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ON THE CHARACTERIZATION OF NON-LINEAR MIXED BI-SKEW JORDAN TRIPLE DERIVATIONS ON $\ast$ -ALGEBRAS

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ABSTRACT. In this article, it is shown that a map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ (not necessarily linear) satisfies $\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C)$ holds for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $\ast$ -derivation where $\mathfrak{A}$ a unital $\ast$ -algebra over the complex fields $\mathbb{C}$. As applications, we apply our main result to some special classes of unital $\ast$ -algebras such as prime $\ast$ -algebras, standard operator algebras, factor von Neumann algebras and von Neumann algebras with no central summands of type I_1 .

1. INTRODUCTION

Let $\mathfrak{A}$ be a $\ast$ -algebra over the complex field $\mathbb{C}$. A map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ is called an additive derivation if $\xi(A + B) = \xi(A) + \xi(B)$ and $\xi(AB) = \xi(A)B + A\xi(B)$ holds for all $A, B \in \mathfrak{A}$. Moreover, ξ is said to be an additive $\ast$ -derivation if it is an additive derivation and $\xi(A^\ast) = \xi(A)^\ast$ holds for all $A \in \mathfrak{A}$. For $A, B \in \mathfrak{A}$, define the Jordan product and bi-skew Jordan product of A and B by $A \circ B = AB + BA$ and $A \bullet B = AB^\ast + BA^\ast$, respectively. A map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ (not necessarily linear) is said to be nonlinear Jordan derivation (resp. nonlinear Jordan triple derivation) if $\xi(A \circ B) = \xi(A) \circ B + A \circ \xi(B)$ (resp. $\xi((A \circ B) \circ C) = (\xi(A) \circ B) \circ C + (A \circ \xi(B)) \circ C + (A \circ B) \circ \xi(C)$) holds for all $A, B, C \in \mathfrak{A}$. Analogously, a map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ (not necessarily linear) is called a nonlinear bi-skew Jordan derivation (resp. nonlinear bi-skew Jordan triple derivation) if $\xi(A \bullet B) = \xi(A) \bullet B + A \bullet \xi(B)$ (resp. $\xi((A \bullet B) \bullet C) = (\xi(A) \bullet B) \bullet C + (A \bullet \xi(B)) \bullet C + (A \bullet B) \bullet \xi(C)$) holds for all $A, B, C \in \mathfrak{A}$. In

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the last decade, many mathematicians devoted themselves to the study of mappings involving new products on various kind of rings and algebras. These kind of new products are playing an increasingly important role in some research topics, and their study has attracted the attention of many authors (see [1, 2, 4–8, 12, 13, 15–18]). In 2016, Zhang [17] studied nonlinear skew Jordan derivations on factor von Neumann algebras and proved that every nonlinear skew Jordan derivation on a factor von Neumann algebra is an additive $*$ -derivation. Later, this result has been extended to skew Jordan triple derivation and skew Jordan-type derivations on $*$ -algebras in [8, 18], respectively. Khan [5] proved that every multiplicative bi-skew Jordan triple derivations on a prime $*$ -algebra is an additive $*$ -derivations. This result has been generalized in [2] by Ashraf et al. where they proved that every nonlinear bi-skew Jordan-type derivation on a factor von Nuemann algebra is an additive $*$ -derivation. Recently, many researchers paid more attention to the study of Lie (Jordan) mappings involving two different kinds of products at the same time in the nonlinear settings (see, for examples [9–11, 19, 20]). In [20], Zhou et al. proved that every nonlinear mixed Lie triple derivation on prime $*$ -algebra is an additive $*$ -derivation. Rehman et al. in [11] proved that every nonlinear mixed Jordan triple derivation on a $*$ -algebra is an additive $*$ -derivation. Similar kinds of problems has been investigated in [9, 10, 19]. Motivated by the above cited works, in this article, we combine the Jordan product and the bi-skew Jordan product in order to get the mixed bi-skew Jordan triple product $(A \circ B) \bullet C$, where $A, B, C \in \mathfrak{A}$. Correspondingly, a map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ (not necessarily linear) is called a nonlinear mixed bi-skew Jordan triple derivation if

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C)$$

holds for all $A, B, C \in \mathfrak{A}$. The aim of this article is to find the relationship between nonlinear mixed bi-skew Jordan triple derivations and additive $*$ -derivation on arbitrary $*$ -algebras. More precisely, we show that, under mild assumptions, every nonlinear mixed bi-skew Jordan triple derivation on a unital $*$ -algebra is an additive $*$ -derivation.

2. THE MAIN RESULTS

The main result of this article states as follows.

Theorem 2.1. *Let $\mathfrak{A}$ be a $*$ -algebra with the unity I and containing a nontrivial projection P_1 . Write $P_2 = I - P_1$ and assume that $\mathfrak{A}$ satisfies*

$$(2.1) \quad X\mathfrak{A}P_k = 0 \Rightarrow X = 0 \quad (k = 1, 2),$$

where $X \in \mathfrak{A}$. Then a map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ (not necessarily linear) satisfies

$$(2.2) \quad \xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $*$ -derivation.

Proof. Let us choose an arbitrary nontrivial projection P_1 and write $P_2 = I - P_1$. Then, $\mathfrak{A}$ can be written as $\mathfrak{A} = P_1\mathfrak{A}P_1 + P_1\mathfrak{A}P_2 + P_2\mathfrak{A}P_1 + P_2\mathfrak{A}P_2$.

Let $\mathfrak{A}^+ = \{H \in \mathfrak{A} : H^* = H\}$ and $\mathfrak{A}^- = \{S \in \mathfrak{A} : S^* = -S\}$. Further write $\mathfrak{A}_{ii}^+ = P_i \mathfrak{A}^+ P_i$, $i = 1, 2$ and $\mathfrak{A}_{12}^+ = \{P_1 H P_2 + P_2 H P_1 : H \in \mathfrak{A}^+\}$. Then, any $H \in \mathfrak{A}^+$ can be written as $H = H_{11} + H_{12} + H_{22}$, where $H_{ii} \in \mathfrak{A}_{ii}^+$, $i = 1, 2$ and $H_{12} \in \mathfrak{A}_{12}^+$. It is easy to verify that if ξ is an additive $*$ -derivation, then it satisfies (2.2). Thus, here we only need to prove the necessity part, which will be established by checking the following series of lemmas. Taking $A = B = C = 0$ in (2.2), the following lemma is easy to obtain. $\square$

Lemma 2.1. $\xi(0) = 0$.

Lemma 2.2. $\xi(H)^* = \xi(H)$ for every $H \in \mathfrak{A}^+$.

Proof. For any $H \in \mathfrak{A}^+$, we have $H = (H \circ \frac{I}{2}) \bullet \frac{I}{2}$. Thus, we obtain

$$\begin{aligned} \xi(H) &= \xi\left(\left(H \circ \frac{I}{2}\right) \bullet \frac{I}{2}\right) \\ &= \left(\left(\xi(H) \circ \frac{I}{2}\right) \bullet \frac{I}{2}\right) + \left(\left(H \circ \xi\left(\frac{I}{2}\right)\right) \bullet \frac{I}{2}\right) + \left(\left(H \circ \frac{I}{2}\right) \bullet \xi\left(\frac{I}{2}\right)\right) \\ &= \xi(H) \frac{I}{2} + \frac{I}{2} \xi(H)^* + \left(H \xi\left(\frac{I}{2}\right) + \xi\left(\frac{I}{2}\right) H\right) \frac{I}{2} + \frac{I}{2} \left(\xi\left(\frac{I}{2}\right)^* H + H \xi\left(\frac{I}{2}\right)^*\right) \\ &\quad + H \xi\left(\frac{I}{2}\right)^* + \xi\left(\frac{I}{2}\right) H, \end{aligned}$$

and hence,

$$\begin{aligned} \xi(H)^* &= \xi(H) \frac{I}{2} + \frac{I}{2} \xi(H)^* + \left(H \xi\left(\frac{I}{2}\right) + \xi\left(\frac{I}{2}\right) H\right) \frac{I}{2} \\ &\quad + \frac{I}{2} \left(\xi\left(\frac{I}{2}\right)^* H + H \xi\left(\frac{I}{2}\right)^*\right) + H \xi\left(\frac{I}{2}\right)^* + \xi\left(\frac{I}{2}\right) H. \end{aligned}$$

That is, $\xi(H) = \xi(H)^*$ for all $H \in \mathfrak{A}^+$. $\square$

Lemma 2.3. For any $H_{11} \in \mathfrak{A}_{11}^+$, $H_{12} \in \mathfrak{A}_{12}^+$ and $H_{22} \in \mathfrak{A}_{22}^+$, we have

$$\xi(H_{11} + H_{12}) = \xi(H_{11}) + \xi(H_{12}) \quad \text{and} \quad \xi(H_{12} + H_{22}) = \xi(H_{12}) + \xi(H_{22}).$$

