Kragujevac Journal of Mathematics
Volume 50(7) (2026), Pages 1105–1117.

ON THE JACOBSON SEMISIMPLE SEMIRINGS

A. K. BHUNIYA AND PUJA SARKAR

Abstract. Based on the minimal and simple representations, we introduce two types of Jacobson semisimplicity, m-semisimplicity and s-semisimplicity, of a semiring S. Every $m(s)$-semisimple semiring is a subdirect product of $m(s)$-primitive semirings. It is shown that a commutative s-primitive semiring is either a two element Boolean algebra or a field. Every s-primitive semiring is isomorphic to a 1-fold transitive subsemiring of the semiring of all endomorphisms of a semimodule over a division semiring.

1. Introduction

A semiring is an algebraic structure satisfying all the axioms of a ring, but one that every element has an additive inverse. The absence of additive inverses forces a semiring to deviate radically from behaving like a ring. For example, ideals are not in bijection with the congruences on a semiring. Further, the presence of additively idempotent semirings makes the class of the semirings abundant. Now semirings have become a part of mainstream mathematics for their importance in theoretical computer science [30], graph theory [16] and automata theory [9, 11, 13]; and for the surprising 'characteristic one analogy' of the usual algebra over fields [10, 12, 25, 26]; and for the role of additively idempotent semirings in tropical mathematics [1, 7, 14, 19, 20, 27].

In a recent paper [24], Katsov and Nam considered two Jacobson type semisimple semirings - J-semisimple semirings and J_s-semisimple semirings. They characterized J-semisimple semirings within the class of all additively cancellative semirings and additively idempotent J_s-semisimple semirings. Besides characterizing the semirings S such that $J(S) = J_s(S)$, a problem stated in [24], Mai and Tuyen [28] extended the

Key words and phrases. Semiring, Jacobson semisimple, faithful, primitive, transitive semiring
2020 *Mathematics Subject Classification.* Primary: 16Y60. Secondary: 16N99, 16D99.
https://doi.org/10.46793/KgJMat2607.1105B
Received: March 07, 2024.
Accepted: September 27, 2024.

study of the J_s-semisimple semirings to the class of all zerosumfree semirings. They proved that every J_s-semisimple zerosumfree commutative semiring is semiisomorphic to a subdirect product of its maximal entire quotients.

This article is a continuation of [6] under the project to develop a radical theory of semirings that can be used to study semiring in general. Based on the minimal and simple representations of a semiring, the present authors introduced and studied two Jacobson type radicals - Jacobson m-radical and s-radical of a semiring in [6]. Here, we define and study two types of Jacobson semisimple semirings - Jacobson m-semisimple and s-semisimple semirings.

Since ideals are not in bijection with the congruences on a semiring, it is not surprising that replacing ideals with the more general notion of congruences on semirings exhibits many excellent properties and several analogies with classical results on the rings [3, 22, 23]. In our approach, the annihilator $ann_S(M)$ of an S-semimodule M is considered as a congruence on S. Similarly to the semirings, subsemimodules are not in bijection with the congruences on a semimodule; which produces three variants of 'irreducibility' of semimodules – minimal semimodules, elementary semimodules, and simple semimodules [8, 21]. In [6], the authors of the present article introduced and characterized two Jacobson type Hoehnke radicals, namely, m-radical and s-radical of a semiring S as two congruences on S. Here we introduce m-semisimple, s-semisimple, m-primitive and s-primitive semirings in an obvious way. Considering radical as a congruence makes it easy to represent these semisimple semirings as a subdirect product of suitable class of primitive semirings. The two element Boolean algebra and the fields are the only commutative s-primitive semirings. Hence the study of commutative s-primitive semirings characterizes the subdirect products of the copies of the two element Boolean algebra and fields.

This paper is organized as follows. Section 2 briefly recaps the necessary definitions and associated facts on semirings and semimodules. Section 3 introduces m-primitive and s-primitive semirings and characterizes Jacobson semisimple semirings as subdirect products of primitive semirings. Every commutative $(s)m$-primitive semiring is a (congruence simple) semifield. Since every congruence simple semifield S with $|S| > 2$ is a field, every commutative s-semisimple semiring is a subdirect product of a family of semirings, each of which is either the 2-element Boolean algebra or a field. Finally, every s-primitive semiring is represented as a 1-fold transitive subsemiring of the semiring of all endomorphisms of a semimodule over a division semiring.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is a nonempty set S with two binary operations '+' and '·' satisfying:

- $(S, +)$ is a commutative monoid with identity element 0;
- $(S, \cdot)$ is a semigroup;
- $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

Moreover, we assume that the additive identity element 0 is absorbing, i.e., $0s = s0 = 0$ for all $s \in S$. There is no consensus whether every semiring contains a multiplicative unity 1. In this article, we follow the convention of Hebisch and Weinert [17], that a semiring does not contain 1, in general. Also, there are many articles on semirings where the existence of unity is not assumed [4, 5, 18, 21, 31, 32]. On the other hand, in Golan [15], it is assumed that every semiring contains 1. If a semiring S with multiplicative identity is such that every nonzero element has a multiplicative inverse, then S is called a *division semiring*. A commutative division semiring is called a *semifield*. A semiring S is said to be an *additively idempotent semiring* if $a + a = a$ for all $a \in S$. Both the two element Boolean algebra $\mathbb{B}$ and the max-plus algebra $\mathbb{R}_{max}$ are additively idempotent semifields.

