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UNIFORMLY CONVERGENT TIME-FRACTIONAL REACTION-DIFFUSION EQUATION

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Abstract. In this paper, singularly perturbed time-fractional parabolic reaction-diffusion of initial boundary value problem is studied. The time-fractional derivative is applied in the Caputo fractional sense and handled by implicit Euler method. The spatial derivative is approximated by fitted cubic B-spline collocation method on a uniform mesh. Convergence analysis of the scheme is conducted and it is accurate of order $O(h^2 + (\Delta t^{2-\alpha}))$. To test the effectiveness of proposed method two model examples are considered. The results from the experiment confirm that the scheme is uniformly convergent and has twin layers at the end spatial domain.

1. Introduction

Fractional differential-equations are generalizations of the standard integer order derivatives to an arbitrary (non-integer) order that can be obtained either in space or time variable. Due to the ability to model complex phenomena fractional differential equations currently attracted several scholars in the fields of engineering, physics, chemistry, biology and other fields [1, 2].

Alike character of classical differential equation, the solution of parameter dependent fractional-differential equations also invariably characterized by large oscillation as the perurbation parameter approaches to zero. Solving such fractional differential equations using classical numerical methods on a uniform mesh may arise large oscillations in the entire domain of interest due to the boundary layer behavior. To overcome this, appropriate numerical method whose accuracy doesn't depend on the perturbation

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parameter will be applied. Hence, there has been a lot of effort in developing numerical methods for the solution of singular perturbed fractional-differential equations [3–11]. This study concerns with the singularly perturbed time-fractional parabolic reaction-diffusion of initial boundary value problem in the domain $D = (0,1) \times (0,T]$:

$$\left(\frac{\partial^\alpha}{\partial t^\alpha}u - \varepsilon\frac{\partial^2}{\partial x^2}u + bu\right)(x,t) = f(x,t), \quad (x,t) \in D, \tag{1.1}$$

with an initial condition

$$u(x,0) = \psi(x), \quad x \in [0,1], \tag{1.2}$$

and boundary conditions

$$u(0,t) = \varphi_l(t), \quad u(1,t) = \varphi_r(t), \quad t \in [0,T], \tag{1.3}$$

where ε is a small perturbation parameter that satisfies $0 < \varepsilon < 1$ and $b(x,t) \geq \vartheta > 0$ is a smooth function. Under sufficient smoothness and compatibility conditions imposed on the functions $\psi(x)$, $\varphi_l(t)$, $\varphi_r(t)$ and $f(x,t)$) the initial-boundary value problem admits a unique solution $u(x,t)$ which exhibits twin boundary layer of width $O(\sqrt{\varepsilon})$ neighboring the boundaries $x = 0$ and $x = 1$.

Reaction-diffusion equations that time-derivative can be replaced by non-integer fractional order is known as time-fractional reaction diffusion equations. Such differential equations are applied in modeling of oil reservoir simulation, the flow of fluid in porous media, global water production, and many numerous natural phenomena [12–14]. Because of the fractional order, obtaining the exact solutions of such problems is difficult. Therefore, developing reliable and effective numerical methods for the solution of such equations is increasingly important.

Even though there have been several investigations into the case of integer order singularly perturbed parabolic-reaction diffusion equations [15–22] and time fractional parabolic reaction-diffusion [23], as the best of author knowledge no one can be applied the numerical scheme on singularly perturbed time-fractional parabolic reaction-diffusion equation. Motivating on this, fitted cubic B-spline collocation method for solving initial boundary value problem of one-dimensional singularly perturbed time-fractional parabolic reaction-diffusion equation is presented in this paper.

The sequences of this study is: the preliminaries and continuous solutions is discussed in Section 2. In Section 3 and 4 numerical formulation and convergence analysis are presented. Discussion of numerical results and conclusions are respectively given under Section 5 and 6.

2. Preliminaries and Continuous Solutions

Definition 2.1. Let Z be a complex number such that $\text{Re}(z) > 0$, then the gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} \exp(-x)x^{z-1}dx.$$

Definition 2.2. When the function $h(t)$ possesses lowest bound of zero, then Caputo fractional derivative is defined as

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t h^{(n)}(\xi)(t-\xi)^{n-\alpha-1}d\xi, \quad n-1<\alpha<n.$$

Definition 2.3. The Caputo fractional differentiation of a function $u(x,t)$ with respect to t is defined as

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\partial^n u(x,\xi)}{\partial \xi^n}(t-\xi)^{n-\alpha-1}d\xi, & \text{if } \alpha\in(n-1,n), \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \text{if } \alpha = n. \end{cases}$$

Lemma 2.1. *Let $t_0 \in (0,1)$ is the minimum value of the function g, where $g \in C^1[0,1]$. Then,*

$$\partial_c^\alpha g(t_0) \le \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}\left(g(t_0)-g(0)\right) \le 0,$$

where $0<\alpha<1$ and ∂_C^α represents the Caputo fractional derivative.