Proof. Assume that $T = \xi(H_{11} + H_{12}) - \xi(H_{11}) - \xi(H_{12})$. It is easy to observe that $T^* = T$ by Lemma 2.2. Our target is to show that $T = 0$. Using the fact that $(H_{11} \circ P_2) \bullet P_2 = 0$ and making use of Lemma 2.1, we have

$$\begin{aligned} &\xi\left(\left((H_{11} + H_{12}) \circ P_2\right) \bullet P_2\right) \\ &= \xi\left(\left(H_{11} \circ P_2\right) \bullet P_2\right) + \xi\left(\left(H_{12} \circ P_2\right) \bullet P_2\right) \\ &= \left(\left(\xi(H_{11}) + \xi(H_{12})\right) \circ P_2\right) \bullet P_2 \\ &\quad + \left(\left(H_{11} + H_{12}\right) \circ \xi(P_2)\right) \bullet P_2 + \left(\left(H_{11} + H_{12}\right) \circ P_2\right) \bullet \xi(P_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\xi\left(\left((H_{11} + H_{12}) \circ P_2\right) \bullet P_2\right) &= (\xi(H_{11} + H_{12}) \circ P_2) \bullet P_2 + ((H_{11} + H_{12}) \circ \xi(P_2)) \bullet P_2 \\ &\quad + ((H_{11} + H_{12}) \circ P_2) \bullet \xi(P_2).\end{aligned}$$

On comparing the above two expressions for $\xi\left(\left((H_{11} + H_{12}) \circ P_2\right) \bullet P_2\right)$, we get $((T \circ P_2) \bullet P_2) = 0$. Application of the fact that $T^* = T$ leads us to $T_{12} = T_{21} = T_{22} = 0$. Invoking the fact $(H_{12} \circ (P_2 - P_1)) \bullet P_1 = 0$ and Lemma 2.1, we find that

$$\begin{aligned}&\xi\left(\left((H_{11} + H_{12}) \circ (P_2 - P_1)\right) \bullet P_1\right) \\ &= \xi\left(\left(H_{11} \circ (P_2 - P_1)\right) \bullet P_1\right) + \xi\left(\left(H_{12} \circ (P_2 - P_1)\right) \bullet P_1\right) \\ &= \left(\left(\xi(H_{11}) + \xi(H_{12})\right) \circ (P_2 - P_1)\right) \bullet P_1 + ((H_{11} + H_{12}) \circ \xi(P_2 - P_1)) \bullet P_1 \\ &\quad + ((H_{11} + H_{12}) \circ (P_2 - P_1)) \bullet \xi(P_1).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\xi\left(\left((H_{11} + H_{12}) \circ (P_2 - P_1)\right) \bullet P_1\right) \\ &= (\xi(H_{11} + H_{12}) \circ (P_2 - P_1)) \bullet P_1 + ((H_{11} + H_{12}) \circ \xi(P_2 - P_1)) \bullet P_1 \\ &\quad + ((H_{11} + H_{12}) \circ (P_2 - P_1)) \bullet \xi(P_1).\end{aligned}$$

From the last two expressions for $\xi\left(\left((H_{11} + H_{12}) \circ (P_2 - P_1)\right) \bullet P_1\right)$, we obtain $((T \circ (P_2 - P_1)) \bullet P_1) = 0$. Using the fact $T^* = T$ and simplifying gives $T_{11} = 0$. Hence, $T = 0$, that is, $\xi(H_{11} + H_{12}) = \xi(H_{11}) + \xi(H_{12})$.

Symmetrically, one can prove that $\xi(H_{12} + H_{22}) = \xi(H_{12}) + \xi(H_{22})$. $\square$

Lemma 2.4. *For any $H_{11} \in \mathfrak{A}_{11}^+$, $H_{12} \in \mathfrak{A}_{12}^+$ and $H_{22} \in \mathfrak{A}_{22}^+$, we have*

$$\xi(H_{11} + H_{12} + H_{22}) = \xi(H_{11}) + \xi(H_{12}) + \xi(H_{22}).$$

Proof. Let $T = \xi(H_{11} + H_{12} + H_{22}) - \xi(H_{11}) - \xi(H_{12}) - \xi(H_{22})$. It is easy to observe that $T^* = T$ by Lemma 2.2. We show that $T = 0$. Using the fact that $(H_{11} \circ P_2) \bullet P_2 = 0$ and Lemmas 2.1 and 2.3, we have

$$\begin{aligned}&\xi\left(\left((H_{11} + H_{12} + H_{22}) \circ P_2\right) \bullet P_2\right) \\ &= \xi\left(\left(H_{11} \circ P_2\right) \bullet P_2\right) + \xi\left(\left((H_{12} + H_{22}) \circ P_2\right) \bullet P_2\right) \\ &= \left(\left(\xi(H_{11}) + \xi(H_{12}) + \xi(H_{22})\right) \circ P_2\right) \bullet P_2 + ((H_{11} + H_{12} + H_{22}) \circ \xi(P_2)) \bullet P_2 \\ &\quad + ((H_{11} + H_{12} + H_{22}) \circ P_2) \bullet \xi(P_2).\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}&\xi\left(\left((H_{11} + H_{12} + H_{22}) \circ P_2\right) \bullet P_2\right) \\ &= \left(\xi(H_{11} + H_{12} + H_{22}) \circ P_2\right) \bullet P_2 + ((H_{11} + H_{12} + H_{22}) \circ \xi(P_2)) \bullet P_2 \\ &\quad + ((H_{11} + H_{12} + H_{22}) \circ P_2) \bullet \xi(P_2).\end{aligned}$$

Comparing the above two expressions for $\xi\left(\left((H_{11} + H_{12} + H_{22}) \circ P_2\right) \bullet P_2\right)$, we find that $(T \circ P_2) \bullet P_2 = 0$, which in turn implies that $T_{12} = T_{21} = T_{22} = 0$.

Invoking the fact $(H_{22} \circ P_1) \bullet P_1 = 0$ and using Lemmas 2.1 and 2.3, we find that

$$\begin{aligned} & \xi\left(\left((H_{11} + H_{12} + H_{22}) \circ P_1\right) \bullet P_1\right) \\ &= \xi\left(\left((H_{11} + H_{12}) \circ P_2\right) \bullet P_1\right) + \xi\left(\left(H_{22} \circ P_1\right) \bullet P_1\right) \\ &= \left(\left(\xi(H_{11}) + \xi(H_{12}) + \xi(H_{22})\right) \circ P_1\right) \bullet P_1 + \left(\left(H_{11} + H_{12} + H_{22}\right) \circ \xi(P_2)\right) \bullet P_1 \\ & \quad + \left(\left(H_{11} + H_{12} + H_{22}\right) \circ P_1\right) \bullet \xi(P_1). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \xi\left(\left((H_{11} + H_{12} + H_{22}) \circ P_1\right) \bullet P_1\right) \\ &= \left(\xi(H_{11} + H_{12} + H_{22}) \circ P_1\right) \bullet P_1 + \left(\left(H_{11} + H_{12} + H_{22}\right) \circ \xi(P_1)\right) \bullet P_1 \\ & \quad + \left(\left(H_{11} + H_{12} + H_{22}\right) \circ P_1\right) \bullet \xi(P_1). \end{aligned}$$

Comparing the above two expressions for $\xi\left(\left((H_{11} + H_{12} + H_{22}) \circ P_1\right) \bullet P_1\right)$, we obtain that $(T \circ P_1) \bullet P_1 = 0$ which further implies that $T_{11} = 0$. Hence, $T = 0$, that is,

$$\xi(H_{11} + H_{12} + H_{22}) = \xi(H_{11}) + \xi(H_{12}) + \xi(H_{22}).$$

□

Lemma 2.5. *For any $H_{12}, H'_{12} \in \mathfrak{A}_{12}^+$, we have $\xi(H_{12} + H'_{12}) = \xi(H_{12}) + \xi(H'_{12})$.*