Definition of the ideals, congruences and homomorphisms of semirings are as usual. Here it is assumed that every semiring homomorphism $\phi : S_1 \to S_2$ satisfies $\phi(0_1) = 0_2$. The kernel of a semiring homomorphism $\phi : S_1 \to S_2$ is defined by $\ker \phi = \{(a, b) \in S_1 \times S_2 \mid \phi(a) = \phi(b)\}$. Then $\ker \phi$ is a congruence on S_1 and $S_1/\ker \phi \simeq \phi(S_1)$.

A semiring S is said to be *congruence-simple* if it has no congruences other than the equality congruence $\Delta_S = \{(s, s) \mid s \in S\}$ and the universal congruence $\nabla_S = S \times S$.

Let I be a (left, right) ideal of a semiring S and μ be a (left, right) congruence relation on S. Then, I is said to be a *μ-saturated (left, right) ideal* of S if for every $s \in S$ and $i \in I$, $(s, i) \in \mu$ implies that $s \in I$. Thus, an ideal I is μ-saturated if and only if $I = \cup_{a \in I}[a]_\mu$.

A *right S-semimodule* is a commutative monoid $(M, +, 0_M)$ equipped with a right action $M \times S \to M$ that satisfies for all $m, m_1, m_2 \in M$ and $s, s_1, s_2 \in S$:

- $(m_1 + m_2)s = m_1 s + m_2 s$;
- $m(s_1 + s_2) = ms_1 + ms_2$;
- $m(s_1 s_2) = (ms_1)s_2$;
- $m0 = 0_M = 0_M s$.

Unless stated otherwise, by an *S-semimodule M*, we mean a right S-semimodule.

The *annihilator* of an S-semimodule M is defined by

$$ann_S(M) = \{(s_1, s_2) \in S \times S \mid ms_1 = ms_2 \text{ for all } m \in M\}.$$

Then $ann_S(M)$ is a congruence on S. If $\psi : S \to End_S(M)$ is the representation of S induced by the right action of S on the semimodule M, then $\ker \psi = ann_S(M)$.

If M is a right S-semimodule then for every congruence ρ on S with $\rho \subseteq ann_S(M)$, the scalar multiplication $m[s]_\rho = ms$ makes M an S/ρ-semimodule.

The following result can be proved easily, and so we omit the proof.

Lemma 2.1. *Let S be a semiring and ρ be a congruence on S.*

(a) *If M is an S/ρ-semimodule, then M becomes an S-semimodule under the scalar multiplication $ms = m[s]$. Moreover, $\rho \subseteq ann_S(M)$.*

(b) *Let M be an S-semimodule and $\rho \subseteq ann_S(M)$. Then, $ann_{S/\rho}(M) = ann_S(M)/\rho$.*

An S-semimodule M is said to be *faithful* if $ann_S(M) = \Delta_S$.

Following Chen et al. [8] we define the following.

Definition 2.1. Let M be an S-semimodule such that $MS \neq 0$. Then M is called

(i) minimal if M has no subsemimodules other than (0) and M;
(ii) simple if it is minimal and the only congruences on M are Δ_M and ∇_M where Δ_M is the equality relation on M and $\nabla_M = M \times M$.

In [21], simple semimodules have been termed as irreducible semimodules. We denote the class of all minimal and simple S-semimodules by $\mathcal{M}(S)$ and $\mathcal{S}(S)$, respectively. If R is a ring then $\mathcal{M}(S) = \mathcal{S}(S)$ [6].

In this paper, we will have many occasions to use the following characterization [6, 21] of the minimal semimodules.

Lemma 2.2. *A nonzero S-semimodule M is minimal if and only if $M = mS$ for all $m(\neq 0) \in M$.*

The classes $\mathcal{M}(S)$ and $\mathcal{S}(S)$ of minimal and simple representations of a semiring S induces the following two notions of Jacobson type radicals of a semiring [6].

Definition 2.2. Let S be a semiring. We define

(a) m-radical of S by $rad_m(S) = \cap_{M \in \mathcal{M}(S)} ann_S(M)$;
(b) s-radical of S by $rad_s(S) = \cap_{M \in \mathcal{S}(S)} ann_S(M)$.

If there are no minimal semimodules over S, then we define $rad_m(S) = \nabla_S$. Similarly, we define $rad_s(S) = \nabla_S$ if there is no simple S-semimodules.

Both the assignments $S \mapsto rad_m(S)$ and $S \mapsto rad_s(S)$ are Hoehnke radicals on S [6].

A right congruence μ on S is said to be a *regular right congruence* if there exists $e \in S$ such that $(es, s) \in \mu$ for every $s \in S$. If μ is a regular right congruence on S, then $M = S/\mu$ is a right S-semimodule such that $MS \neq 0$.

The subsequent two results characterize the regular congruences μ on S such that the quotient semimodule S/μ is a minimal or a simple semimodule over S.