Proof. Define the auxiliary function, $k(t) = g(t) - g(t_0)$. Then, $k(t) \ge 0$ and $k(t_0) = g(t_0) - g(t_0) = 0$. Now,

$$\partial_C^\alpha k(t_0) = \frac{1}{\Gamma(n-\alpha)}\int_0^{t_0}(t_0-\xi)^{-\alpha}k'(\xi)d\xi.$$

Applying integration by parts we obtain

$$\begin{aligned} \partial_C^\alpha k(t_0) &= \frac{1}{\Gamma(n-\alpha)}\left(-t_0^{-\alpha}k(0) - \alpha\int_0^{t_0}(t_0-\xi)^{-\alpha-1}k'(\xi)d\xi\right), \\ &\le \frac{1}{\Gamma(n-\alpha)}(-t_0^{-\alpha}k(0)), \\ &\le \frac{1}{\Gamma(n-\alpha)}(t_0^{\alpha}(g(t_0)-g(0))), \\ &\le 0. \end{aligned}$$

□

The existence and uniqueness for a solution (1.1), (1.2), (1.3) can be established under the assumption that the data $b(x,t)$, $f(x,t)$ are Holder's continuous and satisfying the compatibility conditions at the corner points $(0,0)$ and $(1,0)$. That is

$$\begin{cases} \varphi_b(0,0) = \varphi_l(0), \\ \varphi_b(1,0) = \varphi_r(0), \\ \frac{\partial^\alpha \varphi_l(0)}{\partial t^\alpha} - \varepsilon\frac{\partial^2\varphi_b(0,0)}{\partial x^2} + b(0,0)\varphi_b(0,0) = f(0,0), \\ \frac{\partial^\alpha \varphi_r(0)}{\partial t^\alpha} - \varepsilon\frac{\partial^2\varphi_b(1,0)}{\partial x^2} + b(1,0)\varphi_b(1,0) = f(1,0). \end{cases} \tag{2.1}$$

Lemma 2.2 (Continuous Maximum Principle)**.** *Under the assumption of Lemma 2.1 the following inequality holds. Let $\Lambda(x,t)$ be sufficiently smooth function which satisfies $\Lambda(x,t)\ge 0$ on ∂D. Then, $\mathcal{L}_\varepsilon\Lambda(x,t) > 0$ on D implies that $\Lambda(x,t) \ge 0$, for all $(x,t)\in \bar{D}$.*

Proof. Let (λ, ω) be a point that satisfy

$$\Lambda(\lambda, \omega) = \min_{(x,t)\in\bar{D}} \Lambda(x, t),$$

and $\Lambda(\lambda, \omega) < 0$. Then, $\Lambda(\lambda, \omega) \notin \partial D$. Then, we have

$$\mathcal{L}_\varepsilon \Lambda(\lambda, \omega) = \varepsilon \Lambda_{xx}(\lambda, \omega) - b(\lambda, \omega)\Lambda(\lambda, \omega) - \frac{\partial^\alpha \Lambda(\lambda, \omega)}{\partial t^\alpha} \leq 0.$$

Since $\Lambda_{xx}(\lambda, \omega) \geq 0$ and $\frac{\partial^\alpha \Lambda(\lambda,\omega)}{\partial t^\alpha} = 0$, then $\mathcal{L}_\varepsilon \Lambda(\lambda, \omega) \leq 0$, which contradicts the initial assumption. Therefore, $\Lambda(\lambda, \omega) \geq 0$, for all $(x, t) \in \bar{D}$. □

Lemma 2.3 (Stability Estimate)**.** *Let $u(x, t)$ be the solution of the continuous problem of* (1.1)*. Then,*

$$\|u(x, t)\| \leq \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\},$$

where $\bar{\psi} = \max \psi(x)$*, for all* $x \in [0, 1]$*,* $\varphi_l = \max \varphi_l(t)$*,* $\varphi_r = \max \varphi_r(t)$*, for all* $t \in [0, T]$*.*

Proof. Define barer functions $\Psi^\pm$ as

$$\Psi^\pm(x, t) = \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm u(x, t).$$

At initial and boundaries we have

$$\begin{aligned}
\Psi^\pm(x, 0) &= \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm \psi(x) \geq 0,\\
\Psi^\pm(0, t) &= \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm \varphi_l \geq 0,\\
\Psi^\pm(1, t) &= \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm \varphi_r \geq 0.
\end{aligned}$$

Now, considering the differential operator $\mathcal{L}_\varepsilon$ on the domain D we have

$$\left(\mathcal{L}_\varepsilon + \frac{\partial^\alpha}{\partial t^\alpha}\right)\Psi^\pm(x, t) = -\varepsilon\frac{\partial^2 \Psi^\pm(x, t)}{\partial x^2} + b(x, t)\Psi^\pm(x, t) + \frac{\partial^\alpha \Psi^\pm(x, t)}{\partial t^\alpha}.$$