Proof. For any $X_{12}, Y_{12} \in \mathfrak{A}_{12}$, let $H_{12} = X_{12} + X_{12}^*$ and $H'_{12} = Y_{12} + Y_{12}^*$. Then, it is easy to calculate that

$$\begin{aligned} & \left((P_1 + H_{12}) \circ (P_2 + H'_{12})\right) \bullet \frac{I}{2} \\ &= \left((P_1 + X_{12} + X_{12}^*) \circ (P_2 + Y_{12} + Y_{12}^*)\right) \bullet \frac{I}{2} \\ &= (X_{12} + X_{12}^*) + (Y_{12} + Y_{12}^*) + (X_{12}Y_{12}^* + Y_{12}X_{12}^*) + (X_{12}^*Y_{12} + Y_{12}^*X_{12}) \\ &= H_{12} + H'_{12} + H_{11} + H_{22}, \end{aligned}$$

where $H_{11} = X_{12}Y_{12}^* + Y_{12}X_{12}^* \in \mathfrak{A}_{11}^+$ and $H_{22} = X_{12}^*Y_{12} + Y_{12}^*X_{12} \in \mathfrak{A}_{22}^+$. Therefore, using Lemmas 2.3 and 2.4, we have

$$\begin{aligned} & \xi(H_{12} + H'_{12}) + \xi(H_{11}) + \xi(H'_{22}) \\ &= \xi(H_{12} + H'_{12} + H_{11} + H'_{22}) \\ &= \xi\left(\left((P_1 + X_{12} + X_{12}^*) \circ (P_2 + Y_{12} + Y_{12}^*)\right) \bullet \frac{I}{2}\right) \\ &= \left(\left(\xi(P_1) + \xi(X_{12} + X_{12}^*)\right) \circ (P_2 + Y_{12} + Y_{12}^*)\right) \bullet \frac{I}{2} \\ & \quad + \left(\left((P_1 + X_{12} + X_{12}^*) \circ (\xi(P_2) + \xi(Y_{12} + Y_{12}^*))\right) \bullet \frac{I}{2}\right) \end{aligned}$$

$$\begin{aligned}
& + \left(((P_1 + X_{12} + X_{12}^*) \circ (P_2 + Y_{12} + Y_{12}^*)) \bullet \xi\left(\frac{I}{2}\right) \right) \\
& = \xi\left((P_1 \circ P_2) \bullet \frac{I}{2}\right) + \xi\left((P_1 \circ (Y_{12} + Y_{12}^*)) \bullet \frac{I}{2}\right) \\
& \quad + \xi\left(((X_{12} + X_{12}^*) \circ P_2) \bullet \frac{I}{2}\right) + \xi\left(((X_{12} + X_{12}^*) \circ (Y_{12} + Y_{12}^*)) \bullet \frac{I}{2}\right) \\
& = \xi(H_{12}) + \xi(H'_{12}) + \xi(H_{11}) + \xi(H'_{22}).
\end{aligned}$$

Hence, $\xi(H_{12} + H'_{12}) = \xi(H_{12}) + \xi(H'_{12})$ for any $H_{12} \in \mathfrak{A}_{12}^+$ and $H'_{12} \in \mathfrak{A}_{12}^+$. $\square$

Lemma 2.6. *For any $H_{ii}, H'_{ii} \in \mathfrak{A}_{ii}^+$ for $i = 1, 2$, we have*

$$\xi(H_{11} + H'_{11}) = \xi(H_{11}) + \xi(H'_{11}) \quad \text{and} \quad \xi(H_{22} + H'_{22}) = \xi(H_{22}) + \xi(H'_{22}).$$

Proof. Let $T = \xi(H_{11} + H'_{11}) - \xi(H_{11}) - \xi(H'_{11})$. It is easy to observe that $T^* = T$ by Lemma 2.2. We show that $T = 0$.

Using the fact that $(H_{11} \circ P_2) \bullet P_2 = (H'_{11} \circ P_2) \bullet P_2 = 0$ and Lemma 2.1, we obtain

$$\begin{aligned}
& \xi(((H_{11} + H'_{11}) \circ P_2) \bullet P_2) \\
& = \xi((H_{11} \circ P_2) \bullet P_2) + \xi((H'_{11} \circ P_2) \bullet P_2) \\
& = ((\xi(H_{11}) + \xi(H'_{11})) \circ P_2) \bullet P_2 + ((H_{11} + H'_{11}) \circ \xi(P_2)) \bullet P_2 \\
& \quad + ((H_{11} + H'_{11}) \circ P_2) \bullet \xi(P_2).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \xi(((H_{11} + H'_{11}) \circ P_2) \bullet P_2) = (\xi((H_{11} + H'_{11}) \circ P_2) \bullet P_2 + ((H_{11} + H'_{11}) \circ \xi(P_2)) \bullet P_2 \\
& \quad + ((H_{11} + H'_{11}) \circ P_2) \bullet \xi(P_2)).
\end{aligned}$$

Comparing the above two expressions for $\xi(((H_{11} + H'_{11}) \circ P_2) \bullet P_2)$, we find that $(T \circ P_2) \bullet P_2 = 0$, which in turn gives $T_{12} = T_{21} = T_{22} = 0$.

Next, we show that $T_{11} = 0$. For this, assume that $H_{12} = X_{12} + X_{12}^*$ for some $X_{12} \in \mathfrak{A}_{12}$, then $H_{12} \in \mathfrak{A}_{12}^+$ and it is easy to observe that

$$\left((H_{11} \circ H_{12}) \bullet \frac{I}{2} \right), \left((H'_{11} \circ H_{12}) \bullet \frac{I}{2} \right) \in \mathfrak{A}_{12}^+.$$

Thus, using Lemma 2.5, we find that

$$\begin{aligned}
& \xi\left(((H_{11} + H'_{11}) \circ H_{12}) \bullet \frac{I}{2}\right) \\
& = \xi\left((H_{11} \circ H_{12}) \bullet \frac{I}{2}\right) + \xi\left((H'_{11} \circ H_{12}) \bullet \frac{I}{2}\right) \\
& = ((\xi(H_{11}) + \xi(H'_{11})) \circ H_{12}) \bullet \frac{I}{2} + ((H_{11} + H'_{11}) \circ \xi(H_{12})) \bullet \frac{I}{2}
\end{aligned}$$

$$+ ((H_{11} + H'_{11}) \circ H_{12}) \bullet \xi\left(\frac{I}{2}\right).$$

On the other hand, we have

$$\begin{aligned} \xi\left((H_{11} + H'_{11}) \circ H_{12}\right) \bullet \frac{I}{2} &= (\xi(H_{11} + H'_{11}) \circ H_{12}) \bullet \frac{I}{2} + ((H_{11} + H'_{11}) \circ \xi(H_{12})) \bullet \frac{I}{2} \\ &\quad + ((H_{11} + H'_{11}) \circ H_{12}) \bullet \xi\left(\frac{I}{2}\right). \end{aligned}$$

From the last two expressions for $\xi\left((H_{11} + H'_{11}) \circ H_{12}\right) \bullet \frac{I}{2}$, we get $(T \circ H_{12}) \bullet \frac{I}{2} = 0$ and using the fact that $T^* = T$, we get $TX_{12} + TX_{12}^* + X_{12}T + X_{12}^*T = 0$. Multiplying it by P_1 and P_2 from left and right, we obtain $T_{11}XP_2 + P_1XT_{22} = 0$. Since $T_{22} = 0$, we obtain $T_{11}XP_2 = 0$. Application of the condition (2.1) yields $T_{11} = 0$. Hence, $T = 0$, that is, $\xi(H_{11} + H'_{11}) = \xi(H_{11}) + \xi(H'_{11})$. Symmetrically, one can prove that $\xi(H_{22} + H'_{22}) = \xi(H_{22}) + \xi(H'_{22})$. $\square$

Using Lemmas 2.4, 2.5 and 2.6, the following lemma is easy to obtain.

Lemma 2.7. ξ is additive on $\mathfrak{A}^+$.

Lemma 2.8. $\xi(I) = 0$.

Proof. It follows from $\frac{I}{2} = \left(\frac{I}{2} \circ \frac{I}{2}\right) \bullet \frac{I}{2}$ that

$$\begin{aligned} \xi\left(\frac{I}{2}\right) &= \xi\left(\left(\frac{I}{2} \circ \frac{I}{2}\right) \bullet \frac{I}{2}\right) \\ &= \left(\xi\left(\frac{I}{2}\right) \circ \frac{I}{2}\right) \bullet \frac{I}{2} + \left(\frac{I}{2} \circ \xi\left(\frac{I}{2}\right)\right) \bullet \frac{I}{2} + \left(\frac{I}{2} \circ \frac{I}{2}\right) \bullet \xi\left(\frac{I}{2}\right). \end{aligned}$$

Simplifying with the help of Lemma 2.2, we obtain

$$\xi\left(\frac{I}{2}\right) = \xi\left(\frac{I}{2}\right) + \xi\left(\frac{I}{2}\right) + \xi\left(\frac{I}{2}\right).$$

Hence, $\xi\left(\frac{I}{2}\right) = 0$. Since ξ is additive on $\mathfrak{A}^+$, we get $\xi(I) = \xi\left(\frac{I}{2}\right) + \xi\left(\frac{I}{2}\right) = 0$. $\square$

Lemma 2.9. For any $S \in \mathfrak{A}^-$, we have $\xi(S)^* = -\xi(S)$.

Proof. Since $0 = \left((S \circ I) \bullet \frac{I}{2}\right)$ for any $S \in \mathfrak{A}^-$, using the facts that $\xi(I) = \xi\left(\frac{I}{2}\right) = 0$, we obtain

$$0 = \xi(0) = \xi\left((S \circ I) \bullet \frac{I}{2}\right) = \left((\xi(S) \circ I) \bullet \frac{I}{2}\right),$$

which implies that $\xi(S)^* = -\xi(S)$ for any $S \in \mathfrak{A}^-$. $\square$

Lemma 2.10. $\xi(iI) = 0$.