Lemma 2.3 ([6])**.** *Let M be an S-semimodule. Then, M is minimal if and only if there exists a regular right congruence μ on S such that $S/\mu \simeq M$ and $[0]_\mu$ is a maximal μ-saturated right ideal in S.*

Every simple semimodule is congruence-simple. Therefore, for every right congruence μ on S, if S/μ is simple, then μ is maximal. Hence, the following result follows.

Lemma 2.4 ([6])**.** *Let M be a S-semimodule. Then M is simple if and only if there exists a maximal regular right congruence μ on S such that $S/\mu \simeq M$ and $[0]_\mu$ is a maximal μ-saturated right ideal in S.*

A regular right congruence μ on S is said to be *m-regular* if $[0]_\mu$ is a maximal μ-saturated right ideal in S; and *s-regular* if it is a maximal regular right congruence such that $[0]_\mu$ is a maximal μ-saturated right ideal in S. We denote the set of all m-regular right congruences on S by $\mathcal{RC}_m(S)$ and the set of all s-regular right congruences on S by $\mathcal{RC}_s(S)$.

The following internal characterization of the m-radical and the s-radical of a semiring was proved in [6].

Lemma 2.5 ([6])**.** *Let S be a semiring. Then,*

(i) $rad_m(S) = \bigcap_{\mu\in\mathcal{RC}_m(S)}\mu$;
(ii) $rad_s(S) = \bigcap_{\mu\in\mathcal{RC}_s(S)}\mu$.

The following result has useful applications.

Lemma 2.6 ([6])**.** *Let R and S be two semirings. Then, $rad_m(R \times S) = rad_m(R) \times rad_m(S)$ and $rad_s(R \times S) = rad_s(R) \times rad_s(S)$.*

The reader is referred to [6] for more details on the m-radical and the s-radical of a semiring and to [17] for the undefined terms and notions concerning semirings and semimodules over semirings.

3. Jacobson Semisimple Semirings

In this section, we introduce and study the Jacobson m-semisimple and s-semisimple semirings.

Definition 3.1. A semiring S is said to be *m-semisimple* if $rad_m(S) = \Delta_S$; and *s-semisimple* if $rad_s(S) = \Delta_S$.

Since $\mathcal{RC}_s(S) \subseteq \mathcal{RC}_m(S)$, it follows that $rad_m(S) \subseteq rad_s(S)$. Therefore, every s-semisimple semiring is m-semisimple. The following example shows that the converse does not hold in general.

Example 3.1. Let $\mathbb{H}$ be the ring of all real quaternions. Since $\mathbb{H}$ is a division ring, $rad_m(\mathbb{H}) = rad_s(\mathbb{H}) = \Delta_{\mathbb{H}}$. Also, $rad_m(\mathbb{R}_{max}) = \Delta_{\mathbb{R}_{max}}$ and $\rho = \{(-\infty, -\infty)\} \cup \{(r, s) \mid r, s \in \mathbb{R}\}$ is the only s-regular congruence on $\mathbb{R}_{max}$ implies that $rad_s(\mathbb{R}_{max}) = \rho$ [6, Example 3.9]. Consider the semiring $S = \mathbb{H} \times \mathbb{R}_{max}$. Then, by the Lemma 2.6, we have

$$rad_m(S) = rad_m(\mathbb{H}) \times rad_m(\mathbb{R}_{max}) = \Delta_{\mathbb{H}} \times \Delta_{\mathbb{R}_{max}} = \Delta_S,$$
$$rad_s(S) = rad_s(\mathbb{H}) \times rad_s(\mathbb{R}_{max}) = \Delta_{\mathbb{H}} \times \rho.$$

Therefore, S is m-semisimple but not s-semisimple.

Now we give an example of an s-semisimple semiring.

Example 3.2. Both the ring $\mathbb{H}$ of all real quaternions and the semiring $\mathbb{N}^0$ of all nonnegative integers are s-semisimple. Hence, it follows from the Lemma 2.6 that the semiring $S = \mathbb{H} \times \mathbb{N}^0$ is both s-semisimple and m-semisimple.

If S is a Jacobson m-semisimple semiring, then $\cap_{M\in\mathcal{M}(S)} ann_S(M) = \Delta_S$ implies that every m-semisimple semiring is a subdirect product of the family of semirings $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$. Also, for every $M \in \mathcal{M}(S)$, by Lemma 2.1, implies that M is a minimal and faithful $S/ann_S(M)$-semimodule. Similarly, every s-semisimple semiring S is a subdirect product of the family of semirings $\{S/ann_S(M) \mid M \in \mathcal{S}(S)\}$ where each quotient semiring $S/ann_S(M)$ has a faithful and simple semimodule M. Intending to characterize the structure of semisimple semirings, we introduce the following two notions.

Definition 3.2. Let S be a semiring. Then, S is called

(i) m-primitive if there is a faithful minimal S-semimodule M;
(ii) s-primitive if there is a faithful simple S-semimodule M.

If S is an m-primitive semiring, then there is a minimal S-semimodule M such that $ann_S(M) = \Delta_S$. Hence $rad_m(S) = \cap_{M\in\mathcal{M}(S)} ann_S(M) = \Delta_S$ and so S is m-semisimple. Similarly, every s-primitive semiring is s-semisimple.