Now, using the above barer function we obtain

$$\begin{aligned}
\left(\mathcal{L}_\varepsilon + \frac{\partial^\alpha}{\partial t^\alpha}\right)\Psi^\pm(x, t) = &- \varepsilon\frac{\partial^2 \pm u(x, t)}{\partial x^2} + \frac{\partial^\alpha \Psi^\pm(x, t)}{\partial t^\alpha}\\
&+ b(x, t)\left(\vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right)\\
&\pm f(x, t) + \vartheta\left(\vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right).
\end{aligned}$$

Since $\|f\| \geq f(x, t)$, then we have

$$\begin{aligned}
\left(\mathcal{L}_\varepsilon + \frac{\partial^\alpha}{\partial t^\alpha}\right)\Psi^\pm(x, t) &= \pm f(x, t) + \vartheta\left(\vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right),\\
&\geq \vartheta\left(\max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}\right) \geq 0.
\end{aligned}$$

It follows that from Lemma 2.2 we have $\Psi^{\pm}(x,t) \geq 0$, for all $(x,t) \in [0,1] \times [0,T]$. Hence,

$$\vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\} \pm u(x,t) \geq 0,$$

which implies

$$|u(x,t)| \leq \vartheta^{-1} \|f\| + \max\left\{\left|\bar{\psi}\right|, \max(|\varphi_l|, |\varphi_r|)\right\}. \qquad \square$$

3. Numerical Scheme Formulation

3.1. **Temporal Discretization.** To discretize the time domain $[0,T]$ into M sub-intervals on a uniform step size $t_j = j\Delta t$, $j = 0,1,\ldots,M$, $\Delta t = \frac{T}{M}$. The time fractional derivative is approximated by Caputo sense that obtained from the quadrature formula at the point (x, t_{j+1}) as

$$\begin{aligned}
\partial_t^{\alpha} U(x,t_{j+1}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} \frac{\partial U(x,s)}{\partial s} (t_{j+1}-s)^{-\alpha} ds,\\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \left(\frac{U(x,t_{k+1}) - u(x,t_k)}{\Delta t} \right) \int_{t_k}^{t_{k+1}} (t_{j+1}-s)^{-\alpha} ds + e_{\Delta t}^{j+1},\\
&= \beta \sum_{k=0}^{j} w_k \left(U(x,t_{j-k+1}) - U(x,t_{j-k}) \right) + e_{\Delta t}^{j+1},
\end{aligned}$$

where $\beta = \frac{(\Delta t)^{-\alpha}}{\Gamma(1-\alpha)}$, $w_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}$ and $e_{\Delta t}^{j+1} = \frac{(\Delta t)}{\Gamma(1-\alpha)} \int_{t_k}^{t_{k+1}} (t_{j+1}-s)^{-\alpha} ds$.

Therefore, the Caputo fractional derivative $\partial_t^{\alpha} u(x,t)$ combined with implicit Euler's method at the point (x, t_{j+1}) is

(3.1)
$$\partial_t^{\alpha} U(x,t_{j+1}) = \beta\Big(\left(U(x,t_{j+1}) - U(x,t_j) \right) + \sum_{k=1}^{j-1} w_k \Big(U(x,t_{j-k+1}) - U(x,t_{j-k}) \Big) \Big).$$

Now, using (3.1) into (1.1) we obtain the time semi-discrete equation

$$\begin{aligned}
&\beta \left(\frac{U(x,t_{j+1}) - U(x,t_j)}{2^{1-\alpha}} \right) + \beta \left(\sum_{k=1}^{j} \Big(U(x,t_{j-k+1}) - U(x,t_{j-k}) \Big) w_k \right)\\
&\quad - \varepsilon U_{xx}^{j+1}(x) + b^{j+1} U^{j+1}(x) = f^{j+1}(x),
\end{aligned} \tag{3.2}$$

which is rearranged as

$$\left(\beta + \mathcal{L}_{\varepsilon}^{\Delta t} \right) U^{j+1}(x) = R_i, \tag{3.3}$$

where

$$\begin{aligned}
\left(\beta + \mathcal{L}_{\varepsilon}^{\Delta t} \right) U^{j+1}(x) &= -\varepsilon U_{xx}^{j+1}(x) + (\beta + b^{j+1}(x) U^{j+1}(x),\\
R_i &= \beta U^j(x) - \beta \left(\sum_{j}^{k=1} w_k \left(U^{j-k+1}(x) - U^{j-k}(x) \right) \right) + f^{j+1}(x).
\end{aligned}$$

Lemma 3.1. *An error estimate associated with* (3.3) *satisfies the bounds*
$$\left|e_{\Delta}^{j+1}\right| \leq C(\Delta t)^{(2-\alpha)}.$$