Proof. We have $\xi(iI)^* = -\xi(iI)$, by Lemma 2.9. Since $(iI \circ iI) \bullet I = -4I$, invoking Lemmas 2.7 and 2.8, we obtain that

$$0 = -4\xi(I) = \xi(-4I) = \xi((iI \circ iI) \bullet I) = (\xi(iI) \circ iI) \bullet I + (iI \circ \xi(iI)) \bullet I = \xi(iI).$$

□

Lemma 2.11. *For any $H \in \mathfrak{A}^+$, we have $\xi(iH) = i\xi(H)$.*

Proof. Observe that $(iH \circ iI) \bullet I = -4H$ for any $H \in \mathfrak{A}^+$. Using Lemmas 2.7, 2.8, 2.9 and 2.10, we find that

$$-4\xi(H) = \xi(-4H) = \xi((iH \circ iI) \bullet I) = (\xi(iH) \circ iI) \bullet I = 4i\xi(iH),$$

which implies that $\xi(iH) = i\xi(H)$. □

Lemma 2.12. *For any $A \in \mathfrak{A}$, we have $\xi(A^*) = \xi(A)^*$.*

Proof. Let $A = H_1 + iH_2 \in \mathfrak{A}$ for some $H_1, H_2 \in \mathfrak{A}^+$. Since $4H_1 = ((H_1 + iH_2) \circ I) \bullet I$, and making use of Lemmas 2.7 and 2.8, we obtain

$$\begin{aligned} 4\xi(H_1) &= \xi(4H_1) = \xi(((H_1 + iH_2) \circ I) \bullet I) \\ &= (\xi(H_1 + iH_2) \circ I) \bullet I = 2\{\xi(H_1 + iH_2) + \xi(H_1 + iH_2)^*\}. \end{aligned}$$

Thus, we have

$$(2.3) \quad 2\xi(H_1) = \xi(H_1 + iH_2) + \xi(H_1 + iH_2)^*.$$

On the other hand, we have $4H_2 = ((H_1 + iH_2) \circ I) \bullet iI$. Invoking Lemmas 2.7, 2.8 and 2.10, we obtain

$$\begin{aligned} 4\xi(H_1) &= \xi(4H_1) = \xi(((H_1 + iH_2) \circ I) \bullet iI) \\ &= (\xi(H_1 + iH_2) \circ I) \bullet iI = 2(\xi(H_1 + iH_2) - \xi(H_1 + iH_2)^*). \end{aligned}$$

Therefore, we have

$$(2.4) \quad 2i\xi(H_2) = \xi(H_1 + iH_2) - \xi(H_1 + iH_2)^*.$$

The addition of (2.2) and (2.3) yields

$$(2.5) \quad \xi(H_1 + iH_2) = \xi(H_1) + i\xi(H_2).$$

Since ξ is additive on $\mathfrak{A}^+$ and $\xi(0) = 0$, we have $\xi(-H) = -\xi(H)$ for any $H \in \mathfrak{A}^+$. By using (2.4) and Lemmas 2.2 and 2.7, we have

$$\begin{aligned} \xi(A)^* &= \xi(H_1 + iH_2)^* = (\xi(H_1) + i\xi(H_2))^* = \xi(H_1) - i\xi(H_2) \\ &= \xi(H_1 - iH_2) = \xi(A^*), \end{aligned}$$

for all $A \in \mathfrak{A}$. □

Lemma 2.13. *ξ is additive on $\mathfrak{A}$.*

Proof. Let us consider any two arbitrary elements $A_1 = H_1 + iH_2, A_2 = H'_1 + iH'_2 \in \mathfrak{A}$ for some $H_1, H_2, H'_1, H'_2 \in \mathfrak{A}^+$. Using (2.4) and Lemma 2.7, we find that

$$\begin{aligned}\xi(A_1 + A_2) &= \xi((H_1 + iH_2) + (H'_1 + iH'_2)) = \xi((H_1 + H'_1) + i(H_2 + H'_2)) \\ &= \xi(H_1 + H'_1) + i\xi(H_2 + H'_2) = \xi(H_1) + \xi(H'_1) + i\xi(H_2) + i\xi(H'_2) \\ &= \xi(H_1) + i\xi(H_2) + \xi(H'_1) + i\xi(H'_2) = \xi(H_1 + iH_2) + \xi(H'_1 + iH'_2) \\ &= \xi(A_1) + \xi(A_2). \quad \square\end{aligned}$$

Lemma 2.14. *For any $A \in \mathfrak{A}$, we have $\xi(iA) = i\xi(A)$.*

Proof. Let $A = H_1 + iH_2 \in \mathfrak{A}$ for some $H_1, H_2 \in \mathfrak{A}^+$. Invoking Lemmas 2.11 and 2.13, we obtain

$$\begin{aligned}\xi(iA) &= \xi(iH_1 - H_2) = \xi(iH_1) - \xi(H_2) = i\xi(H_1) - \xi(H_2) \\ &= i(\xi(H_1) + i\xi(H_2)) = i(\xi(H_1 + iH_2)) = i\xi(A). \quad \square\end{aligned}$$

Lemma 2.15. *ξ is an additive $*$ -derivation on $\mathfrak{A}$.*

Proof. In view of Lemmas 2.12 and 2.13, it is sufficient to show that $\xi(A_1A_2) = \xi(A_1)A_2 + A_1\xi(A_2)$ for all $A_1, A_2 \in \mathfrak{A}$. First, we show that

$$\xi(HH') = \xi(H)H' + H\xi(H'), \quad \text{for all } H, H' \in \mathfrak{A}^+.$$

Using Lemmas 2.2, 2.7 and 2.8, we have

$$\begin{aligned}2\xi(HH' + H'H) &= \xi(2(HH' + H'H)) = \xi((H \circ H') \bullet I) \\ &= (\xi(H) \circ H') \bullet I + (H \circ \xi(H')) \bullet I \\ &= 2(\xi(H)H' + H'\xi(H) + H\xi(H') + \xi(H')H).\end{aligned}$$

Thus, we have

$$(2.6) \quad \xi(HH') + \xi(H'H) = \xi(H)H' + H'\xi(H) + H\xi(H') + \xi(H')H.$$

Also, in view of Lemma 2.11, we have

$$\begin{aligned}2i\xi(H'H - HH') &= \xi(2i(H'H - HH')) = \xi((H \circ I) \bullet iH') \\ &= (\xi(H) \circ I) \bullet iH' + (H \circ I) \bullet \xi(iH') \\ &= 2i(\xi(H)H' - H'\xi(H) + H\xi(H') - \xi(H')H).\end{aligned}$$

Consequently, we get

$$(2.7) \quad \xi(HH') - \xi(H'H) = \xi(H)H' - H'\xi(H) + H\xi(H') - \xi(H')H.$$

The addition of (2.5) and (2.6) yields

$$(2.8) \quad \xi(HH') = \xi(H)H' + H\xi(H'), \quad \text{for all } H, H' \in \mathfrak{A}^+.$$

Now, let $A_1 = H_1 + iH_2$ and $A_2 = H'_1 + iH'_2$ be any two arbitrary elements of $\mathfrak{A}$, where $H_1, H_2, H'_1, H'_2 \in \mathfrak{A}^+$. In view of (2.7) and Lemmas 2.13 and 2.14, we find that

$$\begin{aligned}\xi(A_1 A_2) &= \xi\left((H_1 H'_1 - H_2 H'_2) + i(H_1 H'_2 + H_2 H'_1)\right) \\ &= \xi(H_1 H'_1 - H_2 H'_2) + i\xi(H_1 H'_2 + H_2 H'_1) \\ &= \xi(H_1 H'_1) - \xi(H_2 H'_2) + i\xi(H_1 H'_2) + i\xi(H_2 H'_1) \\ &= \xi(H_1)H'_1 + H_1\xi(H'_1) - \xi(H_2)H'_2 - H_2\xi(H'_2) + i\xi(H_1)H'_2 + iH_1\xi(H'_2) \\ &\quad + i\xi(H_2)H'_1 + iH_2\xi(H'_1)\end{aligned}$$

and

$$\begin{aligned}\xi(A_1)A_2 + A_1\xi(A_2) &= \xi(H_1 + iH_2)(H'_1 + iH'_2) + (H_1 + iH_2)\xi(H'_1 + iH'_2) \\ &= (\xi(H_1) + i\xi(H_2))(H'_1 + iH'_2) + (H_1 + iH_2)(\xi(H'_1) + i\xi(H'_2)) \\ &= \xi(H_1)H'_1 + H_1\xi(H'_1) - \xi(H_2)H'_2 - H_2\xi(H'_2) + i\xi(H_1)H'_2 \\ &\quad + iH_1\xi(H'_2) + i\xi(H_2)H'_1 + iH_2\xi(H'_1).\end{aligned}$$

Therefore, we have

$$\xi(A_1 A_2) = \xi(A_1)A_2 + A_1\xi(A_2), \quad \text{for all } A_1, A_2 \in \mathfrak{A}.$$

Hence, the proof of Theorem 2.1 is complete. $\square$

3. APPLICATIONS

In this section, we apply Theorem 2.1 to certain special classes of $*$ -algebras, namely prime $*$ -algebras, standard operator algebras, factor von Neumann algebras and von Neumann algebras with no central summands of type I_1 . Recall that an algebra $\mathfrak{A}$ is prime if for any $A, B \in \mathfrak{A}$, $A\mathfrak{A}B = \{0\}$ implies that either $A = 0$ or $B = 0$. It is easy to verify that every prime $*$ -algebra satisfies (2.1). Therefore, as a direct consequence of Theorem 2.1, we have the following result.

