(S)$ such that $(\rho : \nabla_S) = \Delta_S$.*

A division semiring is a noncommutative generalization of a semifield. The following result shows that primitive semirings are another class of noncommutative generalizations of semifields. The m-primitive semirings generalize the semifields, whereas the s-primitive semirings generalize the congruence-simple semifields.

Theorem 3.1. *Let S be a commutative semiring. Then, S is*

(a) *m-primitive if and only if it is a semifield;*
(b) *s-primitive if and only if it is a congruence-simple semifield.*

Proof. (a) Let S be a commutative m-primitive semiring. Then, by Corollary 3.1, there is a regular right congruence ρ in $\mathcal{RC}_m(S)$ such that $(\rho : \nabla_S) = \Delta_S$. Since S is a commutative semiring, ρ becomes a congruence on S and so $\rho = (\rho : \nabla_S) = \Delta_S$. Therefore $\Delta_S \in \mathcal{RC}_m(S)$ and there is an element $e \in S$ such that $es = s = se$ for all $s \in S$. Thus, e is a multiplicative identity in S. Also $\rho = \Delta_S \in \mathcal{RC}_m(S)$ implies that (0) is maximal Δ_S-saturated ideal in S. Since every ideal in S is Δ_S-saturated, it follows that (0) and S are the only two ideals in S. Now for each non-zero element $a \in S$, aS is a non-zero ideal in S. Hence, $aS = S$, which implies that there exists an element $b \in S$ such that $ab = e = ba$. Thus, S is a semifield.

Conversely, let S be a semifield. Then $M = S$ is a minimal S-semimodule and $ann_S(M) = \{(s_1, s_2) \in S \times S \mid ss_1 = ss_2 \text{ for all } s \in S\} = \Delta_S$. Therefore, S is m-primitive.

(b) Let S be a commutative s-primitive semiring. Then S is m-primitive, and so, by (a), it is a semifield. Also, by Corollary 3.1, there exists a right congruence $\rho \in \mathcal{RC}_s(S)$ such that $(\rho : \nabla_S) = \Delta_S$. Since S is commutative, it follows that $(\rho : \nabla_S) = \rho$. Hence $\Delta_S = \rho \in \mathcal{RC}_s(S)$ which implies that $M = S/\rho \simeq S$ is a simple S-semimodule. Hence, the semifield S is congruence-simple.

Conversely, if S is a congruence-simple semifield, then S itself is a faithful simple S-semimodule. Hence, S is s-primitive. □

Theorem 3.1 tells us that the congruence-simple semifields constitute an important subclass of the semifields. Similarly to the fields, the Krull-dimension of a congruence-simple semifield is 0, whereas there are semifields, say, for example, $\mathbb{R}_{max}$ having the Krull-dimension 1 [22]. A semiring S is called *zerosumfree* if for every $a, b \in S$, we have $a+b = 0$ implies that $a = 0$ and $b = 0$. It is well known that a semifield S is either zerosumfree or is a field [15, Proposition 4.34]. Every field is a congruence-simple semifield. If S is zerosumfree, then $\rho = \{(s,t) \in S \times S \mid s \neq 0 \neq t\} \cup \{(0,0)\}$ is a congruence on S. So for S to be congruence-simple, we must have $|S| = 2$. Then S is the 2-element Boolean algebra $\mathbb{B}$. Thus, a congruence-simple semifield is either the 2-element Boolean algebra $\mathbb{B}$ or a field.

However, in the following, we include an independent proof.

Theorem 3.2. *Let S be a semiring with $|S| > 2$. Then, S is a congruence-simple semifield if and only if it is a field.*

Proof. First, assume that S is a congruence-simple semifield. Denote $Z(S) = \{x \in S \mid x + y = 0$ for some $y \in S\}$. Then, $Z(S)$ is an ideal of S; and so $Z(S)$ is either $\{0\}$ or S. If $Z(S) = \{0\}$, then S is zerosumfree. So, $\{(0,0)\} \cup \{(s,t) \in S \times S \mid s \neq 0 \neq t\}$ induces a nontrivial congruence on S, which contradicts that S is congruence-simple. Hence, $Z(S) = S$ which implies that $(S, +)$ is a group. Thus, S is a field.

The converse follows trivially. □

Thus a zerosumfree semifield S with $|S| > 2$ cannot be congruence-simple. So, in particular, we have the following example.

Example 3.3. The max-plus algebra $\mathbb{R}_{max}$ is a semifield but not congruence-simple. Hence, $\mathbb{R}_{max}$ is an m-primitive semiring but not s-primitive.

The 2-element Boolean algebra $\mathbb{B}$ and the field $\mathbb{Z}_2$ of all integers modulo 2 are the only semifields of order two up to isomorphism. Hence, it turns out to be the following specific characterization of the commutative s-primitive semirings.

Corollary 3.2. *A commutative semiring S is s-primitive if and only if it is either the 2-element Boolean algebra $\mathbb{B}$ or a field.*

A semiring S is called a *subdirect product* of a family $\{S_\alpha\}_\Delta$ of semirings if there is an one-to-one semiring homomorphism $\phi : S \to \prod_\Delta S_\alpha$ such that for each $\alpha \in \Delta$, the composition $\pi_\alpha \circ \phi : S \to S_\alpha$ is onto where $\pi_\alpha : \prod_\Delta S_\alpha \to S_\alpha$ is the projection mapping.

It is well known that a semiring S is a subdirect product of a family $\{S_\alpha\}_\Delta$ of semirings if and only if there is a family $\{\rho_\alpha\}_\Delta$ of congruences on S such that $S/\rho_\alpha \simeq S_\alpha$ for every $\alpha \in \Delta$ and $\cap_\Delta \rho_\alpha = \Delta_S$.

Theorem 3.3. *A semiring S is m-semisimple (s-semisimple) if and only if it is a subdirect product of m-primitive (s-primitive) semirings.*

Proof. We prove the result for m-semisimple semirings. The proof for s-semisimple semirings is similar.

First, assume that S is a m-semisimple semiring. Then $rad_m(S) = \cap_{M \in \mathcal{M}(S)} ann_S(M) = \Delta_S$. Hence, S is a subdirect product of the family $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$ of semirings. Lemma 3.1 implies that $ann_S(M)$ is an m-primitive congruence on S for every minimal S-semimodule M. Therefore, every semiring in the family $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$ is an m-primitive semiring, and so S is a subdirect product of m-primitive semirings.

Conversely, let S be a subdirect product of a family of m-primitive semirings $\{S_i \mid \text{for all } i \in \Lambda\}$. Then there exists a one-to-one homomorphism $\phi : S \to \Pi_{i\in\Lambda} S_i$ such that the mapping $\pi_i \circ \phi : S \to S_i$ is onto for all $i \in \Lambda$. Thus $S/\ker(\pi_i \circ \phi) \cong S_i$ for all $i \in \Lambda$. Let M_i be a faithful minimal S_i-semimodule for each $i \in \Lambda$. Then, by the Lemma 2.1, M_i is a minimal S-semimodule where $ms = m\pi_i \circ \phi(s)$ for all $s \in S$ and $m \in M_i$. Hence, $\cap_{M\in\mathcal{M}(S)} ann_S(M) \subseteq \cap_{i\in\Lambda} ann_S(M_i)$. Now $(a, b) \in ann_S(M_i)$ implies that $m\pi_i \circ \phi(a) = m\pi_i \circ \phi(b)$ for all $m \in M_i$; and so $(\pi_i \circ \phi(a), \pi_i \circ \phi(b)) \in ann_{S_i}(M)$. Since M_i is faithful over S_i, it follows that $\pi_i \circ \phi(a) = \pi_i \circ \phi(b)$. Hence, $\cap_{i\in\Lambda} ann_S(M_i) = \Delta_S$ which implies that $rad_m(S) = \cap_{M\in\mathcal{M}(S)} ann_S(M) = \Delta_S$. Thus, S is a m-semisimple semiring. $\square$

Now, taken together the structure of an s-semisimple semiring characterized in Theorem 3.3 and the characterization of the commutative s-primitive semirings in Corollary 3.2 turn out to be an characterization of the commutative s-semisimple semirings.

Corollary 3.3. *Let S be a commutative semiring. Then, S is an s-semisimple semiring if and only if it is a subdirect product of a family of semirings that are either the 2-element Boolean algebra $\mathbb{B}$ or fields.*

Mischell and Fenoglio [29] and Basir et al. [2] independently proved that a commutative semiring S with $|S| \geqslant 2$ is congruence-simple if and only if it is either a field or the 2-element Boolean algebra $\mathbb{B}$. Hence, it follows that a commutative semiring is s-semisimple if and only if it is a subdirect product of congruence-simple commutative semirings. A semiring homomorphism $f : S_1 \to S_2$ is said to be *semi-isomorphism* if, for every $a \in S_1$, we have $f(a) = 0$ only for $a = 0$. Katsov and Nam [24] proved that a commutative semiring S is Brown-McCoy semisimple if and only if S is semi-isomorphic to a subdirect product of a family of semirings that are either the 2-element Boolean algebra $\mathbb{B}$ or fields. Hence, every commutative s-semisimple semiring is Brown-McCoy semisimple in the sense of Katsov and Nam.

Example 3.4. Consider the semiring $\mathbb{N}$ of all nonnegative integers. Then, for every prime p, the Bourne congruence $\sigma_{p\mathbb{N}}$ is a maximal regular congruence on $\mathbb{N}$ with $[0]_{\sigma_{p\mathbb{N}}} = p\mathbb{N}$. If J is a $\sigma_{p\mathbb{N}}$-saturated ideal in $\mathbb{N}$ with $p\mathbb{N} \subsetneq J$, then there exists $a \in J$ such that $0 < a < p$. By the Fermat's little theorem, we have $a^{p-1} \equiv 1 \pmod p$ which implies that $1 \in J$ and so $J = \mathbb{N}$. Thus, $p\mathbb{N} = [0]_{\sigma_{p\mathbb{N}}}$ is a maximal $\sigma_{p\mathbb{N}}$-saturated ideal in $\mathbb{N}$ and it follows that $\sigma_{p\mathbb{N}} \in \mathcal{RC}_s(\mathbb{N})$. Hence $rad_s(\mathbb{N}) \subseteq \cap\sigma_{p\mathbb{N}} = \Delta_{\mathbb{N}}$; and so $\mathbb{N}$ is an s-semisimple semiring.

Also, $\cap\sigma_{p\mathbb{N}} = \Delta_{\mathbb{N}}$ implies that $\mathbb{N}$ is a subdirect product of the family of fields $\mathbb{N}_p = \mathbb{N}/\sigma_{p\mathbb{N}}$, where p is a prime.

We conclude this section with a representation of s-primitive semirings as a semiring of endomorphisms on a semimodule over a division semiring.

The opposite semiring S^{op} of a semiring $(S, +, \cdot)$ is defined by $(S, +, \circ)$, where $a \circ b = b \cdot a$ for all $a, b \in S$. Hence, a semiring S is a division semiring if and only if the opposite semiring S^{op} is so.

Definition 3.3. Let M be a semimodule over a division semiring D. Then a subsemiring T of the endomorphism semiring $End_D(M)$ is called 1-fold transitive if for every non-zero $m \in M$ and $n \in M$ there exists $\alpha \in T$ such that $\alpha(m) = n$.

In the context of semirings, Schur's lemma [21] states that if M is a simple S-semimodule, then the endomorphism semiring $End_S(M)$ is a division semiring.

Let M be a right S-semimodule and $E = End_S(M)$. Then for the division semiring $D = E^{op}$, M is a right semimodule over D where the scalar multiplication is defined by $m \cdot \alpha = \alpha(m)$ for all $m \in M$ and $\alpha \in D$.

Theorem 3.4. *If S is a right s-primitive semiring, then S^{op} is isomorphic to a 1-fold transitive subsemiring of the semiring $End_D(M)$ of all endomorphisms on a semimodule M over a division semiring D.*

Proof. Let M be a faithful simple right S-semimodule. By Schur's Lemma for semimodules [21], the semiring $E = End_S(M)$ is a division semiring. Hence, $D = E^{op}$ is a division semiring, and so M as a right D-semimodule where $m \cdot \alpha \mapsto \alpha(m)$.

For every $a \in S$, define a mapping $\psi_a : M \to M$ by $\psi_a(m) = ma$. Then for every $\alpha \in D$, we have $\psi_a(m.\alpha) = \psi_a(\alpha(m)) = \alpha(m)a = \alpha(ma) = (ma) \cdot \alpha = \psi_a(m) \cdot \alpha$. In fact, ψ_a is an endomorphism on M considered a D-semimodule.

Also, the mapping $\psi : S^{op} \to End_D(M)$ defined by $\psi(a) = \psi_a$ is a semiring homomorphism. Moreover $\ker \psi = ann_S(M) = \Delta_S$ implies that ψ is an injective homomorphism; and so S^{op} is isomorphic to the subsemiring $T = \{\psi_a \mid a \in S\}$ of $End_D(M)$.

Since M is a simple right S-semimodule, by Lemma 2.2, for every $m(\neq 0) \in M$, $mS = M$. Then for every $n \in M$ there exists $a \in S$ such that $ma = n$ and so $\psi_a(m) = n$. Thus, T is a 1-fold transitive subsemiring of $End_D(M)$. $\square$

It follows from Corollary 3.2 that the semifield $F = \mathbb{R}_{max}$ is not an s-primitive semiring. Since F contains 1, every F-endomorphism on F is of the form $\psi_a : F \to F$ given by $\psi_a(m) = am$. Hence, $F \simeq End_F(F)$ which implies that $End_F(F)$ is not s-primitive; whereas $End_F(F)$ is a 1-fold transitive subsemiring of itself. Thus, the converse of the Theorem 3.4 does not hold. However, the converse holds in the following weaker form.

Theorem 3.5. *Let D be a division semiring and M be a right D-semimodule. If T is a 1-fold transitive subsemiring of $End_D(M)$, then T^{op} is a right m-primitive semiring.*

Proof. Define $M \times T^{op} \to M$ by $m \cdot \alpha \mapsto \alpha(m)$. Then M is a right T^{op}-semimodule. Let m be a non-zero element in M. Then, for every $n \in M$, there exists $\alpha \in T$ such that $m \cdot \alpha = n$. Therefore, $mT^{op} = M$ which implies that M is minimal, by

Lemma 2.2. Now

$$\begin{aligned} ann_{T^{op}}(M) &= \{(\alpha, \beta) \in T \times T \mid m \cdot \alpha = m \cdot \beta \text{ for all } m \in M\} \\ &= \{(\alpha, \beta) \in T \times T \mid \alpha(m) = \beta(m) \text{ for all } m \in M\} \\ &= \{(\alpha, \beta) \in T \times T \mid \alpha = \beta\} \\ &= \Delta_S \end{aligned}$$

and so M is a faithful minimal T^{op}-semimodule. Therefore, T^{op} is a m-primitive semiring. □

4. Conclusion

In [6], based on the notions of minimal semimodule and simple semimodule, the Jacobson m-radical and s-radical of a semiring S have been considered as a congruence on S. In Section 3 of this article, we introduce the m-semisimple and s-semisimple semirings as the semiring that has the trivial Jacobson m-radical and s-radical, respectively. These two notions of semisimplicity effectively characterize the structure of semirings, including the additively idempotent semirings. The m-semisimple (s-semisimple) are isomorphic to a subdirect product of m-primitive (s-primitive) semirings. In particular, a commutative semiring is s-primitive if and only if it is a subdirect product of the fields and copies of the two element Boolean algebra. Finally, every s-primitive semiring is represented as a suitable subsemiring of the semiring $End_D(M)$ of all endomorphisms on a semimodule M over a division semiring D.

There is another notion of simplicity of semimodules, namely the congruence simple semimodules which are known as elementary semimodules [8]. An attempt may be taken to characterize the e-semisimple semirings which are defined based on the class of elementary semimodules.

References

[1] M. Akian, R. Bapat and S. Gaubert, *Max-plus algebra*, in: L. Hogben, R. Brualdi, A. Greenbaum, R. Mathias (Eds.), *Handbook of Linear Algebra*, Chapman & Hall, London, 2006.

[2] R. El Basir, J Hurt, A. Jančařik and T. Kepka, *Simple commutative semirings*, J. Algebra **236** (2001), 277–306. `https://doi.org/10.1006/jabr.2000.8483`

[3] A. Bertram and R. Easton, *The tropical Nullstellensatz for congruences*, Adv. Math. **308** (2017), 36–82. `https://doi.org/10.1016/j.aim.2016.12.004`

[4] A. K. Bhuniya and T. Mondal, *Distributive lattice decompositions of semirings with a semilattice additive reduct*, Semigroup Forum **80** (2010), 293–301. `https://doi.org/10.1007/s00233-009-9205-6`

[5] A. K. Bhuniya and T. Mondal, *On the least distributive lattice congruence on a semiring with a semilattice additive reduct*, Acta Math. Hungar. **147** (2015), 189–204. `https://doi.org/10.1007/s10474-015-0526-5`

[6] A. K. Bhuniya and P. Sarkar, *On the Jacobson radical of a semiring*, J. Algebra Appl. `https://doi.org/10.1142/S0219498825503694`

[7] D. Castella, *Eléments d'algèbre linéaire tropicale*, Linear Algebra Appl. **432** (2010), 1460–1474. `https://doi.org/10.1016/j.laa.2009.11.005`

[8] C. Chen et al., *Extreme representations of semirings*, Serdica Math. J. **44**(3/4) (2018), 365–412. https://doi.org/10.48550/arXiv.1806.06501
[9] J. H. Conway, *Regular Algebra and Finite Machines*, Chapman & Hall, London, 1971.
[10] A. Connes and C. Consani, *Schemes over* $\mathbf{F_1}$ *and zeta functions*, Compos. Math. **146**(6) (2010), 1383–1415. https://doi.org/10.1112/S0010437X09004692
[11] N. Damljanović, M. Ćirić and J. Ignjatović, *Bisimulations for weighted automata over an additively idempotent semirings*, Theoret. Comput. Sci. **534** (2014), 86–100. https://doi.org/10.1016/j.tcs.2014.02.032
[12] A. Deitmar, *F*1*-schemes and toric varieties*, Beitr. Algebra Geom. **49**(2) (2008), 517–525.
[13] M. Droste, W. Kuich and H. Vogler (Eds.), *Handbook of Weighted Automata*, Springer, Berlin, 2009.
[14] A. Gathmann, *Tropical algebraic geometry*, Jahresbericht der Deutschen Mathematiker-Vereinigung **108**(1) (2006), 3–32.
[15] J. S. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, 1999.
[16] M. Gondran and M. Minoux, *Graphs, Dioids and Semirings*, Springer, New York, 2008.
[17] U. Hebisch and H. J. Weinert, *Semirings: Algebraic Theory and Applications in Computer Science*, World Scientific, 1993.
[18] S. N. Il'in, *On the homological classification of semirings*, J. Math. Sci. (N.Y.) **256**(2) (2021), 125–143. https://doi.org/10.1007/s10958-021-05423-1
[19] I. Itenberg, G. Mikhalkin and E. Shustin, *Tropical Algebraic Geometry*, Birkhäuser Verlag, Basel, 2009.
[20] Z. Izhakian and L. Rowen, *Supertropical algebra*, Adv. Math. **225**(4) (2010), 2222–2286. https://doi.org/10.1016/j.aim.2010.04.007
[21] Z. Izhakian, J. Rhodes and B. Steinberg, *Representation theory of finite semigroups over semirings*, J. Algebra **336** (2011), 139–157. https://doi.org/10.1016/j.jalgebra.2011.02.048
[22] D. Joó and K. Mincheva, *On the dimension of polynomial semirings*, J. Algebra **507** (2018), 103–119. https://doi.org/10.1016/j.jalgebra.2018.04.007
[23] D. Joó and K. Mincheva, *Prime congruences of additively idempotent semirings and a Nullstellensatz for tropical polynomials*, Selecta Math. (N.S.) **24** (2018), 2207–2233. https://doi.org/10.1007/s00029-017-0322-x
[24] Y. Katsov and T. G. Nam, *On radicals of semirings and related problems*, Comm. Algebra **42** (2014), 5065–5099. https://doi.org/10.1080/00927872.2013.833208
[25] P. Lescot, *Absolute algebra II - Ideals and spectra*, J. Pure Appl. Algebra **215** (2011), 1782–1790. https://doi.org/10.1016/j.jpaa.2010.10.019
[26] P. Lescot, *Absolute algebra III - The saturated spectrum* J. Pure Appl. Algebra **216** (2012), 1004–1015. https://doi.org/10.1016/j.jpaa.2011.10.031
[27] G. L. Litvinov, *The Maslov dequantization, and idempotent and tropical mathematics: a brief introduction*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **326** (2005), 145–182 (Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 13); translation in: J. Math. Sci. (N.Y.) **140**(3) (2007), 426–444.
[28] L. H. Mai and N. X. Tuyen, *Some remarks on the Jacobson radical types of semirings and related problems*, Vietnam J. Math. **45** (2017), 493–506. https://doi.org/10.1007/s10013-016-0226-7
[29] S. S. Mitchell and P. B. Fenoglio, *Congruence-free commutative semirings*, Semigroup Forum **31** (1988), 79–91. https://doi.org/10.1007/BF02573125
[30] M. P. Schützenberger, *On the definition of a family of automata*, Information and Control **4** (1961), 245–270. https://doi.org/10.1016/S0019-9958(61)80020-X
[31] M. K. Sen and A. K. Bhuniya, *On semirings whose additive reduct is a semilattice*, Semigroup Forum **82** (2011), 131–140. https://doi.org/10.1007/s00233-010-9271-9

[32] D. Wilding, M. Johnson and M. Kambites, *Exact rings and semirings*, J. Algebra **388** (2013), 324–337. https://doi.org/10.1016/j.jalgebra.2013.05.005

Department of Mathematics,
Visva Bharati,
Santiniketan - 731235, India
Email address: anjankbhuniya@gmail.com
ORCID iD: https://orcid.org/0000-0002-8692-0718

Department of Mathematics,
Visva Bharati,
Santiniketan - 731235, India
Email address: puja.vb.math@gmail.com
ORCID iD: https://orcid.org/0009-0002-1860-4454

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1119–1133.

HESITANT FUZZY SET THEORY APPLIED TO HILBERT ALGEBRAS

AIYARED IAMPAN[1,*], S. YAMUNADEVI[2], P. MARAGATHA MEENAKSHI[3], AND NEELAMEGARAJAN RAJESH,[4]

Abstract. The concept of hesitant fuzzy sets (HFSs) was first introduced by Torra (V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst. **25** (2010), 529–539). In this paper, the concept of HFSs to subalgebra, ideals, and deductive systems of Hilbert algebras is introduced. The relationships between hesitant fuzzy subalgebras (HF subalgebras), hesitant fuzzy ideals (HF ideals), and hesitant fuzzy deductive systems (HF deductive systems) and their level subsets are provided.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [21]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration of fuzzy sets and uncertainty theories, such as soft sets and rough sets, has been discussed in [1, 2, 5].

In 2009-2010, Torra and Narukawa [19, 20] introduced the notion of hesitant fuzzy sets, which is, a function from a reference set to a power set of the unit interval. The notion of hesitant fuzzy sets is the other generalization of the notion of fuzzy sets. The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the notion of hesitant fuzzy sets by Torra and Narukawa [19, 20], several researches were conducted on the generalizations of the notion of hesitant fuzzy sets and application

Key words and phrases. Hilbert algebra, hesitant fuzzy subalgebra, hesitant fuzzy ideal, hesitant fuzzy deductive system.

2020 *Mathematics Subject Classification.* Primary: 20N05. Secondary: 94D05, 03E72.

https://doi.org/10.46793/KgJMat2607.1119I

Received: May 08, 2022.

Accepted: September 30, 2024.

to many logical algebras, such as: in 2012, Zhu et al. [23] introduced the notion of dual hesitant fuzzy sets, which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin [15] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [16] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. In 2015, Jun and Song [17] introduced the notions of hesitant fuzzy prefilters (resp., filters) and positive implicative hesitant fuzzy prefilters (resp., filters) of EQ-algebras. In 2016, Jun and Ahn [14] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras. Iampan [11] introduced a new algebraic structure called a UP-algebra, and Mosrijai et al. [18] introduced the notion of hesitant fuzzy sets on UP-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy ideals play an important role in studying the many logical algebras. Diego proved [7] that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [3, 4] and Jun [13], and some of their filters forming deductive systems were recognized. Dudek [8] considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras. In 2022, Iampan et al. [12] introduced the concepts of anti-hesitant fuzzy subalgebras, ideals, and deductive systems of Hilbert algebras.

This paper introduces the concept of HFSs to subalgebra, ideals, and deductive systems of Hilbert algebras, which provides a generalization of the concept of HF fuzzy subalgebras/ideals/deductive systems. The relationship between HF subalgebras/ideals/deductive systems and their level subsets is provided. In the future, our research team hopes to apply these concepts to solving decision-making problems.

2. Preliminaries

Before we begin, let us go through the concept of Hilbert algebras as described by Diego [7] in 1966.

Definition 2.1 ([7])**.** A *Hilbert algebra* is a triplet $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of X such that the following axioms hold:

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

The following result was proved in [8].

Lemma 2.1. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then,*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,
(5) $(\forall x, y, z \in X)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2 ([22])**.** A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *subalgebra* of X if $x \cdot y \in D$ for all $x, y \in D$.

Definition 2.3 ([6])**.** A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an *ideal* of X if the following conditions hold:

(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

Definition 2.4 ([10])**.** A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *deductive system* of X if

(1) $1 \in D$,
(2) $(\forall x, y \in X)(x, x \cdot y \in D \Rightarrow y \in D)$.

A *fuzzy set* [21] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.5 ([8])**.** A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.6 ([9])**.** A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy ideal* of X if the following conditions hold:

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,
(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.7 ([8])**.** A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy deductive system* of X if the following conditions hold:

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(y) \geq \min\{\mu(x \cdot y), \mu(x)\})$.

Definition 2.8 ([19])**.** Let X be a reference set. A *hesitant fuzzy set* (HFS) on X is a mapping $h : X \to \mathcal{P}([0, 1])$, where $\mathcal{P}([0, 1])$ means the power set of $[0, 1]$.

We will review the definition of the characteristic HFS, which is an important and convenient tool for investigating various properties of HF subalgebras/ideals/deductive systems of Hilbert algebras.

Let X be a reference set. If $Y \subseteq X$, the characteristic HFS h_Y on X is a function of X into $\mathcal{P}([0,1])$ defined as follows:

$$(\forall x \in X)\left(h_Y(x) = \begin{cases} [0,1], & \text{if } x \in Y, \\ \emptyset, & \text{otherwise} \end{cases}\right). \tag{2.1}$$

By the definition of characteristic HFSs, h_Y is a function of a nonempty set X into $\{\emptyset, [0,1]\}$. Hence, h_Y is an HFS on X.

Definition 2.9 ([19])**.** Let h be an HFS on a nonempty set X. The HFS h is defined by $\overline{h}(x) = [0,1] - h(x)$ for all $x \in X$ which is said to be the *complement* of h on X.

3. Hesitant Fuzzy Subalgebras/Ideals/Deductive System of Hilbert Algebras

In this section, we introduce the concepts of HF subalgebras/ideals/deductive systems of Hilbert algebras and investigate some related properties.

Definition 3.1. An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is called a *hesitant fuzzy subalgebra* (HF subalgebra) of X if it satisfies the following property:

$$(\forall x, y \in X)(h(x \cdot y) \supseteq h(x) \cap h(y)). \tag{3.1}$$

Example 3.1. Let $X = \{a, b, c, d, 1\}$ with the following Cayley table:

$\cdot$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	c	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

Then, X is a Hilbert algebra. We define an HFS h on X as follows:

$$h(1) = \{0.5, 0.2\}, \quad h(a) = h(b) = h(c) = h(d) = \{0.2\}.$$

Then, h is an HF subalgebra of X.

Proposition 3.1. *If h is an HF subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x \in X)(h(1) \supseteq h(x)). \tag{3.2}$$

Proof. For any $x \in X$, we have $h(1) = h(x \cdot x) \supseteq h(x) \cap h(x) = h(x)$. □

Lemma 3.1. *The constant 1 of a Hilbert algebra $X = (X, \cdot, 1)$ is in a nonempty subset S of X if and only if $h_S(1) \supseteq h_S(x)$ for all $x \in X$.*

Proof. If $1 \in S$, then $h_S(1) = [0,1]$. Thus, $h_S(1) = [0,1] \supseteq h_S(x)$ for all $x \in X$.

Conversely, assume that $h_S(1) \supseteq h_S(x)$ for all $x \in X$. Since S is a nonempty subset of X, we have $a \in S$ for some $a \in X$. Thus, $h_S(1) \supseteq h_S(a) = [0,1]$, so $h_S(1) = [0,1]$. Hence, $1 \in S$. □

Theorem 3.1. *A nonempty subset S of a Hilbert algebra $X = (X, \cdot, 1)$ is a subalgebra of X if and only if the characteristic HFS h_S is an HF subalgebra of X.*

Proof. Assume that S is a subalgebra of X. Let $x, y \in X$.

Case 1: $x, y \in S$. Then, $h_S(x) = [0,1]$ and $h_S(y) = [0,1]$. Thus, $h_S(x) \cap h_S(y) = [0,1]$. Since S is a subalgebra of X, we have $x \cdot y \in S$ and so $h_S(x \cdot y) = [0,1]$. Therefore, $h_S(x \cdot y) = [0,1] \supseteq [0,1] = h_S(x) \cap h_S(y)$.

Case 2: $x \in S$ and $y \notin S$. Then, $h_S(x) = [0,1]$ and $h_S(y) = \emptyset$. Thus, $h_S(x) \cap h_S(y) = \emptyset$. Therefore, $h_S(x \cdot y) \supseteq \emptyset = h_S(x) \cap h_S(y)$.

Case 3: $x \notin S$ and $y \in S$. Then, $h_S(x) = \emptyset$ and $h_S(y) = [0,1]$. Thus, $h_S(x) \cap h_S(y) = \emptyset$. Therefore, $h_S(x \cdot y) \supseteq \emptyset = h_S(x) \cap h_S(y)$.

Case 4: $x \notin S$ and $y \notin S$. Then, $h_S(x) = \emptyset$ and $h_S(y) = \emptyset$. Thus, $h_S(x) \cap h_S(y) = \emptyset$. Therefore, $h_S(x \cdot y) \supseteq \emptyset = h_S(x) \cap h_S(y)$.