Proof. We have
$$\begin{aligned}
e_{\Delta}^{j+1} &= \frac{O(\Delta t)}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t^{k+1}} (t_{j+1}-s)\,ds \\
&= \frac{O(\Delta t)}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \left(\frac{(j-k+1)^{1-\alpha} - (j-k)^{1-\alpha}}{1-\alpha} \right) (\Delta t)^{1-\alpha} \\
&= \frac{O((\Delta t)^{2-\alpha})}{\Gamma(2-\alpha)} \left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha} \right) \\
&= \frac{O((\Delta t)^{2-\alpha})}{\Gamma(2-\alpha)} (j+1)^{1-\alpha} \\
&\leq C(\Delta t)^{2-\alpha},
\end{aligned}$$
which implies $\left|e_{\Delta}^{j+1}\right| \leq C(\Delta t)^{(2-\alpha)}$, where C is a constant independent of ε and Δt. □

Lemma 3.2. *The semi-discretize solution $U(x, t_{j+1})$ and its derivative satisfy the following bounds*
$$\left| \frac{\partial^k U(x, t_{j+1})}{\partial x^k} \right| \leq C \left[1 + \varepsilon^{\frac{i}{2}} \left(\exp\left(\frac{-x}{\sqrt{\varepsilon}} \right) + \exp\left(\frac{-(1-x)}{\sqrt{\varepsilon}} \right) \right) \right],$$
with $0 \leq k \leq 4$.

3.2. **Spatial Discretization.** Consider a uniform mesh size in spatial domain $[0, 1]$ into N sub-intervals as $0 = x_0 < x_1 < \cdots < x_N = 1$, with $h = x_{i+1} - x_i$, $i = 0, 1, \ldots, N-1$. Let $B_i(x)$ be the cubic B-spline basis function defined by:
(3.4)
$$B_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x_{i-2} \leq x \leq x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \leq x \leq x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \leq x \leq x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \leq x \leq x_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

The values of $B_i(x)$ and its first and second derivatives at the knot points x_{i-2}, x_{i-1}, x_i, x_{i+1} and x_{i+2} are given in Table 1. Let $\Xi(x)$ be cubic B-spline collocation

Table 1. Coefficients of cubic B-splines and its derivatives at knots

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i^{'}(x)$	0	$3/h$	0	$-3/h$	0
$B_i^{"}(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

approximation to the solution $U(x,t_{j+1})$ given in (3.4) defined as

$$\Xi(x) \approx \sum_{i=-1}^{N+1} \gamma_i B_i(x). \tag{3.5}$$

Differentiating (3.5) twice and using Table 1 we obtain an approximate value at the knot point x_i as:

$$\begin{cases} \Xi_i = \gamma_{i-1} + 4\gamma_i + \gamma_{i+1}, \\ \Xi_i' = \frac{3}{h}\left(\gamma_{i+1} - \gamma_{i-1}\right), \\ \Xi_i" = \frac{6}{h^2}\left(\gamma_{i-1} - 2\gamma_i + \gamma_{i+1}\right). \end{cases} \tag{3.6}$$

To control the effect of perturbation parameter on the solution behavior, we introduce the fitting factor $(\sigma_i = \sigma(x_i,\varepsilon))$ in the way that the approximate solutions uniformly converges to the exact solution of (1.1) and using (3.6) into (3.3) we obtain an $(N+3)\times(N+3)$ systems of linear equation:

$$\left(-\frac{6\sigma_i}{h^2} + \beta + b_i^{j+1}\right)\gamma_{i-1}^{j+1} + \left(\frac{12\sigma_i}{h^2} + 4\beta + 4b_i^{j+1}\right)\gamma_i^{j+1} + \left(-\frac{6\sigma_i}{h^2} + \beta + b_i^{j+1}\right)\gamma_{i+1}^{j+1} = \bar{R}_i, \tag{3.7}$$

that can be written as a three term recurrence relation of the form:

$$E_i\gamma_{i-1}^{j+1} + F_i\gamma_i^{j+1} + G_i\gamma_{i+1}^{j+1} = H_i, \quad i = 0, 1, \ldots, N, \tag{3.8}$$

where

$$\begin{aligned} E_i &= G_i = -6\sigma_i + \left(\beta + b_i^{j+1}\right)h^2, \\ F_i &= 12\sigma_i + 4\left(\beta + b_i^{j+1}\right)h^2, \\ H_i &= h^2\bar{R}_i, \end{aligned}$$

and

$$\sigma_i = \frac{h^2 b_i^{j+1}}{6}\left(\frac{1 + 2\cosh^2(\rho_i h)}{2\sinh^2 \rho_i h}\right), \qquad \rho_i = \sqrt{\frac{b_i^{j+1}}{\varepsilon}}, \tag{3.9}$$

for $|\sigma_i - \varepsilon| \le Ch^2$ with C is independent of ε and h [24, 25].