A congruence σ on S is said to be an *m-primitive* (*s-primitive*) *congruence* if the quotient semiring S/σ is an m-primitive (s-primitive) semiring. Thus, σ is m-primitive(s-primitive) if and only if there exists a faithful minimal (simple) S/σ-semimodule M.

If ρ is a right congruence on S, we define

$$(\rho : \nabla_S) = \{(x, y) \in \nabla_S \mid (sx, sy) \in \rho \text{ for all } s \in S\}.$$

Then, $(\rho : \nabla_S)$ is a congruence on S.

Lemma 3.1. *Let σ be a congruence on S. Then, the following conditions are equivalent:*

(a) *σ is m-primitive (s-primitive);*
(b) *$\sigma = ann_S(M)$ for some minimal (simple) S-semimodule M;*
(c) *$\sigma = (\rho : \nabla_S)$ for some $\rho \in \mathcal{RC}_m(S)$ ($\rho \in \mathcal{RC}_s(S)$).*

Proof. We prove the result for m-primitive congruences. The other cases are similar.

(a) $\Rightarrow$ (b) Let σ be an m-primitive congruence on S. Then there exists a faithful minimal S/σ-semimodule M. Hence, by Lemma 2.1, M is also a minimal S-semimodule such that $\sigma \subseteq ann_S(M)$ and $\Delta_{S/\sigma} = ann_{S/\sigma}(M) = ann_S(M)/\sigma$, i.e., $\sigma = ann_S(M)$.

(b) $\Rightarrow$ (c) Let M be a minimal S-semimodule and $\sigma = ann_S(M)$. Then M is a minimal and faithful right S/σ-semimodule. Theorem 2.3 implies that there exists $\rho \in \mathcal{RC}_m(S)$ such that $M \simeq S/\rho$; and hence $ann_S(M) = (\rho : \nabla_S)$. Then $ann_{(S/\sigma)}(M) = ann_S(M)/\sigma = \Delta_{(S/\sigma)}$ implies that $ann_S(M) = \sigma$. Hence, $(\rho : \nabla_S) = ann_S(M) = \sigma$.

(c) $\Rightarrow$ (a) Let $\rho \in \mathcal{RC}_m(S)$ and $\sigma = (\rho : \nabla_S)$. Then S/ρ is a minimal right S-semimodule and $ann_S(S/\rho) = (\rho : \nabla_S) = \sigma$. Since σ is a semiring congruence on S and $\sigma = (\rho : \nabla_S)$, it follows that S/ρ is a minimal right S/σ-semimodule. Also, by Lemma 2.1, we have $ann_{S/\sigma}(S/\rho) = ann_S(S/\rho)/\sigma = \Delta_{S/\sigma}$. Hence, S/ρ is a minimal and faithful right S/σ-semimodule, so σ is a m-primitive congruence on S. □

From the definition, it follows that a semiring S is an m-primitive (s-primitive) semiring if and only if Δ_S is an m-primitive (s-primitive) congruence on S. Thus, we have the following.

Corollary 3.1. *Let S be a semiring. Then, S is*

(a) *m-primitive if and only if there exists $\rho \in \mathcal{RC}_m(S)$ such that $(\rho : \nabla_S) = \Delta_S$;*
(b) *s-primitive if and only if there exists $\rho \in \mathcal{RC}_s(S)$ such that $(\rho : \nabla_S) = \Delta_S$.*

A division semiring is a noncommutative generalization of a semifield. The following result shows that primitive semirings are another class of noncommutative generalizations of semifields. The m-primitive semirings generalize the semifields, whereas the s-primitive semirings generalize the congruence-simple semifields.

Theorem 3.1. *Let S be a commutative semiring. Then, S is*

(a) *m-primitive if and only if it is a semifield;*
(b) *s-primitive if and only if it is a congruence-simple semifield.*

Proof. (a) Let S be a commutative m-primitive semiring. Then, by Corollary 3.1, there is a regular right congruence ρ in $\mathcal{RC}_m(S)$ such that $(\rho : \nabla_S) = \Delta_S$. Since S is a commutative semiring, ρ becomes a congruence on S and so $\rho = (\rho : \nabla_S) = \Delta_S$. Therefore $\Delta_S \in \mathcal{RC}_m(S)$ and there is an element $e \in S$ such that $es = s = se$ for all $s \in S$. Thus, e is a multiplicative identity in S. Also $\rho = \Delta_S \in \mathcal{RC}_m(S)$ implies that (0) is maximal Δ_S-saturated ideal in S. Since every ideal in S is Δ_S-saturated, it follows that (0) and S are the only two ideals in S. Now for each non-zero element $a \in S$, aS is a non-zero ideal in S. Hence, $aS = S$, which implies that there exists an element $b \in S$ such that $ab = e = ba$. Thus, S is a semifield.

Conversely, let S be a semifield. Then $M = S$ is a minimal S-semimodule and $ann_S(M) = \{(s_1, s_2) \in S \times S \mid ss_1 = ss_2 \text{ for all } s \in S\} = \Delta_S$. Therefore, S is m-primitive.