Hence, h_S is an HF subalgebra of X.

Conversely, assume that h_S is an HF subalgebra of X. Since $h_S(1) \supseteq h_S(x)$ for all $x \in X$, it follows from Lemma 3.1 that $1 \in S$. Let $x, y \in S$. Then, $h_S(x) = [0,1]$ and $h_S(y) = [0,1]$. Thus, $h_S(x \cdot y) \supseteq h_S(x) \cap h_S(y) = [0,1]$, so $h_S(x \cdot y) = [0,1]$. Hence, $x \cdot y \in S$ and so S is a subalgebra of X. □

Definition 3.2. An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is called a *hesitant fuzzy ideal* (HF ideal) of X if it satisfies the following properties:

$$(\forall x \in X)(h(1) \supseteq h(x)), \tag{3.3}$$

$$(\forall x, y \in X)(h(x \cdot y) \supseteq h(y)), \tag{3.4}$$

$$(\forall x, y_1, y_2 \in X)(h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2)). \tag{3.5}$$

Example 3.2. Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then, X is a Hilbert algebra. We define an HFS h on X as follows:

$$h(1) = \{0.4, 0.5, 0.7\}, \quad h(x) = \{0.4, 0.5\}, \quad h(y) = \{0.5\}, \quad h(z) = h(0) = \emptyset.$$

Then, h is an HF ideal of X.

Proposition 3.2. *If h is an HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(h((y \cdot x) \cdot x) \supseteq h(y)). \tag{3.6}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (3.5), we have $h((y \cdot x) \cdot x) \supseteq h(y) \cap h(1) = h(y)$ for all $x, y \in X$. □

Lemma 3.2. *If h is an HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(x \leq y \Rightarrow h(x) \subseteq h(y)). \tag{3.7}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then, $x \cdot y = 1$ and so

$$\begin{aligned} h(y) =& h(1 \cdot y) = h(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ \supseteq& h(x \cdot y) \cap h(x) \\ =& h(1) \cap h(x) = h(x). \end{aligned}$$

□

Theorem 3.2. *Every HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an HF subalgebra of X.*

Proof. Let h be an HF ideal of X and let $x, y \in X$. Since $y \leq x \cdot y$ and by Lemma 3.2, we have $h(y) \supseteq h(x \cdot y)$. It follows from (3.4) that

$$h(x \cdot y) \supseteq h(y) \supseteq h(x \cdot y) \cap h(x) \supseteq h(x) \cap h(y).$$

Hence, h is an HF subalgebra of X. □

Definition 3.3. An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is called a *hesitant fuzzy deductive system* (HF deductive system) of X if it satisfies the following properties:

$$(\forall x \in X)(h(1) \supseteq h(x)), \tag{3.8}$$

$$(\forall x, y \in X)(h(y) \supseteq h(x \cdot y) \cap h(x)). \tag{3.9}$$

Proposition 3.3. *Every HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an HF deductive system of X.*

Proof. Let h be an HF ideal of X and let $x, y \in X$. If $y_1 = x \cdot y$ and $y_2 = x$, then by Lemma 2.1 and (3.5), we have

$$h(y) = h(1 \cdot y) = h(((x \cdot y) \cdot (x \cdot y)) \cdot y) \supseteq h(x \cdot y) \cap h(x).$$

Hence, h is an HF deductive system of X. □

Lemma 3.3. *If h is an HF deductive system of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y, z \in X)(z \leq x \cdot y \Rightarrow h(y) \supseteq h(x) \cap h(z)). \tag{3.10}$$

Proof. Let $x, y, z \in X$ be such that $z \leq x \cdot y$. Then, $z \cdot (x \cdot y) = 1$ and so

$$\begin{aligned} h(y) \supseteq& h(x \cdot y) \cap h(x) \supseteq h(z \cdot (x \cdot y)) \cap h(z) \cap h(x) \\ =& h(1) \cap h(z) \cap h(x) = h(x) \cap h(z). \end{aligned}$$

□

Lemma 3.4. *If h is an HF deductive system of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(x \leq y \Rightarrow h(y) \supseteq h(x)). \tag{3.11}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then, $x \cdot y = 1$ and so $h(y) \supseteq h(x \cdot y) \cap h(x) = h(1) \cap h(x) = h(x)$. □

Theorem 3.3. *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is a deductive system of X if and only if the characteristic HFS h_D is an HF deductive system of X.*

Proof. Assume that D is a deductive system of X. Since $1 \in D$, it follows from Lemma 3.1 that $h_D(1) \supseteq h_D(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1: $x, y \in D$. Then, $h_D(x) = [0,1]$ and $h_D(y) = [0,1]$. Thus, $h_D(y) = [0,1] \supseteq h_D(x \cdot y) = h_D(x \cdot y) \cap h_D(x)$.

Case 2: $x \notin D$ and $y \in D$. Then, $h_D(x) = \emptyset$ and $h_D(y) = [0,1]$. Thus, $h_D(y) = [0,1] \supseteq \emptyset = h_D(x \cdot y) \cap h_D(x)$.

Case 3: $x \in D$ and $y \notin D$. Then, $h_D(x) = [0,1]$ and $h_D(y) = \emptyset$. Since D is a deductive system of X, we have $x \cdot y \notin D$ or $x \notin D$. But $x \in D$, so $x \cdot y \notin D$. Then, $h_D(x \cdot y) = \emptyset$. Thus, $h_D(y) = \emptyset \supseteq \emptyset = h_D(x \cdot y) \cap h_D(x)$.

Case 4: $x \notin D$ and $y \notin D$. Then, $h_D(x) = \emptyset$ and $h_D(y) = \emptyset$. Thus, $h_D(y) = \emptyset \supseteq \emptyset = h_D(x \cdot y) \cap h_D(x)$.

Hence, h_D is an HF deductive system of X.

Conversely, assume that h_D is an HF deductive system of X. Since $h_D(1) \supseteq h_D(x)$ for all $x \in X$, it follows from Lemma 3.1 that $1 \in D$. Next, let $x, y \in X$ be such that $x \cdot y \in D$ and $x \in D$. Then, $h_D(x \cdot y) = [0,1]$ and $h_D(x) = [0,1]$. Thus, $h_D(y) \supseteq h_D(x \cdot y) \cap h_D(x) = [0,1]$, so $h_D(y) = [0,1]$. Therefore, $y \in D$ and so D is a deductive system of X. □

Theorem 3.4. *A nonempty subset I of a Hilbert algebra $X = (X, \cdot, 1)$ is an ideal of X if and only if the characteristic HFS h_I is an HF ideal of X.*

Proof. Assume that I is an ideal of X. Since $1 \in I$, it follows from Lemma 3.1 that $h_I(1) \supseteq h_I(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1: $x, y \in I$. Then $h_I(x) = [0,1]$ and $h_I(y) = [0,1]$. Thus, $h_I(x \cdot y) = [0,1] \supseteq h_I(y)$.

Case 2: $x \notin I$ and $y \in I$. Then, $h_I(x) = \emptyset$ and $h_I(y) = [0,1]$. Thus, $h_I(x \cdot y) = [0,1] \supseteq h_I(y)$.

Case 3: $x \in I$ and $y \notin I$. Then, $h_I(x) = [0,1]$ and $h_I(y) = \emptyset$. Since I is an ideal of X, we have $x \cdot y \notin I$ or $x \notin I$. But $x \in I$, so $x \cdot y \notin I$. Then, $h_I(x \cdot y) = \emptyset$. Thus, $h_I(x \cdot y) = \emptyset \supseteq \emptyset = h_I(y)$.

Case 4: $x \notin I$ and $y \notin I$. Then, $h_I(x) = \emptyset$ and $h_I(y) = \emptyset$. Thus, $h_I(x \cdot y) = \emptyset \supseteq \emptyset = h_I(y)$.

Now, let $x, y_1, y_2 \in X$.

Case 1: $x, y_1, y_2 \in I$. Then, $h_I(x) = [0,1]$, $h_I(y_1) = [0,1]$, and $h_I(y_2) = [0,1]$. Since I is an ideal of X, we have $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = [0,1] \supseteq h_I(y_1) \cap h_I(y_2)$.

Case 2: $x \notin I$ and $y_1, y_2 \in I$. Then, $h_I(x) = \emptyset$ and $h_I(y_1) = h_I(y_2) = [0,1]$. Since I is an ideal of X, we have $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) = [0,1] \supseteq h_I(y_1) \cap h_I(y_2)$.

Case 3: $x \in I$, and $y_1 \notin I$ or $y_2 \notin I$. Then, $h_I(x) = [0,1]$ and $h_I(y_1) = \emptyset$ or $h_I(y_2) = \emptyset$. Since I is an ideal of X, we have $y_1 \cdot (y_2 \cdot x) \cdot x \notin I$. Then, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset \supseteq \emptyset = h_I(y_1) \cap h_I(y_2)$.

Case 4: $x \notin I$, and $y_1 \notin I$ or $y_2 \notin I$. Then, $h_I(x) = \emptyset$, and $h_I(y_1) = \emptyset$ or $h_I(y_2) = \emptyset$. Since I is an ideal of X, we have $(y_1 \cdot (y_2 \cdot x)) \cdot x \notin I$. Then, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset \supseteq \emptyset = h_I(y_1) \cap h_I(y_2)$.

Hence, h_I is an HF ideal of X.

Conversely, assume that h_I is an HF ideal of X. Since $h_I(1) \supseteq h_I(x)$ for all $x \in X$, it follows from Lemma 3.1 that $1 \in I$. Let $x \in X$ and $y \in I$. Then, $h_I(y) = [0,1]$. Thus, $h_I(x \cdot y) \supseteq h_I(y) = [0,1]$, so $h_I(x \cdot y) = [0,1]$. Hence, $x \cdot y \in I$. Next, let $x, y_1, y_2 \in X$ be such that $y_1 \in I$ and $y_2 \in I$. Then, $h_I(y_1) = h_I(y_2) = [0,1]$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h_I(y_1) \cap h_I(y_2) = [0,1]$, so $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = [0,1]$. Hence, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I$ and so I is an ideal of X. □

4. Level Subsets of an HFS on Hilbert Algebras

In this section, we provide the relationship between HF subalgebras, ideals, and deductive systems and their level subsets.

Definition 4.1. Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. For any $\pi \in \mathcal{P}([0,1])$, the sets $U(h,\pi) = \{x \in X \mid h(x) \supseteq \pi\}$ and $U^+(h,\pi) = \{x \in X \mid h(x) \supset \pi\}$ are called an upper π-level subset and an upper π-strong level subset of h, respectively. The sets $L(h,\pi) = \{x \in X \mid h(x) \subseteq \pi\}$ and $L^-(h,\pi) = \{x \in X \mid h(x) \subset \pi\}$ are called a lower π-level subset and a lower π-strong level subset of h, respectively. The set $E(h,\pi) = \{x \in X \mid h(x) = \pi\}$ is called an equal π-level subset of h. Then, $U(h,\pi) = U^+(h,\pi) \cup E(h,\pi)$ and $L(h,\pi) = L^-(h,\pi) \cup E(h,\pi)$.

Theorem 4.1. *An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is an HF subalgebra of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a subalgebra of X.*

Proof. Assume that h is an HF subalgebra of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h,\pi) \neq \emptyset$ and let $x \in U(h,\pi)$. Then, $h(x) \supseteq \pi$. Since h is an HF subalgebra of X, we have $h(1) \supseteq h(x) \supseteq \pi$. Thus, $1 \in U(h,\pi)$. Let $x, y \in U(h,\pi)$. Then, $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Since h is an HF subalgebra of X, we have $h(x \cdot y) \supseteq h(x) \cap h(y) \supseteq \pi$ and thus, $x \cdot y \in U(h,\pi)$. Hence, $U(h,\pi)$ is a subalgebra of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a subalgebra of X. Let $x, y \in X$. Choose $\pi = h(x) \cap h(y) \in \mathcal{P}([0,1])$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Thus, $x, y \in U(h,\pi) \neq \emptyset$. By assumption, $U(h,\pi)$ is a subalgebra of X and thus $x \cdot y \in U(h,\pi)$. So, $h(x \cdot y) \supseteq \pi = h(x) \cap h(y)$. Hence, h is an HF subalgebra of X. □

Theorem 4.2. *An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is an HF ideal of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is an ideal of X.*

Proof. Assume that h is an HF ideal of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h,\pi) \neq \emptyset$ and let $x \in U(h,\pi)$. Then, $h(x) \supseteq \pi$. Since h is an HF ideal of X, we have $h(1) \supseteq h(x) \supseteq \pi$. Thus, $1 \in U(h,\pi)$. Next, let $x, y \in X$ be such that $y \in U(h,\pi)$. Then, $h(y) \supseteq \pi$. Since h is an HF ideal of X, we have $h(x \cdot y) \supseteq h(y) \supseteq \pi$. So, $x \cdot y \in U(h,\pi)$. Let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in U(h,\pi)$. Then, $h(y_1) \supseteq \pi$ and $h(y_2) \supseteq \pi$. Since h is an HF ideal of X, we have $h((y_1 \cdot (y_2 \cdot x)) \cdot x)) \supseteq h(y_1) \cap h(y_2) \supseteq \pi$. So, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(h,\pi)$. Hence, $U(h,\pi)$ is an ideal of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is an ideal of X. Let $x \in X$. Then $h(x) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then, $h(x) \supseteq \pi$. Thus, $x \in U(h,\pi) \neq \emptyset$. By assumption, we have $U(h,\pi)$ is an ideal of X and thus, $1 \in U(h,\pi)$. So, $h(1) \supseteq \pi = h(x)$. Next, let $x, y \in X$. Then, $h(x), h(y) \in \mathcal{P}([0,1])$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(y) \supseteq \pi$, so $x \in X$ and $y \in U(h,\pi) \neq \emptyset$. By assumption, we have $U(h,\pi)$ is an ideal of X and then $x \cdot y \in U(h,\pi)$. Thus, $h(x \cdot y) \supseteq \pi = h(y)$. Let $x, y_1, y_2 \in X$. Then, $h(x), h(y_1), h(y_2) \in \mathcal{P}([0,1])$. Choose $\pi = h(y_1) \cap h(y_2) \in \mathcal{P}([0,1])$. Then, $h(y_1) \supseteq \pi$ and $h(y_2) \supseteq \pi$, so $x \in X$ and $y_1, y_2 \in U(h,\pi) \neq \emptyset$. By assumption, we have $U(h,\pi)$ is an ideal of X and then $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(h,\pi)$. Thus, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq \pi = h(y_1) \cap h(y_2)$. Hence, h is an HF ideal of X. □

Theorem 4.3. *An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is an HF deductive system of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a deductive system of X.*

Proof. Assume that h is an HF deductive system of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h,\pi) \neq \emptyset$ and let $x \in U(h,\pi)$. Then, $h(x) \supseteq \pi$. Since h is an HF deductive system of X, we have $h(1) \supseteq h(x) \supseteq \pi$. Thus, $1 \in U(h,\pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in U(h,\pi)$. Then, $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. Since h is an HF deductive system of X, we have $h(y) \supseteq h(x \cdot y) \cap h(x) \supseteq \pi$. So, $y \in U(h,\pi)$. Hence, $U(h,\pi)$ is a deductive system of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a deductive system of X. Let $x \in X$. Then $h(x) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then $h(x) \supseteq \pi$. Thus, $x \in U(h,\pi) \neq \emptyset$. By assumption, $U(h,\pi)$ is a deductive system of X and thus $1 \in U(h,\pi)$. So, $h(1) \supseteq \pi = h(x)$. Let $x, y \in X$. Then, $h(x), h(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \cap h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$, so $x, x \cdot y \in U(h,\pi) \neq \emptyset$. By assumption, $U(h,\pi)$ is a deductive system of X and then $y \in U(h,\pi)$. Thus, $h(y) \supseteq \pi = h(x) \cap h(x \cdot y)$. Hence, h is an HF deductive system of X. □

Theorem 4.4. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

(1) *If h is an HF subalgebra of X, then for all $\pi \in \mathcal{P}([0,1])$, $U^+(h,\pi)$ is a subalgebra of X if $U^+(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
(2) *If $\operatorname{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a subalgebra of X, then h is an HF subalgebra of X.*

Proof. (1) It is straightforward by Theorem 4.1.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a subalgebra of X. Assume that there exist $x, y \in X$ such that $h(x \cdot y) \not\supseteq h(x) \cap h(y)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(x \cdot y) \subset h(x) \cap h(y)$. Then, $h(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(x) \supset \pi$ and $h(y) \supset \pi$. Thus, $x, y \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is a subalgebra of X and thus $x \cdot y \in U^+(h,\pi)$. So, $h(x \cdot y) \supset \pi = h(x \cdot y)$, a contradiction. Hence, $h(x \cdot y) \supseteq h(x) \cap h(y)$ for all $x, y \in X$. Therefore, h is an HF subalgebra of X. □

Theorem 4.5. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

1. *If h is an HF ideal of X, then for all $\pi \in \mathcal{P}([0,1])$, $U^+(h,\pi)$ is an ideal of X if $U^+(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
2. *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is an ideal of X, then h is an HF ideal of X.*

Proof. (1) It is straightforward by Theorem 4.2.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is an ideal of X. Assume that there exists $x \in X$ such that $h(1) \not\supseteq h(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(1) \subset h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then $h(x) \supset h(1) = \pi$. Thus, $x \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is an ideal of X and thus $1 \in U^+(h,\pi)$. So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $h(1) \supseteq h(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $h(x \cdot y) \not\supseteq h(y)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(x \cdot y) \subset h(y)$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(y) \supset \pi$. Thus $y \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is an ideal of X and thus $x \cdot y \in U^+(h,\pi)$. So, $h(x \cdot y) \supset \pi = h(x \cdot y)$, a contradiction. Hence, $h(x \cdot y) \supseteq h(y)$ for all $x, y \in X$. Assume that there exist $x, y_1, y_2 \in X$ such that $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \not\supseteq h(y_1) \cap h(y_2)$. Since $\mathrm{Im}(h)$ is a chain, we have $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subset h(y_1) \cap h(y_2)$. Choose $\pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x) \in \mathcal{P}([0,1])$. Then, $h(y_1) \supset \pi$ and $h(y_2) \supset \pi$. Thus, $y_1, y_2 \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is an ideal of X and thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U^+(h,\pi)$. So, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supset \pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x)$, a contradiction. Hence, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2)$ for all $x, y_1, y_2 \in X$. Therefore, h is an HF ideal of X. □

Theorem 4.6. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

1. *If h is an HF deductive system of X, then for all $\pi \in \mathcal{P}([0,1])$, $U^+(h,\pi)$ is a deductive system of X if $U^+(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
2. *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a deductive system of X, then h is an HF deductive system of X.*

Proof. (1) It is straightforward by Theorem 4.3.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a deductive system of X. Assume that there exists $x \in X$ such that

$h(1) \not\supseteq h(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(1) \subset h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then, $h(x) \supset h(1) = \pi$. Thus, $x \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is a deductive system of X and thus, $1 \in U^+(h,\pi)$. So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $h(1) \supseteq h(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $h(y) \not\supseteq h(x \cdot y) \cap h(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(y) \subset h(x \cdot y) \cap h(x)$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(x \cdot y) \supset \pi$ and $h(x) \supset \pi$. Thus, $x \cdot y, x \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is a deductive system of X and thus $y \in U^+(h,\pi)$. So, $h(y) \supset \pi = h(y)$, a contradiction. Hence, $h(y) \supseteq h(x \cdot y) \cap h(x)$ for all $x, y \in X$. Therefore, h is an HF deductive system of X. $\square$

Theorem 4.7. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then, $\overline{h}$ is an HF subalgebra of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is a subalgebra of X.*

Proof. Assume that $\overline{h}$ is an HF subalgebra of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(h,\pi) \neq \emptyset$ and let $x, y \in L(h,\pi)$. Then $h(x) \subseteq \pi$ and $h(y) \subseteq \pi$. Since $\overline{h}$ is an HF subalgebra of X, we have $\overline{h}(x \cdot y) \supseteq \overline{h}(x) \cap \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \supseteq ([0,1] - h(x)) \cap ([0,1] - h(y)) = [0,1] - (h(x) \cup h(y))$. Thus, $h(x \cdot y) \subseteq h(x) \cup h(y) \subseteq \pi$. So, $x \cdot y \in L(h,\pi)$. Hence, $L(h,\pi)$ is a subalgebra of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is a subalgebra of X. Let $x, y \in X$. Choose $\pi = h(x) \cup h(y) \in \mathcal{P}([0,1])$. Then $h(x) \subseteq \pi$ and $h(y) \subseteq \pi$. Thus, $x, y \in L(h,\pi) \neq \emptyset$. By assumption, we have $L(h,\pi)$ is a subalgebra of X and thus $x \cdot y \in L(h,\pi)$. So, $h(x \cdot y) \subseteq \pi = h(x) \cup h(y)$. Thus, $\overline{h}(x \cdot y) = [0,1] - h(x \cdot y) \supseteq [0,1] - (h(x) \cup h(y)) = ([0,1] - h(x)) \cap ([0,1] - h(y)) = \overline{h}(x) \cap \overline{h}(y)$. Hence, $\overline{h}$ is an HF subalgebra of X. $\square$

Theorem 4.8. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then $\overline{h}$ is an HF ideal of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is an ideal of X.*

Proof. Assume that $\overline{h}$ is an HF ideal of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(h,\pi) \neq \emptyset$ and let $a \in L(h,\pi)$. Then, $h(a) \subseteq \pi$. Since $\overline{h}$ is an HF ideal of X, we have $\overline{h}(1) \supseteq \overline{h}(a)$. Thus, $[0,1] - h(1) \supseteq [0,1] - h(a)$. So, $h(1) \subseteq h(a) \subseteq \pi$. Hence, $1 \in L(h,\pi)$. Next, let $x, y \in X$ be such that $y \in L(h,\pi)$. Then, $h(y) \subseteq \pi$. Since $\overline{h}$ is an HF ideal of X, we have $\overline{h}(x \cdot y) \supseteq \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \supseteq [0,1] - h(y)$. Thus, $h(x \cdot y) \subseteq h(y) \subseteq \pi$. So, $x \cdot y \in L(h,\pi)$. Also, let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in L(h,\pi)$. Then, $h(y_1) \subseteq \pi$ and $h(y_2) \subseteq \pi$. Since $\overline{h}$ is an HF ideal of X, we have $\overline{h}((y_1 \cdot (y_2 \cdot x) \cdot x)) \supseteq \overline{h}(y_1) \cap \overline{h}(y_2)$ and so $[0,1] - h(((y_1 \cdot (y_2 \cdot x)) \cdot x)) \supseteq ([0,1] - h(y_1)) \cap ([0,1] - h(y_2)) = [0,1] - (h(y_1) \cup h(y_2))$. Thus, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq h(x) \cup h(y) \subseteq \pi$. So, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(h,\pi)$. Hence, $L(h,\pi)$ is an ideal of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is an ideal of X. Let $x \in X$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then, $h(x) \subseteq \pi$, so $x \in L(h,\pi) \neq \emptyset$. By assumption, we have $L(h,\pi)$ is an ideal of X. Thus, $1 \in L(h,\pi)$. So, $h(1) \subseteq \pi = h(x)$. Hence, $\overline{h}(1) = [0,1] - h(1) \supseteq [0,1] - h(x) =$

$\overline{h}(x)$. Next, let $x, y \in X$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(y) \subseteq \pi$. Thus, $y \in L(h,\pi) \neq \emptyset$. By assumption, $L(h,\pi)$ is an ideal of X and thus, $x \cdot y \in L(h,\pi)$. So, $h(x \cdot y) \subseteq \pi = h(y)$. Thus, $\overline{h}(x \cdot y) = [0,1] - h(x \cdot y) \supseteq [0,1] - h(y) = \overline{h}(y)$. Also, let $x, y_1, y_2 \in X$. Choose $\pi = h(y_1) \cup h(y_2) \in \mathcal{P}([0,1])$. Then $h(y_1) \subseteq \pi$ and $h(y_2) \subseteq \pi$. Thus, $y_1, y_2 \in L(h,\pi) \neq \emptyset$. By assumption, $L(h,\pi)$ is an ideal of X and thus, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(h,\pi)$. So, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq \pi = h(y_1) \cup h(y_2)$. Thus, $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) = [0,1] - h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq [0,1] - (h(y_1) \cup h(y_2)) = ([0,1] - h(y_1)) \cap ([0,1] - h(y_2)) = \overline{h}(y_1) \cap \overline{h}(y_2)$. Hence, $\overline{h}$ is an HF ideal of X. ☐

Theorem 4.9. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then $\overline{h}$ is an HF deductive system of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is a deductive system of X.*

Proof. Assume that $\overline{h}$ is an HF deductive system of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(h,\pi) \neq \emptyset$ and let $a \in L(h,\pi)$. Then $h(a) \subseteq \pi$. Since $\overline{h}$ is an HF deductive system of X, we have $\overline{h}(1) \supseteq \overline{h}(a)$. Thus $[0,1] - h(1) \supseteq [0,1] - h(a)$. So, $h(1) \subseteq h(a) \subseteq \pi$. Hence, $1 \in L(h,\pi)$. Next, let $x, y \in X$ be such that $x \cdot y, x \in L(h,\pi)$. Then, $h(x \cdot y) \subseteq \pi$ and $h(x) \subseteq \pi$. Since $\overline{h}$ is an HF deductive system of X, we have $\overline{h}(y) \supseteq \overline{h}(x \cdot y) \cap \overline{h}(y)$ and so $[0,1] - h(y) \supseteq ([0,1] - h(x \cdot y)) \cap ([0,1] - h(x)) = [0,1] - (h(x \cdot y) \cup h(x))$. Thus, $h(y) \subseteq h(x \cdot y) \cup h(x) \subseteq \pi$. So, $y \in L(h,\pi)$. Hence, $L(h,\pi)$ is a deductive system of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is a deductive system of X. Let $x \in X$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then, $h(x) \subseteq \pi$, so $x \in L(h,\pi) \neq \emptyset$. By assumption, $L(h,\pi)$ is a deductive system of X. Thus, $1 \in L(h,\pi)$. So, $h(1) \subseteq \pi = h(x)$. Hence, $\overline{h}(1) = [0,1] - h(1) \supseteq [0,1] - h(x) = \overline{h}(x)$. Next, let $x, y \in X$. Choose $\pi = h(x \cdot y) \cup h(x) \in \mathcal{P}([0,1])$. Then, $h(x \cdot y) \subseteq \pi$ and $h(x) \subseteq \pi$. Thus, $x \cdot y, x \in L(h,\pi) \neq \emptyset$. By assumption, $L(h,\pi)$ is a deductive system of X and thus $y \in L(h,\pi)$. So, $h(y) \subseteq \pi = h(x \cdot y) \cup h(x)$. Thus, $\overline{h}(y) = [0,1] - h(y) \supseteq [0,1] - (h(x \cdot y) \cup h(x)) = ([0,1] - h(x \cdot y)) \cap ([0,1] - h(x)) = \overline{h}(x \cdot y) \cap \overline{h}(x)$. Hence, $\overline{h}$ is an HF deductive system of X. ☐

Theorem 4.10. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then, the following statements hold.*

(1) *If $\overline{h}$ is an HF subalgebra of X, then for all $\pi \in \mathcal{P}([0,1])$, $L^-(h,\pi)$ is a subalgebra of X if $L^-(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
(2) *If $\operatorname{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h,\pi)$ of X is a subalgebra of X, then $\overline{h}$ is an HF subalgebra of X.*

Proof. (1) It is straightforward by Theorem 4.7.

(2) Assume that $\operatorname{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h,\pi)$ of X is a subalgebra of X. Assume that there exist $x, y \in X$ such that $\overline{h}(x \cdot y) \not\supseteq \overline{h}(x) \cap \overline{h}(y)$. Since $\operatorname{Im}(h)$ is a chain, we have $\overline{h}(x \cdot y) \subset \overline{h}(x) \cap \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \subset ([0,1] - h(x)) \cap ([0,1] - h(y)) = [0,1] - (h(x) \cup h(y))$. Then, $h(x \cdot y) \supset h(x) \cup h(y)$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then $h(x) \subset \pi$ and $h(y) \subset \pi$.