Corollary 3.1. *Let $\mathfrak{A}$ be a unital prime $*$ -algebra containing a nontrivial projection. Then a map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ satisfies*

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $*$ -derivation.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ denote the subalgebra of all bounded finite rank operators. A subalgebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. Now we have the following result.

Corollary 3.2. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I . Suppose that $\mathfrak{A}$ is*

closed under the adjoint operation. Then, a map $\xi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ satisfies

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ if and only if ξ is a linear $*$ -derivation. Moreover, there exists an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\xi(A) = AT - TA$ for all $A \in \mathfrak{A}$, that is, ξ is inner.

Proof. Since $\mathfrak{A}$ is a unital prime $*$ -algebra containing nontrivial projections, then by Corollary 3.1, we see that ξ is an additive $*$ -derivation. It follows from [14] that ξ is an linear inner derivation, that is, there exists an operator $S \in \mathcal{B}(\mathcal{H})$ such that $\xi(A) = AS - SA$ for all $A \in \mathfrak{A}$. Using the fact that $\xi(A^*) = \xi(A)^*$, we have

$$A^*S - SA^* = \xi(A^*) = \xi(A)^* = S^*A^* - A^*S^*,$$

for any $A \in \mathfrak{A}$. This leads to $A^*(S + S^*) = (S + S^*)A^*$. Hence, $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Letting $T = S - \frac{1}{2}\lambda I$, one can check that $T + T^* = 0$ and $\xi(A) = AT - TA$ for all $A \in \mathfrak{A}$. $\square$

A von Neumann algebra $\mathfrak{A}$ is a weakly closed, self adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I . A von Neumann algebra $\mathfrak{A}$ is a factor von Neumann algebra if its center contains only the scalar operators. It is well known that a factor von Neumann algebra is prime, thus, it always satisfies (2.1). Hence, as an immediate consequence of Corollary 3.1, we get the following.

Corollary 3.3. *Let $\mathfrak{A}$ be a factor von Neumann algebra with $\dim(\mathfrak{A}) \geq 2$. Then, a map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ satisfies*

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $*$ -derivation.

Further, it is well-known that every von Neumann algebra with no central summands of type I_1 satisfies (2.1) (see [3, 6] for details). Therefore, applying Theorem 2.1, we have the following result.

Corollary 3.4. *Let $\mathfrak{A}$ be a von Neumann algebra having no central summands of type I_1 . Then a map $\xi : \mathfrak{A} \rightarrow \mathfrak{A}$ satisfies*

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $*$ -derivation.

4. CONCLUSION

In this paper, we have studied the relationship between nonlinear mixed bi-skew Jordan triple derivations and additive $*$ -derivations on arbitrary $*$ -algebras. In fact, it is shown that, under certain assumptions, every nonlinear mixed bi-skew Jordan triple derivation on a unital $*$ -algebra is an additive $*$ -derivation.

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ON GENERALIZED MADDUX SPACES VIA DIFFERENCE OPERATOR

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ABSTRACT. In the present work, we begin our investigation with some dynamic properties of new generalized difference operator $\Delta_h^{\alpha,\beta,\gamma}$, defined in Baliarsingh [5]. Combining this operator with the well known Cesàro operator, we also introduce new classes of generalized Cesàro summable difference sequence spaces $w(\Delta_h^{\alpha,\beta,\gamma}, p)$, $w_0(\Delta_h^{\alpha,\beta,\gamma}, p)$ and $w_\infty(\Delta_h^{\alpha,\beta,\gamma}, p)$, which are the natural extension of the spaces w^p , w_0^p and w_∞^p defined in [12] and $w_0(p)$, $w(p)$, and $w_\infty(p)$ defined by Maddox [21]. We establish various topological properties on these spaces along with some inclusion relations with other basic sequence spaces. Further, our investigation is carried out to determine α^* - and β^* - duals and characterize matrix transformations on these spaces.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

One of the potential fields of sequence spaces as well as the growing achievements of summability theory is the difference sequence spaces, which enriched with the existence of sequence spaces via various difference operators providing wide range of applications in both pure and applied area of mathematics. Moreover, ideal concepts of difference operators and the related spaces are stimulating diverse fields of research like approximation theory [11, 19], spectral theory and linear algebra [1, 7, 15], numerical analysis [8], compact operator theory [25, 31], fractional calculus [5, 6] and matrix theory [13, 26, 29], etc.

A sequence space is a subspace of linear space w (the space of all real or complex valued sequences), for instance, c , c_0 and ℓ_∞ are the spaces of all convergent, null,

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and bounded sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. As the initial development of this theory, in 1981, the idea of difference sequence spaces $X(\Delta) = \{x = (x_k) \in w : \Delta x \in X\}$, where $X \in \{c_0, c, \ell_\infty\}$ had been considered by Kizmaz [20], by using the forward difference operator

$$(1.1) \quad \Delta x_k = x_k - x_{k+1}, \quad k \in \mathbb{N}_0 \text{ (the set of all non negative integers).}$$

Recently, the difference space bv_p consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been studied in the case $0 < p < 1$ by Altay and Başar [2], and in the case $1 \leq p \leq +\infty$ by Başar and Altay [13]. The reader also refer to the recent monograph [29], and references therein, devoted to the matrix transformations and related topics.

In 1995, these spaces had been extended to the case of integer order 'm' by Et and Çolak [17], using the difference operator Δ^m , defined by

$$(1.2) \quad \Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}, \quad k \in \mathbb{N}_0.$$

Latter, Baliarsingh [4] (see, also [8]) extended these spaces by using fractional order difference operator, Δ^α for a positive proper fraction α , i.e.,

$$(1.3) \quad \Delta^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i! \Gamma(\alpha+1-i)} x_{k+i}, \quad k \in \mathbb{N}_0,$$

where $\Gamma(\alpha)$ denotes the Euler gamma function. Quite recently, the idea has been extended to the case of arbitrary order α, β, γ by Baliarsingh [5] by defining

$$(1.4) \quad \Delta_h^{\alpha, \beta, \gamma} x_k = \sum_{i=0}^{\infty} \frac{(-\alpha)_i (-\beta)_i}{(-\gamma)_i i! h^{\alpha+\beta-\gamma}} x_{k+i},$$

where $(\sigma)_k$ is the Pochhammer symbol or shifted factorial for $\sigma \in \mathbb{R}$, defined by

$$(\sigma)_k = \begin{cases} 1, & \sigma = 0 \text{ or } k = 0, \\ \sigma(\sigma+1) \cdots (\sigma-k+1), & k \in \mathbb{N}. \end{cases}$$

In particular, if $h = 1$ the above operators include the cases, such as the operator Δ^1 ([20]) for $\alpha = 1, \beta = \gamma$, the operator $\Delta^{(m)}$ ([17]) for $\alpha = m \in \mathbb{N}, \gamma = \beta$ and the operator Δ^α ([4]) for $\beta = \gamma$. Note that matrix representation of the operators $\Delta = (\delta_{nk})$, $\Delta^m = (\delta_{nk}^m)$, $\Delta^\alpha = (\delta_{nk}^\alpha)$ and $\Delta_h^{\alpha, \beta, \gamma} = (\delta_{h,nk}^{a,b,c})$ (defined in (1.1), (1.2), (1.3) and (1.4)), respectively, are as follows:

$$\delta_{nk} = \begin{cases} 1, & k = n, \\ -1, & k = n-1, \\ 0, & k > n, \end{cases} \quad \delta_{nk}^m = \begin{cases} 1, & k = n, \\ (-1)^{n-k} \binom{m}{n-k}, & 0 \leq k < n, \\ 0, & k > n, \end{cases}$$

$$\delta_{nk}^\alpha = \begin{cases} 1, & k = n, \\ \frac{(-\alpha)_{n-k}}{(n-k)!}, & 0 \leq k < n, \\ 0, & k > n, \end{cases} \quad \text{and} \quad \delta_{h,nk}^{a,b,c} = \begin{cases} 1, & k = n, \\ \frac{(-\alpha)_{n-k} (-\beta)_{n-k}}{(-\gamma)_{n-k} (n-k)!}, & 0 \leq k < n, \\ 0, & k > n. \end{cases}$$

Several authors including Mursaleen, Altay and Başar [27], Mursaleen and Başar [29], Dutta and Baliarsingh [6], Tripathy and Sarma [30], Aydin and Polat [26], Vardwaj and Gupta [14], Et [16], etc., have given their valuable attempts to construct new sequence spaces by combining the notion of difference operators with some functions (like Modulus function, Orlicz function, ϕ function, etc.), and certain means (Cesàro mean, Nörlund mean, Riesz mean, Euler mean, etc.). In this regard, various authors studied the topological properties on defined sequence spaces along with their matrix transformations, inclusion relations and also determined their dual spaces. It has been remarked that the above works will be more convenient if the used difference operators defined by (1.1), (1.2), (1.3) and (1.4) are well defined and the related difference sequences are convergent. In fact, the behavior of the operators mentioned earlier on any arbitrary sequences is completely dynamic in nature. At this stage, we want to emphasize this dynamic nature by following examples.