(b) Let S be a commutative s-primitive semiring. Then S is m-primitive, and so, by (a), it is a semifield. Also, by Corollary 3.1, there exists a right congruence $\rho \in \mathcal{RC}_s(S)$ such that $(\rho : \nabla_S) = \Delta_S$. Since S is commutative, it follows that $(\rho : \nabla_S) = \rho$. Hence $\Delta_S = \rho \in \mathcal{RC}_s(S)$ which implies that $M = S/\rho \simeq S$ is a simple S-semimodule. Hence, the semifield S is congruence-simple.

Conversely, if S is a congruence-simple semifield, then S itself is a faithful simple S-semimodule. Hence, S is s-primitive. □

Theorem 3.1 tells us that the congruence-simple semifields constitute an important subclass of the semifields. Similarly to the fields, the Krull-dimension of a congruence-simple semifield is 0, whereas there are semifields, say, for example, $\mathbb{R}_{max}$ having the Krull-dimension 1 [22]. A semiring S is called *zerosumfree* if for every $a, b \in S$, we have $a+b = 0$ implies that $a = 0$ and $b = 0$. It is well known that a semifield S is either zerosumfree or is a field [15, Proposition 4.34]. Every field is a congruence-simple semifield. If S is zerosumfree, then $\rho = \{(s,t) \in S \times S \mid s \neq 0 \neq t\} \cup \{(0,0)\}$ is a congruence on S. So for S to be congruence-simple, we must have $|S| = 2$. Then S is the 2-element Boolean algebra $\mathbb{B}$. Thus, a congruence-simple semifield is either the 2-element Boolean algebra $\mathbb{B}$ or a field.

However, in the following, we include an independent proof.

Theorem 3.2. *Let S be a semiring with $|S| > 2$. Then, S is a congruence-simple semifield if and only if it is a field.*

Proof. First, assume that S is a congruence-simple semifield. Denote $Z(S) = \{x \in S \mid x + y = 0 \text{ for some } y \in S\}$. Then, $Z(S)$ is an ideal of S; and so $Z(S)$ is either $\{0\}$ or S. If $Z(S) = \{0\}$, then S is zerosumfree. So, $\{(0, 0)\} \cup \{(s, t) \in S \times S \mid s \neq 0 \neq t\}$ induces a nontrivial congruence on S, which contradicts that S is congruence-simple. Hence, $Z(S) = S$ which implies that $(S, +)$ is a group. Thus, S is a field.

The converse follows trivially. □

Thus a zerosumfree semifield S with $|S| > 2$ cannot be congruence-simple. So, in particular, we have the following example.

Example 3.3. The max-plus algebra $\mathbb{R}_{max}$ is a semifield but not congruence-simple. Hence, $\mathbb{R}_{max}$ is an m-primitive semiring but not s-primitive.

The 2-element Boolean algebra $\mathbb{B}$ and the field $\mathbb{Z}_2$ of all integers modulo 2 are the only semifields of order two up to isomorphism. Hence, it turns out to be the following specific characterization of the commutative s-primitive semirings.

Corollary 3.2. *A commutative semiring S is s-primitive if and only if it is either the 2-element Boolean algebra $\mathbb{B}$ or a field.*

A semiring S is called a *subdirect product* of a family $\{S_\alpha\}_\Delta$ of semirings if there is an one-to-one semiring homomorphism $\phi : S \to \prod_\Delta S_\alpha$ such that for each $\alpha \in \Delta$, the composition $\pi_\alpha \circ \phi : S \to S_\alpha$ is onto where $\pi_\alpha : \prod_\Delta S_\alpha \to S_\alpha$ is the projection mapping.

It is well known that a semiring S is a subdirect product of a family $\{S_\alpha\}_\Delta$ of semirings if and only if there is a family $\{\rho_\alpha\}_\Delta$ of congruences on S such that $S/\rho_\alpha \simeq S_\alpha$ for every $\alpha \in \Delta$ and $\cap_\Delta \rho_\alpha = \Delta_S$.

Theorem 3.3. *A semiring S is m-semisimple (s-semisimple) if and only if it is a subdirect product of m-primitive (s-primitive) semirings.*

Proof. We prove the result for m-semisimple semirings. The proof for s-semisimple semirings is similar.

First, assume that S is a m-semisimple semiring. Then $rad_m(S) = \cap_{M \in \mathcal{M}(S)} ann_S(M) = \Delta_S$. Hence, S is a subdirect product of the family $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$ of semirings. Lemma 3.1 implies that $ann_S(M)$ is an m-primitive congruence on S for every minimal S-semimodule M. Therefore, every semiring in the family $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$ is an m-primitive semiring, and so S is a subdirect product of m-primitive semirings.