By imposing the boundary conditions (1.3) into (3.6) it yields

$$\begin{aligned} \gamma_{-1} &= \varphi_l(t_{j+1}) - 4\gamma_0 - \gamma_1, \\ \gamma_{N+1} &= \varphi_r(t_{j+1}) - \gamma_{N-1} - 4\gamma_N. \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8) we obtain an $(N+1)\times(N+1)$ systems of linear equations:

$$\begin{aligned} \left(F_0 - 4E_0\right)\gamma_0^{j+1} + \left(G_0 - E_0\right)\gamma_1^{j+1} &= H_0 - E_0\varphi_l(t_{j+1}), \\ E_i\gamma_{i-1}^{j+1} + F_i\gamma_i^{j+1} + G_i\gamma_{i+1}^{j+1} &= H_i, \quad i =, 1, \ldots, N-1, \\ \left(E_N - G_N\right)\gamma_{N-1}^{j+1} + \left(F_N - 4G_N\right)\gamma_N^{j+1} &= H_N - G_N\varphi_r(t_{j+1}). \end{aligned} \tag{3.11}$$

4. Convergence Analysis

Lemma 4.1. *The collocation cubic B-spline* $\{B_{-1}(x), B_0(x), \dots, B_N(x), B_{N+1}(x)\}$ *defined in* (3.4) *satisfy the inequality*

$$\sum_{i=-1}^{N+1} |B_i(x)| \leq 10, \quad x \in [0,1]. \tag{4.1}$$

Proof. From triangular inequality we have

$$\left|\sum_{i=-1}^{N+1} B_i(x)\right| \leq \sum_{i=-1}^{N+1} |B_i(x)|.$$

The cubic B-spline $B_i(x)$ is non-zero at only three nodal points. Thus, at any nodal point x_i, we have

$$\sum_{i=-1}^{N+1} |B_i(x)| = |B_{i-1}(x)| + |B_i(x)| + |B_{i+1}(x)| = 1 + 4 + 1 = 6 < 10.$$

For $x \in [x_{i-1}, x_i]$,

$$\sum_{i=-1}^{N+1} |B_i(x_i)| \leq 4 \quad \text{and} \quad \sum_{i=-1}^{N+1} |B_{i-1}(x_{i-1})| \leq 4,$$

and similarly for $x \in [x_{i-1}, x_i]$,

$$\sum_{i=-1}^{N+1} |B_{i+1}(x_i)| \leq 1 \quad \text{and} \quad \sum_{i=-1}^{N+1} |B_{i-2}(x_{i-1})| \leq 1.$$

Therefore, for any point $x \in [x_{i-1}, x_i]$, we obtain

$$\sum_{i=-1}^{N+1} |B_i(x)| = |B_{i-2}(x)| + |B_{i-1}(x)| + |B_i(x)| + |B_{i+1}(x)| \leq 10.$$ □

Lemma 4.2. *Let* $\Xi(x)$ *be the collocation approximation to the solution* $\bar{u}(x)$*of the boundary value problem of* (3.3)*. If* $R \in C^2[0,1]$*, then the parameter uniform error estimate is given by*

$$\|u(x_i, t_{j+1}) - \Xi(x_i)\|_\infty \leq C\left(h^2\right), \tag{4.2}$$

for sufficiently small h *and* C *is a positive constant.*

Proof. Assume $\bar{\Xi}(x)$ be a unique spline that interpolate $\bar{u}(x)$ of the boundary value problem (3.3) at the $(j+1)^{th}$ time level given by

$$\bar{\Xi}(x) \approx \sum_{i=-1}^{N+1} \bar{\gamma}_i B_i(x). \tag{4.3}$$

If $b(x), R_i \in C^2[0,1]$, then $\bar{u}(x) \in C^4[0,1]$, for all j, $0 \le j \le M$, and hence from error bound estimate [26] we have

$$\left\| D^n \left(\bar{u}(x) - \bar{\Xi}(x)\right)\right\|_\infty \le \varsigma_n \left\| \frac{d^4\bar{u}(x)}{dx^4}\right\|_\infty h^{4-n}, \quad n = 0, 1, 2, \tag{4.4}$$

and ς_n is constant independent of term h and N.

Using triangular inequality we have

$$\|\bar{u}(x_i) - \Xi(x_i)\|_\infty \le \left\|\bar{u}(x_i) - \bar{\Xi}(x_i)\right\|_\infty + \left\|\bar{\Xi}(x_i) - \Xi(x_i)\right\|_\infty . \tag{4.5}$$

Let $\left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right)\Xi(x_i) = H_i$ and $\left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right)\bar{\Xi}(x_i) = \bar{H}_i$ that satisfies the boundary conditions $\bar{\Xi}(x_0) = \varphi_l(t_{j+1})$ and $\bar{\Xi}(x_N) = \varphi_r(t_{j+1})$. Then,

(4.6)
$$\left|\left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right)\Xi(x_i) - \left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right)\bar{\Xi}(x_i)\right| \le |\sigma_i - \varepsilon|\,\varsigma_2 \left\|\frac{d^2\bar{u}(x_i)}{dx^2}\right\|_\infty + |\varepsilon|\,\varsigma_2 \left\|\frac{d^2\bar{u}(x_i)}{dx^2}\right\|_\infty + \varsigma_0\|b(x_i)\|_\infty \|\bar{u}(x_i)\|_\infty .$$

Using the bound of artifial viscocity $|\sigma_i - \varepsilon| \le Ch^2$, and Lemma 3.2 into (4.6) it gives

$$\max_{x\in D}\left|\left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right)u(x_i, t_{j+1}) - \left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right)\bar{\Xi}(x_i)\right| \le \varsigma h^2, \tag{4.7}$$

where $\varsigma = \varepsilon\varsigma_2 + (C\varsigma_2 + \varsigma_0\|b(x_i, t_{j+1})\|_\infty)\,h^2$.