Example 1.1. Let $\alpha > \gamma > 0$, $\beta = 0$, $h = 1$ and define a sequence $x = (x_k) = (\frac{1}{5^k})$, $k \in \mathbb{N}_0$. Then, we have

$$\begin{aligned}\Delta_h^{\alpha, \beta, \gamma} x_k &= \sum_{i=0}^{+\infty} \frac{(-\alpha)_i}{(-\gamma)_i i!} x_{k+i} = \frac{1}{5^k} \sum_{i=0}^{+\infty} \frac{(-\alpha)_i}{(-\gamma)_i i!} \cdot \frac{1}{5^i} \\ &= \frac{1}{5^k} \sum_{i=0}^{+\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-i+1)}{\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-i+1)i!} \cdot \frac{1}{5^i} \\ &= \frac{1}{5^k} \sum_{i=0}^{+\infty} t_i.\end{aligned}$$

By applying D’Almbert’s ratio test, we obtain the right hand side of above series

$$\lim_{i \rightarrow +\infty} \left| \frac{t_{i+1}}{t_i} \right| = \lim_{i \rightarrow +\infty} \left| \frac{(\alpha-i)}{(\gamma-i)(i+1)5} \right| = 0.$$

This ultimately makes the sequence $\Delta_h^{\alpha, \beta, \gamma} x$ converges. So, the above series is convergent.

Example 1.2. Let $x = e = (1, 1, 1, \dots)$ and $\alpha > \gamma > 0$ and $\beta = 0$, $h = 1$. Then, clearly, the sequence $\Delta_1^{\alpha, \beta, \gamma} x$ converges. But, for $\beta = \gamma$, $h = 1$, it can be observed that

$$\Delta^{-\alpha} x_k = \alpha + \frac{\alpha(\alpha+1)}{2} + \cdots \rightarrow +\infty, \quad \text{as } k \rightarrow +\infty,$$

which is divergent.

Example 1.3. Let the sequence $x = (x_k)$ be defined by $x_k = k^{\frac{5}{2}}$, $k \in \mathbb{N}_0$. Then, it can be easily observed that

$$\Delta x_k = k^{\frac{5}{2}} - (k+1)^{\frac{5}{2}} = -\frac{5}{2} - \mathcal{O}(k^{\frac{1}{2}}) \rightarrow -\infty, \quad \text{as } k \rightarrow +\infty,$$

which is divergent. But, for $\alpha = 3$, $\beta = \gamma$, $h = 1$ we obtain

$$\Delta^3 x_k = k^{\frac{5}{2}} - 3(k+1)^{\frac{5}{2}} + 3(k+2)^{\frac{5}{2}} - (k+3)^{\frac{5}{2}}$$

$$\begin{aligned}
&= k^{\frac{5}{2}} - 3 \left\{ k^{\frac{5}{2}} + \frac{5}{2} k^{\frac{3}{2}} + \frac{15}{8} k^{\frac{1}{2}} + \mathcal{O}(k^{-\frac{1}{2}}) \right\} \\
&\quad + 3 \left\{ k^{\frac{5}{2}} + 5k^{\frac{3}{2}} + \frac{15}{2} k^{\frac{1}{2}} + \mathcal{O}(k^{-\frac{1}{2}}) \right\} - \left\{ k^{\frac{5}{2}} + \frac{15}{2} k^{\frac{3}{2}} + \frac{135}{2} k^{\frac{1}{2}} + \mathcal{O}(k^{-\frac{1}{2}}) \right\} \rightarrow 0, \\
&\text{as } k \rightarrow +\infty.
\end{aligned}$$

As a result, we conclude that the sequence $\Delta^3 x$ is convergent.

Example 1.4. Let $x = (x_k)$ be the oscillating sequence defined by $x_k = \frac{1}{3} + \frac{(-1)^k}{6}$, $k \in \mathbb{N}_0$. Now, after applying the operator Δ^α on x we have,

$$\Delta^\alpha x_k = \begin{cases} \frac{2^{\alpha-1}}{3}, & k \text{ is even,} \\ -\frac{2^{\alpha-1}}{3}, & k \text{ is odd,} \end{cases}$$

which is a divergent sequence.

Furthermore, the study on convergence of difference sequences up to the case of fractional order, is found in the work of Baliarsingh [6]. But, convergence analysis in general case is quite challenging, which to be discussed in this sections. Moreover, our aim is to construct some new class of difference sequence spaces by combining the generalized difference operator with Cesàro operator of order one and also investigate their topological properties with some inclusion relations. Usually, the Cesàro operator C_1 of order one, and its inverse $(C_1)^{-1}$, respectively, in matrix form are given by $C_1 = (c_{nk})$, and $C_1^{-1} = (s_{nk})$, where

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \leq k \leq n, \\ 0, & \text{elsewhere,} \end{cases} \quad \text{and} \quad s_{nk} = \begin{cases} n+1, & k = n, \\ -(k+1), & k = n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

Now, combining the generalized difference operator $\Delta_{1,-}^{\alpha,\beta,\gamma}$ and Cesàro operator C_1 , we have the generalized Cesàro difference operator $(C_1 \Delta_{1,-}^{\alpha,\beta,\gamma})$ as

$$(C_1 \Delta_{1,-}^{\alpha,\beta,\gamma})_{nk} = \begin{cases} \frac{1}{n+1} \sum_{j=0}^{n-k} \frac{(-\alpha)_j (-\beta)_j}{(-\gamma)_j j!}, & \text{if } 0 \leq k < n, \\ \frac{1}{n+1}, & k = n, \\ 0, & k > n. \end{cases}$$

Using Theorem 2 and Remark 1 of [9], we determine the inverse of $C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}$ explicitly as

$$(C_1 \Delta_{1,-}^{a,b,c})_{nk}^{-1} = \begin{cases} (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}), & \text{if } 0 \leq k < n, \\ n+1, & k = n, \\ 0, & k > n, \end{cases}$$

where

$$(1.5) \quad \bar{D}_n^k(C_1\Delta_{1,-}^{\alpha,\beta,\gamma}) = \begin{vmatrix} \frac{1}{k+2} \sum_{j=k}^{k+1} \frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{k+2} & 0 & \cdots \\ \frac{1}{k+3} \sum_{j=k}^{k+2} \frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{k+3} \sum_{j=k}^{k+1} \frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{k+3} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} \sum_{j=k}^n \frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{n+1} \sum_{j=k}^{n-1} \frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \cdots & \frac{1}{n+1} \sum_{j=k}^{k+1} \frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} \end{vmatrix}.$$

This follows from the fact that $(C_1\Delta_{1,-}^{\alpha,\beta,\gamma})^{-1} = (\Delta_{1,-}^{\alpha,\beta,\gamma})^{-1}C_1^{-1}$. Note that C_1^{-1} is as stated above and

$$(\Delta_{1,-}^{\alpha,\beta,\gamma})_{nk}^{-1} = \begin{cases} 1, & k = n, \\ (-1)^{n-k} \tilde{D}_{n-k}^k(\Delta_{1,-}^{\alpha,\beta,\gamma}), & 0 \leq k \leq n-1, \\ 0, & k > n, \end{cases}$$

where

$$\tilde{D}_n^k(\Delta_{1,-}^{\alpha,\beta,\gamma}) = \begin{vmatrix} \tilde{d}_{10} & \tilde{d}_{11} & \cdots & \tilde{d}_{1,n-1} \\ \tilde{d}_{20} & \tilde{d}_{21} & \cdots & \tilde{d}_{2,n-1} \\ \tilde{d}_{30} & \tilde{d}_{31} & \cdots & \tilde{d}_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{n0} & \tilde{d}_{n1} & \cdots & \tilde{d}_{n,n-1} \end{vmatrix},$$

where $\tilde{d}_{10} = -\frac{\alpha\beta}{\gamma}$, $\tilde{d}_{11} = 1$, $\tilde{d}_{1,n-1} = 0$, $\tilde{d}_{20} = \frac{\alpha(\alpha-1)\beta(\beta-1)}{2!\gamma(\gamma-1)}$, $\tilde{d}_{21} = -\frac{\alpha\beta}{\gamma}$, $\tilde{d}_{2,n-1} = 0$,

$$\tilde{d}_{30} = \frac{-\alpha(\alpha-1)(\alpha-2)\beta(\beta-1)(\beta-2)}{3!\gamma(\gamma-1)(\gamma-2)},$$

$$\tilde{d}_{31} = \frac{\alpha(\alpha-1)\beta(\beta-1)}{2!\gamma(\gamma-1)}, \quad \tilde{d}_{3,n-1} = 0,$$