Conversely, let S be a subdirect product of a family of m-primitive semirings $\{S_i \mid \text{for all } i \in \Lambda\}$. Then there exists a one-to-one homomorphism $\phi : S \to \Pi_{i\in\Lambda} S_i$ such that the mapping $\pi_i \circ \phi : S \to S_i$ is onto for all $i \in \Lambda$. Thus $S/\ker(\pi_i \circ \phi) \cong S_i$ for all $i \in \Lambda$. Let M_i be a faithful minimal S_i-semimodule for each $i \in \Lambda$. Then, by the Lemma 2.1, M_i is a minimal S-semimodule where $ms = m\pi_i \circ \phi(s)$ for all $s \in S$ and $m \in M_i$. Hence, $\cap_{M\in\mathcal{M}(S)} ann_S(M) \subseteq \cap_{i\in\Lambda} ann_S(M_i)$. Now $(a, b) \in ann_S(M_i)$ implies that $m\pi_i \circ \phi(a) = m\pi_i \circ \phi(b)$ for all $m \in M_i$; and so $(\pi_i \circ \phi(a), \pi_i \circ \phi(b)) \in ann_{S_i}(M)$. Since M_i is faithful over S_i, it follows that $\pi_i \circ \phi(a) = \pi_i \circ \phi(b)$. Hence, $\cap_{i\in\Lambda} ann_S(M_i) = \Delta_S$ which implies that $rad_m(S) = \cap_{M\in\mathcal{M}(S)} ann_S(M) = \Delta_S$. Thus, S is a m-semisimple semiring. $\square$

Now, taken together the structure of an s-semisimple semiring characterized in Theorem 3.3 and the characterization of the commutative s-primitive semirings in Corollary 3.2 turn out to be an characterization of the commutative s-semisimple semirings.

Corollary 3.3. *Let S be a commutative semiring. Then, S is an s-semisimple semiring if and only if it is a subdirect product of a family of semirings that are either the 2-element Boolean algebra $\mathbb{B}$ or fields.*

Mischell and Fenoglio [29] and Basir et al. [2] independently proved that a commutative semiring S with $|S| \geqslant 2$ is congruence-simple if and only if it is either a field or the 2-element Boolean algebra $\mathbb{B}$. Hence, it follows that a commutative semiring is s-semisimple if and only if it is a subdirect product of congruence-simple commutative semirings. A semiring homomorphism $f : S_1 \to S_2$ is said to be *semi-isomorphism* if, for every $a \in S_1$, we have $f(a) = 0$ only for $a = 0$. Katsov and Nam [24] proved that a commutative semiring S is Brown-McCoy semisimple if and only if S is semi-isomorphic to a subdirect product of a family of semirings that are either the 2-element Boolean algebra $\mathbb{B}$ or fields. Hence, every commutative s-semisimple semiring is Brown-McCoy semisimple in the sense of Katsov and Nam.

Example 3.4. Consider the semiring $\mathbb{N}$ of all nonnegative integers. Then, for every prime p, the Bourne congruence $\sigma_{p\mathbb{N}}$ is a maximal regular congruence on $\mathbb{N}$ with $[0]_{\sigma_{p\mathbb{N}}} = p\mathbb{N}$. If J is a $\sigma_{p\mathbb{N}}$-saturated ideal in $\mathbb{N}$ with $p\mathbb{N} \subsetneq J$, then there exists $a \in J$ such that $0 < a < p$. By the Fermat's little theorem, we have $a^{p-1} \equiv 1 \pmod{p}$ which implies that $1 \in J$ and so $J = \mathbb{N}$. Thus, $p\mathbb{N} = [0]_{\sigma_{p\mathbb{N}}}$ is a maximal $\sigma_{p\mathbb{N}}$-saturated ideal in $\mathbb{N}$ and it follows that $\sigma_{p\mathbb{N}} \in \mathcal{RC}_s(\mathbb{N})$. Hence $rad_s(\mathbb{N}) \subseteq \cap\sigma_{p\mathbb{N}} = \Delta_{\mathbb{N}}$; and so $\mathbb{N}$ is an s-semisimple semiring.

Also, $\cap\sigma_{p\mathbb{N}} = \Delta_{\mathbb{N}}$ implies that $\mathbb{N}$ is a subdirect product of the family of fields $\mathbb{N}_p = \mathbb{N}/\sigma_{p\mathbb{N}}$, where p is a prime.

We conclude this section with a representation of s-primitive semirings as a semiring of endomorphisms on a semimodule over a division semiring.

The opposite semiring S^{op} of a semiring $(S, +, \cdot)$ is defined by $(S, +, \circ)$, where $a \circ b = b \cdot a$ for all $a, b \in S$. Hence, a semiring S is a division semiring if and only if the opposite semiring S^{op} is so.

Definition 3.3. Let M be a semimodule over a division semiring D. Then a subsemiring T of the endomorphism semiring $End_D(M)$ is called 1-fold transitive if for every non-zero $m \in M$ and $n \in M$ there exists $\alpha \in T$ such that $\alpha(m) = n$.

In the context of semirings, Schur's lemma [21] states that if M is a simple S-semimodule, then the endomorphism semiring $End_S(M)$ is a division semiring.

Let M be a right S-semimodule and $E = End_S(M)$. Then for the division semiring $D = E^{op}$, M is a right semimodule over D where the scalar multiplication is defined by $m \cdot \alpha = \alpha(m)$ for all $m \in M$ and $\alpha \in D$.

Theorem 3.4. *If S is a right s-primitive semiring, then S^{op} is isomorphic to a 1-fold transitive subsemiring of the semiring $End_D(M)$ of all endomorphisms on a semimodule M over a division semiring D.*

Proof. Let M be a faithful simple right S-semimodule. By Schur's Lemma for semimodules [21], the semiring $E = End_S(M)$ is a division semiring. Hence, $D = E^{op}$ is a division semiring, and so M as a right D-semimodule where $m \cdot \alpha \mapsto \alpha(m)$.