Thus, $x, y \in L^-(h, \pi) \neq \emptyset$. By assumption, we have $L^-(h, \pi)$ is a subalgebra of X and thus $x \cdot y \in L^-(h, \pi)$. So, $h(x \cdot y) \subset \pi = h(x \cdot y)$, a contradiction. Hence, $\overline{h}(x \cdot y) \supseteq \overline{h}(x) \cap \overline{h}(y)$ for all $x, y \in X$. Therefore, $\overline{h}$ is an HF subalgebra of X. $\square$

Theorem 4.11. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

(1) *If $\overline{h}$ is an HF ideal of X, then for all $\pi \in \mathcal{P}([0,1])$, $L^-(h, \pi)$ is an ideal of X if $L^-(h, \pi)$ is nonempty and $E(h, \pi)$ is empty.*

(2) *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h, \pi)$ of X is an ideal of X, then $\overline{h}$ is an HF ideal of X.*

Proof. (1) It is straightforward by Theorem 4.8.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h, \pi)$ of X is an ideal of X. Assume that there exists $x \in X$ such that $\overline{h}(1) \not\supseteq \overline{h}(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $\overline{h}(1) \subset \overline{h}(x)$. Then, $[0,1] - h(1) \subset [0,1] - h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then, $h(x) \subset h(1) = \pi$. Thus, $x \in L^-(h, \pi) \neq \emptyset$. By assumption, $L^-(h, \pi)$ is an ideal of X and thus $1 \in L^-(h, \pi)$. So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $\overline{h}(1) \supseteq \overline{h}(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $\overline{h}(x \cdot y) \not\supseteq \overline{h}(y)$. Since $\mathrm{Im}(h)$ is a chain, $\overline{h}(x \cdot y) \subset \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \subset [0,1] - h(y)$. Then, $h(x \cdot y) \supset h(y)$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(y) \subset \pi$. Thus, $y \in L^-(h, \pi) \neq \emptyset$. By assumption, $L^-(h, \pi)$ is an ideal of X and thus $x \cdot y \in L^-(h, \pi)$. So, $h(x \cdot y) \supset \pi = h(x \cdot y)$, a contradiction. Hence, $\overline{h}(x \cdot y) \supseteq \overline{h}(y)$ for all $x, y \in X$. Assume that there exist $x, y_1, y_2 \in X$ such that $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) \not\supseteq \overline{h}(y_1) \cap \overline{h}(y_2)$. Since $\mathrm{Im}(h)$ is a chain, $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) \subset \overline{h}(y_1) \cap \overline{h}(y_2)$ and so $[0,1] - h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subset ([0,1] - h(y_1)) \cap ([0,1] - h(y_2)) = [0,1] - (h(y_1) \cup h(y_2))$. Then. $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supset h(y_1) \cup h(y_2)$. Choose $\pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x) \in \mathcal{P}([0,1])$. Then, $h(y_1) \subset \pi$ and $h(y_2) \subset \pi$. Thus, $y_1, y_2 \in L^-(h, \pi) \neq \emptyset$. By assumption, $L^-(h, \pi)$ is an ideal of X and thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L^-(h, \pi)$. So, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supset \pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x)$, a contradiction. Hence, $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq \overline{h}(y_1) \cap \overline{h}(y_2)$ for all $x, y_1, y_2 \in X$. Therefore, $\overline{h}$ is an HF ideal of X. $\square$

Theorem 4.12. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

(1) *If h is an HF deductive system of X, then for all $\pi \in \mathcal{P}([0,1])$, $L^-(h, \pi)$ is a deductive system of X if $L^-(h, \pi)$ is nonempty and $E(h, \pi)$ is empty.*

(2) *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h, \pi)$ of X is a deductive system of X, then $\overline{h}$ is an HF deductive system of X.*

Proof. (1) It is straightforward by Theorem 4.9.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h, \pi)$ of X is a deductive system of X. Assume that there exists $x \in X$ such that $\overline{h}(1) \not\supseteq \overline{h}(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $\overline{h}(1) \subset \overline{h}(x)$. Then $[0,1] - h(1) \subset [0,1] - h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then $h(x) \subset h(1) = \pi$. Thus, $x \in L^-(h, \pi) \neq \emptyset$. By assumption, we have $L^-(h, \pi)$ is a deductive system of X and thus, $1 \in L^-(h, \pi)$.

So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $\overline{h}(1) \supseteq \overline{h}(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $\overline{h}(y) \not\supseteq \overline{h}(x \cdot y) \cap \overline{h}(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $\overline{h}(y) \subset \overline{h}(x \cdot y) \cap \overline{h}(x)$ and so $[0,1] - h(y) \subset ([0,1] - h(x \cdot y)) \cap ([0,1] - h(x)) = [0,1] - (h(x \cdot y) \cup h(x))$. Then, $h(y) \supset h(x \cdot y) \cup h(x)$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(x \cdot y) \subset \pi$ and $h(x) \subset \pi$. Thus, $x \cdot y, x \in L^{-}(h, \pi) \neq \emptyset$. By assumption, $L^{-}(h, \pi)$ is a deductive system of X and thus, $y \in L^{-}(h, \pi)$. So, $h(y) \supset \pi = h(y)$, a contradiction. Hence, $\overline{h}(y) \supseteq \overline{h}(x \cdot y) \cap \overline{h}(x)$ for all $x, y \in X$. Therefore, $\overline{h}$ is an HF deductive system of X. □

Acknowledgements. This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

References

[1] B. Ahmad ad A. Kharal, *On fuzzy soft sets*, Adv. Fuzzy Syst. **2009** (2009), Article ID 586507, 6 pages. https://doi.org/10.1155/2009/586507

[2] M. Atef, M. I. Ali and T. Al-Shami, *Fuzzy soft covering-based multi-granulation fuzzy rough sets and their applications*, Comput. Appl. Math. **40**(4) (2021), Article ID 115. https://doi.org/10.1007/s40314-021-01501-x

[3] D. Busneag, *A note on deductive systems of a Hilbert algebra*, Kobe J. Math. **2** (1985), 29–35.

[4] D. Busneag, *Hilbert algebras of fractions and maximal Hilbert algebras of quotients*, Kobe J. Math. **5** (1988), 161–172.

[5] N. Cağman, S. Enginoğlu and F. Citak, *Fuzzy soft set theory and its application*, Iran. J. Fuzzy Syst. **8**(3) (2011), 137–147. https://doi.org/10.22111/ijfs.2011.292

[6] I. Chajda and R. Halas, *Congruences and ideals in Hilbert algebras*, Kyungpook Math. J. **39**(2) (1999), 429–429.

[7] A. Diego, *Sur les algébres de Hilbert*, Collection de Logique Math. Ser. A (Ed. Hermann, Paris) **21** (1966), 1–52.

[8] W. A. Dudek, *On fuzzification in Hilbert algebras*, Contrib. Gen. Algebra **11** (1999), 77–83.

[9] W. A. Dudek and Y. B. Jun, *On fuzzy ideals in Hilbert algebra*, Novi Sad J. Math. **29**(2) (1999), 193–207.

[10] W. A. Dudek, *On ideals in Hilbert algebras*, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **38** (1999), 31–34.

[11] A. Iampan, *A new branch of the logical algebra: UP-algebras*, J. Algebra Relat. Topics **5**(1) (2017), 35–54. https://doi.org/10.22124/jart.2017.2403

[12] A. Iampan, S. Yamunadevi, P. M. Meenakshi and N. Rajesh, *Anti-hesitant fuzzy subalgebras, ideals and deductive systems of Hilbert algebras*, Math. Stat. **10**(5) (2022), 1121–1126. https://doi.org/10.13189/ms.2022.100523

[13] Y. B. Jun, *Deductive systems of Hilbert algebras*, Math. Japon. **43** (1996), 51–54.

[14] Y. B. Jun and S. S. Ahn, *Hesitant fuzzy set theory applied to BCK/BCI-algebras*, J. Comput. Anal. Appl. **20**(4) (2016), 635–646.

[15] Y. B. Jun, S. S. Ahn and G. Muhiuddin, *Hesitant fuzzy soft subalgebras and ideals in BCK/BCI-algebras*, Sci. World J. **2014** (2014), Article ID 763929, 7 pages. https://doi.org/10.1155/2014/763929

[16] Y. B. Jun and S.-Z. Song, *Hesitant fuzzy set theory applied to filters in MTL-algebras*, Honam Math. J. **36**(4) (2014), 813–830. https://doi.org/10.5831/HMJ.2014.36.4.813

[17] Y. B. Jun and S.-Z. Song, *Hesitant fuzzy prefilters and filters of EQ-algebras*, Appl. Math. Sci. **9**(11) (2015), 515–532. https://doi.org/10.12988/ams.2015.411962
[18] P. Mosrijai, W. Kamti, A. Satirad and A. Iampan, *Hesitant fuzzy sets on UP-algebras*, Konuralp J. Math. **5**(2) (2017), 268–280. https://doi.org/10.5937/MatMor1802029M
[19] V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst. **25** (2010), 529–539. https://doi.org/10.1002/int.20418
[20] V. Torra and Y. Narukawa, *On hesitant fuzzy sets and decision*, 18th IEEE Int. Conf. Fuzzy Syst. (2009), 1378–1382. https://doi.org/10.1109/FUZZY.2009.5276884
[21] L. A. Zadeh, *Fuzzy sets*, Inf. Control **8**(3) (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
[22] J. Zhan and Z. Tan, *Intuitionistic fuzzy deductive systems in Hibert algebra*, Southeast Asian Bull. Math. **29**(4) (2005), 813–826.
[23] B. Zhu, Z. Xu and M. Xia, *Dual hesitant fuzzy sets*, J. Appl. Math. **2012** (2012), Article ID 879629, 13 pages. https://doi.org/10.1155/2012/879629

[1]Department of Mathematics,
School of Science,
University of Phayao,
Mae Ka, Mueang, Phayao 56000, Thailand
Email address: aiyared.ia@up.ac.th
ORCID iD: https://orcid.org/0000-0002-0475-3320

[2]Department of Mathematics,
Vivekanandha College of Arts and Science for Women,
Tiruchengode 637205, Tamilnadu, India
Email address: yamunawin10@gmail.com

[3]Department of Mathematics,
Thanthai Periyar Government Arts and Science College
(affiliated to Bharathidasan University),
Tiruchirappalli 624024, Tamilnadu, India
Email address: maragathameenakship@gmail.com
ORCID iD: https://orcid.org/0009-0003-3558-3875

[4]Department of Mathematics,
Rajah Serfoji Government College,
Thanjavur 613005, Tamilnadu, India
Email address: nrajesh_topology@yahoo.co.in

*Corresponding Author

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1135–1147.

INEQUALITIES FORMULATED BY A SPECIAL CLASS OF BAZILEVIČ FUNCTIONS COMBINING THE BELL SERIES

G. MURUGUSUNDARAMOORTHY[1] AND RABHA W. IBRAHIM[2]

Abstract. We study a family of inequalities formed by the Fekete-Szegö design, making use of the normalized analytic functions in the open unit disk. We investigate the following functional:

$$\Psi(z) := \frac{z^{1-\vartheta}\psi'(z)}{\psi^{1-\vartheta}(z)},$$

where $\vartheta \geq 0$ acts on a domain having the starlike with respect to the boundary of the unit disk and symmetric with respect to the real axis. In addition, various presentations of the central result for functions formulated by convolution are investigated. As a special instance of this result, Fekete-Szegö issue associated with Special functions (differential operators) is studied. Moreover, by using bounds of the initial Taylor coefficients, we discussed Second Hankel determinant results.

1. Introduction

We deal with the class of normalized analytic functions denoting by Λ taking the construction series

$$\psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} \mid |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of Λ owing the univalent functions. Moreover, there is another class of analytic function in Λ taking the series

$$\varphi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + \cdots \quad (L_1 > 0),$$

Key words and phrases. Analytic functions, Bazilevic functions, subordination and superordination, Fekete-Szego inequality, fractional calculus.

2020 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30C55.

https://doi.org/10.46793/KgJMat2607.1135M

Received: August 13, 2021.

Accepted: October 14, 2024.

such that $\varphi(0) = 1$, $\varphi'(0) > 0$, which maps the unit disk Δ onto a starlike domain, which is symmetric with respect to the real axis. A subclass of $\mathcal{S}$ symbolized by $S^*(\varphi)$ and recognized by

$$\frac{z\psi'(z)}{\psi(z)} \prec \varphi(z) \quad (z \in \Delta).$$

Furthermore, a subclass of $\mathcal{S}$ symbolized by $C(\varphi)$ and defined by

$$1 + \frac{z\psi''(z)}{\psi'(z)} \prec \varphi(z) \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were formulated and investigated by Ma and Minda [14]. They have found the Fekete-Szegö inequality for the function in the class $C(\varphi)$. Since

$$\psi \in C(\varphi) \;\Leftrightarrow\; z\psi'(z) \in S^*(\varphi),$$

we get the Fekete-Szegö inequality for functions in the class $S^*(\varphi)$. A brief explanation of the Fekete-Szegö problem for the class of starlike, convex, and close-to-convex functions can be found in the most recent publication by Srivastava et al. [22]. Motivated by the classes defined above, we consider a function associated with the Bell numbers.

The Bell numbers (BNs) $\flat_n$ having the following binomial coefficients $\flat_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \flat_k$. Clearly, [3–6, 25–27]

$$\flat_0 = \flat_1 = 1, \quad \flat_2 = 2, \quad \flat_3 = 5, \quad \flat_4 = 15, \quad \flat_5 = 52 \quad \text{and} \quad \flat_6 = 203.$$

Cho et al. [9] and Kumar et al. [12] considered the function

$$\phi(z) := e^{e^z} - 1 = \sum_{n=0}^{+\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots \quad (z \in \Delta), \tag{1.2}$$

which is starlike and its coefficients generate the BNs and established the first order differential subordination relations between functions with a positive real part and starlike functions related to the Bell numbers. We now consider the function $\phi(z) := e^{e^z} - 1$ with its domain of definition as the open unit disk Δ. We shall see that the function ϕ_1, defined by

$$\phi_1(z) = z \exp\left(\int_0^z \frac{\phi(t) - 1}{t} \mathrm{d}t\right) = z + z^2 + z^3 + \frac{17}{18}z^4 - \frac{245}{288}z^5 + \cdots,$$

would serve as an extreme function in many problems.

The Fekete-Szegö inequality is obtained in this study for functions in a more extended class $\mathfrak{B}^{\vartheta}(\phi)$ of Bazilevič functions, which we describe below. Furthermore, we provide our findings with implementations to specific functions specified by the convolution class.

Definition 1.1. Let $\phi(z)$ be a starlike function given by (1.2). A function $\psi \in \Lambda$ belongs to the class $\mathfrak{B}^{\vartheta}(\phi)$ if

$$\Psi(z) = \frac{z^{1-\vartheta}\psi'(z)}{[\psi(z)]^{1-\vartheta}} \prec \phi(z) \quad (\vartheta \geq 0).$$

By fixing $\vartheta = 0$ and $\vartheta = 1$ we state the following.

Example 1.1. Let $\phi(z)$ be given by (1.2) and starlike function. A function $\psi \in \Lambda$ belongs to the class $\mathfrak{S}(\phi)$ if

$$\frac{z\psi'(z)}{\psi(z)} \prec \phi(z).$$

Example 1.2. Let $\phi(z)$ be a starlike function given by (1.2). A function $\psi \in \Lambda$ belongs to the class $\mathfrak{Q}(\phi)$ if

$$\psi'(z) \prec \phi(z) \quad (\vartheta \geq 0).$$

We request the next result.

Lemma 1.1 ([13])**.** *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, with $\operatorname{Re}(p(z)) > 0$, then the following sharp estimate holds*

$$\begin{aligned} |c_n| &\leq 2 \quad (n = 1, 2, 3, \ldots), \\ |c_2 - v c_1^2| &\leq 2 \max\{1, |2v - 1|\}, \end{aligned} \tag{1.3}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 1.2 ([14])**.** *Suppose that $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ. Then,*

- *for $v < 0$ or $v > 1$, the equality*

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2, & \text{if } v \leq 0, \\ 2, & \text{if } 0 \leq v \leq 1, \\ 4v - 2, & \text{if } v \geq 1, \end{cases}$$

  holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations;
- *for $0 < v < 1$, then equality holds if and only if*

$$p_1(z) = \frac{1+z^2}{1-z^2}$$

  or one of its rotations;
- *for $v = 0$, the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

  or one of its rotations;

- *for $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.*

Furthermore, the top bound above is sharp; it may be made better when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 0.5)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (0.5 < v \leq 1).$$

2. Coefficient Bounds and Fekete-Szegö Problem

Our main result is the following.

Theorem 2.1. *Let $\phi(z)$ be given by (1.2). If $\psi(z)$ given by (1.1) belongs to $\mathfrak{B}^{\vartheta}(\phi)$, then*

$$\begin{aligned} |a_2| \leq & \frac{1}{\vartheta+1}, \\ |a_3| \leq & \frac{1}{\vartheta+2} \max\left\{1, \left|\frac{\vartheta^2+3\vartheta+4}{2(\vartheta+1)^2}\right|\right\}. \end{aligned}$$

Further,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2+\vartheta} - \frac{\mu}{(1+\vartheta)^2} + \frac{1-\vartheta}{2(1+\vartheta)^2}, & \text{if } \mu \leq \sigma_1, \\ \frac{1}{2(2+\vartheta)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{2+\vartheta} + \frac{\mu}{(1+\vartheta)^2} - \frac{1-\vartheta}{2(1+\vartheta)^2}, & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \sigma_1 :=& \frac{(1-\vartheta)(2+\vartheta)}{2(2+\vartheta)}, \\ \sigma_2 :=& \frac{2(1+\vartheta)^2 + (1-\vartheta)(2+\vartheta)}{2(2+\vartheta)}. \end{aligned}$$

The result is sharp.

Proof. Since $\psi \in \mathfrak{B}^{\vartheta}(\phi)$ there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\frac{z^{1-\vartheta}\psi'(z)}{(\psi(z))^{1-\vartheta}} = \phi(w(z)). \tag{2.1}$$

Define the function p_1 by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

or, equivalently

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \cdots\right]. \tag{2.2}$$

Given $p_1(0) = 1$ and $\mathrm{Re}\,(p_1(z)) > 0$, p_1 is analytic in Δ. Obviously, we have

$$\phi(w(z)) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{c_1}{2}z + \frac{c_2}{2}z^2 + \frac{1}{2}\left(c_3 - \frac{11}{24}c_1^3\right)z^3 + \cdots . \tag{2.3}$$

Since

$$\begin{aligned}\frac{z^{1-\vartheta}\psi'(z)}{(\psi(z))^{1-\vartheta}} =& 1 + a_2(\vartheta+1)z + \left(\frac{(\vartheta-1)(\vartheta+2)}{2}a_2^2 + (\vartheta+2)a_3\right)z^2 \\ &+ \left\{a_4(\vartheta+3) + (\vartheta+1)(\vartheta-3)a_2a_3 + \frac{(\vartheta-1)(\vartheta-2)(\vartheta+3)}{6}a_2^3\right\}z^3 + \cdots .\end{aligned} \tag{2.4}$$

Equating the coefficients of z, z^2 in (2.3) and (2.4), we get

$$a_2 = \frac{c_1}{2(\vartheta+1)}, \tag{2.5}$$

$$\begin{aligned}a_3 =& \frac{1}{8(\vartheta+2)}\left[4c_2 - \frac{(\vartheta-1)(\vartheta+2)}{(\vartheta+1)^2}c_1^2\right] \\ =& \frac{1}{2(\vartheta+2)}\left[c_2 - c_1^2\frac{(\vartheta-1)(\vartheta+2)}{4(\vartheta+1)^2}\right].\end{aligned} \tag{2.6}$$

In view of Lemma 1.1, we have

$$|a_2| \le \frac{1}{\vartheta+1}$$

and

$$|a_3| \le \frac{1}{\vartheta+2}\max\left\{1, \left|\frac{\vartheta^2+3\vartheta+4}{2(\vartheta+1)^2}\right|\right\}.$$

Further, we have

$$a_3 - \mu a_2^2 = \frac{1}{2(2+\vartheta)}\left\{c_2 - vc_1^2\right\}, \tag{2.7}$$

where

$$v := \frac{(2+\vartheta)(2\mu+\vartheta-1)}{4(1+\vartheta)^2}.$$

Our result now follows by Lemma 1.2. The upper bounds are sharp in terms of the following conclusion. Formulate the functional $K^{\vartheta}_{\phi n}$, $n = 2, 3, \ldots$, by

$$\frac{z^{1-\vartheta}[K^{\vartheta}_{\phi n}]'(z)}{\left[K^{\vartheta}_{\phi n}(z)\right]^{1-\vartheta}} = \phi(z^{n-1}), \quad K^{\vartheta}_{\phi n}(0) = 0 = [K^{\vartheta}_{\phi n}]'(0) - 1,$$

and the function F^{ϑ}_{λ} and G^{ϑ}_{λ}, $0 \le \lambda \le 1$, by

$$\frac{z^{1-\vartheta}[F^{\vartheta}_{\lambda}]'(z)}{\left[F^{\vartheta}_{\lambda}(z)\right]^{1-\vartheta}} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F_{\lambda}(0) = 0 = F'_{\lambda}(0) - 1$$

and

$$\frac{z^{1-\vartheta}[G_\lambda^\vartheta]'(z)}{\left[G_\lambda^\vartheta(z)\right]^{1-\vartheta}} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G_\lambda(0) = 0 = G'_\lambda(0) - 1.$$

Clearly, the functions $K^\vartheta_{\phi n}, F^\vartheta_\lambda, G^\vartheta_\lambda \in \mathfrak{B}^\vartheta(\phi)$. Also, we write $K^\vartheta_\phi := K^\vartheta_{\phi 2}$. We have the following inequalities

- $\mu < \sigma_1$ or $\mu > \sigma_2$ if and only if $\psi \in K^\vartheta_\phi$ or its rotations;
- $\sigma_1 < \mu < \sigma_2$ if and only if $\psi \in K^\vartheta_{\phi 3}$ or its rotations;
- $\mu = \sigma_1$ if and only if $\psi \in F^\vartheta_\lambda$ or its rotations;
- $\mu = \sigma_2$ if and only if $\psi \in G^\vartheta_\lambda$ or its rotations.

□

Remark 2.1. • For $\sigma_1 \le \mu \le \sigma_2$, then, in view of Lemma 1.2, Theorem 2.1 can be modified.

- For σ_3, we have

$$\sigma_3 := \frac{(1+\vartheta)^2 + (1-\vartheta)(2+\vartheta)}{2(2+\vartheta)}.$$

- For $\sigma_1 \le \mu \le \sigma_3$, we get

$$|a_3 - \mu a_2^2| + \frac{(1+\vartheta)^2}{(2+\vartheta)}\left[\frac{(2\mu+\vartheta-1)(2+\vartheta)}{2(1+\vartheta)}\right]|a_2|^2 \le \frac{1}{2+\vartheta}.$$

- If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\vartheta)^2}{(2+\vartheta)}\left[2 - \frac{(2\mu+\vartheta-1)(2+\vartheta)}{2(1+\vartheta)}\right]|a_2|^2 \le \frac{1}{2+\vartheta}.$$

Theorem 2.2. *Assume that $\phi(z)$ is defined by* (1.2). *If $\psi(z)$ as in* (1.1) *belongs to $\mathfrak{B}^\vartheta(\phi)$, then for a complex number μ, we have*

$$|a_3 - \mu a_2^2| = \frac{1}{2+\vartheta}\max\left\{1, \left|\frac{(2+\vartheta)\mu}{(1+\vartheta)^2} - \frac{\vartheta^2+3\vartheta+4}{2(1+\vartheta)^2}\right|\right\}. \tag{2.8}$$

In particular,

$$|a_3 - a_2^2| = \frac{1}{2+\vartheta}\max\left\{1, \left|\frac{\vartheta}{2(1+\vartheta)}\right|\right\}.$$

Proof. Using (2.5), (2.6) and (2.7) we have

$$a_3 - \mu a_2^2 = \frac{1}{2(2+\vartheta)}\left\{c_2 - v c_1^2\right\},$$

where

$$v := \left[\frac{(2+\vartheta)(2\mu+\vartheta-1)}{4(1+\vartheta)^2}\right].$$

In view of Lemma 1.1, we have the desired assertion. □

3. Application to Functions associated with Special functions

Let $\psi(z) = z + \sum_{n=2}^{+\infty} \psi_n z^n$, $\psi_n > 0$. Since $f(z) = z + \sum_{n=2}^{+\infty} a_n z^n \in B_g^{\vartheta}(\phi)$ if and only if $(f * \psi) = z + \sum_{n=2}^{+\infty} \psi_n a_n z^n \in \mathfrak{B}_{\psi}^{\vartheta}(\phi)$, we obtain the coefficient estimate for functions in the class $\mathfrak{B}_{\psi}^{\vartheta}(\phi)$ from the corresponding estimate for functions in the class $\mathfrak{B}^{\vartheta}(\phi)$. Applying Theorem 2.1 to the function $(f * \psi)(z) = z + \psi_2 a_2 z^2 + \psi_3 a_3 z^3 + \cdots$, we get the following results, Theorem 3.1 after an obvious change of the parameter μ. For various choices of $\psi(z)$ we get different operators, which are listed below.

(a) For $\psi(z) = z + \sum_{n=2}^{+\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1},\ldots,(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1},\ldots,(\beta_s)_{n-1}(1)_{n-1}} z^n$, we get the Dziok–Srivastava operator $H_{q,s}(\alpha)f(z)$ introduced by Dziok and Srivastava [11].

(b) For $\psi(z) = \phi(a,c,z) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(c)_n} z^n$, we get the Carlson-Shaffer operator $L(a,c)f(z)$ introduced by Carlson-Shaffer [7].

(c) For $\varphi(z) = \frac{z}{(1-z)^{\lambda+1}}$, we get the Ruscheweyh operator $D^{\lambda} f(z)$ introduced by Ruschweyh [20].

(d) For $\psi(z) = z + \sum_{n=2}^{+\infty} n^m z^n$, $m \geq 0$, we get the Sălăgean operator $D^m f(z)$ introduced by Sălăgean [21].

(e) For $\psi(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, $\lambda \geq 0$, $k \in \mathbb{Z}$, we get the multiplier transformation $I(\lambda,k)$ introduced by Cho and Srivastava [10].

(f) For $\varphi(z) = z + \sum_{n=2}^{+\infty} n \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, $\lambda \geq 0$, $k \in \mathbb{Z}$, the multiplier transformation $J(\lambda,k)$ introduced by Cho and Kim [8].

(g) For $\psi(z) = z + \sum_{n=2}^{+\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n$, where $z, \alpha, \beta \in \mathbb{C}$, $\beta \neq 0, -1, -2, \ldots$ and $\mathrm{Re}\,(\beta) > 0$, $\mathrm{Re}\,(\alpha) > 0$, the Mittag-Leffler function denoted by $E_{\alpha,\beta}(\zeta)$ (see [2,17]).

(h) For $\lambda \neq 2,3,4,\ldots$, let $\psi(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n$, we get fractional derivatives and fractional integrals operator $(\Omega^{\lambda} f)(z)$ (also see [23,24].

(i) A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0,1,2,3,\ldots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}, \ldots$, respectively, where m is called the parameter. Thus, $P(\mathcal{X} = r) = \frac{m^r e^{-m}}{r!}$, $r = 0,1,2,3,\ldots$. In [18], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m,z) = z + \sum_{n=2}^{+\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n \quad (z \in \Delta),$$

where $m > 0$. By ratio test the radius of convergence of above series is infinity. Due to Porwal [18] (see [15,16,19]) we have

$$f * \mathcal{K}_m(z) = z + \sum_{n=2}^{+\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n \quad (z \in \Delta).$$

(j) The symmetric differential operator [28]

$$(\eth\psi)_\alpha^m(z) = z + \sum_{k=2}^{+\infty}[k(\alpha + (1-\alpha)(-1)^k)]^m \psi_k z^k.$$

(k) The conformable differential operator [29]

$$\begin{aligned}(\Omega\psi)^{\wp}(z) =& \frac{\kappa_1(\wp, z)}{\kappa_1(\wp, z) + \kappa_0(\wp, z)}\psi(z) + \frac{\kappa_0(\wp, z)}{\kappa_1(\wp, z) + \kappa_0(\wp, z)}\left(z\psi'(z)\right)\\ =& z + \sum_{k=2}^{+\infty}\left(\frac{\kappa_1(\wp, z) + k\kappa_0(\wp, z)}{\kappa_1(\wp, z) + \kappa_0(\wp, z)}\right)\psi_k z^k.\end{aligned}$$

(l) The hybrid fractional integro-differential operator is given by [30]

$$(\Upsilon\psi)^{\wp}(z) = z + \sum_{k=2}^{+\infty}\mathcal{K}_k(\wp)\left(\frac{\Gamma(3-\wp)\Gamma(k+1)}{\Gamma(k+2-\wp)}\right)\psi_k z^k,$$

where $\mathcal{K}_k(\wp) := \frac{\kappa_1(\wp)+k\kappa_0(\wp)}{\kappa_1(\wp)+\kappa_0(\wp)}$.

Theorem 3.1. *Assume that $\phi(z)$ is defined by* (1.2). *If $f(z)$ as in* (1.1) *belongs to $\mathfrak{B}_\psi^\vartheta(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{\psi_3}\left[\frac{1}{2+\vartheta} - \frac{\mu\psi_3}{(1+\vartheta)^2\psi_2^2} + \frac{(1-\vartheta)}{2(1+\vartheta)^2}\right], & \text{if } \mu \leq \sigma_1,\\ \frac{1}{\psi_3}\cdot\frac{1}{2(2+\vartheta)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2,\\ \frac{1}{\psi_3}\left[-\frac{1}{2+\vartheta} + \frac{\mu\psi_3}{(1+\vartheta)^2\psi_2^2} - \frac{(1-\vartheta)}{2(1+\vartheta)^2}\right], & \text{if } \mu \geq \sigma_2,\end{cases}$$

where

$$\begin{aligned}\sigma_1 :=& \frac{\psi_2^2}{\psi_3}\cdot\frac{(1-\vartheta)(2+\vartheta)}{2(2+\vartheta)},\\ \sigma_2 :=& \frac{\psi_2^2}{\psi_3}\cdot\frac{2(1+\vartheta)^2 + (1-\vartheta)(2+\vartheta)}{2(2+\vartheta)}.\end{aligned}$$

The result is sharp.

Theorem 3.2. *Let $\phi(z)$ be given by* (1.2). *If $f(z)$ given by* (1.1) *belongs to $\mathfrak{B}_\psi^\vartheta(\phi)$, then for complex μ we have*

$$|a_3 - \mu a_2^2| = \frac{1}{(2+\vartheta)\psi_3}\max\left\{1, \left|\frac{(2+\vartheta)\psi_3\mu}{(1+\vartheta)^2\psi_2^2} - \frac{\vartheta^2 + 3\vartheta + 4}{2(1+\vartheta)^2}\right|\right\}. \tag{3.1}$$

4. Second Hankel Determinant of Analytic Functions

Here, the Second Hankel determinant of analytic functions $\psi \in \mathfrak{B}^\vartheta(\phi)$.

Lemma 4.1 ([13])**.** *Let $p \in \mathcal{P}$ with $c_1 \geq 0$. Then it satisfies the series*

$$\begin{aligned}2c_2 =& c_1^2 + x(4 - c^2),\\ 4c_3 =& c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2 z),\end{aligned} \tag{4.1}$$

for some x, z, with $|x| \leq 1$ and $|z| \leq 1$.

Theorem 4.1. *Let the function $f \in \mathfrak{B}^{\vartheta}(\phi)$ be given by* (1.1).