$$\tilde{d}_{n0} = (-1)^n \frac{\alpha(\alpha-1)(\alpha-2) \cdots (\alpha-n+1)\beta(\beta-1)(\beta-2) \cdots (\beta-n+1)}{\gamma(\gamma-1)(\gamma-2) \cdots (\gamma-n+1)n!},$$

$$\tilde{d}_{n,1} = (-1)^{n-1} \frac{\alpha(\alpha-1)(\alpha-2) \cdots (\alpha-n+2)\beta(\beta-1)(\beta-2) \cdots (\beta-n+2)}{\gamma(\gamma-1)(\gamma-2) \cdots (\gamma-n+2)(n-1)!},$$

$$\tilde{d}_{n,n-1} = \frac{-\alpha\beta}{\gamma}.$$

Let $p = (p_k)$ be bounded sequence of positive real numbers and $M = \sup_k \{1, p_k\}$ and $0 < \inf p_k$. Then, the spaces $w_0(p)$, $w(p)$ and $w_\infty(p)$ of all sequences that are strongly summable to zero, summable and bounded sequences, respectively defined in [21, 22] as follows:

$$w_0(p) = \left\{ x = (x_k) \in w : \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n |x_k|^{p_k} = 0 \right\},$$

$$w(p) = \left\{ x = (x_k) \in w : \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C} \right\},$$

$$w_\infty(p) = \left\{ x = (x_k) \in w : \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n |x_k|^{p_k} \right) < +\infty \right\},$$

which are complete paranormed space with respect to the paranorm [12]

$$g(x) = \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n |x_k|^{p_k} \right)^{\frac{1}{M}}.$$

Motivated by the idea above, we define following new spaces by combining the generalized difference operator $\Delta_{h,-}^{\alpha,\beta,\gamma}$ with Cesàro operator of order one C_1 as

$$\begin{aligned} w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) &= \left\{ x = (x_k) \in w : \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = 0 \right\}, \\ w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) &= \left\{ x = (x_k) \in w : \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k - l|^{p_k} = 0, \right. \\ &\quad \left. \text{for some } l \in \mathbb{C} \right\}, \\ w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) &= \left\{ x = (x_k) \in w : \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} \right) < +\infty \right\}, \end{aligned}$$

where $\Delta_{h,-}^{\alpha,\beta,\gamma} x_k = \sum_{i=0}^{+\infty} \frac{(-\alpha)_i (-\beta)_i}{(-\gamma)_i i! h^{\alpha+\beta-\gamma}} x_{k-i}$. In particular, these spaces includes the following spaces.

- (a) For $\alpha = 0$, $\beta = \gamma$, $h = 1$, the above spaces are reduced to the basic spaces defined by [21].
- (b) For $(p_k) = (p)$, where $1 \leq p < +\infty$, $\alpha = 0$, $\beta = \gamma$, $h = 1$, the above spaces are reduce to the basic spaces w_∞^p , w_0^p and w^p defined by [12].

Now, we provide certain definitions which are used in the sequel.

- Let g be a function from a linear space to $\mathbb{R}$, is said to be a paranorm if it satisfies the following axioms
 - (a) $g(x) = 0$ if $x = 0$;
 - (b) $g(-x) = g(x)$;
 - (c) $g(x+y) \leq g(x) + g(y)$;
 - (d) If (μ_n) be a sequence of scalars with $\mu_n \rightarrow \mu$ as $n \rightarrow +\infty$, and $(x_n)_{n \in \mathbb{N}}$, $x \in X$. Then, $g(\mu_n x_n - \mu x) \rightarrow 0$ as $n \rightarrow +\infty$.

Moreover, (X, g) is called the paranormed space (cf. [12])

- A sequence space E is said to be solid, if $(\lambda_k x_k) \in E$, for all sequence of scalar $|\lambda_k| \leq 1$, for $(x_k) \in E$.
- A sequence space E is said to be separable if there exists a countable dense subspace for this space.

2. MAIN RESULTS

This section includes some results associated with convergence of generalized difference sequence $\Delta_{h,-}^{\alpha,\beta,\gamma}x$, and topological properties of newly defined spaces. Before proceeding further, we required the following definition (cf. [6]),

A sequence $x = (x_k) \in w$ is said to be of order $\nu > 0$, i.e., $x = \mathcal{O}(k^\nu)$ if there exists a positive constant $\mathcal{C}$ such that

$$|x_k| \leq \mathcal{C}k^\nu, \quad k = 0, 1, 2, \dots$$

Motivated by the work in [6], we generalize the following theorem.

Theorem 2.1. *Let $\alpha, \beta, \gamma \in \mathbb{R}^+$ (the set of positive real numbers) and $x = (x_k)$, be a sequence of order ν such that $\alpha + \beta - \gamma > \nu$. Then, the difference sequence $(\Delta_{h,-}^{\alpha,\beta,\gamma}x_k)$ is absolutely convergent.*

Proof. Suppose $x = (x_k)$ is a sequence of order ν . Then there exists a positive constant $\mathcal{M}$, such that,

$$|x_k| \leq \mathcal{M}k^\nu, \quad k \in \mathbb{N}_0.$$

Now, the series defined in equation (1.4) becomes

$$\begin{aligned} |\Delta_{h,-}^{\alpha,\beta,\gamma}x_k| &= \left| \sum_{i=0}^{+\infty} \frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}} x_{k-i} \right| \\ &\leq \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}} x_{k-i} \right| \\ &\leq \mathcal{M} \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}} \right| \cdot |k-i|^\nu \\ &= \mathcal{M} \sum_{i=0}^{+\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-(i-1))\beta(\beta-1)\cdots(\beta-(i-1))}{i!h^{\alpha+\beta-\gamma}\gamma(\gamma-1)\cdots(\gamma-(i-1))} \right| \cdot |k-i|^\nu \\ &= \frac{\mathcal{M}}{h^{a+b-c}} \sum_{i=0}^{+\infty} \left| \frac{(i-(\alpha+1))(i-(\alpha+2))\cdots(i-(\alpha+i))}{i!} \right| \\ &\quad \times \left| \frac{(i-(\beta+1))(i-(\beta+2))\cdots(i-(\beta+i))}{(i-(\gamma+1))(i-(\gamma+2))\cdots(i-(\gamma+i))} \right| \cdot |k-i|^\nu \\ &\leq \frac{\mathcal{M}}{h^{\alpha+\beta-\gamma}} \sum_{i=0}^{+\infty} \left| \frac{\mathcal{O}(i^{i-(\alpha+1)})\mathcal{O}(i^{i-(\beta+1)})}{i!\mathcal{O}(i^{i-(\gamma+1)})} \right| \cdot |k-i|^\nu \\ &= \frac{\mathcal{M}}{h^{\alpha+\beta-\gamma}} \sum_{i=0}^{+\infty} \left| \frac{\mathcal{O}(i^{2i-(\alpha+\beta)-2})}{\mathcal{O}(i^{2i-(\gamma+1)})} \right| i^\nu \left| \frac{k}{i} - 1 \right|^\nu \\ &= \frac{M}{h^{\alpha+\beta-\gamma}} \sum_{i=0}^{+\infty} \left| \mathcal{O}(i^{\nu-(\alpha+\beta)+\gamma-1}) \right| \cdot \left| \frac{k}{i} - 1 \right|^\nu. \end{aligned}$$

Note that for $i \rightarrow +\infty$, $\left| \frac{k}{i} - 1 \right|^\alpha \rightarrow 1$, and the right hand side converges if $\nu - (\alpha + \beta) + \gamma - 1 < -1$ which implies $\nu < \alpha + \beta - \gamma$.

For more clarity, we mention the plots of $\Delta_{h,-}^{\alpha,\beta,\gamma} x_k$, for the set of sequences $x_k = k^{1.05}, k^{1.75}, k^{1.25}, k^{1.15}, k^{1.95}, k^2$, with $\alpha = 1.58, \beta = 3.11, \gamma = 3.12, h = 1$ as below. $\square$

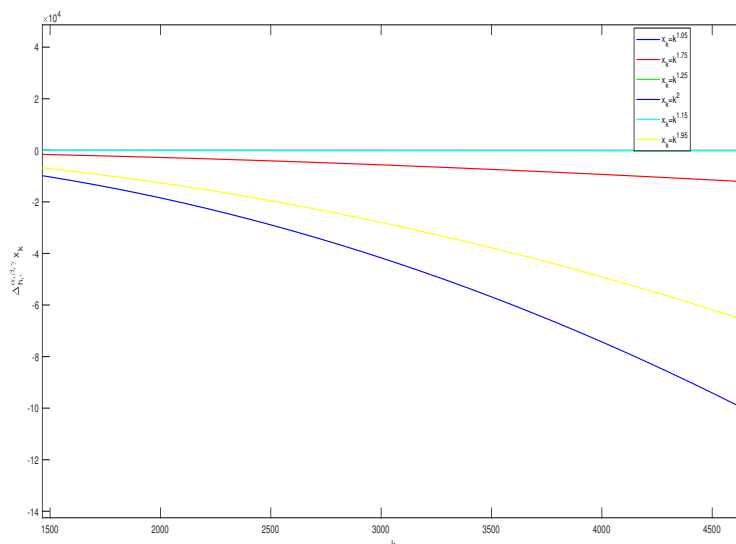


FIGURE 1. Convergence of $(\Delta_{h,-}^{\alpha,\beta,\gamma} x_k)$, using the condition as in Theorem 1

Theorem 2.2. For every convergent sequence $x = (x_k)$, the difference sequence $(\Delta_{h,-}^{\alpha,\beta,\gamma} x_k)$ converges if $\alpha + \beta > \gamma$.