For every $a \in S$, define a mapping $\psi_a : M \to M$ by $\psi_a(m) = ma$. Then for every $\alpha \in D$, we have $\psi_a(m.\alpha) = \psi_a(\alpha(m)) = \alpha(m)a = \alpha(ma) = (ma) \cdot \alpha = \psi_a(m) \cdot \alpha$. In fact, ψ_a is an endomorphism on M considered a D-semimodule.

Also, the mapping $\psi : S^{op} \to End_D(M)$ defined by $\psi(a) = \psi_a$ is a semiring homomorphism. Moreover $\ker \psi = ann_S(M) = \Delta_S$ implies that ψ is an injective homomorphism; and so S^{op} is isomorphic to the subsemiring $T = \{\psi_a \mid a \in S\}$ of $End_D(M)$.

Since M is a simple right S-semimodule, by Lemma 2.2, for every $m(\neq 0) \in M$, $mS = M$. Then for every $n \in M$ there exists $a \in S$ such that $ma = n$ and so $\psi_a(m) = n$. Thus, T is a 1-fold transitive subsemiring of $End_D(M)$. □

It follows from Corollary 3.2 that the semifield $F = \mathbb{R}_{max}$ is not an s-primitive semiring. Since F contains 1, every F-endomorphism on F is of the form $\psi_a : F \to F$ given by $\psi_a(m) = am$. Hence, $F \simeq End_F(F)$ which implies that $End_F(F)$ is not s-primitive; whereas $End_F(F)$ is a 1-fold transitive subsemiring of itself. Thus, the converse of the Theorem 3.4 does not hold. However, the converse holds in the following weaker form.

Theorem 3.5. *Let D be a division semiring and M be a right D-semimodule. If T is a 1-fold transitive subsemiring of $End_D(M)$, then T^{op} is a right m-primitive semiring.*

Proof. Define $M \times T^{op} \to M$ by $m \cdot \alpha \mapsto \alpha(m)$. Then M is a right T^{op}-semimodule. Let m be a non-zero element in M. Then, for every $n \in M$, there exists $\alpha \in T$ such that $m \cdot \alpha = n$. Therefore, $mT^{op} = M$ which implies that M is minimal, by

Lemma 2.2. Now

$$\begin{aligned} ann_{T^{op}}(M) &= \{(\alpha,\beta) \in T \times T \mid m \cdot \alpha = m \cdot \beta \text{ for all } m \in M\} \\ &= \{(\alpha,\beta) \in T \times T \mid \alpha(m) = \beta(m) \text{ for all } m \in M\} \\ &= \{(\alpha,\beta) \in T \times T \mid \alpha = \beta\} \\ &= \Delta_S \end{aligned}$$

and so M is a faithful minimal T^{op}-semimodule. Therefore, T^{op} is a m-primitive semiring. □

4. Conclusion

In [6], based on the notions of minimal semimodule and simple semimodule, the Jacobson m-radical and s-radical of a semiring S have been considered as a congruence on S. In Section 3 of this article, we introduce the m-semisimple and s-semisimple semirings as the semiring that has the trivial Jacobson m-radical and s-radical, respectively. These two notions of semisimplicity effectively characterize the structure of semirings, including the additively idempotent semirings. The m-semisimple (s-semisimple) are isomorphic to a subdirect product of m-primitive (s-primitive) semirings. In particular, a commutative semiring is s-primitive if and only if it is a subdirect product of the fields and copies of the two element Boolean algebra. Finally, every s-primitive semiring is represented as a suitable subsemiring of the semiring $End_D(M)$ of all endomorphisms on a semimodule M over a division semiring D.

There is another notion of simplicity of semimodules, namely the congruence simple semimodules which are known as elementary semimodules [8]. An attempt may be taken to characterize the e-semisimple semirings which are defined based on the class of elementary semimodules.

References

[1] M. Akian, R. Bapat and S. Gaubert, *Max-plus algebra*, in: L. Hogben, R. Brualdi, A. Greenbaum, R. Mathias (Eds.), *Handbook of Linear Algebra*, Chapman & Hall, London, 2006.

[2] R. El Basir, J Hurt, A. Jančařik and T. Kepka, *Simple commutative semirings*, J. Algebra **236** (2001), 277–306. https://doi.org/10.1006/jabr.2000.8483

[3] A. Bertram and R. Easton, *The tropical Nullstellensatz for congruences*, Adv. Math. **308** (2017), 36–82. https://doi.org/10.1016/j.aim.2016.12.004

[4] A. K. Bhuniya and T. Mondal, *Distributive lattice decompositions of semirings with a semilattice additive reduct*, Semigroup Forum **80** (2010), 293–301. https://doi.org/10.1007/s00233-009-9205-6

[5] A. K. Bhuniya and T. Mondal, *On the least distributive lattice congruence on a semiring with a semilattice additive reduct*, Acta Math. Hungar. **147** (2015), 189–204. https://doi.org/10.1007/s10474-015-0526-5

[6] A. K. Bhuniya and P. Sarkar, *On the Jacobson radical of a semiring*, J. Algebra Appl. https://doi.org/10.1142/S0219498825503694