(1) *If* $\vartheta \leq 0$,

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| + \frac{4\vartheta-36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)} \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{4(\vartheta+1)^2}.$$

(2) *If* $\vartheta \geq 0$,

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| - \left(\frac{36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right) \geq 0$$

or the conditions $\vartheta \leq 0$,

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| + \frac{4\vartheta-36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)} \geq 0,$$

then the second Hankel determinant satisfies

$$\begin{aligned}&\left|a_2a_4-a_3^2\right| \\ \leq &\frac{1}{12}\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| \\ &+ \frac{96\vartheta(\vartheta+4)-32\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}.\end{aligned}$$

(3) *If* $\vartheta > 0$,

$$\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{44\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right| - \left(\frac{36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right) \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{12}\cdot\frac{3\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right|+\frac{88\vartheta-108}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}}{\left|\frac{10}{(\vartheta+1)(\vartheta+3)}+\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}-\frac{12}{(\vartheta+2)^2}\right|-\frac{36}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}}.$$

Proof. Since $f \in \mathfrak{B}^{\vartheta}(\phi)$ and equating the coefficients of z, z^2, z^3 in (2.3) and (2.4) we get

$$a_2 = \frac{c_1}{2(\vartheta+1)}, \quad a_3 = \frac{1}{2(\vartheta+2)}\left[c_2 - \frac{c_1^2}{4}\left\{\frac{(\vartheta-1)(\vartheta+2)}{(\vartheta+1)^2}\right\}\right],$$

and further by using the above, we get

$$\begin{aligned}
a_4 = & \frac{1}{48(\vartheta+3)}\left[\left\{-1+\frac{(\vartheta-1)}{(\vartheta+1)^3}(-(\vartheta-2)(\vartheta+3)+3(\vartheta+1)(\vartheta-3))\right\}c_1^3\right. \\
& \left.-\left(\frac{12(\vartheta-3)}{(\vartheta+2)}\right)c_1c_2+24c_3\right] \\
= & \frac{1}{48(\vartheta+3)}\left[\left\{\frac{(\vartheta-1)(2\vartheta^2-7\vartheta-3)}{(\vartheta+1)^3}-1\right\}c_1^3-\left(\frac{12(\vartheta-3)}{(\vartheta+2)}\right)c_1c_2+24c_3\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_2a_4-a_3^2 = & \frac{1}{96}\left[c_1^4\left\{\frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{2(\vartheta+1)^4(\vartheta+3)}-\frac{6}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right.\right. \\
& \left.+\frac{5}{(\vartheta+1)(\vartheta+3)}-\frac{6}{(\vartheta+2)^2}\right\}+\left(\frac{24\vartheta}{(\vartheta+1)^2(\vartheta+2)(\vartheta+3)}\right)2c_2c_1^2 \\
& \left.+\frac{24c_1c_3}{(\vartheta+1)(\vartheta+3)}-\frac{24c_2^2}{(\vartheta+2)^2}\right].
\end{aligned}$$

Let

$$\begin{aligned}
d_1 = & \frac{24}{(\vartheta+1)(\vartheta+3)}, \\
d_2 = & \frac{48\vartheta(\vartheta+2)}{(\vartheta+1)^2(\vartheta+3)(\vartheta+2)^2}, \\
d_3 = & \frac{-24}{(\vartheta+2)^2}, \quad T=\frac{1}{96}, \\
d_4 = & \frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{2(\vartheta+1)^4(\vartheta+3)}-\frac{6}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}+\frac{5}{(\vartheta+1)(\vartheta+3)}-\frac{6}{(\vartheta+2)^2} \\
= & \frac{(\vartheta-1)(\vartheta^2-20\vartheta+3)}{2(\vartheta+1)^4(\vartheta+3)}-\frac{1}{(\vartheta+1)(\vartheta+3)}. \qquad (4.2)
\end{aligned}$$

Then,

$$|a_2a_4-a_3^2| = T|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|, \qquad (4.3)$$

since the function $p(e^{i\theta})$, $\theta \in R$, belongs to the class P for any $p \in P$, there is no loss of generality in assuming $c_1 \geq 0$. Write $c_1 = c$, $c \in [0,2]$. Substituting the values of c_2 and c_3, respectively, from (1.4) and (1.5) in (2.2), we obtain

$$\begin{aligned}
|a_2a_4-a_3^2| = & \frac{T}{4}\left|c^4(d_1+2d_2+d_3+4d_4)+2xc^2(4-c^2)(d_1+d_2+d_3)\right. \\
& \left.+(4-c^2)x^2(-d_1c^2+d_3(4-c^2))+2d_1c(4-c^2)(1-|x|^2)z\right|.
\end{aligned}$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (2.6) yield

$$\begin{aligned}
|a_2a_4-a_3^2| \leq & \frac{T}{4}\left[c^4\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}\right.\right. \\
& \left.+\frac{48\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)^2}-\frac{24}{(\vartheta+2)^2}\right|+\frac{48c(4-c^2)(1-\mu^2)}{(\vartheta+1)(\vartheta+3)}
\end{aligned}$$

$$
\begin{aligned}
(4.4)\qquad & +4\mu c^2(4-c^2)\left(\frac{12+24\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right) \\
& -\mu^2(4-c^2)\left(\frac{24c^2+96(\vartheta^2+4\vartheta+3)}{(\vartheta+1)(\vartheta+3)(\vartheta+2)^2}\right)\Bigg] \\
& = T\left[\frac{c^4}{4}\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}\right.\right. \\
& \left.+\frac{96\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)^2}-\frac{24}{(\vartheta+2)^2}\right|+\frac{12c(4-c^2)}{(\vartheta+1)(\vartheta+3)} \\
& +\mu c^2(4-c^2)\left(\frac{12+2\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right) \\
& \left.+\frac{6\mu^2(4-c^2)}{(\vartheta+1)(\vartheta+3)(\vartheta+2)^2}[(c-2)(-2\vartheta(\vartheta+4)+(c-6))]\right] \\
(4.5)\qquad & \equiv F(c,\mu,\vartheta).
\end{aligned}
$$

Note that for $(c,\mu,\vartheta)\in[0,2]\times[0,1]$, differentiating $F(c,\mu,\vartheta)$ in (2.8) partially with respect to μ yields

(4.6)

$$
\frac{\partial F}{\partial\mu}=T\left[\frac{c^2(4-c^2)}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}(12-2\vartheta)+\frac{(c-2)(-2\vartheta(\vartheta+4)+(c-6))}{(\vartheta+1)(\vartheta+3)(\vartheta+2)^2}[12\mu(4-c^2)]\right],
$$

then for $0<\mu<1$ and for any fixed c with $0<c<2$, it is clear from (2.9) that $\frac{\partial F}{\partial\mu}>0$, that is, $F(c,\mu,\vartheta)$ is an increasing function of μ. Therefore, for some special fixed parameters, we get $\max F(c,\mu,\vartheta)=F(c,1,\vartheta)\equiv G(c)$. In addition, we have

$$
\begin{aligned}
G(c)=\frac{1}{96}&\left[\frac{c^4}{4}\left\{\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}+\frac{88\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right.\right.\right. \\
&\left.\left.-\frac{24}{(\vartheta+2)^2}\right|-\left(\frac{72}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right)\right\}+4c^2\left(\frac{2\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right) \\
&\left.+\frac{24(12+4\vartheta(\vartheta+4))}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right].
\end{aligned}
$$

Moreover, by the conclusion

$$
\begin{aligned}
P&=\frac{1}{4}\left\{\left|\frac{2(\vartheta-1)(\vartheta^2-20\vartheta+3)}{(\vartheta+1)^4(\vartheta+3)}+\frac{20}{(\vartheta+1)(\vartheta+3)}+\frac{88\vartheta}{(\vartheta+1)^2(\vartheta+2)^2(\vartheta+3)}\right.\right. \\
&\qquad\left.\left.-\frac{24}{(\vartheta+2)^2}\right|-\left(\frac{72}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right)\right\}, \\
Q&=4\left(\frac{2\vartheta}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}\right), \\
(4.7)\qquad R&=\frac{24(12+4\vartheta(\vartheta+4))}{(\vartheta+1)(\vartheta+2)^2(\vartheta+3)}.
\end{aligned}
$$

We have,

$$|a_2a_4 - a_3^2| \leq \frac{1}{96} \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}, \\ 16P + 4Q + R, & Q \leq 0, P \geq -\frac{Q}{4}, \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8}, \end{cases}$$

where P, Q and R are given by (4.7). □

5. Conclusion

In this investigation, we define a new subclass $\mathfrak{B}^{\vartheta}(\phi)$ of normalized analytic functions in the open unit disk Δ, which is associated with Bell Numbers. We then successfully investigate several properties and characteristics, such as estimates for the first few Taylor-Maclaurin coefficients, the Fekete-Szegö problem, and the second-order Hankel determinant $H_2(2)$. Finally, we indicate a number of known operators (or special functions) listed in Section 3, that are already available in the literature on the subject and their application. The appropriate approximation for functions in the class $\mathfrak{B}^{\vartheta}(\phi)$ is used to estimate the coefficients for functions in $\mathfrak{B}^{\vartheta}_{\psi}(\phi)$.

References

[1] R. M. Ali and V. Ravichandran, *Integral operators on Ma-Minda type starlike and convex functions*, Math. Comput. Model. **53** (2011), 581–586.

[2] D. Bansal and J. K. Prajapat, *Certain geometric properties of the Mittag-Leffler functions*, Complex Var. Elliptic Equ. **61**(3) (2016), 338–350.

[3] E. T. Bell, *The iterated exponential integers*. Ann. Math. **39**(1938), 539–557.

[4] E. T. Bell, *Exponential polynomials*, Ann. Math. **35** (1934), 258–277.

[5] B. C. Berndt, *Ramanujan reaches his hand from his grave to snatch your theorems from you*, Asia Pac. Math. Newsl. **1** (2011), 8–13.

[6] E. R. Canfield, *Engel's inequality for Bell numbers*. J. Combin. Theory Ser. A **72** (1995), 184–187.

[7] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), 737–745.

[8] N. E. Cho and T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc. **40**(3) (2003), 399–410.

[9] N. E. Cho, S. Kumar, V. Kumar, V. Ravichandran and H. M. Srivastava, *Starlike functions related to the Bell numbers*, Symmetry **11**(2) (2019), Article ID 219. `https://doi.org/10.3390/sym11020219`

[10] N. E. Cho and H. M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Mod. **37**(1–2) (2003), 39–49.

[11] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **103**(1) (1999), 1–13.

[12] V. Kumar, N. E. Cho, V. Ravichandran and H. M. Srivastava, *Sharp coefficient bounds for starlike functions associated with the Bell numbers*, Math. Slovaca **69**(5) (2019), 1053–1064.

[13] R. J. Libera and E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85**(2) (1982), 225–230.

[14] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in: Z. Li, F. Ren, L. Yang and S. Zhang (Eds.), *Proceedings of the Conference on Complex Analysis*, Int. Press, 1994, 157–169.

[15] G. Murugusundaramoorthy, *Subclasses of starlike and convex functions involving Poisson distribution series*, Afr. Mat. **28** (2017), 1357–1366.

[16] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, *Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series,* Hacettepe J. Math. Stat. **45**(4) (2016), 1101–1107.

[17] S. Noreen, M. Raza and S. N. Malik, *Certain geometric properties of Mittag-Leffler functions*, J. Ineq. Appl. **94** (2019), 1–15.

[18] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal. (2014), Article ID 984135, 3 pages.

[19] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat. **27**(5) (2016), 1021–1027.

[20] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[21] G. Ş. Sălăgean, *Subclasses of univalent functions*, in: *Complex Analysis-fifth Romanian-Finnish Seminar, Part* 1*,* Bucharest, 1981, Springer, Berlin, 362–372.

[22] H. M. Srivastava, A. K. Mishra and M. K. Das, *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Variables, Theory Appl. **44** (2001), 145–163.

[23] H. M. Srivastava and S. Owa, *An application of the fractional derivative*, Math. Japon. **29** (1984), 383–389.

[24] H. M. Srivastava and S. Owa, *Univalent functions, Fractional Calculus, and Their Applications*, Halsted Press/John Wiley and Songs, Chichester/New York, 1989.

[25] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Ltd., Chichester, UK, 1984.

[26] F. Qi, *An explicit formula for the Bell numbers in terms of the Lah and Stirling numbers*. Mediterr. J. Math. **13** (2016), 2795–2800.

[27] F. Qi, *Some inequalities for the Bell numbers*. Proc. Indian Acad. Sci. Math. Sci. **127** (2017), 551–564.

[28] R. W. Ibrahim and M. Darus, *New symmetric differential and integral operators defined in the complex domain*, Symmetry **11**(7) (2019), Article ID 906.

[29] R. W. Ibrahim and J. M. Jahangiri. *Conformable differential operator generalizes the Briot-Bouquet differential equation in a complex domain*, AIMS Mathematics **4.6** (2019), 1582–1595.

[30] R. W. Ibrahim and D. Baleanu, *On a combination of fractional differential and integral operators associated with a class of normalized functions*, AIMS Math **6**(4) (2021), 4211–4226.

[1]Department of Mathematics,
Vellore Institute of Technology
Email address: `gmsmoorthy@yahoo.com`
ORCID iD: https://orcid.org/0000-0001-8285-6619

[2] Information and Communication Technology Research Group,
Scientific Research Center,
Al-Ayen University, Thi-Qar, Iraq
Email address: `rabhaibrahim@yahoo.com`
ORCID iD: https://orcid.org/0000-0001-9341-025X

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1149–1159.

FRACTIONAL CALCULUS PERTAINING TO MULTIVARIABLE I-FUNCTION

DINESH KUMAR[1] AND FRÉDÉRIC AYANT[2,3]

Abstract. In this paper, we study and investigate unified and extended fractional integral operator involving the multivariable I-function defined by Prasad, Raizada's generalized polynomial set and general class of multivariable polynomials. During the present study, we derive five theorems pertaining to Mellin transforms of these operators. Furthermore, on account of the general nature of the functions involved herein, many known and (presumably) new fractional integral operators involved simpler functions can be obtained. We also give the special case concerning the multivariable H-function.

1. Introduction and Preliminaries

Fractional calculus is a branch of mathematical analysis that deals with derivatives and integrals of arbitrary orders. Recently, it has been shown many phenomena in physics, mechanics, biology, chemistry and other sciences can be described successfully by models using mathematical tools from fractional calculus. Baleanu et al. [2] have given generalized fractional integrals for the product of two H-functions and a general class of polynomials. Certain unified fractional integrals and derivatives for a product of Aleph function and a general class of multivariable polynomials have been studied by Choi and Kumar [5]. Chaurasia and Srivastava [3], Daiya et al. [6], Kumar and Daiya [12], Kumar et al. [13] and others have studied the fractional calculus pertaining to the multivariable H-function [24]. Recently, Kumar and Ayant [9] have given formulas for fractional calculus pertaining to the multivariable I-function defined by

Key words and phrases. Multivariable I-function, generalized set polynomial, fractional integral, general class of multivariable polynomials, Mellin transform, multivariable H-function.

2020 *Mathematics Subject Classification.* Primary: 33C60, 33C99. Secondary: 44A20.

https://doi.org/10.46793/KgJMat2607.1149K

Received: June 16, 2023.

Accepted: October 14, 2024.

Prathima. The eeader can also refer to recent work on multivariable special functions, for example, see [1, 10, 11, 18].

The explicit form of Raizada's generalized polynomial set [16, Eqn.(2.3.4), p.7] is defined and represented by

$$\begin{aligned} &S_n^{\alpha,\beta,\tau}[x; r, s, q, A, B, m, k, l] \\ &= B^{qn} x^{l(m+n)} (1-\tau x^r)\, l^{m+n} \sum_{v=0}^{m+n} \sum_{e=0}^{v} \sum_{\delta=0}^{m+n} \sum_{p=0}^{\delta} \frac{(-1)^\delta (-\delta)_p (\alpha)_\delta}{p!\,\delta!\,e!\,v!} \\ &\times \frac{(-v)_e (-\alpha-qn)_p}{(1-\alpha-\delta)_p} \left(-\frac{\beta}{\tau} - sn\right)_v \left(\frac{p+k+re}{l}\right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r}\right)^v \left(\frac{Ax}{B}\right)^\delta . \end{aligned} \tag{1.1}$$

It may be pointed out here that the polynomial set defined by (1.1) is very general in nature and it unifies and extends a number of classical polynomials.

The generalized polynomials defined by Srivastava [23], are given in the following manner:

$$\begin{aligned} &S_{N_1,\dots,N_s}^{M_1,\dots,M_s}[y_1,\dots,y_s] \\ &= \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \times \cdots \times \frac{(-N_s)_{M_sK_s}}{K_s!} A[N_1,K_1;\dots;N_s,K_s]\, y_1^{K_1} \cdots y_s^{K_s}, \end{aligned}$$

where $M_1, \dots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \dots; N_s, K_s]$ are arbitrary constants, real or complex.

The multivariable I-function [15] is an extension of the multivariable H-function [24]. It is defined in term of multiple Mellin-Barnes type integral, given by

$$\begin{aligned} I(z_1,\dots,z_r) = &I^{0,n_2;0,n_3;\dots;0,n_r:m',n';\dots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\dots;p_r,q_r:p',q';\dots;p^{(r)},q^{(r)}} \left(\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{l} \left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};\dots; \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};\dots; \end{array} \right. \\ &\left. \begin{array}{l} \left(a_{rj};\alpha'_{rj},\dots,\alpha^{(r)}_{rj}\right)_{1,p_r} : \left(a'_j,\alpha'_j\right)_{1,p'};\dots;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \\ \left(b_{rj};\beta'_{rj},\dots,\beta^{(r)}_{rj}\right)_{1,q_r} : \left(b'_j,\beta'_j\right)_{1,q'};\dots;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \end{array} \right), \\ = &\frac{1}{(2\pi\omega)^r} \int_{\mathfrak{L}_1} \cdots \int_{\mathfrak{L}_r} \phi(s_1,\dots,s_r) \prod_{i=1}^{r} \theta_i(s_i)\, z_i^{s_i}\, \mathrm{d}s_1 \cdots \mathrm{d}s_r. \end{aligned} \tag{1.2}$$

For the existence and convergence conditions of defined integral of the above function, see Prasad [15].

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.2) can be obtained by extension of the corresponding conditions for multivariable H-function, given by

$$|\arg z_i| < \frac{1}{2}\Omega_i\,\pi,$$

where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right)$$
$$+ \cdots + \left(\sum_{k=1}^{n_r} \alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r} \alpha_{rk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r} \beta_{rk}^{(i)} \right),$$

where $i = 1, \ldots, r$ and $z_i \in \mathbb{C} \setminus \{0\}$.

Throughout the present paper, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may express the asymptotic expansion in the following convenient form:

$$I(z_1, \ldots, z_r) = 0\left(|z_1|^{\gamma_1'}, \ldots, |z_r|^{\gamma_r'} \right), \quad \max\{|z_1|, \ldots, |z_r|\} \to 0,$$

$$I(z_1, \ldots, z_r) = 0\left(|z_1|^{\beta_1'}, \ldots, |z_r|^{\beta_s'} \right), \quad \min\{|z_1|, \ldots, |z_r|\} \to +\infty,$$

where $k = 1, \ldots, z,\ \alpha_k' = \min\left[\mathrm{Re}\left(b_j^{(k)} / \beta_j^{(k)} \right) \right],\ j = 1, \ldots, m_k$, and

$$\beta_k' = \max\left[\mathrm{Re}\left(\left(a_j^{(k)} - 1 \right) / \alpha_j^{(k)} \right) \right], \quad j = 1, \ldots, n_k.$$

In this paper, we also use the following notations:

$$A = (a_{2k}, \alpha_{2k}', \alpha_{2k}'')_{1,p_2}; \ldots; \left(a_{(r-1)k}, \alpha_{(r-1)k}', \alpha_{(r-1)k}'', \ldots, \alpha_{(r-1)k}^{(r-1)} \right)_{1,p_{r-1}},$$
$$B = (b_{2k}, \beta_{2k}', \beta_{2k}'')_{1;q_2}; \ldots; \left(b_{(r-1)k}, \beta_{(r-1)k}', \beta_{(r-1)k}'', \ldots, \beta_{(r-1)k}^{(r-1)} \right)_{1,q_{r-1}},$$
$$\mathbb{A} = (a_{sk}; \alpha_{rk}', \alpha_{rk}'', \ldots, \alpha_{rk}^{r})_{1,p_r}; \quad \mathbb{B} = (b_{rk}; \beta_{rk}', \beta_{rk}'', \ldots, \beta_{rk}^{r})_{1,q_r},$$
$$A' = (a_k', \alpha_k')_{1,p'}; \ldots; \left(a_k^{(r)}, \alpha_k^{(r)} \right)_{1,p^{(r)}}; \quad B' = (b_k', \beta_k')_{1,q'}; \ldots; \left(b_k^{(r)}, \beta_k^{(r)} \right)_{1,q^{(r)}}.$$

The Mellin transform of $f(x)$ will be denoted by $M[f(x)]$ or $F(s)$. If p and y are real, we write $s = p^{-1} + iy$. If $p \geq 1,\ f(x) \in \mathcal{L}_p(0, +\infty)$, then, for $p = 1$, we have

$$M[f(x)] = F(s) = \int_0^{+\infty} x^{s-1} f(x) \mathrm{d}x \quad \text{and} \quad f(x) = \frac{1}{2i\pi} \int_{\mathcal{L}} F(s) x^{-s} \mathrm{d}s.$$

For $p > 1$,

$$M[f(x)] = F(s) = l.i.m. \int_{1/x}^{x} x^{s-1} f(x) \mathrm{d}x,$$

where $l.i.m.$ denotes the usual limit in the mean for $\mathcal{L}_p$-spaces.

2. Definitions

In the context of fractional calculus, extended fractional integral operators can enhance our understanding and application of fractional integrals, especially when dealing with functions that exhibit complex behavior.

The pair of new extended fractional integral operators are defined by the following equations:

$$Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right] = tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta} I\left(\begin{array}{c|c} \gamma_1 \upsilon_1 & A;\mathbb{A}:A' \\ \vdots & \\ \gamma_n \upsilon_n & B;\mathbb{B}:B' \end{array}\right)$$
$$\times \prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}; r_j, s_j, q_j, A_j, B_j, m_j, k_j, l_j\right]$$
$$\times \prod_{j=1}^{r} S_{N_1^{(j)},\dots,N_s^{(j)}}^{M_1^{(j)},\dots,M_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}} \end{array}\right)\Psi\left(\frac{y^t}{x^t}\right) f(y)\,\mathrm{d}y \tag{2.1}$$

and

$$R_{\gamma_n}^{\rho,\beta}\left[f(x)\right] = tx^{\rho}\int_x^{+\infty} y^{-\rho-t\beta-1}\left(y^t-x^t\right)^{\beta} I\left(\begin{array}{c|c} \gamma_1 \mu_1 & A;\mathbb{A}:A' \\ \vdots & \\ \gamma_n \mu_n & B;\mathbb{B}:B' \end{array}\right)$$
$$\times \prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}; r_j, s_j, q_j, A_j, B_j, m_j, k_j, l_j\right]$$
$$\times \prod_{j=1}^{r} S_{N_1^{(j)},\dots,N_s^{(j)}}^{M_1^{(j)},\dots,M_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}} \end{array}\right)\Psi\left(\frac{x^t}{y^t}\right) f(y)\,\mathrm{d}y, \tag{2.2}$$

where $\upsilon_i = \left(\frac{y^t}{x^t}\right)^{u_i}\left(1-\frac{y^t}{x^t}\right)^{v_i}$, $\mu_i = \left(\frac{x^t}{y^t}\right)^{u_i}\left(1-\frac{x^t}{y^t}\right)^{v_i}$, t, u_i and v_i, $g_i^{(j)}, h_i^{(j)}$, a_j and b_j are positive numbers.

The kernels $\Psi\left(\frac{y^t}{x^t}\right)$ and $\Psi\left(\frac{x^t}{y^t}\right)$ appearing in (2.1) and (2.2), respectively, are assumed to be continuous functions such that integrals make sense for a wide class of functions f.

The existence conditions of these operators are given as follows:

(a) $f \in \mathcal{L}_p(0,+\infty)$;

(b) $1 \le p, q < +\infty,\ p^{-1}+q^{-1}=1$;

(c) $\mathrm{Re}\left[\alpha + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} u_i \min_{1\le j\le m^{(i)}} \mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1}$;

(d) $\mathrm{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1}$;

(e) $\mathrm{Re}\left[\rho + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -p^{-1}$,

where $i = 1,\dots,k$. Condition (a) ensures that both operators defined in (2.1) and (2.2) exist and belong to $\mathcal{L}_p(0,+\infty)$. These operators are extensions of fractional integral operators defined and studied by several authors like Erdélyi [7], Love [14], Saigo et al. [19], Saxena and Kiryakova [20], Saxena and Kumbhat [21,22], etc.

3. Main Results

In this section, we investigate a unified and extended fractional integral operator involving the multivariable I-function defined by Prasad, Raizada's generalized polynomial set and a general class of multivariable polynomials. Here, we derive results pertaining to the Mellin transforms of these operators.

Theorem 3.1. *If $f \in \mathcal{L}_p(0,+\infty)$, $1 \leq p \leq 2$, or $f \in \mathcal{L}_p(0,+\infty)$, $p > 2$, and the following conditions are satisfied: $p^{-1}+q^{-1}=1$,*

$$\operatorname{Re}\left[\alpha + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} u_i \min_{1\leq j\leq m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\leq j\leq m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

and the integrals involved are absolutely convergent, then

$$M\left\{Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\right\} = M\left\{f(x)\right\}\, R_{\gamma_n}^{\alpha-s+1,\beta}\left[1\right], \tag{3.1}$$

where $M_p(0,+\infty)$ stands for the class of all functions f from $\mathcal{L}_p(0,+\infty)$ with $p>2$, which are the inverse Mellin-transforms of functions from $\mathcal{L}_p(-\infty,+\infty)$.

Proof. By making use of the Mellin transform of (2.1), we get

$$\begin{aligned}
M\left\{Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\right\} = \int_0^{+\infty} x^{s-1}\Bigg\{ & tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta} I\left(\begin{array}{c|c} \gamma_1 v_1 & A;\mathbb{A}:A' \\ \vdots & \\ \gamma_n v_n & B;\mathbb{B}:B'\end{array}\right) \\
& \times \prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}; r_j, s_j, q_j, A_j, B_j, m_j, k_j, l_j\right] \\
& \times \prod_{j=1}^{r} S_{N_1^{(j)},\ldots,N_s^{(j)}}^{M_1^{(j)},\ldots,M_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}}\end{array}\right)\Psi\left(\frac{y^t}{x^t}\right)f(y)\mathrm{d}y\Bigg\}\,\mathrm{d}x.
\end{aligned}$$

By interchanging the order of integration, which is permissible under the conditions, the result (3.1) follows in view of (2.2). □

Theorem 3.2. *If $f \in \mathcal{L}_p(0,+\infty)$, $1 \leq p \leq 2$, or $f \in \mathbb{L}_p(0,+\infty)$, $p > 2$, and the following conditions are satisfied : $p^{-1}+q^{-1}=1$,*

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\leq j\leq m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

$$\mathrm{Re}\left[\rho + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}v_i\min_{1\le j\le m^{(i)}}\mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-p^{-1},$$

and the integrals involved are absolutely convergent, then

$$M\left\{R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]\right\}=M\left\{f(x)\right\}\,Q_{\gamma_n}^{\rho+s-1,\beta}\left[1\right]. \tag{3.2}$$

Proof. By making use of Mellin transform of (2.2), we get

$$\begin{aligned}M\left\{R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]\right\}=\int_0^{+\infty}x^{s-1}\Bigg\{&tx^{\rho}\int_x^{+\infty}y^{-\rho-t\beta-1}\left(y^t-x^t\right)^{\beta}I\left(\begin{array}{c|c}\gamma_1\mu_1 & A;\mathbb{A}:A'\\ \vdots & \\ \gamma_n\mu_n & B;\mathbb{B}:B'\end{array}\right)\\ &\times\prod_{j=1}^{k}S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j};r_j,s_j,q_j,A_j,B_j,m_j,k_j,l_j\right]\\ &\times\prod_{j=1}^{r}S_{N_1^{(j)},\dots,N_s^{(j)}}^{M_1^{(j)},\dots,M_s^{(j)}}\begin{pmatrix}z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}}\\ \vdots\\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}}\end{pmatrix}\Psi\left(\frac{x^t}{y^t}\right)f(y)\,\mathrm{d}y\Bigg\}\,\mathrm{d}x.\end{aligned}$$

By interchanging the order of integration, which is permissible under the conditions, the result (3.2) follows in view of (2.1). $\square$

Theorem 3.3. *If $f\in\mathcal{L}_p\left(0,+\infty\right)$, $v\in\mathcal{L}_p\left(0,+\infty\right)$, and the following conditions are satisfied:*

$$\mathrm{Re}\left[\alpha + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}u_i\min_{1\le j\le m^{(i)}}\mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1},$$

$$\mathrm{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}v_i\min_{1\le j\le m^{(i)}}\mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1},$$

$p^{-1}+q^{-1}=1$, *and the integrals involved are absolutely convergent, then*

$$\int_0^{+\infty}v(x)\,Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\mathrm{d}x=\int_0^{+\infty}f(x)\,R_{\gamma_n}^{\alpha,\beta}\left[v(x)\right]\mathrm{d}x. \tag{3.3}$$

Proof. The result of (3.3) can be obtained in view of (2.1) and (2.2). $\square$

4. Inversion Formulae

In this section, we we provide inversion formulas for the main results.