Using (4.3) and (3.8) the i-th nodal point of $\left|\left(\beta + \mathcal{L}_\varepsilon^{\Delta t,h}\right)\left(\bar{u}(x_i) - \bar{\Xi}(x_i)\right)\right|$ yields

$$36\sigma_0\mu_0 = Y_0, \tag{4.8}$$

$$(-6\sigma_i + h^2(\beta + b_i))(\mu_{i-1} + \mu_{i+1}) + (12\sigma_i + 4h^2(\beta + b_i))\mu_i = Y_i, \quad i = 1, 2, \ldots, N-1, \tag{4.9}$$

$$36\sigma_N\mu_N = Y_N, \tag{4.10}$$

where $\mu_i = \gamma_i - \bar{\gamma}_i$, $i = -1, 0, \ldots, N+1$ and $Y_i = H_i - \bar{H}_i$, $i = 0, 1, \ldots, N$. Then, (4.9) given as

$$(12\sigma_i + 4h^2(\beta + b_i))\mu_i = Y_i - (-6\sigma_i + h^2(\beta + b_i))(\mu_{i-1} + \mu_{i+1}). \tag{4.11}$$

Define

$$Y = \max_{1\le i\le N-1} Y_i, \quad \mu = \max_{1\le i\le N-1}\mu_i,$$

and using the condition $0 < \eta \le b_i$ into (4.11) we get

$$\left(12\varepsilon + (4(\beta + \eta) + 12C)\,h^2\right)\mu \le Y + 2\mu\left(6\varepsilon + (\beta + \eta + 6C)\,h^2\right),$$

which gives

$$\mu \le \frac{\varsigma h^2}{(\beta + \eta)}, \quad \mu_0 \le \frac{\varsigma h^2}{36C} \quad \text{and} \quad \mu_N \le \frac{\varsigma h^2}{36C}.$$

The values of μ_{-1} and μ_{N+1} is evaluated from (3.6) gives

$$\mu_{-1} \leq \varsigma h^2 \left(\frac{1}{9C} + \frac{1}{C(\beta+\eta)} \right), \quad \mu_{N+1} \leq \varsigma h^2 \left(\frac{1}{9C} + \frac{1}{C(\beta+\eta)} \right).$$

With values of ς, there exists a constant ϖ, independent of h and ε such that

$$\mu = \max_{-1\leq i\leq N+1} \{\mu_i\} \leq \varpi h^2. \tag{4.12}$$

Results from (4.12) and Lemma 4.1 gives

$$\|\Xi(x_i) - \bar{\Xi}(x_i)\|_\infty \leq 10\varpi h^2. \tag{4.13}$$

Thus,

$$\|\bar{u}(x_i) - \Xi(x_i)\|_\infty \leq Ch^2,$$

where $C = 10\varpi + \varsigma_0 h^2$. ☐

Theorem 4.1. *Let $u(x,t)$ be the solution of* (1.1) *and $U(x_i, t_{j+1})$ is the solution of the total discretized equation. Under the hypothesis of Lemma* 3.1 *and Lemma* 4.2, *then the ε-uniform estimate holds*

$$\max_{1\leq i,j\leq N,M-1} |U(x_i, t_{j+1}) - \Xi(x_i)| \leq C\left(h^2 + (\Delta t)^{2-\alpha}\right), \tag{4.14}$$

where C is the constant independent of ε, h and (Δt).

Proof. The proof is applying triangular inequality. ☐

5. Numerical Results and Discussion

The results of maximum point-wise error is evaluated by double mesh principle. i.e.

$$E_\varepsilon^{N,M} = \max_{1\leq i\leq N-1} \left| U_i^{N,M} - U_i^{2N,2M} \right|.$$

The ε-uniform errors are approximated using

$$E^{N,M} = \max_{\varepsilon} E_\varepsilon^{N,M},$$

and ε-uniform rate of convergence is obtained by the formula

$$r_\varepsilon^{N,M} = \frac{\log(E_\varepsilon^{N,M}) - \log(E_\varepsilon^{2N,2M})}{\log(2)}.$$

Example 5.1.

$$\frac{\partial^\alpha}{\partial t^\alpha} u - \varepsilon \frac{\partial^2 u}{\partial x^2} + (1 + x^2 + t^2 e^t) u = e^t - 1 + \sin(\pi x), \quad (x,t) \in D,$$

subject to the initial and boundary conditions

$$u(x,0) = 0, \quad x \in \Omega = [0,1],$$

and

$$u(0,t) = 0 = u(1,t) = 0, \quad t \in [0,1].$$

Example 5.2.