[7] D. Castella, *Eléments d'algèbre linéaire tropicale*, Linear Algebra Appl. **432** (2010), 1460–1474. https://doi.org/10.1016/j.laa.2009.11.005

[8] C. Chen et al., *Extreme representations of semirings*, Serdica Math. J. **44**(3/4) (2018), 365–412. https://doi.org/10.48550/arXiv.1806.06501
[9] J. H. Conway, *Regular Algebra and Finite Machines*, Chapman & Hall, London, 1971.
[10] A. Connes and C. Consani, *Schemes over* $\mathbf{F_1}$ *and zeta functions*, Compos. Math. **146**(6) (2010), 1383–1415. https://doi.org/10.1112/S0010437X09004692
[11] N. Damljanović, M. Ćirić and J. Ignjatović, *Bisimulations for weighted automata over an additively idempotent semirings*, Theoret. Comput. Sci. **534** (2014), 86–100. https://doi.org/10.1016/j.tcs.2014.02.032
[12] A. Deitmar, *F1-schemes and toric varieties*, Beitr. Algebra Geom. **49**(2) (2008), 517–525.
[13] M. Droste, W. Kuich and H. Vogler (Eds.), *Handbook of Weighted Automata*, Springer, Berlin, 2009.
[14] A. Gathmann, *Tropical algebraic geometry*, Jahresbericht der Deutschen Mathematiker-Vereinigung **108**(1) (2006), 3–32.
[15] J. S. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, 1999.
[16] M. Gondran and M. Minoux, *Graphs, Dioids and Semirings*, Springer, New York, 2008.
[17] U. Hebisch and H. J. Weinert, *Semirings: Algebraic Theory and Applications in Computer Science*, World Scientific, 1993.
[18] S. N. Il'in, *On the homological classification of semirings*, J. Math. Sci. (N.Y.) **256**(2) (2021), 125–143. https://doi.org/10.1007/s10958-021-05423-1
[19] I. Itenberg, G. Mikhalkin and E. Shustin, *Tropical Algebraic Geometry*, Birkhäuser Verlag, Basel, 2009.
[20] Z. Izhakian and L. Rowen, *Supertropical algebra*, Adv. Math. **225**(4) (2010), 2222–2286. https://doi.org/10.1016/j.aim.2010.04.007
[21] Z. Izhakian, J. Rhodes and B. Steinberg, *Representation theory of finite semigroups over semirings*, J. Algebra **336** (2011), 139–157. https://doi.org/10.1016/j.jalgebra.2011.02.048
[22] D. Joó and K. Mincheva, *On the dimension of polynomial semirings*, J. Algebra **507** (2018), 103–119. https://doi.org/10.1016/j.jalgebra.2018.04.007
[23] D. Joó and K. Mincheva, *Prime congruences of additively idempotent semirings and a Nullstellensatz for tropical polynomials*, Selecta Math. (N.S.) **24** (2018), 2207–2233. https://doi.org/10.1007/s00029-017-0322-x
[24] Y. Katsov and T. G. Nam, *On radicals of semirings and related problems*, Comm. Algebra **42** (2014), 5065–5099. https://doi.org/10.1080/00927872.2013.833208
[25] P. Lescot, *Absolute algebra II - Ideals and spectra*, J. Pure Appl. Algebra **215** (2011), 1782–1790. https://doi.org/10.1016/j.jpaa.2010.10.019
[26] P. Lescot, *Absolute algebra III - The saturated spectrum* J. Pure Appl. Algebra **216** (2012), 1004–1015. https://doi.org/10.1016/j.jpaa.2011.10.031
[27] G. L. Litvinov, *The Maslov dequantization, and idempotent and tropical mathematics: a brief introduction*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **326** (2005), 145–182 (Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 13); translation in: J. Math. Sci. (N.Y.) **140**(3) (2007), 426–444.
[28] L. H. Mai and N. X. Tuyen, *Some remarks on the Jacobson radical types of semirings and related problems*, Vietnam J. Math. **45** (2017), 493–506. https://doi.org/10.1007/s10013-016-0226-7
[29] S. S. Mitchell and P. B. Fenoglio, *Congruence-free commutative semirings*, Semigroup Forum **31** (1988), 79–91. https://doi.org/10.1007/BF02573125
[30] M. P. Schützenberger, *On the definition of a family of automata*, Information and Control **4** (1961), 245–270. https://doi.org/10.1016/S0019-9958(61)80020-X
[31] M. K. Sen and A. K. Bhuniya, *On semirings whose additive reduct is a semilattice*, Semigroup Forum **82** (2011), 131–140. https://doi.org/10.1007/s00233-010-9271-9

[32] D. Wilding, M. Johnson and M. Kambites, *Exact rings and semirings*, J. Algebra **388** (2013), 324–337. https://doi.org/10.1016/j.jalgebra.2013.05.005

Department of Mathematics,
Visva Bharati,
Santiniketan - 731235, India
Email address: anjankbhuniya@gmail.com
ORCID iD: https://orcid.org/0000-0002-8692-0718

Department of Mathematics,
Visva Bharati,
Santiniketan - 731235, India
Email address: puja.vb.math@gmail.com
ORCID iD: https://orcid.org/0009-0002-1860-4454