Theorem 4.1. *If $f\in\mathcal{L}_p\left(0,+\infty\right)$, $1\le p\le 2$, or $f(x)\in\mathcal{L}_p\left(0,+\infty\right)$, $p>2$, and the following conditions are satisfied: $p^{-1}+q^{-1}=1$,*

$$\mathrm{Re}\left[\alpha + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right]+t\sum_{i=1}^{n}u_i\min_{1\le j\le m^{(i)}}\mathrm{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right]>-q^{-1},$$

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

and the integrals involved are absolutely convergent and

$$Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right] = v(x), \tag{4.1}$$

then

$$f(x) = \int_0^{+\infty} y^{-1}\left[v(y)\right]\left[h\left(\frac{x}{y}\right)\right] \mathrm{d}y, \tag{4.2}$$

where

$$\begin{aligned} h(x) =& \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} y^{-1}\frac{x^{-s}}{R(s)}\,\mathrm{d}s, \\ R(s) =& R_{\gamma_n}^{\alpha-s+1,\beta}\left[1\right]. \end{aligned} \tag{4.3}$$

Proof. By taking the Mellin transform of (4.1) and then applying Theorem 3.1, we get

$$M\left\{f(x)\right\} = \frac{M\left\{v(x)\right\}}{R(s)},$$

which on inverting leads to

$$f(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{M\left\{v(x)\right\}}{R(s)}\,\mathrm{d}s = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{R(s)}\left\{\int_0^{+\infty}\left[v(y)\right]\mathrm{d}y\right\}\mathrm{d}s.$$

Upon interchanging the order of integration, we have

$$f(x) = \int_0^{+\infty}\frac{v(y)}{y}\left\{\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^{s}\frac{1}{R(s)}\,\mathrm{d}s\right\}\mathrm{d}y.$$

Now in view of (4.3), we obtain the desired result (4.2). □

Theorem 4.2. *If $f \in \mathcal{L}_p(0,+\infty)$, $1 \le p \le 2$, or $f \in \mathcal{L}_p(0,+\infty)$, $p > 2$, and the following conditions are satisfied: $p^{-1}+q^{-1}=1$,*

$$\operatorname{Re}\left[\beta + tb_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} v_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -q^{-1},$$

$$\operatorname{Re}\left[\rho + ta_i\left(l_i\left(m_i+n_i\right)+e_i+r_is_in_i\right)\right] + t\sum_{i=1}^{n} u_i \min_{1\le j\le m^{(i)}} \operatorname{Re}\left[\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > -p^{-1},$$

and the integrals involved are absolutely convergent and

$$R_{\gamma_n}^{\rho,\beta}\left[f(x)\right] = w(x), \tag{4.4}$$

then

$$f(x) = \int_0^{+\infty} y^{-1}\left[w(y)\right]\left[g\left(\frac{x}{y}\right)\right] \mathrm{d}y, \tag{4.5}$$

(4.6)
$$g(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} y^{-1} \frac{x^{-s}}{T(s)} \, \mathrm{d}s$$

and

$$T(s) = Q_{\gamma_n}^{\rho+s-1,\beta}[1].$$

Proof. By taking the Mellin transform into account of (4.4) and then applying Theorem 3.2, we get

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{M\{w(x)\}}{T(s)} \, \mathrm{d}s = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{T(s)} \left\{ \int_0^{+\infty} [w(y)] \, \mathrm{d}y \right\} \mathrm{d}s,$$

on interchanging the order of integration, then we have

$$f(x) = \int_0^{+\infty} \frac{w(y)}{y} \left\{ \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{y}\right)^s \frac{1}{T(s)} \, \mathrm{d}s \right\} \mathrm{d}y.$$

Now in view of (4.6), we obtain the desired result (4.5). □

5. General Properties

The properties given below are consequences of Definitions 2.1 and 2.2

$$\begin{aligned}
x^{-1} Q_{\gamma_n}^{\alpha,\beta}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] &= R_{\gamma_n}^{\alpha,\beta}[f(x)], \\
x^{-1} R_{\gamma_n}^{\rho,\beta}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] &= Q_{\gamma_n}^{\rho,\beta}[f(x)], \\
x^{\mu} Q_{\gamma_n}^{\alpha,\beta}[f(x)] &= Q_{\gamma_n}^{\alpha-\mu,\beta}[x^{\mu} f(x)], \\
x^{\mu} R_{\gamma_n}^{\rho,\beta}[f(x)] &= R_{\gamma_n}^{\rho+\mu,\beta}[x^{\mu} f(x)].
\end{aligned}$$

The properties given below express the homogeneity of the operators Q and R, respectively. If $Q_{\gamma_n}^{\alpha,\beta}[f(x)] = v(x)$, then $Q_{\gamma_n}^{\alpha,\beta}[f(cx)] = v(cx)$, and $R_{\gamma_n}^{\rho,\beta}[f(x)] = w(x)$, then $R_{\gamma_n}^{\rho,\beta}[f(cx)] = w(cx)$.

6. Multivariable H-Function

In this section, we give special case concerning multivariable H-function.

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad reduces to multivariable H-function [4, 6, 13], see also [8, 17]. We obtain the two following operators.

$$\begin{aligned}
Q_{\gamma_n}^{\alpha,\beta}[f(x)] =& t x^{-\alpha-t\beta-1} \int_0^x y^{\alpha} (x^t - y^t)^{\beta} H \left(\begin{array}{c|c} \gamma_1 v_1 & \mathbb{A} : A' \\ \vdots & \\ \gamma_n v_n & \mathbb{B} : B' \end{array} \right) \\
&\times \prod_{j=1}^{k} S_{n_j}^{\alpha_j,\beta_j,\tau_j}\left[z_j \left(\frac{y^t}{x^t}\right)^{a_j} \left(1 - \frac{y^t}{x^t}\right)^{b_j} ; r_j, s_j, q_j, A_j, B_j, m_j, k_j, l_j \right]
\end{aligned}$$

$$\times \prod_{j=1}^{r} S^{M_1^{(j)},\dots,M_s^{(j)}}_{N_1^{(j)},\dots,N_s^{(j)}} \begin{pmatrix} z_1^{(j)} \left(\frac{y^t}{x^t}\right)^{g_1^{(j)}} \left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{y^t}{x^t}\right)^{g_s^{(j)}} \left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}} \end{pmatrix} \Psi\left(\frac{y^t}{x^t}\right) f(y)\,\mathrm{d}y,$$

under the same notations and conditions that (2.1) with $U = V = A = B = 0$.

$$R^{\rho,\beta}_{\gamma_n}[f(x)] = tx^{\rho} \int_x^{+\infty} y^{-\rho-t\beta-1} \left(y^t - x^t\right)^{\beta} H \left(\begin{array}{c|c} \gamma_1\mu_1 & \mathbb{A} : A' \\ \vdots & \\ \gamma_n\mu_n & \mathbb{B} : B' \end{array} \right)$$

$$\times \prod_{j=1}^{k} S^{\alpha_j,\beta_j,\tau_j}_{n_j} \left[z_j \left(\frac{x^t}{y^t}\right)^{a_j} \left(1-\frac{x^t}{y^t}\right)^{b_j} ; r_j, s_j, q_j, A_j, B_j, m_j, k_j, l_j \right]$$

$$\times \prod_{j=1}^{r} S^{M_1^{(j)},\dots,M_s^{(j)}}_{N_1^{(j)},\dots,N_s^{(j)}} \begin{pmatrix} z_1^{(j)} \left(\frac{x^t}{y^t}\right)^{g_1^{(j)}} \left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{x^t}{y^t}\right)^{g_s^{(j)}} \left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}} \end{pmatrix} \Psi\left(\frac{x^t}{y^t}\right) f(y)\,\mathrm{d}y,$$

under the same notations and conditions as given in (2.2) with $U = V = A = B = 0$. We can obtain the similar theorems and properties for multivariable H-function concerning multivariable I-function.

7. Concluding Remarks

Fractional calculus involving multivariable I-functions is a rich area of study that combines advanced mathematical concepts with practical applications. Understanding the definitions and methods of fractional differentiation and integration is essential for effectively leveraging these tools in real-world problems. The functions involved in the results established in the present paper are of a unified and general nature, hence a large number of known results lying in the literature follow as particular cases. Further, on suitable specifications of the parameters involved, many new results involving simpler functions can also be derived.

Acknowledgements. The first author (DK) would like to thank Agriculture University Jodhpur for supporting and encouraging this work.

References

[1] F. Y. Ayant and D. Kumar, *Generating relations and multivariable Aleph-function*, Analysis **38**(3) (2018), 137–143. https://doi.org/10.1515/anly-2017-0054

[2] D. Baleanu, D. Kumar and S. D. Purohit, *Generalized fractional integrals of product of two H-functions and a general class of polynomials*, Int. J. Comput. Math. **93**(8) (2016), 1320–1329. https://doi.org/10.1080/00207160.2015.1045886

[3] V. B. L. Chaurasia and A. Srivastava, *A unified approach to fractional calculus pertaining to H-functions*, Soochow J. Math. **33**(2) (2007), 211–221.

[4] J. Choi, J. Daiya, D. Kumar and R. K. Saxena, *Fractional differentiation of the product of Appell function F_3 and multivariable H-function*, Commun. Korean Math. Soc. **31**(1) (2016), 115–129. https://doi.org/10.4134/CKMS.2016.31.1.115

[5] J. Choi and D. Kumar, *Certain unified fractional integrals and derivatives for a product of Aleph function and a general class of multivariable polynomials*, J. Inequal. Appl. **2014** (2014), 1–15. https://doi.org/10.1186/1029-242X-2014-499

[6] J. Daiya, J. Ram and D. Kumar, *The multivariable H-function and the general class of Srivastava polynomials involving the generalized Mellin-Barnes contour integrals*, Filomat **30**(6) (2016), 1457–1464. https://doi.org/10.2298/FIL1606457D

[7] A. Erdélyi, *On some functional transformations*, Univ. Politec. Torino Rend. Sem. Mat. **10** (1950-51), 217–234.

[8] R. K. Gupta, B. S. Shaktawat and D. Kumar, *On generalized fractional differentials involving product of two H-functions and a general class of polynomials*, J. Rajasthan Acad. Phys. Sci. **15**(4) (2016), 327–344.

[9] D. Kumar and F. Y. Ayant, *Fractional calculus pertaining to multivariable I-function defined by Prathima*, J. Appl. Math. Stat. Inform. **15**(2) (2019), 61–73. https://doi.org/10.2478/jamsi-2019-0009

[10] D. Kumar and F. Y. Ayant, *Some double integrals involving multivariable I-function*, Acta Univ. Apulensis Math. Inform. **58**(2) (2019), 35–43. https://doi.org/10.17114/j.aua.2019.58.03

[11] D. Kumar, F. Y. Ayant and D. Kumar, *A new class of integrals involving generalized hypergeometric function and multivariable Aleph-function*, Kragujevac J. Math. **44**(4) (2020), 539–550. https://doi.org/10.46793/KgJMat2004.539K

[12] D. Kumar and J. Daiya, *Fractional calculus pertaining to generalized H-functions*, Global Journal of Science Frontier Research: F, Mathematics and Decision Sciences **14**(3) (2014), 25–35.

[13] D. Kumar, S. D. Purohit and J. Choi, *Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials*, J. Nonlinear Sci. Appl. **9** (2016), 8–21. https://doi.org/10.1007/s13370-021-00885-7

[14] E. R. Love, *Some integral equations involving hypergeometric functions*, Proc. Edinb. Math. Soc. **3**(15) (1967), 169–198.

[15] Y. N. Prasad, *Multivariable I-function*, Vijnana Parishad Anusandhan Patrika **29** (1986), 231–237.

[16] S. K. Raizada, *A study of unified representation of special functions of mathematical physics and their use in statistical and boundary value problem*, Ph.D. Thesis, Bundlkhand Univ., India 1991.

[17] J. Ram and D. Kumar, *Generalized fractional integration involving Appell hypergeometric of the product of two H-functions*, Vijanana Parishad Anusandhan Patrika **54**(3) (2011), 33–43.

[18] N. Sahni, D. Kumar, F. Y. Ayant and S. Singh, *A transformation involving basic multivariable I-function of Prathima*, Journal of Ramanujan Society of Mathematics and Mathematical Sciences **8**(2), (2021), 95–108.

[19] M. Saigo, R. K. Saxena and J. Ram, *On the fractional calculus operator associated with the H-function*, Ganita Sandesh **6**(1) (1992), 36–47.

[20] R. K. Saxena and V. S. Kiryakova, *On relation between the two-dimensional H-transforms in terms of Erdélyi-Kober operators*, Math. Balkanica **6** (1992), 133–140.

[21] R. K. Saxena and R. K. Kumbhat, *Fractional integration operators of two variables*, Proc. Indian Acad. Sci. **78** (1973), 177–186.

[22] R. K. Saxena and R. K. Kumbhat, *Integral operators involving H-function*, Indian J. Pure Appl. Math. **5** (1974), 1–6.

[23] H. M. Srivastava, *A multi-linear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial*, Pacific. J. Math. **177** (1985), 183–191.
[24] H. M. Srivastava and R. Panda, *Some bilateral generating functions for a class of generalized hypergeometric polynomials*, J. Reine Angew. Math. **283/284** (1976), 265–274.

[1]Department of Applied Sciences,
College of Agriculture-Jodhpur,
Agriculture University Jodhpur, Jodhpur-342304, INDIA
Email address: dinesh_dino03@yahoo.com
ORCID iD: https://orcid.org/0000-0001-5415-1777

[2]Collége Jean L'herminier,
Allée des Nymphéas, 83500 La Seyne-sur-Mer, FRANCE
[3]Department VAR, Avenue Joseph Raynaud Le parc Fleuri,
83140 Six-Fours-les-Plages, FRANCE
Email address: fredericayant@gmail.com
ORCID iD: https://orcid.org/0000-0003-0651-294X

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1161–1169.

MOLECULAR TREES WITH EXTREMAL VALUES OF THE SECOND SOMBOR INDEX

ZIKAI TANG[1] AND HANYUAN DENG[1*]

Abstract. A new geometric background of graph invariants was introduced by Gutman, of which the simplest is the second Sombor index SO_2, defined as $SO_2 = SO_2(G) = \sum_{uv\in E} \frac{|d_G^2(u)-d_G^2(v)|}{d_G^2(u)+d_G^2(v)}$, where $G = (V, E)$ is a simple graph and $d_G(v)$ denotes the degree of v in G. In this paper, the chemical applicability of the second Sombor index is investigated and it is shown that the second Sombor index is useful in predicting physicochemical properties with high accuracy compared to some well-established and often used indices. Also, we obtain a bound for the second Sombor index among all (molecular) trees with fixed numbers of vertices, and characterize those molecular trees achieving the extremal value.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$, and denote by $n = |V(G)|$ and $m = |E(G)|$ the number of vertices and edges, respectively. The degree of a vertex v in G, denoted by $d_G(v)$ or $d(u)$, is the number of its neighbors. If the vertices u and v are adjacent, then the edge connecting them is labeled by $e = uv$. In the mathematical and chemical literature, several dozens of vertex-degree-based (VDB) graph invariants (usually referred to as ”topological indices“) have been introduced and extensively studied [1–5]. Their general formula is

$$TI(G) = \sum_{uv\in E} F(d_G(u), d_G(v)),$$

where $F(x, y)$ is a function with the property $F(x, y) = F(y, x)$.

Key words and phrases. Second Sombor index, tree, molecular tree, extremal value.
2020 *Mathematics Subject Classification.* Primary: 05C35. Secondary: 05C92, 05C05, 05C07.
https://doi.org/10.46793/KgJMat2607.1161T
Received: December 04, 2022.
Accepted: October 16, 2024.

TABLE 1. Experimental values of the second Sombor index of for octane isomers

Molecule	SO_2	Molecule	SO_2
octane	1.2	2-methyl-heptane	2.5846
3-methyl-heptane	2.7692	4-methyl-heptane	2.7692
3-ethyl-hexane	2.9538	2,2-dimethyl-hexane	3.8471
2,3-dimethyl-hexane	3.3846	2,4-dimethyl-hexane	4.1538
2,5-dimethyl-hexane	3.9692	3,3-dimethyl-hexane	4.1647
3,4-dimethyl-hexane	3.5692	2-methyl-3-ethyl-pentane	3.5692
3-methyl-3-ethyl-pentane	4.4824	2,2,3-trimethyl-pentane	4.7117
2,2,4-trimethyl-pentane	5.2317	2,3,3-trimethyl-pentane	4.8447
2,3,4-trimethyl-pentane	4	2,2,3,3-tetramethylbutane	5.2941

Recently in [2] Gutman showed that geometry-based reasonings reveal the geometric background of several classical topological indices (Zagreb, Albertson) and introduced a series of new Sombor-index-like VDB invariants, denoted below by $SO_1, SO_2, \ldots, SO_6$. The second Sombor index SO_2 of a graph G is defined as

$$SO_2 = SO_2(G) = \sum_{uv \in E} \frac{|d_G^2(u) - d_G^2(v)|}{d_G^2(u) + d_G^2(v)}. \tag{1.1}$$

Also, Gutman pointed out at the end of the paper [2] that it would be interesting to examine the properties of these geometry-based topological indices, and see if these are usable in applications.

A molecular tree is a tree of maximum degree at most four. In this paper, the chemical applicability of the second Sombor index is investigated and it is shown that the second Sombor index is useful in predicting physicochemical properties with high accuracy compared to some well-established and often used indices. Also, a bound for the second Sombor index among all (molecular) trees with a fixed number of vertices is obtained, and those molecular trees achieving the extremal value are characterized.

2. The Chemical Applicability of the Second Sombor Index

In this section, the chemical applicability of the second Sombor index is investigated. We consider the data set of octane isomers for such testing and corresponding experimental values of physico-chemical properties are collected from http: //www. moleculardescriptors.eu/dataset/dataset. htm. First, we give experimental values of the second Sombor index of octane isomers, which are listed in Table 1, where there are two pairs of octane isomers with identical values of the second Sombor index since they have the same degree coordinates.

By the experimental values of AcenFac, S, SNar and HNar of octane isomers (from http://www.moleculardescriptors.eu/dataset/dataset.htm) and Table 1, we find the correlation of AcenFac, S, SNar and HNar with the second Sombor index SO_2 for

TABLE 2. The square of correlation coefficient of the second Sombor index with AcenFac, S, SNar and HNar.

	AcenFac	S	SNar	HNar
SO_2	0.9202	0.8433	0.9356	0.9512

octane isomers. The data related to octanes are listed in Table 2. The following equations give the regression models for the second Sombor index.

$$\begin{aligned} AcenFac =& 0.4536 - 0.0314 \times SO_2, \\ S =& 119.1755 - 3.6697 \times SO_2, \\ SNar =& 4.6576 - 0.3003 \times SO_2, \\ HNar =& 1.7137 - 0.0815 \times SO_2. \end{aligned}$$

Figure 1 (1)–(4) reveal the strength of structure property relationship between SO_2 and acentric factor, entropy, SHar and HNar, respectively.

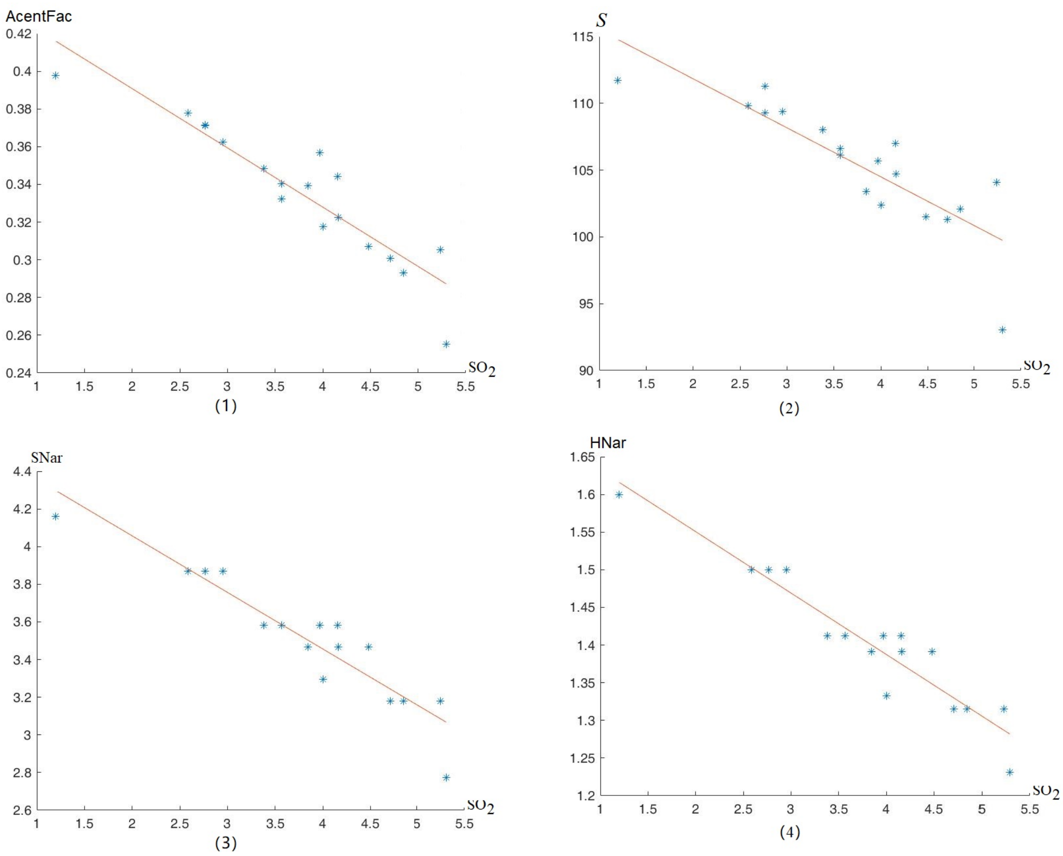

FIGURE 1. Linear fittings of the second Sombor index (SO_2) with AcenFac, S, SNar and HNar for octane isomers.

Table 3. The square of correlation coefficient of SO_2 with some existing indices.

	SO	M_1	M_2	F	R	SCI	SDD	M_N
SO_2	0.918	0.9201	0.8679	0.9022	0.8969	0.9150	0.8820	0.9123

Abbreviations: SO, Sombor index; SCI, sum connectivity index; SDD, symmetric division degree index.

By corresponding experimental values [6], we get the correlations of SO_2 with some existing indices like Sombor index(SO), first (M_1) and second (M_2) Zagreb indices, forgotten topological index (F), connectivity index (R), sum connectivity index (SCI), symmetric division degree index (SDD) and neighborhood Zagreb index (M_N), which are listed in Table 3.

All these facts above show that the second Sombor index is useful in predicting physicochemical properties.

3. Extremal Value of the Second Sombor Index of (Molecular) Trees

Let $G=(V,E)$ be a graph with order n and size m. By Definition 1.1 of the second Sombor index of a graph, we have

$$SO_2(G)=\sum_{uv\in E}\frac{\frac{\max\{d_G^2(u),d_G^2(v)\}}{\min\{d_G^2(u),d_G^2(v)\}}-1}{\frac{\max\{d_G^2(u),d_G^2(v)\}}{\min\{d_G^2(u),d_G^2(v)\}}+1}=m-\sum_{uv\in E}\frac{2}{\frac{\max\{d_G^2(u),d_G^2(v)\}}{\min\{d_G^2(u),d_G^2(v)\}}+1}. \tag{3.1}$$

Note that $1\leq\frac{\max\{d_G^2(u),d_G^2(v)\}}{\min\{d_G^2(u),d_G^2(v)\}}\leq\frac{\Delta^2}{\delta^2}$, where δ and Δ are the minimum and maximum degree in graph G, respectively. From (3.1), we straightforwardly obtain the following.

Theorem 3.1. *Let G be a graph with m edges and the maximal degree Δ and the minimal degree δ. Then,*

$$0\leq SO_2(G)\leq m\frac{\Delta^2-\delta^2}{\Delta^2+\delta^2},$$

the left equality holds if and only if every component of G is regular and the right equality holds if and only if $\frac{\max\{d_G(u),d_G(v)\}}{\min\{d_G(u),d_G(v)\}}=\frac{\Delta}{\delta}$ for any edge $uv\in E$.

Theorem 3.2. *Let T be a tree with $n>1$ vertices. Then,*

$$\frac{6}{5}\leq SO_2(T)\leq\frac{(n^2-2n)(n-1)}{n^2-2n+2},$$

the left equality holds if and only if T is the path P_n and the right equality holds if and only if T is the star S_n.

Proof. From (3.1), we have

$$SO_2(T)=n-1-\sum_{uv\in E}\frac{2}{\frac{\max\{d_T^2(u),d_T^2(v)\}}{\min\{d_T^2(u),d_T^2(v)\}}+1},$$

and $\frac{\max\{d_T(u),d_T(v)\}}{\min\{d_T(u),d_T(v)\}} \leq n-1$, so

$$SO_2(T) \leq \frac{(n^2-2n)(n-1)}{n^2-2n+2},$$

with equality if and only if $T \cong S_n$.

On the other hand, we know that $f(x) = \frac{x^2-1}{x^2+1}$ is increasing for $x > 0$. Let E_1 be the set of pendant edges in T and $E_2 = E - E_1$. Then, $|E_1| \geq 2$, $\frac{\max\{d_T(u),d_T(v)\}}{\min\{d_T(u),d_T(v)\}} \geq 2$ for $uv \in E_1$ and $\frac{\max\{d_T(u),d_T(v)\}}{\min\{d_T(u),d_T(v)\}} \geq 1$ for $uv \in E_2$. By (3.1), we have

$$\begin{aligned} SO_2(T) &= n-1-\sum_{uv\in E_1}\frac{2}{\frac{\max\{d_T^2(u),d_T^2(v)\}}{\min\{d_T^2(u),d_T^2(v)\}}+1}-\sum_{uv\in E_2}\frac{2}{\frac{\max\{d_T^2(u),d_T^2(v)\}}{\min\{d_T^2(u),d_T^2(v)\}}+1} \\ &\geq n-1-\frac{2}{5}|E_1|-|E_2| = \frac{3}{5}|E_1| \geq \frac{6}{5}, \end{aligned}$$

with equality holds if and only if $T \cong P_n$. $\square$

Let $\mathcal{C}T_n$ be the set of molecular trees with n vertices. Let m_{ij} be the number of edges with end vertices of degree i and j in T.

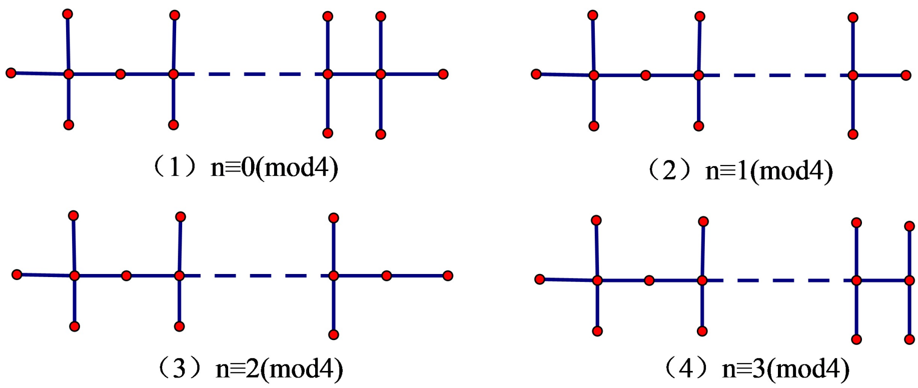

FIGURE 2. Four types molecular trees with n vertices.

Denote by $\mathcal{T}_0$ the set of molecular trees T with $n \equiv 0 \pmod 4$ vertices, where T has no vertex with degree 3, every vertex of degree 2 is adjacent to two vertices of degree 4 in T and there is exactly a pair of vertices with degree 4 adjacent to each other, i.e., T is a tree on n vertices with $m_{14} = \frac{n+4}{2}$, $m_{24} = \frac{n-8}{2}$, $m_{44} = 1$, $m_{12} = m_{13} = m_{22} = m_{23} = m_{33} = m_{34} = 0$ (an example is shown in Figure 2 (1)).

Denote by $\mathcal{T}_1$ the set of molecular trees T with $n \equiv 1 \pmod 4$ vertices, where T has no vertex with degree 3, every vertex of degree 2 is adjacent to two vertices of degree 4 in T and no two vertices of degree 4 are mutually adjacent, i.e., T is a tree with $m_{14} = \frac{n+3}{2}$, $m_{24} = \frac{n-5}{2}$, $m_{12} = m_{13} = m_{22} = m_{23} = m_{33} = m_{34} = m_{44} = 0$ (an example is shown in Figure 2 (2)).

Denote by $\mathcal{T}_2$ the set of molecular trees T with $n \equiv 2 \pmod 4$ vertices, where T has no vertex with degree 3, exactly one of vertices of degree 2 is adjacent to a vertex of degree 4 and a vertex of degree 1, and other vertices of degree 2 are adjacent to two vertices of degree 4, i.e., T is a tree with $m_{14} = \frac{n}{2}$, $m_{24} = \frac{n-4}{2}$, $m_{12} = 1$, $m_{13} = m_{22} = m_{23} = m_{33} = m_{34} = m_{44} = 0$ (an example is shown in Figure 2 (3)).

Denote by $\mathcal{T}_3$ the set of molecular trees T with $n \equiv 3 \pmod 4$ vertices, where T has exactly one vertex of degree 3, which is adjacent to two vertices of degree 1 and one vertex of degree 4 and every vertex of degree 2 is adjacent to two vertices of degree 4, i.e., T is a tree with $m_{14} = \frac{n-1}{2}$, $m_{24} = \frac{n-7}{2}$, $m_{13} = 2$, $m_{34} = 1$, $m_{12} = m_{22} = m_{23} = m_{33} = m_{44} = 0$ (an example is shown in Figure 2 (4)).

Theorem 3.3. *Let $T \in \mathcal{C}T_n$ with $n \geq 5$. Then,*

$$\frac{6}{5} \leq SO_2(T) \leq \begin{cases} \dfrac{126n-108}{170}, & n \equiv 0 \pmod 4, \\ \dfrac{126n-30}{170}, & n \equiv 1 \pmod 4, \\ \dfrac{126n-102}{170}, & n \equiv 2 \pmod 4, \\ \dfrac{315n-281}{425}, & n \equiv 3 \pmod 4, \end{cases}$$

the left equality holds if and only if $T \cong P_n$ and the right equality holds if and only if $T \in \mathcal{T}_i$ for $n \equiv i \pmod 4$, $i = 0, 1, 2, 3$.

Proof. By Theorem 3.2, we have $SO_2(T) \geq \frac{6}{5}$ with equality if and only if $T \cong P_n$.