$$\frac{\partial^\alpha}{\partial t^\alpha}u - \varepsilon\frac{\partial^2 u}{\partial x^2} + (1 + xe^{-t})u = f(x,t), \quad (x,t) \in D,$$

subject to

$$u(x,0) = 0, \quad x \in \Omega = [0,1],$$

and

$$u(0,t) = \varphi_l(t), \quad u(1,t) = \varphi_r(t), \quad t \in [0,1],$$

where the functions $\varphi_l(t)$, $\varphi_r(t)$ and $f(x,t)$ are chosen from the exact solution

$$u(x,t) = t\left(\frac{e^{\frac{-x}{\sqrt{\varepsilon}}} + e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}}{1 + e^{\frac{-1}{\sqrt{\varepsilon}}}} - \cos^2(\pi x)\right).$$

TABLE 2. Maximum absolute error of Example 5.1 for $\alpha = 0.9$ and $N = M$

ε	16	32	64	128	256
2^{-8}	6.5344e-03	3.8082e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-10}	6.5321e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-12}	6.5317e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-14}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-16}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-18}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-20}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
2^{-22}	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
$E^{N,M}$	6.5316e-03	3.8081e-03	2.2185e-03	1.2487e-03	6.8766e-04
$r^{N,M}$	0.77837	0.77949	0.82916	0.85394	

To validate the theoretical findings point-wise maximum absolute error of the test examples is given out in Tables 2 and 3. As observed from the tables, as the perturbation parameter approaches zero and the mesh size increases, the result of numerical examples converges uniformly which confirms an agreement with theoretical findings. The physical behavior of the scheme is demonstrated in Figure 1 and 3 which yields numerical example has twin layers at the end of the spatial domain. In Figure 2 and 4 the boundary layer behavior is displayed and it has a parabolic boundary layer at $x = 0$ and $x = 1$. Finally, in Figure 5 and 6 the log-log scale of maximum absolute error is illustrated and it shows as mesh points increases and $\varepsilon \to 0$, then the maximum absolute error decreases monotonically.

TABLE 3. Maximum absolute error of Example 5.2 for $\alpha = 0.9$ and $N = M$

ε	8	16	32	64	128
2^{-8}	5.9457e-03	3.9889e-03	2.5301e-03	1.5170e-03	9.7828e-04
2^{-10}	7.7475e-03	5.0285e-03	3.1961e-03	1.8801e-03	1.0662e-03
2^{-12}	8.0307e-03	5.7727e-03	3.5375e-03	2.0678e-03	1.1680e-03
2^{-14}	8.0445e-03	5.9251e-03	3.6635e-03	2.1471e-03	1.2139e-03
2^{-16}	8.0477e-03	5.9292e-03	3.7472e-03	2.1816e-03	1.2326e-03
2^{-18}	8.0485e-03	5.9296e-03	3.7489e-03	2.2003e-03	1.2397e-03
2^{-20}	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2425e-03
2^{-22}	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2429e-03
2^{-24}	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2429e-03
$E^{N,M}$	8.0487e-03	5.9297e-03	3.7489e-03	2.2011e-03	1.2429e-03
$r^{N,M}$	0.44080	0.66149	0.76824	0.82451	

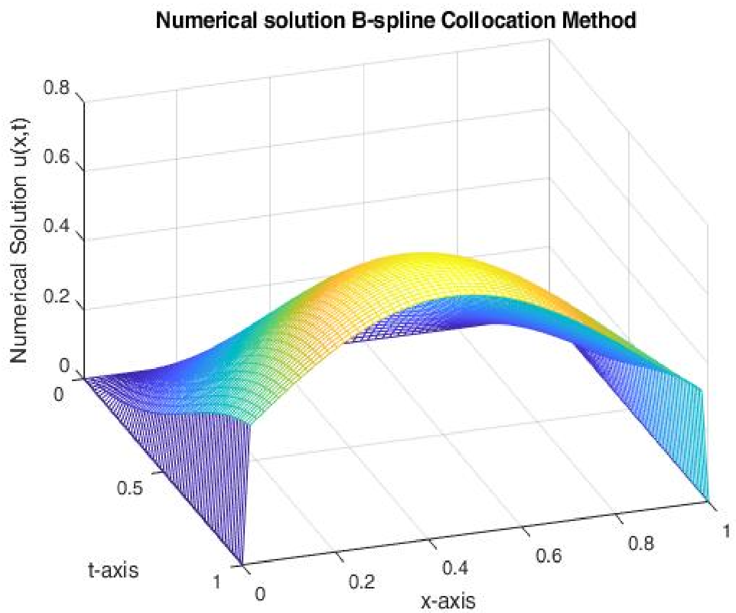

FIGURE 1. Numerical solution of Example 5.1 for $N = M = 64$ and $\varepsilon = 2^{-10}$

6. CONCLUSIONS

A fitted cubic B-spline collocation method is conducted for one-dimensional initial boundary value problem of singularly perturbed time-fractional parabolic reaction-diffusion equations. By using the Caputo fractional sense and employing the implicit Euler technique, the time fractional derivative is addressed. Spatial derivatives are