From the Definition 1.1 of the second Sombor index, we have

$$SO_2(T) = \frac{3}{5}m_{12} + \frac{4}{5}m_{13} + \frac{15}{17}m_{14} + \frac{5}{13}m_{23} + \frac{3}{5}m_{24} + \frac{7}{25}m_{34}. \tag{3.2}$$

Let n_i be the number of vertices of degree i in T, $i \in \{1, 2, 3, 4\}$. We can get the following system of six linear equations which are satisfied by all molecular trees

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = n, \\ n_1 + 2n_2 + 3n_3 + 4n_4 = 2n - 2, \\ m_{12} + m_{13} + m_{14} = n_1, \\ m_{12} + 2m_{22} + m_{23} + m_{24} = 2n_2, \\ m_{13} + m_{23} + 2m_{33} + m_{34} = 3n_3, \\ m_{14} + m_{24} + m_{34} + 2m_{44} = 4n_4. \end{cases} \tag{3.3}$$

Solving this system with unknowns $m_{14}, m_{24}, n_1, n_2, n_3$ and n_4, we can obtain

$$
(3.4)\qquad \begin{cases}
m_{14} = \dfrac{n+3}{2} - \dfrac{3m_{12}}{2} - \dfrac{7m_{13}}{6} - \dfrac{m_{22}}{2} - \dfrac{m_{23}}{6} + \dfrac{m_{33}}{6} + \dfrac{m_{34}}{3} + \dfrac{m_{44}}{2}, \\
m_{24} = \dfrac{n-5}{2} + \dfrac{m_{12}}{2} + \dfrac{m_{13}}{6} - \dfrac{m_{22}}{2} - \dfrac{5m_{23}}{6} - \dfrac{7m_{33}}{6} - \dfrac{4m_{34}}{3} - \dfrac{3m_{44}}{2}, \\
n_1 = \dfrac{n+3}{2} - \dfrac{m_{12}}{2} - \dfrac{m_{13}}{6} - \dfrac{m_{22}}{2} - \dfrac{m_{23}}{6} + \dfrac{m_{33}}{6} + \dfrac{m_{34}}{3} + \dfrac{m_{44}}{2}, \\
n_2 = \dfrac{n-5}{4} + \dfrac{3m_{12}}{4} + \dfrac{7m_{13}}{12} + \dfrac{3m_{22}}{4} + \dfrac{m_{23}}{12} - \dfrac{7m_{33}}{12} - \dfrac{2m_{34}}{3} - \dfrac{3m_{44}}{4}, \\
n_3 = \dfrac{m_{13}}{3} + \dfrac{m_{23}}{3} + \dfrac{2m_{33}}{3} + \dfrac{m_{34}}{3}, \\
n_4 = \dfrac{n-1}{4} - \dfrac{m_{12}}{4} - \dfrac{m_{13}}{4} - \dfrac{m_{22}}{4} - \dfrac{m_{23}}{4} - \dfrac{m_{33}}{4} + \dfrac{m_{44}}{4}.
\end{cases}
$$

Replacing m_{14} and m_{24} in (3.2) by (3.4), we have

$$
(3.5)\qquad \begin{aligned} SO_2(T) = {} & \frac{126n-30}{170} - \frac{36m_{12}}{85} - \frac{11m_{13}}{85} - \frac{64m_{22}}{85} \\ & - \frac{58m_{23}}{221} - \frac{47m_{33}}{85} - \frac{96m_{34}}{425} - \frac{39m_{44}}{85}, \end{aligned}
$$

which is maximal for a fixed number of vertices when the values $m_{12}, m_{13}, m_{22}, m_{23}, m_{33}$, m_{34}, and m_{44} are equal to zero. However, in the case of molecular trees with n vertices, the condition

$$
(3.6)\qquad m_{12} = m_{13} = m_{22} = m_{23} = m_{33} = m_{34} = m_{44} = 0
$$

can be satisfied only if $n \equiv 1 \pmod 4$.

Any molecular tree satisfying (3.6) has no vertices of degree 3, all its vertices of degree 2 are adjacent to two vertices of degree 4, and no two vertices of degree 4 are mutually adjacent (see (2) in Figure 2).

Hence, if $n \equiv 1 \pmod 4$, then for any molecular tree with n vertices,

$$
SO_2(T) \leq \frac{126n-35}{170},
$$

with equality if and only if $T \in \mathcal{T}_1$.

If $n \not\equiv 1 \pmod 4$, then (3.6) cannot be satisfied by any molecular tree on n vertices. In order to find the molecular trees with the maximal SO_2-value, we have to find the values of the parameters $m_{12}, m_{13}, m_{22}, m_{23}, m_{33}, m_{34}$, and m_{44} as close to zero as possible compatible to the existence of a molecular tree, i.e., for which the right-hand sides of (3.4) are integers, and for which a graph exists and we have that $m_{13} + m_{23} + 2m_{33} + m_{34}$ has to be a multiple of 3 from $n_3 = \frac{m_{13}}{3} + \frac{m_{23}}{3} + \frac{2m_{33}}{3} + \frac{m_{34}}{3}$.

By (3.5), we know that there is must be $n_3 = 0$ or $n_3 = 1$ for $n \not\equiv 1 \pmod 4$ if T is a molecular tree with the maximal $SO_2(T)$-value.

TABLE 4. All $(m_{13}, m_{23}, m_{33}, m_{34})$ satisfy $m_{13}+m_{23}+2m_{33}+m_{34}=3$, where $t=\frac{11}{85}m_{13}+\frac{58}{221}m_{23}+\frac{47}{85}m_{33}+\frac{96}{425}m_{34}$.

m_{13}	m_{23}	m_{33}	m_{34}	t	m_{13}	m_{23}	m_{33}	m_{34}	t
1	1	0	1	0.617715	1	0	1	0	0.682341
0	1	1	0	0.815374	0	0	1	1	0.778823
2	1	0	0	0.521233	1	2	0	0	0.654266
1	0	0	2	0.581164	2	0	0	1	0.484682
0	2	0	1	0.750748	0	1	0	2	0.714197

Case 1. If $n_3=0$, then $m_{13}=m_{23}=m_{33}=m_{34}=0$, and

$$SO_2(T)=\frac{126n-30}{170}-\frac{36m_{12}}{85}-\frac{64m_{22}}{85}-\frac{39m_{44}}{85}. \tag{3.7}$$

To find the molecular tree(s) with the maximal $SO_2(T)$-value, we only need to consider $m_{12}+m_{22}+m_{44}=1$.

If $n\equiv 2 \pmod 4$, there are two types of molecular trees such that $m_{12}+m_{22}+m_{44}=1$, i.e., $m_{12}=1$, $m_{22}=m_{44}=0$ or $m_{12}=0$, $m_{22}=1$, $m_{44}=0$. By simply computing and comparing, for any molecular tree T with $n\equiv 1 \pmod 4$ vertices,

$$SO_2(T)\le\frac{126n-102}{170},$$

with equality if and only if $T\in\mathcal{T}_2$.

If $n\equiv 0 \pmod 4$, there is only one type of molecular trees such that $m_{12}+m_{22}+m_{44}=1$, i.e., $m_{12}=m_{22}=0$, $m_{44}=1$. Then, for any molecular tree with $n\equiv 0 \pmod 4$ vertices,

$$SO_2(T)\le\frac{126n-108}{170},$$

with equality if and only if $T\in\mathcal{T}_0$.

If $n\equiv 3 \pmod 4$, then there is no molecular trees such that $m_{12}+m_{22}+m_{44}=1$.

Case 2. If $n_3=1$, then $m_{13}+m_{23}+2m_{33}+m_{34}=3$. To find the molecular trees with the maximal $SO_2(T)$-value, we only need to consider all possible choices of $(m_{13}$, m_{23}, m_{33}, $m_{34})$ such that $m_{13}+m_{23}+2m_{33}+m_{34}=3$ $m_{13}+m_{23}+2m_{33}+m_{34}=3$.

From Table 4 and (3.5), a molecular tree on n vertices and $n_3=1$ with the maximal SO_2-value must satisfy

$$m_{12}=m_{22}=m_{44}=m_{23}=m_{33}=0,\quad m_{13}=2,\quad m_{34}=1, \tag{3.8}$$

and it can be satisfied only if $n\equiv 3 \pmod 4$. Then, for any molecular tree with $n\equiv 3 \pmod 4$ vertices,

$$SO_2(T)\le\frac{315n-281}{425},$$

with equality if and only if $T\in\mathcal{T}_3$. □

Acknowledgements. This work is supported by the National Natural Science Foundation of China (No. 11971164) and the Hunan Provincial Natural Science Foundation of China (2020JJ4423).

References

[1] I. Gutman, *Geometric approach to degree-based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem. **86** (1) (2021) 11–16.

[2] I. Gutman, *Sombor indices-back to geometry*, Open J. Discrete Appl. Math. **5**(2) (2022) 1–5.

[3] I. Gutman, *Degree-based topological indices*, Croatica Chemica Acta **86** (2013), 351–361.

[4] V. R. Kulli, *Graph indices*, in: M. Pal, S. Samanta, A. Pal (Eds.), *Handbook of Research of Advanced Applications of Graph Theory in Modern Society*, Hershey: Global, 2020, 66–91.

[5] R. Todeschini and V. Consonni, *Molecular Descriptors for Chemoinformatics*, Weinheim, Wiley-VCH, 2009.

[6] H. Deng, Z. Tang and R. Wu, *Molecular trees with extremal values of Sombor indices*, Int. J. Quantum Chem. **121** (2021), Article ID e26622.

[1]MOE-LCSM, School of Mathematics and Statistics,
Hunan Normal University,
Changsha, Hunan 410081, P. R. China
Email address: zikaitang@163.com
ORCID iD: https://orcid.org/0000-0002-2577-7890
Email address: hydeng@hunnu.edu.cn
ORCID iD: https://orcid.org/0000-0003-1680-2473

*Corresponding Author

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1171–1182.

GENERALIZATION OF INEQUALITIES FOR DIFFERENT TYPES OF FUNCTIONS VIA ψ-CAPUTO FRACTIONAL DERIVATIVE

ABDELLAH TAQBIBT[1,2], MOHAMED CHAIB[1], AND M'HAMED EL OMARI[1]

Abstract. In this manuscript, we present a collection of inequalities that extend and generalize the important results of the findings from [14], with the latter serving as specific cases within our study. Our research takes many forms of fractional differential inequalities, including the comprehensive structure of the Caputo fractional derivative operator with respect to a function ψ. Furthermore, we demonstrate the practical applications of σ-convex functions in the domain of fractional calculus within this framework.

1. Introduction

The broad applicability of fractional calculus in modeling natural phenomena across various scientific and engineering fields has captured the interest of many researchers [4, 5, 12, 13, 17]. These fields include physics, finance, biology, engineering and chemistry. To facilitate a deeper understanding of these phenomena, various definitions of fractional derivatives have been introduced, including the Caputo derivative. This last derivative, proposed by Italian Caputo in his article [3], is considered as an important contribution to fractional calculus [11], consisting of the introduction of a new fractional derivation which is better adapted to to problems of partial differential equations. This derivative has been generalised in different sense so that the reader can see in [8–10] and the related references therein. In this study, our attention will be directed towards the Caputo fractional operators, aiming to establish crucial fractional inequalities. The concept of convexity is crucial in various domains in both pure and applied sciences. Therefore, the classical concepts of convex sets and convex

Key words and phrases. Convex functions, σ-convex functions, fractional inequalities, ψ-Caputo fractional operators.
2020 *Mathematics Subject Classification.* Primary: 34K37, 26A33.
https://doi.org/10.46793/KgJMat2607.1171T
Received: April 21, 2024.
Accepted: October 24, 2024.

functions have been extended and generalized in different research directions. For more information, we refer the reader to the references [7,14–16]. The close connection between convexity theory and the theory of inequalities has been a significant factor, drawing the attention of various researchers. The theory of mathematical inequalities has seen extensive research through numerous extensions and various generalizations, especially those dealing with Hadamard-type inequalities the reader can see in [2,6] and references therein.

In this section, we recall a few basics properties from the theory of fractional calculs. The Caputo fractional derivatives of order α are defined for the first time by the Italian mathematician Caputo which presented his fractional derivative of order $\alpha > 0$ as in the following definition σ-convex functions.

Definition 1.1. A set $\mathcal{Q} \subset \mathbb{R}$ is defined as a σ-convex set with respect to a strictly monotonic continuous function σ if

$$\mathcal{M}_{[\sigma]}(x,y) := \sigma^{-1}\left(tx + (1-t)y\right) \in \mathcal{Q}, \quad \text{for all } x, y \in \mathcal{Q},\ t \in [0,1].$$

A function $f : \mathcal{Q} \to \mathbb{R}$ is defined as a σ-convex function with respect to a strictly monotonic continuous function σ if

$$f\left(\mathcal{M}_{[\sigma]}(x,y)\right) \leq tf(x) + (1-t)f(y), \quad \text{for all } x, y \in \mathcal{Q},\ t \in [0,1].$$

The set of σ-convex functions will be denoted by $\mathcal{Q}(I)$.

Definition 1.2. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $\mathcal{P}$-function or belongs to $\mathcal{P}(I)$, if it is nonnegative and for all $x, y \in I$ we have

$$f\left(\mathcal{M}_{[\sigma]}\left(\sigma(x), \sigma(y)\right)\right) \leq f(x) + f(y).$$

Definition 1.3. Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is h-convex or that f belongs to the class of $\mathcal{S}X(h, I)$, if f is nonegative and for all $x, y \in I$ $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y).$$

Definition 1.4. Let $\alpha > 0$, $f \in AC^n[a,b]$,

$$\mathcal{D}^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > a, \tag{1.1}$$

and

$$\mathcal{D}^{\alpha}_{b-}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(s)}{(s-t)^{\alpha+1-n}} ds, \quad t < b. \tag{1.2}$$

If $\alpha = n \in \{1, 2, 3, \dots\}$ and usual derivative of order n exists, then Caputo fractional derivative ${}^c\mathcal{D}^{\alpha,\psi}_{a+}f(t)$ coincides with $f^{(n)}(t)$, while ${}^c\mathcal{D}^{\alpha,\psi}_{b-}f(t)$ coincides with $f^{(n)}(t)$ with exactness to a constant multiplier $(-1)^n$.

The Caputo derivatives given in Definition 1.4 have been generalized by the ψ-Caputo fractional derivative, as described below.

Definition 1.5 ([1]). Let $\alpha > 0$, $f \in AC^n[a,b]$. The ψ-Caputo fractional derivatives of order α are defined as follows:

$$ {}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f^{[n]}_{\psi}(s) ds, \quad t > a, \tag{1.3}$$

and

$$ {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \psi'(s)(\psi(s)-\psi(t))^{n-\alpha-1} f^{[n]}_{\psi}(s) ds, \quad t < b, \tag{1.4}$$

where $f^{[n]}_{\psi}(t) = \left(\frac{1}{\psi'(t)} \cdot \frac{d}{dt}\right)^n f(t)$.

Within this section, we formulate new inequalities for various types of functions using ψ-Caputo derivatives.

Lemma 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f^{[n+1]}_{\psi}$ is integrable over $[a,b]$, then the following generalized fractional equality*

$$\begin{aligned} &\left[f^{[n]}_{\psi}(a) + f^{[n]}_{\psi}(b)\right] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \left[{}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n \, {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(a)\right] \\ &= \Big(\psi(b)-\psi(a)\Big) \int_0^1 \left(t^{n-\alpha} - (1-t)^{n-\alpha}\right) f^{[n+1]}_{\psi}\left(\psi^{-1}(t\psi(a) + (1-t)\psi(b))\right) dt \end{aligned} \tag{1.5}$$

holds.

Proof. Using Definition 1.5, we can write

$$\begin{aligned} {}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n \, {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(a) =& \frac{1}{\Gamma(n-\alpha)} \int_a^b \psi'(\tau) f^{[n]}_{\psi}(\tau) \\ & \times \left[\Big(\psi(b)-\psi(\tau)\Big)^{n-\alpha-1} + \Big(\psi(\tau)-\psi(a)\Big)^{n-\alpha-1}\right] d\tau. \end{aligned}$$

Using integration by parts and by change of variable, it follows by taking $u = \frac{1}{\Gamma(n-\alpha)} f^{[n]}_{\psi}(\tau)$ and $v = \psi'(\tau)\left(\psi(b)-\psi(\tau)\right)^{n-\alpha} + \psi'(\tau)\left(\psi(\tau)-\psi(a)\right)^{n-\alpha}$ that

$$\begin{aligned} &{}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n \, {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(a) \\ =& \frac{1}{\Gamma(n-\alpha)} \times \left[- f^{[n]}_{\psi}(\tau) \frac{\Big(\psi(b)-\psi(\tau)\Big)^{n-\alpha+1}}{n-\alpha+1} + f^{[n]}_{\psi}(\tau) \frac{\Big(\psi(\tau)-\psi(a)\Big)^{n-\alpha+1}}{n-\alpha+1} \right]_a^b \\ & - \frac{1}{\Gamma(n-\alpha)} \int_a^b f^{[n+1]}_{\psi}(\tau) \left[- \frac{\Big(\psi(b)-\psi(\tau)\Big)^{n-\alpha+1}}{n-\alpha+1} + \frac{\Big(\psi(\tau)-\psi(a)\Big)^{n-\alpha+1}}{n-\alpha+1} \right] d\tau. \end{aligned}$$

Now, we use the change of variable $\tau = \mathcal{M}_{[\psi]}(a,b) = \psi^{-1}(ta + (1-t)b)$, we obtain

$$\begin{aligned}{}^{c}\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n \, {}^{c}\mathcal{D}_{b-}^{\alpha,\psi} f(a) =& \frac{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}{\Gamma(n-\alpha+1)} \left[f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b) \right] \\ &+ \frac{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \\ &\times \int_0^1 f_{\psi}^{[n+1]}\Big(\mathcal{M}_{[\psi]}(a,b)\Big) \left[t^{n-\alpha} - (1-t)^{n-\alpha} \right] dt.\end{aligned}$$

Finally, we multiply both sides of the previous equality by $\frac{(\psi(b)-\psi(a))^{n-\alpha}}{\Gamma(n-\alpha+1)}$, we obtain

$$\begin{aligned}&\frac{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}{\Gamma(n-\alpha+1)} \left[{}^{c}\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n \, {}^{c}\mathcal{D}_{b-}^{\alpha,\psi} f(a) \right] \\ =& \left[f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b) \right] + \Big(\psi(b)-\psi(a)\Big) \int_0^1 f_{\psi}^{[n+1]}\Big(\mathcal{M}_{[\psi]}(a,b)\Big) \left[t^{n-\alpha} - (1-t)^{n-\alpha} \right] dt.\end{aligned}$$

Hence,

$$\begin{aligned}&\left[f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b) \right] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \left[{}^{c}\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n \, {}^{c}\mathcal{D}_{b-}^{\alpha,\psi} f(a) \right] \\ =& (\psi(b)-\psi(a)) \int_0^1 \Big((1-t)^{n-\alpha} - t^{n-\alpha} \Big) f_{\psi}^{[n+1]} \Big(\mathcal{M}_{[\psi]}(a,b)\Big) \, dt.\end{aligned}$$

□

Theorem 1.1. *Let us consider a differentiable function $f : [a,b] \to \mathbb{R}$ defined over the open interval (a,b). Assume that $|f_{\psi}^{[n+1]}|$ belongs to the class $\mathcal{Q}(I)$. Then, the following inequality*

$$\begin{aligned}&\left| \left[f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b) \right] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \left[{}^{c}\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n \, {}^{c}\mathcal{D}_{b-}^{\alpha,\psi} f(a) \right] \right| \\ \leq& \Big(\psi(b)-\psi(a)\Big) \left\{ \Big(\left| f_{\psi}^{[n+1]}(a) \right| + \left| f_{\psi}^{[n+1]}(b) \right| \Big) \left[B(0, n-\alpha+1) + \frac{1}{n-\alpha+2} \right] \right\}\end{aligned}$$

holds.

Proof. Combining the result of Lemma 1.1 and Definition 1.1, we have

$$\begin{aligned}&\left| \left[f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b) \right] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \left[{}^{c}\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n \, {}^{c}\mathcal{D}_{b-}^{\alpha,\psi} f(a) \right] \right| \\ =& \left| \Big(\psi(b)-\psi(a)\Big) \int_0^1 \Big((1-t)^{n-\alpha} - t^{n-\alpha} \Big) f_{\psi}^{[n+1]} \Big(\mathcal{M}_{[\psi]}(a,b)\Big) \, dt \right| \\ \leq& \Big(\psi(b)-\psi(a)\Big) \int_0^1 \left| (1-t)^{n-\alpha} - t^{n-\alpha} \right| \left[\left| t \, f_{\psi}^{[n+1]}(a) \right| + \left| (1-t) f_{\psi}^{[n+1]}(b) \right| \right] dt.\end{aligned}$$

We expand the product under the integral, and we obtain

$$\begin{aligned}
&\Bigg|\Big[f_\psi^{[n]}(a)+f_\psi^{[n]}(b)\Big]-\frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\Bigg|\\
\leq&\Big(\psi(b)-\psi(a)\Big)\left\{\left|f_\psi^{[n+1]}(a)+f_\psi^{[n+1]}(b)\right|\left[\int_0^1(1-t)^{n-\alpha}\,t^{-1}+t^{n-\alpha+1}\,dt\right]\right\}\\
=&\Big(\psi(b)-\psi(a)\Big)\left\{\left|f_\psi^{[n+1]}(a)\right|+\left|f_\psi^{[n+1]}(b)\right|\left[B(0,n-\alpha+1)+\frac{1}{n-\alpha+2}\right]\right\},
\end{aligned}$$

where B is the beta function, which completes the proof of the theorem. □

Corollary 1.1. *If we take $\psi(x)=x$, the ψ-Caputo fractional operators ${}^c\mathcal{D}_{a+}^{\alpha,\psi}$ and ${}^c\mathcal{D}_{b-}^{\alpha,\psi}$ are reduced respectively to the Caputo fractional operators ${}^c\mathcal{D}_{a+}^{\alpha}$ and ${}^c\mathcal{D}_{b-}^{\alpha}$, and the function $f_\psi^{[n+1]}$ is reduced to $f_\psi^{(n+1)}$, then the inequality* (1.6) *reduces to the following inequality*

$$\begin{aligned}
&\Bigg|\Big[f^{(n)}(a)+f^{(n)}(b)\Big]-\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha}f(a)\right]\Bigg|\\
\leq&\Big(\psi(b)-\psi(a)\Big)\left\{\left|f^{(n+1)}(a)\right|+\left|f^{(n+1)}(b)\right|\left[B(0,n-\alpha+1)+\frac{1}{n-\alpha+2}\right]\right\},
\end{aligned}$$

which it exactly the inequality obtained in [14, Theorem 2.1]*, when we replace the class $\mathcal{Q}(I)$ in* [14] *by that in the Definition* 1.1*.*

Theorem 1.2. *Let us consider a differentiable function $f:[a,b]\to\mathbb{R}$ defined over the open interval (a,b). Assume that $|f_\psi^{[n+1]}|$ belongs to the class $\mathcal{Q}(I)$. Then, the following inequality*

$$\begin{aligned}
&\Bigg|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\,(\psi(b)-\psi(a))^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\Bigg|\\
\leq&\frac{\psi(b)-\psi(a)}{n-\alpha+1}\left\{\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right\}
\end{aligned}\tag{1.6}$$

holds.

Proof. Since $\left|f_\psi^{[n+1]}\right|$ is a ψ-convex function, then in particular we can obtain

$$\left|f_\psi^{[n+1]}\right|\Big(\mathcal{M}_{[\psi]}(x,y)\Big)\leq t\left|f_\psi^{[n+1]}\right|(x)+(1-t)\left|f_\psi^{[n+1]}\right|(y),\quad\text{for all } x,y\in[a,b].\tag{1.7}$$

By the Lemma 1.1, we have

$$\begin{aligned}
&\Bigg|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\,(\psi(b)-\psi(a))^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\Bigg|\\
=&\left|\frac{\psi(b)-\psi(a)}{2}\int_0^1\Big((1-t)^{n-\alpha}-t^{n-\alpha}\Big)\,f_\psi^{[n+1]}\Big(\mathcal{M}_{[\psi]}(a,b)\Big)\,dt\right|.
\end{aligned}$$

Therefore, using inequality (1.7) to the points a and b, we obtain

$$\begin{aligned}
&\left|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\psi(b)-\psi(a)\right)^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\right|\\
\leq&\frac{\psi(b)-\psi(a)}{2}\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[t\left|f_\psi^{[n+1]}\right|(a)+(1-t)\left|f_\psi^{[n+1]}\right|(b)\right]dt\\
\leq&\frac{\psi(b)-\psi(a)}{2}\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right]dt\\
\leq&\frac{\psi(b)-\psi(a)}{2}\Bigg\{\left|f_\psi^{[n+1]}\right|(a)\int_0^1(1-t)^{n-\alpha}dt+\left|f_\psi^{[n+1]}\right|(b)\int_0^1(1-t)^{n-\alpha}dt\\
&+\left|f_\psi^{[n+1]}\right|(a)\int_0^1 t^{n-\alpha}dt+\left|f_\psi^{[n+1]}\right|(b)\int_0^1 t^{n-\alpha}dt\Bigg\}\\
=&\frac{\psi(b)-\psi(a)}{n-\alpha+1}\left[\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right].
\end{aligned}$$

So,

$$\begin{aligned}
&\left|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\psi(b)-\psi(a)\right)^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\right|\\
\leq&\frac{\psi(b)-\psi(a)}{n-\alpha+1}\left[\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right].
\end{aligned}$$

□

Theorem 1.3. *Let f be a function such that $\left|f_\psi^{[n]}\right|$ belongs to the class $\mathcal{Q}(I)$, $a,b\in I$ with $0\leq a<b$ and $\left|f_\psi^{[n]}\right|\in L^1[a,b]$. Then, inequality*

$$\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right|\leq\frac{2\Gamma(n-\alpha+1)}{\left(\psi(b)-\psi(a)\right)^{n-\alpha}}\left|{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right| \tag{1.8}$$

holds for all $\alpha>0$.

Proof. Since $f_\psi^{[n]}$ is σ-convex, then we obtain

$$\left|f_\psi^{[n]}\left(\mathcal{M}_{[\sigma]}(x,y)\right)\right|\leq\lambda\left|f_\psi^{[n]}(x)\right|+(1-\lambda)\left|f_\psi^{[n]}(y)\right|,\qquad\text{for all } x,y\in[a,b],\ \lambda\in[0,1].$$

In particular for $\lambda=\frac{1}{2}$, we get

$$\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)\right|\leq 2\left(\left|f_\psi^{[n]}(x)\right|+\left|f_\psi^{[n]}(y)\right|\right),\qquad\text{for all } x,y\in[a,b].$$

Substituting $\psi(x)=t\psi(a)+(1-t)\psi(b)$ and $\psi(y)=(1-t)\psi(a)+t\psi(b)$, we obtain

$$\begin{aligned}
\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right|\leq&2\Big[f_\psi^{[n]}(\psi^{-1}(t\psi(a)+(1-t)\psi(b)))\\
&+f_\psi^{[n]}(\psi^{-1}((1-t)\psi(a)+t\psi(b)))\Big].
\end{aligned}\tag{1.9}$$

Multiplying both sides of (1.9) by $t^{n-\alpha-1}$, we can get

$$t^{n-\alpha-1}\left|f_{\psi}^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right| \leq 2t^{n-\alpha-1}\Big[f_{\psi}^{[n]}(\psi^{-1}(t\psi(a)+(1-t)\psi(b))) + f_{\psi}^{[n]}(\psi^{-1}((1-t)\psi(a)+t\psi(b)))\Big],$$

and integrating the above inequality with respect to t over the $[0,1]$, we obtain

$$\begin{aligned}
&f_{\psi}^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\int_0^1 t^{n-\alpha-1}dt \leq 2(n-\alpha)\\
&\times\Big[\int_0^1 t^{n-\alpha-1}f_{\psi}^{[n]}\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)dt\\
&+\int_0^1 t^{n-\alpha-1}f_{\psi}^{[n]}\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)dt\Big]\\
\leq&2(n-\alpha)\Big[\int_a^b \psi'(x)\left(\frac{\psi(b)-\psi(x)}{\psi(b)-\psi(a)}\right)^{n-\alpha-1} f_{\psi}^{[n]}(x)\frac{dx}{\psi(b)-\psi(a)}\\
&+\int_a^b \psi'(y)\left(\frac{\psi(y)-\psi(a)}{\psi(b)-\psi(a)}\right)^{n-\alpha-1} f_{\psi}^{[n]}(y)\frac{dy}{\psi(b)-\psi(a)}\Big]\\
\leq&\frac{2(n-\alpha)}{(\psi(b)-\psi(a))^{n-\alpha}}\Big[\int_a^b \psi'(x)\left(\psi(b)-\psi(x)\right)^{n-\alpha-1} f_{\psi}^{[n]}(x)dx\\
&+\int_a^b \psi'(y)\left(\psi(y)-\psi(a)\right)^{n-\alpha-1} f_{\psi}^{[n]}(y)dy\Big]\\
=&\frac{2\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}}\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big].
\end{aligned}$$

This ends the proof. □

Corollary 1.2. *If we take $\psi(x)=x$. Then, the inequality (1.8) reduces to the following inequality*

$$f^{(n)}\Big(\frac{a+b}{2}\Big) \leq \frac{2\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\Big|{}^c\mathcal{D}_{a+}^{\alpha}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha}f(a)\Big|. \tag{1.10}$$

Theorem 1.4. *Let f be a function such that $\left|f_{\psi}^{[n]}\right|$ belongs to the class $\mathcal{P}(I)$, $a,b\in I$ with $0\leq a<b$ and $\left|f_{\psi}^{[n]}\right|\in L^1[a,b]$. Then, inequality*

$$\begin{aligned}
\left|f_{\psi}^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right| \leq& \frac{2\Gamma(n-\alpha+1)}{\left(\psi(b)-\psi(a)\right)^{n-\alpha}}\Big|{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big|\\
\leq&\left|f_{\psi}^{[n]}\right|(a)+\left|f_{\psi}^{[n]}\right|(b)
\end{aligned}\tag{1.11}$$

holds for all $\alpha>0$.

Proof. Using Definition 1.2 and substituting $\psi(x) = t\psi(a) + (1-t)\psi(b)$ and $\lambda = \frac{1}{2}$, we obtain

$$\begin{aligned} \left| f_\psi^{[n]} \left(\psi^{-1} \left(\frac{\psi(a)+\psi(b)}{2} \right) \right) \right| \leq & \left| f_\psi^{[n]} \right| \Big(t\psi(a) + (1-t)\psi(b) \Big) \\ & + \left| f_\psi^{[n]} \right| \Big((1-t)\psi(a) + t\psi(b) \Big), \end{aligned} \tag{1.12}$$

for all $t \in [0,1]$. As in the proof of the last theorem, multiplying both sides of inequality (1.12) by $t^{n-\alpha-1}$ and integrating the resulting inequality with respect to t over the interval $[0,1]$, we get

$$\begin{aligned} \left| f_\psi^{[n]} \Big(\psi^{-1} \Big(\frac{\psi(a)+\psi(b)}{2} \Big) \Big) \right| \int_0^1 t^{n-\alpha-1} dt \leq & \int_0^1 t^{n-\alpha-1} \Big[\left| f_\psi^{[n]} \right| \Big(t\psi(a) + (1-t)\psi(b) \Big) \\ & + \left| f_\psi^{[n]} \right| \Big((1-t)\psi(a) + t\psi(b) \Big) \Big] dt. \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{n-\alpha} \left| f_\psi^{[n]} \Big(\psi^{-1} \Big(\frac{\psi(a)+\psi(b)}{2} \Big) \Big) \right| \leq & \frac{1}{\Big(\psi(b) - \psi(a) \Big)^{n-\alpha}} \int_a^b \Big[\Big(\psi(b) - \psi(x) \Big)^{n-\alpha-1} \\ & + \Big(\psi(x) - \psi(a) \Big)^{n-\alpha-1} \Big] \psi'(x) \left| f_\psi^{[n]} \right| (x) dx. \end{aligned}$$

And thus,

$$\begin{aligned} \left| f_\psi^{[n]} \Big(\psi^{-1} \Big(\frac{\psi(a)+\psi(b)}{2} \Big) \Big) \right| \leq & \frac{n-\alpha}{\Big(\psi(b) - \psi(a) \Big)^{n-\alpha}} \int_a^b \Big[\Big(\psi(b) - \psi(x) \Big)^{n-\alpha-1} \\ & + \Big(\psi(x) - \psi(a) \Big)^{n-\alpha-1} \Big] \psi'(x) \left| f_\psi^{[n]} \right| (x) dx. \end{aligned}$$

Therefore,

$$\left| f_\psi^{[n]} \right| \Big(\psi^{-1} \Big(\frac{\psi(a)+\psi(b)}{2} \Big) \Big) \leq \frac{\Gamma(n-\alpha+1)}{\Big(\psi(b) - \psi(a) \Big)^{n-\alpha}} \Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n \, {}^c\mathcal{D}_{b-}^{\alpha,\psi} f(a) \Big],$$

which completes the proof of the first inequality in (1.11). Since $\left| f_\psi^{[n]} \right|$ belongs to the class $\mathcal{P}(I)$, then we can write

$$\left| f_\psi^{[n]} \Big(\psi^{-1} \Big(t\psi(a) + (1-t)\psi(b) \Big) \Big) \right| \leq \left| f_\psi^{[n]}(a) \right| + \left| f_\psi^{[n]}(b) \right|$$

and

$$\left| f_\psi^{[n]} \Big(\psi^{-1} \Big((1-t)\psi(a) + t\psi(b) \Big) \Big) \right| \leq \left| f_\psi^{[n]}(a) \right| + \left| f_\psi^{[n]}(b) \right|.$$

Combining these two inequalities, provide that the dummy variable choosing in $[0,1]$ for $\mathcal{M}_{[\psi]}$ is t, we obtain

$$\left|f_{\psi}^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(a),\psi(b)\big)\Big)\right| + \left|f_{\psi}^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(b),\psi(a)\big)\Big)\right| \leq 2\Big(\left|f_{\psi}^{[n]}(a)\right| + \left|f_{\psi}^{[n]}(b)\right|\Big). \tag{1.13}$$

If we multiplie both sides of (1.13) by $t^{n-\alpha-1}$ and integrate over the interval $[0,1]$ with respect to the variable t, we get

$$\begin{aligned}&\int_0^1 t^{n-\alpha-1}\Big[f_{\psi}^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(a),\psi(b)\big)\Big) + f_{\psi}^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(b),\psi(a)\big)\Big)\Big]dt\\ \leq& 2\Big(f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b)\Big)\int_0^1 t^{n-\alpha-1}dt,\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{n-\alpha}{\big(\psi(b)-\psi(x)\big)^{n-\alpha}}\int_a^b\Big[\big(\psi(b)-\psi(x)\big)^{n-\alpha-1} + \big(\psi(x)-\psi(b)\big)^{n-\alpha-1}\Big]\psi'(x)\left|f_{\psi}^{[n]}(x)\right|dx\\ \leq& 2\Big(\left|f_{\psi}^{[n]}(a)\right| + \left|f_{\psi}^{[n]}(b)\right|\Big),\end{aligned}$$

and therefore,

$$\begin{aligned}&\frac{\Gamma(n-\alpha+1)}{\big(\psi(b)-\psi(x)\big)^{n-\alpha}}\int_a^b\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b) + (-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big]\psi'(x)\left|f_{\psi}^{[n]}(x)\right|dx\\ \leq& 2\Big(\left|f_{\psi}^{[n]}(a)\right| + \left|f_{\psi}^{[n]}(b)\right|\Big).\end{aligned}$$

Finally, the second inequality of (1.11) is proved, which completes the proof. □

Theorem 1.5. *Assume that $f_{\psi}^{[n]} \in \mathcal{S}X(h,I)$, $a,b \in I$ with $a<b$. Then,*

$$\begin{aligned}\frac{1}{(n-\alpha)h(\frac{1}{2})}f_{\psi}^{[n]}\Big(\frac{a+b}{2}\Big) \leq& \frac{\Gamma(n-\alpha)}{\big(\psi(b)-\psi(a)\big)^{n-\alpha}}\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b) + (-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big]\\ \leq& \Big(f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b)\Big)\int_0^1 t^{n-\alpha-1}[h(t)+h(1-t)]dt.\end{aligned} \tag{1.14}$$

Proof. Applying the inequality in Definition 1.3, provided that $\psi(x) = t\psi(a) + (1-t)\psi(b)$, $\psi(y) = (1-t)\psi(a) + t\psi(b)$, we get

$$f_{\psi}^{[n]}\Big(\frac{\psi(x)+\psi(y)}{2}\Big) \leq h\left(\frac{1}{2}\right)\Big[f_{\psi}^{[n]}\big(\psi(x)\big)f_{\psi}^{[n]}\big(\psi(y)\big)\Big]. \tag{1.15}$$

Multiplying both sides of (1.15) by $t^{n-\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}\frac{1}{h\big(\frac{1}{2}\big)}\int_0^1 t^{n-\alpha-1}f_{\psi}^{[n]}\Big(\frac{\psi(a)+\psi(b)}{2}\Big)dt \leq& \int_0^1 t^{n-\alpha-1}f_{\psi}^{[n]}\big(t\psi(a)+(1-t)\psi(b)\big)dt\\ &+\int_0^1 t^{n-\alpha-1}f_{\psi}^{[n]}\big((1-t)\psi(a)+t\psi(b)\big)dt.\end{aligned}$$

And thus,

$$\begin{aligned}&\frac{1}{h\left(\frac{1}{2}\right)(n-\alpha)} f_\psi^{[n]}\Big(\frac{\psi(a)+\psi(b)}{2}\Big)\\ \leq&\frac{1}{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}\int_a^b\Big\{\Big(\psi(b)-\psi(x)\Big)^{n-\alpha-1} f_\psi^{[n]}(x)\psi'(x)\\ &+\Big(\psi(x)-\psi(a)\Big)^{n-\alpha-1} f_\psi^{[n]}(x)\psi'(x)\Big\}dx.\end{aligned}$$

Consequently,

$$\frac{1}{h\left(\frac{1}{2}\right)(n-\alpha)} f_\psi^{[n]}\Big(\frac{\psi(a)+\psi(b)}{2}\Big)\leq\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big],$$

and the first part of the inequality (1.14) is proved. Since $f_\psi^{[n]}\in\mathcal{S}X(h,I)$, we obtain

$$f_\psi^{[n]}(t\psi(x)+(1-t)\psi(y))\leq h(t)f_\psi^{[n]}(x)+(1-t)f_\psi^{[n]}(y)$$

and

$$f_\psi^{[n]}((1-t)\psi(x)+t\psi(y))\leq h(1-t)f_\psi^{[n]}(x)+h(t)f_\psi^{[n]}(y).$$

Combining these two inequalities, we get:

$$\begin{aligned}&f_\psi^{[n]}(t\psi(x)+(1-t)\psi(y))+f_\psi^{[n]}((1-t)\psi(x)+t\psi(y))\\ \leq&\Big(h(t)+h(1-t)\Big)\Big[f_\psi^{[n]}(x)+f_\psi^{[n]}(y)\Big].\end{aligned}\tag{1.16}$$

If we take $x=a$ and $y=b$ in inequality (1.16), multiplying its both sides by $t^{n-\alpha-1}$ and integrating the resulting inequality over $[0,1]$ with respect to t we obtain

$$\begin{aligned}&\int_0^1 t^{n-\alpha-1}\Big[f_\psi^{[n]}\Big(t\psi(a)+(1-t)\psi(b)\Big)+f_\psi^{[n]}\Big((1-t)\psi(a)+t\psi(b)\Big)\Big]dt\\ \leq&\int_0^1 t^{n-\alpha-1}\Big(h(t)+h(1-t)\Big)\Big[f_\psi^{[n]}(a)+f_\psi^{[n]}(b)\Big]dt.\end{aligned}\tag{1.17}$$

Hence,

$$\begin{aligned}&\frac{\Gamma(n-\alpha)}{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big]\\ \leq&\Big[f_\psi^{[n]}(a)+f_\psi^{[n]}(b)\Big]\int_0^1 t^{n-\alpha-1}\Big(h(t)+h(1-t)\Big)dt.\end{aligned}\tag{1.18}$$

Thus, the second inequality of (1.14) is proved, hence the proof is completed. □

2. Conclusion

We have formulated novel generalizations concerning inequalities of fractional type associated with different class of σ-convex type functions using the ψ-Caputo fractional operator knowing that ψ plays the same role as σ. Once again, we want to emphasize that our major finding, which possesses a broad and general nature, can be adapted to obtain various compelling fractional inequalities.

Acknowledgements. The authors are thankful to the referee for her/his valuable suggestions towards the improvement of the paper.

References

[1] M. S. Abdo, S. K. Panachal and A. M. Saeed, *Fractional boundary value problem with ψ-Caputo fractional derivative*, Proc. Indian Acad. Sci. **12** (9) (2019), Article ID 65. https://doi.org/10.1007/s12044-019-0514-8

[2] M. Bombardelli and S. Varosanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejer inequalities*, Comput. Math. Appl. **58** (2009), 1869–1877. https://doi.org/10.1016/j.camwa.2009.07.073

[3] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophysical Journal International **13**(5) (1967), 529–539.

[4] N. Chefnaj, A. Taqbibt, K. Hilal and S. Melliani, *Study of nonlocal boundary value problems for hybrid differential equations involving ψ-Caputo fractional derivative with measures of noncompactness*, J. Math. Sci. **271** (2023), 1–10.

[5] N. Chefnaj, A. Taqbibt, K. Hilal, S. Melliani and A. Kajouni, *Boundary value problems for differential equations involving the generalized Caputo-Fabrizio fractional derivative in b-metric spaces*, Turkish J. Sci. **8**(1) (2023), 24–36.

[6] S. S. Dragomir, *Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, Proyecciones **34**(4) (2015), 323–341. https://doi.org/10.4067/S0716-09172015000400002

[7] S. Erdas, E. Ünlüyol and S. Salas, *Some new inequalities of operator m-convex functions and applications for Synchronous-Asynchronous functions*, Complex Anal. Oper. Theory **13** (8), 3871–3881.

[8] G. Farid, A. Javed and A. U. Rehman, *On Hadamard inequalities for n-times differentiable functions which are relative convex via Caputo k-fractional derivatives*, Nonlinear Anal. Forum **22**(2) (2017), 17–28.

[9] G. Farid, A. Javed and A. U. Rehman, *Fractional integral inequalities of Hadamard-type for m-convex functions via Caputo k-fractional derivatives*, J. Fract. Calc. Appl. **10**(1) (2019), 120–134.

[10] G. Farid, A. Javed and S. Naqvi, *Hadamard and Fejer Hadamard inequalities and related results via Caputo fractional derivatives*, Bull. Math. Anal. Appl. **9**(3) (2017), 16–30.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[12] A.Taqbibt, N. Chefnaj, K. Hilal and S. Melliani, *ψ-Caputo fractional differential equations with maxima on time scales*, J. Math. Sci. (2024), 1–13. https://doi.org/10.1007/s10958-024-07034-y

[13] A. Taqbibt, M. Elomari and S. Melliani, *Nonlocal semilinear ϕ-Caputo fractional evolution equation with a measure of noncompactness in Banach space*, Filomat **37**(20) (2023), 6877–6890.

[14] D. Ucar, *Inequalities for different type of functions via Caputo fractional derivative*, AIMS Mathematics **7**(7) (2022), 12815–12826.

[15] E. Ünlüyol, Y. Erdas and S. Salas, *Operator (α, m)-convex functions and applications for synchronous and asynchronous functions*, Mathematical Sciences and Applications E-Notes **7**(2) (2019), 225–236.

[16] S. Wu, M. U. Awan, M. A. Noor, K. I. Noor and S. Iftikhar, *On a new class of convex functions and integral inequalities*, J. Inequal. Appl. **2019** (2019), 1–14.

[17] J. Zuo, A. Taqbibt, M. Chaib, M., ELomari and J. V. D. C. Sousa, *An existence and uniqueness of mild solutions of fractional evolution problems*, Comput. Appl. Math. **43**(424) (2024). https://doi.org/10.1007/s40314-024-02943-9

[1]LMACS,
University Sultan Moulay Sliman,
Campus Mghila FST, Béni Mellal

[2]University Sultan Moulay Sliman,
Campus Mghila ENS, Béni Mellal
Email address: abdellah.taqbibt@usms.ma
ORCID iD: https://orcid.org/0000-0001-5650-1988
Email address: mohamed15lmd@gmail.com
ORCID iD: https://orcid.org/0009-0000-0189-3582
Email address: m.elomari@usms.ma
ORCID iD: https://orcid.org/0000-0003-0336-9092

Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1183–1198.

UNIFORMLY CONVERGENT TIME-FRACTIONAL REACTION-DIFFUSION EQUATION

FEYISA EDOSA MERGA[1] AND GEMECHIS FILE DURESSA[1]

Abstract. In this paper, singularly perturbed time-fractional parabolic reaction-diffusion of initial boundary value problem is studied. The time-fractional derivative is applied in the Caputo fractional sense and handled by implicit Euler method. The spatial derivative is approximated by fitted cubic B-spline collocation method on a uniform mesh. Convergence analysis of the scheme is conducted and it is accurate of order $O(h^2 + (\Delta t^{2-\alpha}))$. To test the effectiveness of proposed method two model examples are considered. The results from the experiment confirm that the scheme is uniformly convergent and has twin layers at the end spatial domain.

1. Introduction

Fractional differential-equations are generalizations of the standard integer order derivatives to an arbitrary (non-integer) order that can be obtained either in space or time variable. Due to the ability to model complex phenomena fractional differential equations currently attracted several scholars in the fields of engineering, physics, chemistry, biology and other fields [1, 2].

Alike character of classical differential equation, the solution of parameter dependent fractional-differential equations also invariably characterized by large oscillation as the perurbation parameter approaches to zero. Solving such fractional differential equations using classical numerical methods on a uniform mesh may arise large oscillations in the entire domain of interest due to the boundary layer behavior. To overcome this, appropriate numerical method whose accuracy doesn't depend on the perturbation

Key words and phrases. Time-fractional, parabolic reaction-diffusion, fitted B-spline collocation method, convergence analysis.

2020 *Mathematics Subject Classification.* Primary: 65M22. Secondary: 65M12, 65H10.

https://doi.org/10.46793/KgJMat2607.1183M

Received: March 05, 2024.

Accepted: November 01, 2024.

parameter will be applied. Hence, there has been a lot of effort in developing numerical methods for the solution of singular perturbed fractional-differential equations [3–11]. This study concerns with the singularly perturbed time-fractional parabolic reaction-diffusion of initial boundary value problem in the domain $D = (0,1) \times (0,T]$:

$$\left(\frac{\partial^\alpha}{\partial t^\alpha}u - \varepsilon\frac{\partial^2}{\partial x^2}u + bu\right)(x,t) = f(x,t), \quad (x,t) \in D, \tag{1.1}$$

with an initial condition

$$u(x,0) = \psi(x), \quad x \in [0,1], \tag{1.2}$$

and boundary conditions

$$u(0,t) = \varphi_l(t), \quad u(1,t) = \varphi_r(t), \quad t \in [0,T], \tag{1.3}$$

where ε is a small perturbation parameter that satisfies $0 < \varepsilon < 1$ and $b(x,t) \geq \vartheta > 0$ is a smooth function. Under sufficient smoothness and compatibility conditions imposed on the functions $\psi(x)$, $\varphi_l(t)$, $\varphi_r(t)$ and $f(x,t)$) the initial-boundary value problem admits a unique solution $u(x,t)$ which exhibits twin boundary layer of width $O(\sqrt{\varepsilon})$ neighboring the boundaries $x = 0$ and $x = 1$.

Reaction-diffusion equations that time-derivative can be replaced by non-integer fractional order is known as time-fractional reaction diffusion equations. Such differential equations are applied in modeling of oil reservoir simulation, the flow of fluid in porous media, global water production, and many numerous natural phenomena [12–14]. Because of the fractional order, obtaining the exact solutions of such problems is difficult. Therefore, developing reliable and effective numerical methods for the solution of such equations is increasingly important.

Even though there have been several investigations into the case of integer order singularly perturbed parabolic-reaction diffusion equations [15–22] and time fractional parabolic reaction-diffusion [23], as the best of author knowledge no one can be applied the numerical scheme on singularly perturbed time-fractional parabolic reaction-diffusion equation. Motivating on this, fitted cubic B-spline collocation method for solving initial boundary value problem of one-dimensional singularly perturbed time-fractional parabolic reaction-diffusion equation is presented in this paper.

The sequences of this study is: the preliminaries and continuous solutions is discussed in Section 2. In Section 3 and 4 numerical formulation and convergence analysis are presented. Discussion of numerical results and conclusions are respectively given under Section 5 and 6.

2. Preliminaries and Continuous Solutions

Definition 2.1. Let Z be a complex number such that $\mathrm{Re}(z) > 0$, then the gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} \exp(-x)x^{z-1}dx.$$

Definition 2.2. When the function $h(t)$ possesses lowest bound of zero, then Caputo fractional derivative is defined as

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t h^{(n)}(\xi)(t-\xi)^{n-\alpha-1}d\xi, \quad n-1<\alpha<n.$$

Definition 2.3. The Caputo fractional differentiation of a function $u(x,t)$ with respect to t is defined as

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\partial^n u(x,\xi)}{\partial \xi^n}(t-\xi)^{n-\alpha-1}d\xi, & \text{if } \alpha \in (n-1,n), \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \text{if } \alpha = n. \end{cases}$$

Lemma 2.1. *Let $t_0 \in (0,1)$ is the minimum value of the function g, where $g \in C^1[0,1]$. Then,*

$$\partial_c^\alpha g(t_0) \le \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}(g(t_0)-g(0)) \le 0,$$

where $0<\alpha<1$ and ∂_C^α represents the Caputo fractional derivative.

Proof. Define the auxiliary function, $k(t) = g(t)-g(t_0)$. Then, $k(t)\ge 0$ and $k(t_0) = g(t_0)-g(t_0) = 0$. Now,

$$\partial_C^\alpha k(t_0) = \frac{1}{\Gamma(n-\alpha)}\int_0^{t_0}(t_0-\xi)^{-\alpha}k'(\xi)d\xi.$$

Applying integration by parts we obtain

$$\begin{aligned} \partial_C^\alpha k(t_0) &= \frac{1}{\Gamma(n-\alpha)}\left(-t_0^{-\alpha}k(0) - \alpha\int_0^{t_0}(t_0-\xi)^{-\alpha-1}k'(\xi)d\xi\right), \\ &\le \frac{1}{\Gamma(n-\alpha)}(-t_0^{-\alpha}k(0)), \\ &\le \frac{1}{\Gamma(n-\alpha)}(t_0^{\alpha}(g(t_0)-g(0))), \\ &\le 0. \end{aligned}$$

□

The existence and uniqueness for a solution (1.1), (1.2), (1.3) can be established under the assumption that the data $b(x,t)$, $f(x,t)$ are Holder's continuous and satisfying the compatibility conditions at the corner points $(0,0)$ and $(1,0)$. That is

$$\begin{cases} \varphi_b(0,0) = \varphi_l(0), \\ \varphi_b(1,0) = \varphi_r(0), \\ \frac{\partial^\alpha \varphi_l(0)}{\partial t^\alpha} - \varepsilon\frac{\partial^2\varphi_b(0,0)}{\partial x^2} + b(0,0)\varphi_b(0,0) = f(0,0), \\ \frac{\partial^\alpha \varphi_r(0)}{\partial t^\alpha} - \varepsilon\frac{\partial^2\varphi_b(1,0)}{\partial x^2} + b(1,0)\varphi_b(1,0) = f(1,0). \end{cases} \tag{2.1}$$

Lemma 2.2 (Continuous Maximum Principle)**.** *Under the assumption of Lemma 2.1 the following inequality holds. Let $\Lambda(x,t)$ be sufficiently smooth function which satisfies $\Lambda(x,t)\ge 0$ on ∂D. Then, $\mathcal{L}_\varepsilon\Lambda(x,t) > 0$ on D implies that $\Lambda(x,t)\ge 0$, for all $(x,t)\in \bar{D}$.*

Proof. Let (λ, ω) be a point that satisfy

$$\Lambda(\lambda, \omega) = \min_{(x,t)\in\bar{D}} \Lambda(x,t),$$

and $\Lambda(\lambda, \omega) < 0$. Then, $\Lambda(\lambda, \omega) \notin \partial D$. Then, we have

$$\mathcal{L}_\varepsilon \Lambda(\lambda, \omega) = \varepsilon \Lambda_{xx}(\lambda, \omega) - b(\lambda, \omega)\Lambda(\lambda, \omega) - \frac{\partial^\alpha \Lambda(\lambda, \omega)}{\partial t^\alpha} \leq 0.$$

Since $\Lambda_{xx}(\lambda, \omega) \geq 0$ and $\frac{\partial^\alpha \Lambda(\lambda,\omega)}{\partial t^\alpha} = 0$, then $\mathcal{L}_\varepsilon \Lambda(\lambda, \omega) \leq 0$, which contradicts the initial assumption. Therefore, $\Lambda(\lambda, \omega) \geq 0$, for all $(x,t) \in \bar{D}$. □

Lemma 2.3 (Stability Estimate). *Let $u(x,t)$ be the solution of the continuous problem of* (1.1)*. Then,*

$$\|u(x,t)\| \leq \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\},$$

where $\bar{\psi} = \max \psi(x)$, for all $x \in [0,1]$, $\varphi_l = \max \varphi_l(t)$, $\varphi_r = \max \varphi_r(t)$, for all $t \in [0,T]$.

Proof. Define barer functions $\Psi^\pm$ as

$$\Psi^\pm(x,t) = \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm u(x,t).$$

At initial and boundaries we have

$$\begin{aligned}
\Psi^\pm(x,0) =& \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm \psi(x) \geq 0,\\
\Psi^\pm(0,t) =& \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm \varphi_l \geq 0,\\
\Psi^\pm(1,t) =& \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm \varphi_r \geq 0.
\end{aligned}$$

Now, considering the differential operator $\mathcal{L}_\varepsilon$ on the domain D we have

$$\left(\mathcal{L}_\varepsilon + \frac{\partial^\alpha}{\partial t^\alpha}\right) \Psi^\pm(x,t) = -\varepsilon \frac{\partial^2 \Psi^\pm(x,t)}{\partial x^2} + b(x,t)\Psi^\pm(x,t) + \frac{\partial^\alpha \Psi^\pm(x,t)}{\partial t^\alpha}.$$

Now, using the above barer function we obtain

$$\begin{aligned}
\left(\mathcal{L}_\varepsilon + \frac{\partial^\alpha}{\partial t^\alpha}\right) \Psi^\pm(x,t) =& -\varepsilon \frac{\partial^2 \pm u(x,t)}{\partial x^2} + \frac{\partial^\alpha \Psi^\pm(x,t)}{\partial t^\alpha}\\
&+ b(x,t)\left(\vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right)\\
&\pm f(x,t) + \vartheta\left(\vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right).
\end{aligned}$$

Since $\|f\| \geq f(x,t)$, then we have

$$\begin{aligned}
\left(\mathcal{L}_\varepsilon + \frac{\partial^\alpha}{\partial t^\alpha}\right) \Psi^\pm(x,t) =& \pm f(x,t) + \vartheta\left(\vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right),\\
&\geq \vartheta\left(\max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right) \geq 0.
\end{aligned}$$

It follows that from Lemma 2.2 we have $\Psi^{\pm}(x,t) \geq 0$, for all $(x,t) \in [0,1] \times [0,T]$. Hence,

$$\vartheta^{-1}\|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm u(x,t) \geq 0,$$

which implies

$$|u(x,t)| \leq \vartheta^{-1}\|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}. \qquad \square$$

3. Numerical Scheme Formulation

3.1. **Temporal Discretization.** To discretize the time domain $[0,T]$ into M sub-intervals on a uniform step size $t_j = j\Delta t$, $j = 0,1,\ldots,M$, $\Delta t = \frac{T}{M}$. The time fractional derivative is approximated by Caputo sense that obtained from the quadrature formula at the point (x,t_{j+1}) as

$$\begin{aligned}
\partial_t^{\alpha}U(x,t_{j+1}) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{j+1}} \frac{\partial U(x,s)}{\partial s}(t_{j+1}-s)^{-\alpha}ds, \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\left(\frac{U(x,t_{k+1})-u(x,t_k)}{\Delta t}\right)\int_{t_k}^{t_{k+1}}(t_{j+1}-s)^{-\alpha}ds + e_{\Delta t}^{j+1}, \\
&= \beta\sum_{k=0}^{j} w_k\left(U(x,t_{j-k+1})-U(x,t_{j-k})\right) + e_{\Delta t}^{j+1},
\end{aligned}$$

where $\beta = \frac{(\Delta t)^{-\alpha}}{\Gamma(1-\alpha)}$, $w_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}$ and $e_{\Delta t}^{j+1} = \frac{(\Delta t)}{\Gamma(1-\alpha)}\int_{t_k}^{t_{k+1}}(t_{j+1}-s)^{-\alpha}ds$.

Therefore, the Caputo fractional derivative $\partial_t^{\alpha}u(x,t)$ combined with implicit Euler's method at the point (x,t_{j+1}) is

(3.1)
$$\partial_t^{\alpha}U(x,t_{j+1}) = \beta\Bigg(\left(U(x,t_{j+1})-U(x,t_j)\right) + \sum_{k=1}^{j-1} w_k\Big(U(x,t_{j-k+1})-U(x,t_{j-k})\Big)\Bigg).$$

Now, using (3.1) into (1.1) we obtain the time semi-discrete equation

$$\begin{aligned}
&\beta\left(\frac{U(x,t_{j+1})-U(x,t_j)}{2^{1-\alpha}}\right) + \beta\left(\sum_{k=1}^{j}\Big(U(x,t_{j-k+1})-U(x,t_{j-k})\Big)w_k\right) \\
&\quad - \varepsilon U_{xx}^{j+1}(x) + b^{j+1}U^{j+1}(x) = f^{j+1}(x),
\end{aligned} \tag{3.2}$$

which is rearranged as

$$\left(\beta + \mathcal{L}_{\varepsilon}^{\Delta t}\right)U^{j+1}(x) = R_i, \tag{3.3}$$

where

$$\begin{aligned}
\left(\beta + \mathcal{L}_{\varepsilon}^{\Delta t}\right)U^{j+1}(x) &= -\varepsilon U_{xx}^{j+1}(x) + (\beta + b^{j+1}(x)U^{j+1}(x), \\
R_i &= \beta U^j(x) - \beta\left(\sum_{j}^{k=1} w_k\left(U^{j-k+1}(x) - U^{j-k}(x)\right)\right) + f^{j+1}(x).
\end{aligned}$$

Lemma 3.1. *An error estimate associated with* (3.3) *satisfies the bounds*

$$\left|e_{\Delta}^{j+1}\right| \leq C(\Delta t)^{(2-\alpha)}.$$

Proof. We have

$$\begin{aligned} e_{\Delta}^{j+1} &= \frac{O(\Delta t)}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t^{k+1}} (t_{j+1}-s)\, ds \\ &= \frac{O(\Delta t)}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \left(\frac{(j-k+1)^{1-\alpha} - (j-k)^{1-\alpha}}{1-\alpha} \right) (\Delta t)^{1-\alpha} \\ &= \frac{O((\Delta t)^{2-\alpha})}{\Gamma(2-\alpha)} \left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha} \right) \\ &= \frac{O((\Delta t)^{2-\alpha})}{\Gamma(2-\alpha)} (j+1)^{1-\alpha} \\ &\leq C(\Delta t)^{2-\alpha}, \end{aligned}$$

which implies $\left|e_{\Delta}^{j+1}\right| \leq C(\Delta t)^{(2-\alpha)}$, where C is a constant independent of ε and Δt. □

Lemma 3.2. *The semi-discretize solution* $U(x, t_{j+1})$ *and its derivative satisfy the following bounds*

$$\left| \frac{\partial^k U(x, t_{j+1})}{\partial x^k} \right| \leq C \left[1 + \varepsilon^{\frac{i}{2}} \left(\exp\left(\frac{-x}{\sqrt{\varepsilon}} \right) + \exp\left(\frac{-(1-x)}{\sqrt{\varepsilon}} \right) \right) \right],$$

with $0 \leq k \leq 4$.