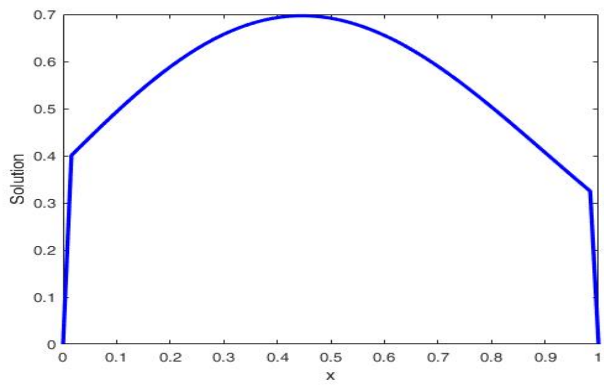

FIGURE 2. Boundary layer formation of Example 5.1

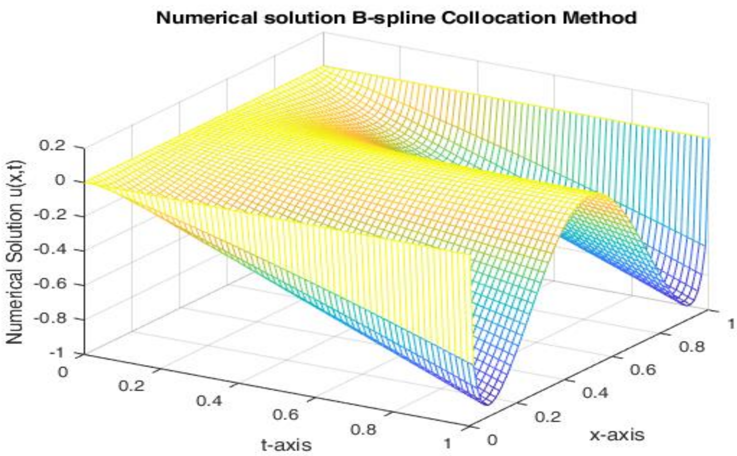

FIGURE 3. Numerical solution of Example 5.2 for $N = M = 64$ and $\varepsilon = 2^{-10}$

dicretized by the fitted cubic B-spline collocation method. The convergence analysis of the scheme is proved and it is accurate of order $O(h^2 + (\Delta t)^{2-\alpha})$. The results from numerical examples confirmed that the scheme is uniformly convergent and has twin layers at $x = 0$ and $x = 1$.

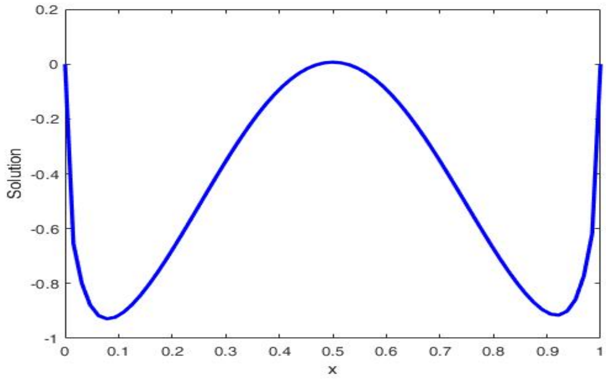

FIGURE 4. Boundary layer formation of Example 5.2

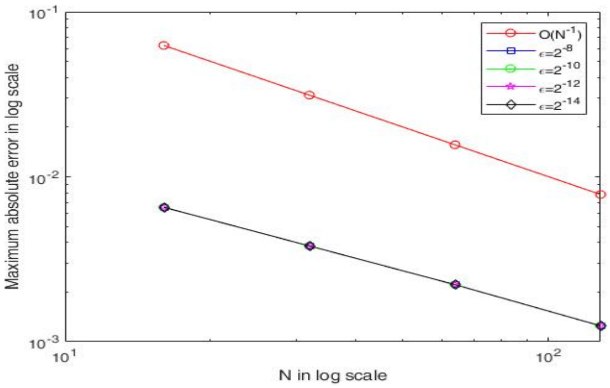

FIGURE 5. Log-log scale plot for Example 5.1

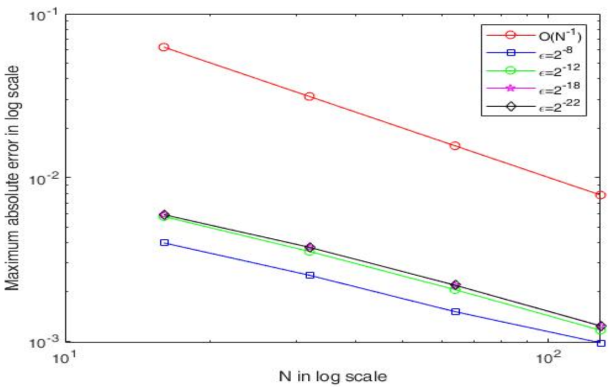

FIGURE 6. Log-log scale plot for Example 5.2

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