3.2. **Spatial Discretization.** Consider a uniform mesh size in spatial domain $[0,1]$ into N sub-intervals as $0 = x_0 < x_1 < \cdots < x_N = 1$, with $h = x_{i+1} - x_i$, $i = 0, 1, \ldots, N-1$. Let $B_i(x)$ be the cubic B-spline basis function defined by:

(3.4)

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x_{i-2} \leq x \leq x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \leq x \leq x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \leq x \leq x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \leq x \leq x_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

The values of $B_i(x)$ and its first and second derivatives at the knot points x_{i-2}, x_{i-1}, x_i, x_{i+1} and x_{i+2} are given in Table 1. Let $\Xi(x)$ be cubic B-spline collocation

TABLE 1. Coefficients of cubic B-splines and its derivatives at knots

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i^{'}(x)$	0	$3/h$	0	$-3/h$	0
$B_i^{"}(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

approximation to the solution $U(x, t_{j+1})$ given in (3.4) defined as

$$\Xi(x) \approx \sum_{i=-1}^{N+1} \gamma_i B_i(x). \tag{3.5}$$

Differentiating (3.5) twice and using Table 1 we obtain an approximate value at the knot point x_i as:

$$\begin{cases} \Xi_i = \gamma_{i-1} + 4\gamma_i + \gamma_{i+1}, \\ \Xi_i' = \frac{3}{h}\left(\gamma_{i+1} - \gamma_{i-1}\right), \\ \Xi_i" = \frac{6}{h^2}\left(\gamma_{i-1} - 2\gamma_i + \gamma_{i+1}\right). \end{cases} \tag{3.6}$$

To control the effect of perturbation parameter on the solution behavior, we introduce the fitting factor ($\sigma_i = \sigma(x_i, \varepsilon)$) in the way that the approximate solutions uniformly converges to the exact solution of (1.1) and using (3.6) into (3.3) we obtain an $(N+3) \times (N+3)$ systems of linear equation:

$$\left(-\frac{6\sigma_i}{h^2} + \beta + b_i^{j+1}\right)\gamma_{i-1}^{j+1} + \left(\frac{12\sigma_i}{h^2} + 4\beta + 4b_i^{j+1}\right)\gamma_i^{j+1} + \left(-\frac{6\sigma_i}{h^2} + \beta + b_i^{j+1}\right)\gamma_{i+1}^{j+1} = \bar{R}_i, \tag{3.7}$$

that can be written as a three term recurrence relation of the form:

$$E_i\gamma_{i-1}^{j+1} + F_i\gamma_i^{j+1} + G_i\gamma_{i+1}^{j+1} = H_i, \quad i = 0, 1, \dots, N, \tag{3.8}$$

where

$$\begin{aligned} E_i &= G_i = -6\sigma_i + \left(\beta + b_i^{j+1}\right)h^2, \\ F_i &= 12\sigma_i + 4\left(\beta + b_i^{j+1}\right)h^2, \\ H_i &= h^2\bar{R}_i, \end{aligned}$$

and

$$\sigma_i = \frac{h^2 b_i^{j+1}}{6}\left(\frac{1 + 2\cosh^2(\rho_i h)}{2\sinh^2 \rho_i h}\right), \qquad \rho_i = \sqrt{\frac{b_i^{j+1}}{\varepsilon}}, \tag{3.9}$$

for $|\sigma_i - \varepsilon| \leq Ch^2$ with C is independent of ε and h [24, 25].

By imposing the boundary conditions (1.3) into (3.6) it yields

$$\begin{aligned} \gamma_{-1} &= \varphi_l(t_{j+1}) - 4\gamma_0 - \gamma_1, \\ \gamma_{N+1} &= \varphi_r(t_{j+1}) - \gamma_{N-1} - 4\gamma_N. \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8) we obtain an $(N+1) \times (N+1)$ systems of linear equations:

$$\begin{aligned} \left(F_0 - 4E_0\right)\gamma_0^{j+1} + \left(G_0 - E_0\right)\gamma_1^{j+1} &= H_0 - E_0\varphi_l(t_{j+1}), \\ E_i\gamma_{i-1}^{j+1} + F_i\gamma_i^{j+1} + G_i\gamma_{i+1}^{j+1} &= H_i, \quad i =, 1, \dots, N-1, \\ \left(E_N - G_N\right)\gamma_{N-1}^{j+1} + \left(F_N - 4G_N\right)\gamma_N^{j+1} &= H_N - G_N\varphi_r(t_{j+1}). \end{aligned} \tag{3.11}$$

4. Convergence Analysis

Lemma 4.1. *The collocation cubic B-spline* $\{B_{-1}(x), B_0(x), \ldots, B_N(x), B_{N+1}(x)\}$ *defined in* (3.4) *satisfy the inequality*

$$\sum_{i=-1}^{N+1} |B_i(x)| \leq 10, \quad x \in [0,1]. \tag{4.1}$$

Proof. From triangular inequality we have

$$\left| \sum_{i=-1}^{N+1} B_i(x) \right| \leq \sum_{i=-1}^{N+1} |B_i(x)|.$$

The cubic B-spline $B_i(x)$ is non-zero at only three nodal points. Thus, at any nodal point x_i, we have

$$\sum_{i=-1}^{N+1} |B_i(x)| = |B_{i-1}(x)| + |B_i(x)| + |B_{i+1}(x)| = 1 + 4 + 1 = 6 < 10.$$

For $x \in [x_{i-1}, x_i]$,

$$\sum_{i=-1}^{N+1} |B_i(x_i)| \leq 4 \quad \text{and} \quad \sum_{i=-1}^{N+1} |B_{i-1}(x_{i-1})| \leq 4,$$

and similarly for $x \in [x_{i-1}, x_i]$,

$$\sum_{i=-1}^{N+1} |B_{i+1}(x_i)| \leq 1 \quad \text{and} \quad \sum_{i=-1}^{N+1} |B_{i-2}(x_{i-1})| \leq 1.$$

Therefore, for any point $x \in [x_{i-1}, x_i]$, we obtain

$$\sum_{i=-1}^{N+1} |B_i(x)| = |B_{i-2}(x)| + |B_{i-1}(x)| + |B_i(x)| + |B_{i+1}(x)| \leq 10. \qquad \square$$

Lemma 4.2. *Let* $\Xi(x)$ *be the collocation approximation to the solution* $\bar{u}(x)$*of the boundary value problem of* (3.3). *If* $R \in C^2[0,1]$, *then the parameter uniform error estimate is given by*

$$\|u(x_i, t_{j+1}) - \Xi(x_i)\|_\infty \leq C\left(h^2\right), \tag{4.2}$$

for sufficiently small h *and* C *is a positive constant.*

Proof. Assume $\bar{\Xi}(x)$ be a unique spline that interpolate $\bar{u}(x)$ of the boundary value problem (3.3) at the $(j+1)^{th}$ time level given by

$$\bar{\Xi}(x) \approx \sum_{i=-1}^{N+1} \bar{\gamma}_i B_i(x). \tag{4.3}$$

If $b(x), R_i \in C^2[0,1]$, then $\bar{u}(x) \in C^4[0,1]$, for all j, $0 \le j \le M$, and hence from error bound estimate [26] we have

$$\left\| D^n \left(\bar{u}(x) - \bar{\Xi}(x) \right) \right\|_\infty \le \varsigma_n \left\| \frac{d^4 \bar{u}(x)}{dx^4} \right\|_\infty h^{4-n}, \quad n = 0, 1, 2, \tag{4.4}$$

and ς_n is constant independent of term h and N.

Using triangular inequality we have

$$\| \bar{u}(x_i) - \Xi(x_i) \|_\infty \le \left\| \bar{u}(x_i) - \bar{\Xi}(x_i) \right\|_\infty + \left\| \bar{\Xi}(x_i) - \Xi(x_i) \right\|_\infty . \tag{4.5}$$

Let $\left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right) \Xi(x_i) = H_i$ and $\left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right) \bar{\Xi}(x_i) = \bar{H}_i$ that satisfies the boundary conditions $\bar{\Xi}(x_0) = \varphi_l(t_{j+1})$ and $\bar{\Xi}(x_N) = \varphi_r(t_{j+1})$. Then,

(4.6)

$$\begin{aligned} \left| \left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right) \Xi(x_i) - \left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right) \bar{\Xi}(x_i) \right| \le |\sigma_i - \varepsilon| \, \varsigma_2 \left\| \frac{d^2 \bar{u}(x_i)}{dx^2} \right\|_\infty + |\varepsilon| \, \varsigma_2 \left\| \frac{d^2 \bar{u}(x_i)}{dx^2} \right\|_\infty \\ + \varsigma_0 \|b(x_i)\|_\infty \|\bar{u}(x_i)\|_\infty . \end{aligned}$$

Using the bound of artifial viscocity $|\sigma_i - \varepsilon| \le Ch^2$, and Lemma 3.2 into (4.6) it gives

$$\max_{x \in D} \left| \left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right) u(x_i, t_{j+1}) - \left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right) \bar{\Xi}(x_i) \right| \le \varsigma h^2, \tag{4.7}$$

where $\varsigma = \varepsilon \varsigma_2 + \left(C\varsigma_2 + \varsigma_0 \|b(x_i, t_{j+1})\|_\infty\right) h^2$.

Using (4.3) and (3.8) the i-th nodal point of $\left| \left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right) \left(\bar{u}(x_i) - \bar{\Xi}(x_i) \right) \right|$ yields

$$36\sigma_0 \mu_0 = Y_0, \tag{4.8}$$

$$\left(-6\sigma_i + h^2 (\beta + b_i)\right)(\mu_{i-1} + \mu_{i+1}) + (12\sigma_i + 4h^2 (\beta + b_i))\mu_i = Y_i, \tag{4.9}$$

$$i = 1, 2, \ldots, N-1,$$

$$36\sigma_N \mu_N = Y_N, \tag{4.10}$$

where $\mu_i = \gamma_i - \bar{\gamma}_i$, $i = -1, 0, \ldots, N+1$ and $Y_i = H_i - \bar{H}_i$, $i = 0, 1, \ldots, N$. Then, (4.9) given as

$$(12\sigma_i + 4h^2 (\beta + b_i))\mu_i = Y_i - (-6\sigma_i + h^2 (\beta + b_i))(\mu_{i-1} + \mu_{i+1}). \tag{4.11}$$

Define

$$Y = \max_{1 \le i \le N-1} Y_i, \quad \mu = \max_{1 \le i \le N-1} \mu_i,$$

and using the condition $0 < \eta \le b_i$ into (4.11) we get

$$\left(12\varepsilon + (4(\beta + \eta) + 12C)\, h^2\right) \mu \le Y + 2\mu \left(6\varepsilon + (\beta + \eta + 6C)\, h^2\right),$$

which gives

$$\mu \le \frac{\varsigma h^2}{(\beta + \eta)}, \quad \mu_0 \le \frac{\varsigma h^2}{36C} \quad \text{and} \quad \mu_N \le \frac{\varsigma h^2}{36C}.$$

The values of μ_{-1} and μ_{N+1} is evaluated from (3.6) gives

$$\mu_{-1} \le \varsigma h^2 \left(\frac{1}{9C} + \frac{1}{C(\beta+\eta)} \right), \quad \mu_{N+1} \le \varsigma h^2 \left(\frac{1}{9C} + \frac{1}{C(\beta+\eta)} \right).$$

With values of ς, there exists a constant ϖ, independent of h and ε such that

$$\mu = \max_{-1\le i \le N+1} \{\mu_i\} \le \varpi h^2. \tag{4.12}$$

Results from (4.12) and Lemma 4.1 gives

$$\|\Xi(x_i) - \bar{\Xi}(x_i)\|_\infty \le 10\varpi h^2. \tag{4.13}$$

Thus,

$$\|\bar{u}(x_i) - \Xi(x_i)\|_\infty \le Ch^2,$$

where $C = 10\varpi + \varsigma_0 h^2$. □

Theorem 4.1. *Let $u(x,t)$ be the solution of* (1.1) *and $U(x_i, t_{j+1})$ is the solution of the total discretized equation. Under the hypothesis of Lemma* 3.1 *and Lemma* 4.2*, then the ε-uniform estimate holds*

$$\max_{1\le i,j\le N,M-1} |U(x_i,t_{j+1}) - \Xi(x_i)| \le C\left(h^2 + (\Delta t)^{2-\alpha}\right), \tag{4.14}$$

where C is the constant independent of ε, h and (Δt).

Proof. The proof is applying triangular inequality. □

5. Numerical Results and Discussion

The results of maximum point-wise error is evaluated by double mesh principle. i.e.

$$E_\varepsilon^{N,M} = \max_{1\le i\le N-1} \left| U_i^{N,M} - U_i^{2N,2M} \right|.$$

The ε-uniform errors are approximated using

$$E^{N,M} = \max_\varepsilon E_\varepsilon^{N,M},$$

and ε-uniform rate of convergence is obtained by the formula

$$r_\varepsilon^{N,M} = \frac{\log(E_\varepsilon^{N,M}) - \log(E_\varepsilon^{2N,2M})}{\log(2)}.$$

Example 5.1.

$$\frac{\partial^\alpha}{\partial t^\alpha} u - \varepsilon \frac{\partial^2 u}{\partial x^2} + (1 + x^2 + t^2 e^t)u = e^t - 1 + \sin(\pi x), \quad (x,t) \in D,$$

subject to the initial and boundary conditions

$$u(x,0) = 0, \quad x \in \Omega = [0,1],$$

and

$$u(0,t) = 0 = u(1,t) = 0, \quad t \in [0,1].$$

Example 5.2.

$$\frac{\partial^\alpha}{\partial t^\alpha}u - \varepsilon\frac{\partial^2 u}{\partial x^2} + (1 + xe^{-t})u = f(x,t), \quad (x,t) \in D,$$

subject to

$$u(x,0) = 0, \quad x \in \Omega = [0,1],$$

and

$$u(0,t) = \varphi_l(t), \quad u(1,t) = \varphi_r(t), \quad t \in [0,1],$$

where the functions $\varphi_l(t)$, $\varphi_r(t)$ and $f(x,t)$ are chosen from the exact solution

$$u(x,t) = t\left(\frac{e^{\frac{-x}{\sqrt{\varepsilon}}} + e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}}{1 + e^{\frac{-1}{\sqrt{\varepsilon}}}} - \cos^2(\pi x)\right).$$

Table 2. Maximum absolute error of Example 5.1 for $\alpha = 0.9$ and $N = M$

ε	16	32	64	128	256
2^{-8}	6.5344e-03	3.8082e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-10}	6.5321e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-12}	6.5317e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-14}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-16}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-18}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-20}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-22}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
$E^{N,M}$	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
$r^{N,M}$	0.77837	0.77949	0.82916	0.85394	

To validate the theoretical findings point-wise maximum absolute error of the test examples is given out in Tables 2 and 3. As observed from the tables, as the perturbation parameter approaches zero and the mesh size increases, the result of numerical examples converges uniformly which confirms an agreement with theoretical findings. The physical behavior of the scheme is demonstrated in Figure 1 and 3 which yields numerical example has twin layers at the end of the spatial domain. In Figure 2 and 4 the boundary layer behavior is displayed and it has a parabolic boundary layer at $x = 0$ and $x = 1$. Finally, in Figure 5 and 6 the log-log scale of maximum absolute error is illustrated and it shows as mesh points increases and $\varepsilon \to 0$, then the maximum absolute error decreases monotonically.

TABLE 3. Maximum absolute error of Example 5.2 for $\alpha = 0.9$ and $N = M$

ε	8	16	32	64	128
2^{-8}	5.9457e-03	3.9889e-03	2.5301e-03	1.5170e-03	9.7828e-04
2^{-10}	7.7475e-03	5.0285e-03	3.1961e-03	1.8801e-03	1.0662e-03
2^{-12}	8.0307e-03	5.7727e-03	3.5375e-03	2.0678e-03	1.1680e-03
2^{-14}	8.0445e-03	5.9251e-03	3.6635e-03	2.1471e-03	1.2139e-03
2^{-16}	8.0477e-03	5.9292e-03	3.7472e-03	2.1816e-03	1.2326e-03
2^{-18}	8.0485e-03	5.9296e-03	3.7489e-03	2.2003e-03	1.2397e-03
2^{-20}	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2425e-03
2^{-22}	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2429e-03
2^{-24}	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2429e-03
$E^{N,M}$	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2429e-03
$r^{N,M}$	0.44080	0.66149	0.76824	0.82451	

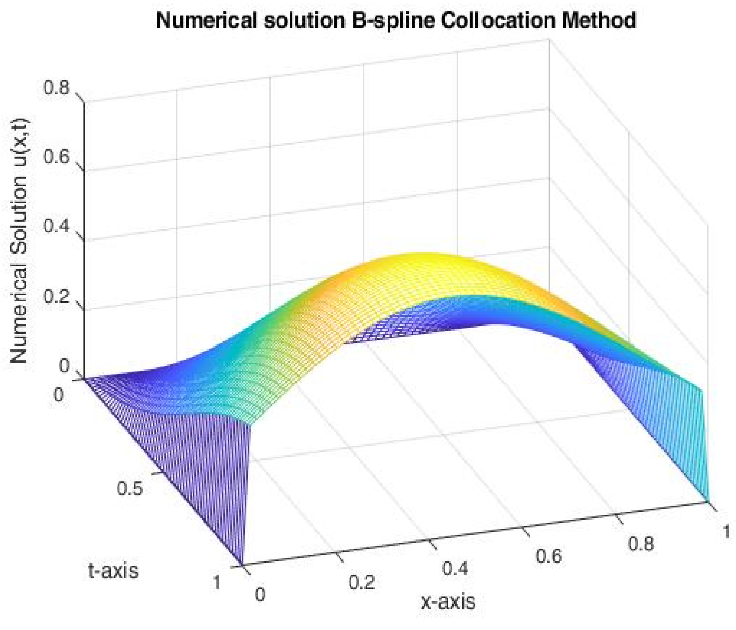

FIGURE 1. Numerical solution of Example 5.1 for $N = M = 64$ and $\varepsilon = 2^{-10}$

6. CONCLUSIONS

A fitted cubic B-spline collocation method is conducted for one-dimensional initial boundary value problem of singularly perturbed time-fractional parabolic reaction-diffusion equations. By using the Caputo fractional sense and employing the implicit Euler technique, the time fractional derivative is addressed. Spatial derivatives are

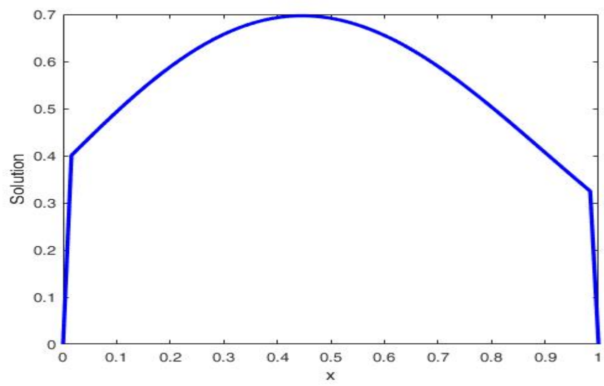

FIGURE 2. Boundary layer formation of Example 5.1

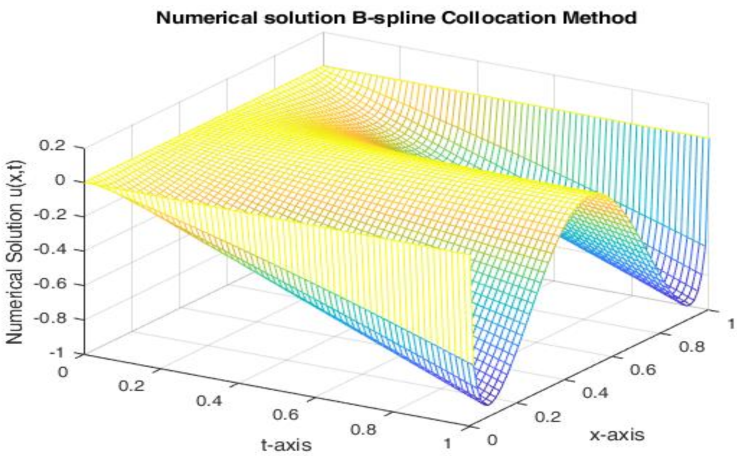

FIGURE 3. Numerical solution of Example 5.2 for $N = M = 64$ and $\varepsilon = 2^{-10}$

dicretized by the fitted cubic B-spline collocation method. The convergence analysis of the scheme is proved and it is accurate of order $O(h^2 + (\Delta t)^{2-\alpha})$. The results from numerical examples confirmed that the scheme is uniformly convergent and has twin layers at $x = 0$ and $x = 1$.

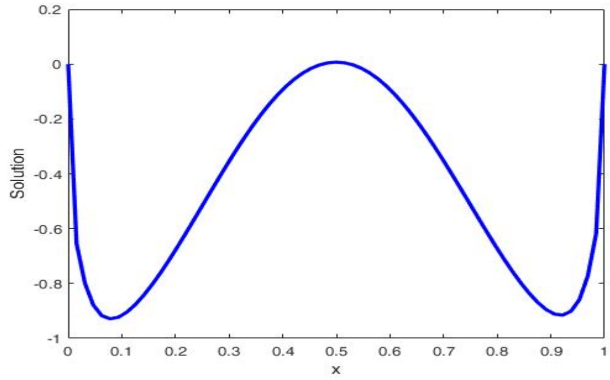

FIGURE 4. Boundary layer formation of Example 5.2

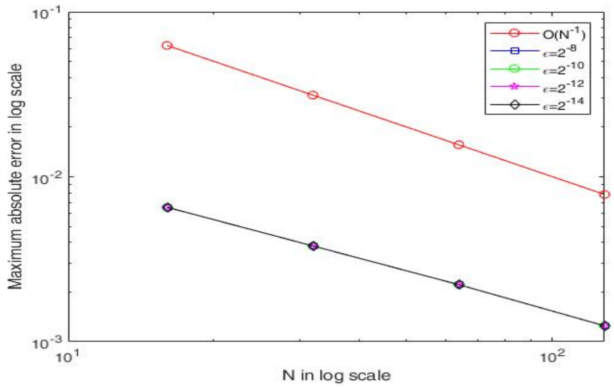

FIGURE 5. Log-log scale plot for Example 5.1

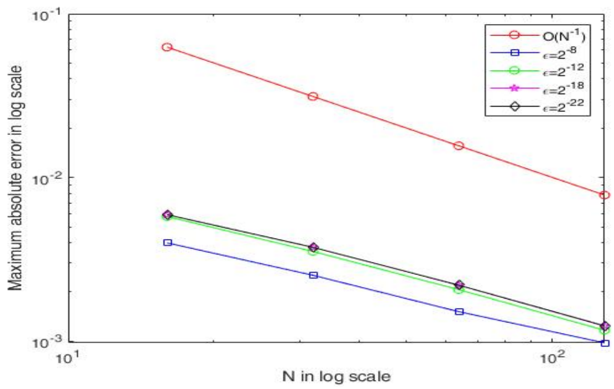

FIGURE 6. Log-log scale plot for Example 5.2

REFERENCES

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier-Mathematics Studies, North-Holland, 2006.

[2] C. Jadhav, T. Dale and S. Dhondge, *A review on applications of fractional differential equations in engineering domain*, MSEA **71**(4) (2022), 7147–7166. https://doi.org/10.17762/msea.v71i4.1331

[3] A. Atangana, *On the singular perturbations for fractional differential equation*, Scientific World Journal **2014** (2014). https://doi.org/10.1155/2014/752371

[4] Y. H. Feng and J. Q. Mo, *Asymptopic solution for singularly perturbed fractional order differential equation*, J. Math. **36**(2) (2016), 239–245.

[5] H. Jin, W. Liu and J. Zhang, *Singularly perturbed fractional Schrödinger equation involving a general critical nonlinearity*, Adv. Nonlinear Stud. **18**(3) (2018), 487–499. https://doi.org/10.1515/ans-2018-2015

[6] A. Atangana and E. F. D. Goufo, *The Caputo-Fabrizio fractional derivative applied to a singular perturbation problem*, International Journal of Mathematical Modelling and Numerical Optimisation **9**(3) (2019), 241–253. https://doi.org/10.1504/IJMMNO.2019.100486

[7] K. Kumar, P. C. P and J. Vigo-Aguiar, *Numerical solution of time-fractional singularly perturbed convection-diffusion problems with a delay in time*, Math. Methods Appl. Sci. **44**(4) (2021), 3080–3097. https://doi.org/10.1002/mma.6477

[8] S. K. Sahoo and V. Gupta, *A robust uniformly convergent finite difference scheme for the time-fractional singularly perturbed convection-diffusion problem*, Comput. Math. Appl. **137** (2023), 126–146. https://doi.org/10.1016/j.camwa.2023.02.016

[9] W. T. Aniley and G. F. Duressa, *Nonstandard finite difference method for time-fractional singularly perturbed convection-diffusion problems with a delay in time*, Results Appl. Math. **21** (2024), Article ID 100432. https://doi.org/10.1016/j.rinam.2024.100432

[10] H. G. Kumie, A. A. Tiruneh and G. A. Derese, *Crank-Nicolson method for solving time-fractional singularly perturbed delay partial differential equations*, Results Appl. Math. **11**(1) (2024), Article ID 2293373. https://doi.org/10.1080/27684830.2023.2293373

[11] S. K. Sahoo, V. Gupta and S. Dubey, *A robust higher-order finite difference technique for a time-fractional singularly perturbed problem*, Math. Comput. Simulation **215** (2024), 43–68. https://doi.org/10.1016/j.matcom.2023.08.013

[12] V. Chuk, B. Datsko and V. Meleshko, *Mathematical modeling of time fractional reaction-diffusion systems*, J. Comput. Appl. Math. **220**(1–2) (2008), 215–225. https://doi.org/10.1016/j.cam.2007.08.011

[13] M. Daoud, E. H. Laamri and A. Baalal, *A class of fractional parabolic reaction-diffusion systems with control of total mass: theory and numerics*, J. Pseudo-Differ. **15**(1) (2024), 18 pages. https://doi.org/10.1007/s11868-023-00576-w

[14] K. Van Bockstal, M. A. Zaky and A. S. Hendy, *On the existence and uniqueness of solutions to a nonlinear variable order time-fractional reaction-diffsion equation with delay*, Commun. Nonlinear Sci. Numer. **115** (2022), Article ID 106755. https://doi.org/10.1016/j.cnsns.2022.106755

[15] J. B. Munyakazi and K. C. Patidar, *A fitted numerical method for singularly perturbed parabolic reaction-diffusion problems*, Comput. Appl. Math. **32** (2013), 509–519. https://doi.org/10.1007/s40314-013-0033-7

[16] C. Clavero and J. L. Gracia, *A higher order uniformly convergent method with Richardson extrapolation in time for singularly perturbed reaction-diffusion parabolic problems*, J. Comput. Appl. Math. **252** (2013), 75–85. https://doi.org/10.1016/j.cam.2012.05.023

[17] S. C. S. Rao, S. Kumar and J. Singh, *A discrete Schwarz waveform relaxation method of higher order for singularly perturbed parabolic reaction-diffusion problems*, J. Math. Chem. **58**(3) (2020), 574–594. https://doi.org/10.1007/s10910-019-01086-1

[18] T. A. Bullo, G. F. Duressa and G. Degla, *Accelerated fitted operator finite difference method for singularly perturbed parabolic reaction-diffusion problems*, Comput. Methods Differ. Equ. **9**(3) (2021), 886–898. https://doi.org/10.22034/cmde.2020.39685.1737

[19] F. W. Gelu and G. F. Duressa, *Computational method for singularly perturbed parabolic reaction-diffusion equations with Robin boundary conditions*, JAMI **40**(1-2) (2022), 25–45. https://doi.org/10.14317/jami.2022.025

[20] N. A. Mbroh and J. B. Munyakazi, *A robust numerical scheme for singularly perturbed parabolic reaction-diffusion problems via the method of lines*, Int. J. Comput. Math. **99**(6) (2022), 1139–1158. https://doi.org/10.1080/00207160.2021.1954621

[21] A. Gupta, A. Kaushik and M. Sharma, *A higher-order hybrid spline difference method on adaptive mesh for solving singularly perturbed parabolic reaction-diffusion problems with Robin-boundary conditions*, Numer. Methods Partial Differ. Equ. **39**(2) (2023), 1220–1250. https://doi.org/10.1002/num.22931

[22] G. F. Duressa, F. W. Gelu and G. D. Kebede, *A robust higher-order fitted mesh numerical method for solving singularly perturbed parabolic reaction-diffusion problems*, Results Appl. Math. **20** (2023), Article ID 100405. https://doi.org/10.1016/j.rinam.2023.100405

[23] J. Zhang and X. Yang, *A class of efficient difference method for time fractional reaction-diffusion equation*, Comput. Appl. Math. **37**(4) (2018), 4376–4396. https://doi.org/10.1007/s40314-018-0579-5

[24] M. K. Kadalbajoo and P. Arora, *B-spline collocation method for a singularly perturbed reaction-diffusion problem using artificial viscosity*, Int. J. Comput. Methods **6**(01) (2009), 23–41.

[25] N. T. Negero, *A robust fitted numerical scheme for singularly perturbed parabolic reaction-diffusion problems with a general time delay*, Results Phys. **51** (2023), Article ID 106724. https://doi.org/10.1016/j.rinp.2023.106724

[26] C. A. Hall, *On error bounds for spline interpolation*, J. Approx. Theory **1**(2) (1968), 209–218. https://doi.org/10.1016/0021-9045(68)90025-7

[1]Department of Mathematics,
Jimma University,
Jimma, Ethiopia
Email address: feyisae.2014@gmail.com
ORCID iD: https://orcid.org/0000-0002-8737-8684
Email address: gammeef@gmail.com
ORCID iD: https://orcid.org/0000-0003-1889-4690

KRAGUJEVAC JOURNAL OF MATHEMATICS

About this Journal

The *Kragujevac Journal of Mathematics* (KJM) is an international journal devoted to research concerning all aspects of mathematics. The journal's policy is to motivate authors to publish original research that represents a significant contribution and is of broad interest to the fields of pure and applied mathematics. All published papers are reviewed and final versions are freely available online upon receipt. Volumes are compiled and published and hard copies are available for purchase. The journal appears in one volume annually, with the publication frequency increasing over time: four issues per annum (2018–2020), six issues per annum (2021–2025), and ten issues per annum commencing from 2026.

During the period 1980–1999 (volumes 1–21) the journal appeared under the name *Zbornik radova Prirodno–matematičkog fakulteta Kragujevac* (Collection of Scientific Papers from the Faculty of Science, Kragujevac), after which two separate journals—the *Kragujevac Journal of Mathematics* and the *Kragujevac Journal of Science*—were formed.

Instructions for Authors

The journal's acceptance criteria are originality, significance, and clarity of presentation. The submitted contributions must be written in English and be typeset in TEX or LATEX using the journal's defined style (please refer to the Information for Authors section of the journal's website `http://kjm.pmf.kg.ac.rs`). Papers should be submitted using the online system located on the journal's website by creating an account and following the submission instructions (the same account allows the paper's progress to be monitored). For additional information please contact the Editorial Board via e-mail (`krag_j_math@kg.ac.rs`).