Proof. Suppose $x = (x_k)$ is a convergent sequence. Then, for each $\epsilon > 0$, there exist $l \in \mathbb{C}$ and $K \in \mathbb{N}_0$ such that

$$|x_k - l| < \epsilon, \quad \text{for all } k \geq K.$$

So, the series,

$$\begin{aligned} |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| &= \left| \sum_{i=0}^{+\infty} \frac{(-\alpha)_i (-\beta)_i}{i! (-\gamma)_i h^{\alpha+\beta-\gamma}} x_{k-i} \right| \\ &= \left| \sum_{i=0}^{+\infty} \frac{(-\alpha)_i (-\beta)_i}{i! (-\gamma)_i h^{\alpha+\beta-\gamma}} x_{k-i} - l + l \right| \\ &\leq \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i (-\beta)_i}{i! (-\gamma)_i h^{\alpha+\beta-\gamma}} \right| \cdot |x_{k-i} - l| + |l| \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i (-\beta)_i}{i! (-\gamma)_i h^{\alpha+\beta-\gamma}} \right| \\ &= (|l| + \epsilon) \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i (-\beta)_i}{i! (-\gamma)_i h^{\alpha+\beta-\gamma}} \right|. \end{aligned}$$

Note that the convergence of the difference sequence depends on the convergence of series in right hand side, above. For simplicity we take $h = 1$. Then, applying Raabe's test in the above series, we obtain

$$\begin{aligned} \lim_{i \rightarrow +\infty} \left| i \left(\frac{a_i}{a_{i+1}} - 1 \right) \right| &= \lim_{i \rightarrow +\infty} \left| \frac{(-\alpha)_i (-\beta)_i}{(-\gamma)_i i!} \cdot \frac{(-\gamma)_{i+1} (i+1)!}{(-\alpha)_{i+1} (-\beta)_{i+1}} \right| \\ &= \lim_{i \rightarrow +\infty} \left| \frac{-\alpha(-\alpha+1) \cdots (-\alpha+i-1)}{\alpha(-\alpha+1)(-\alpha+2) \cdots (-\alpha+i)} \right| \\ &\quad \times \left| \frac{-\beta(-\beta+1) \cdots (-\beta+i-1)(i+1) - \gamma(-\gamma+1) \cdots (-\gamma+i)}{-\beta(-\beta+1) \cdots (-\beta+i) - \gamma(-\gamma+1) \cdots (-\gamma+i-1)} \right| \\ &= \lim_{i \rightarrow +\infty} \left| i \left(\frac{(i+1)(-\gamma+i)}{(-\alpha+i)(-\beta+i)} - 1 \right) \right| \\ &= \lim_{i \rightarrow +\infty} \left| i \frac{-i\gamma + i^2 - \gamma + i - \alpha\beta + i\alpha + i\beta - i^2}{i(-\frac{\alpha}{i} + 1)(-\frac{\beta}{i} + 1)} \right| \\ &= \lim_{i \rightarrow +\infty} \left| \frac{(\alpha + \beta - \gamma + 1) - \frac{\alpha\beta + \gamma}{i}}{(-\frac{\alpha}{i} + 1)(-\frac{\beta}{i} + 1)} \right| = \alpha + \beta - \gamma + 1. \end{aligned}$$

This implies that the difference sequence converges while $\alpha + \beta - \gamma + 1 > 1$, i.e., $\alpha + \beta > \gamma$, which concludes the proof. $\square$

Remark 2.1. It is pertinent to note here that,

$$(\Delta_{1,-}^{\alpha_1, \beta_1, \gamma_1} (\Delta_{1,-}^{\alpha_2, \beta_2, \gamma_2} (x_k))) \neq (\Delta_{1,-}^{\alpha_1 + \alpha_2 + \beta_1 + \beta_2, \gamma_1 + \gamma_2} (x_k)),$$

which can be countered by following example.

Example 2.1. Suppose the sequence $x = (x_k)$ is defined by $x_k = 1$, $k \in \mathbb{N}_0$. Then it can be easily observed that

$$\Delta_{1,-}^{1,2,2} x_k = \Delta_{1,-}^{0+1, -1+3, 5-3} x_k = \Delta(x_k) = 0, \quad \text{as } k \rightarrow +\infty.$$

But, for $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 3$, $\gamma_1 = -3$, $\gamma_2 = 5$ and $h = 1$ and we obtain,

$$\Delta_{1,-}^{1,3,5} x_k = \sum_{i=0}^{+\infty} \frac{(-1)_i (-3)_i}{(-5)_i i!},$$

which is a divergent sequence as $k \rightarrow +\infty$, from Raabe's test of a series. Consequently, the sequence $(\Delta_{1,-}^{0,-1,-3} (\Delta_{1,-}^{1,3,5} x_k))$ is divergent.

Remark 2.2. The spaces $X(\Delta_{h,-}^{\alpha, \beta, \gamma}, p)$, $X \in \{w, w_0, w_\infty\}$, respectively, form linear spaces, provided $\Delta_{h,-}^{\alpha, \beta, \gamma} x_k$ exists for all $k \in \mathbb{N}_0$.

Proof. The proof is trivial, hence omitted. $\square$

Theorem 2.3. The spaces $X(\Delta_{h,-}^{\alpha, \beta, \gamma}, p)$, $X \in \{w, w_0\}$ form complete paranormed spaces with respect to the paranorm

$$g(x) = \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha, \beta, \gamma} x_k \right|^{p_k} \right)^{\frac{1}{M}},$$

where $M = \max\{1, H\}$ and $H = \sup_k p_k$. However, the space $w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ is a paranormed space if and only if $\inf p_k > 0$.

Proof. Clearly, if $x = \theta = (0, 0, \dots)$, then $g(x) = 0$. Now, we prove the theorem for $X = w$.

Again,

$$\begin{aligned} g(x) &= \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| (-1) \Delta_{h,-}^{\alpha,\beta,\gamma}(-x_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ &= \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma} x_k \right|^{p_k} \right)^{\frac{1}{M}} = g(-x). \end{aligned}$$

Also,

$$\begin{aligned} g(x+y) &= \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma}(x_k + y_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ &\leq \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma} x_k \right|^{p_k} \right)^{\frac{1}{M}} + \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma} y_k \right|^{p_k} \right)^{\frac{1}{M}} \\ &= g(x) + g(y). \end{aligned}$$

Now, let (q^n) be a sequence of scalar with $q^n \rightarrow q$ as $n \rightarrow +\infty$ and $x^n \in w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$. Then,

$$\begin{aligned} g(q^n x^n - qx) &= \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma}(q^n x_k^n - qx_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ &\leq \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma}(q^n x_k^n - q^n x_k + q^n x_k - qx_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ &= \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma}(q^n(x_k^n - x_k) + x_k(q^n - q)) \right|^{p_k} \right)^{\frac{1}{M}} \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$. As (q^n) and (x^n) converge to q and x respectively, the right hand side tends to zero.

For the case of $w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ the proofs of condition 1, 2 and 3 of the paranorm are similar, hence we omit these. For last condition, we require the followings.

Let $\lambda = (\lambda_n)$ be the sequence of scalar with $\lambda_n \rightarrow 0$ and x be fixed (implies $\lambda x \rightarrow \theta$ (null sequence)). We need to prove $\inf p_k > 0$ to form a paranormed space.

On contrary, suppose $\inf p_k < 0$ (Theorem 2 in [23]). Then, there exist $k_1, k_2, \dots, k_i, \dots$ such that

$$p_k < \frac{1}{i^2},$$

where $2^1 < k_1 < 2^2, 2^2 < k_2 < 2^3, \dots, 2^r < k_r < 2^{r+1}$. Again consider,

$$h(x) = \sup_r \left\{ \frac{1}{2^r} \sum_{k=0}^{2^r} \left| \Delta_{h,-}^{\alpha,\beta,\gamma} x_k \right|^{p_k} \right\}^{\frac{1}{M}}.$$

Let (x_k) be the sequence such that, $\Delta_{h,-}^{\alpha,\beta,\gamma} x_k \rightarrow 1$ as $k \rightarrow +\infty$. Now, let us define the sequence $\bar{x} = (\bar{x}_k)$ by

$$\bar{x}_k = \begin{cases} \Delta_{h,-}^{\alpha,\beta,\gamma} x_k, & k = k_i, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly, $\bar{x} = (\bar{x}_k) \in w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$. Again,

$$h(\lambda\bar{x}) = \frac{1}{2^{r_i}} \sum_{r_i} |\Delta_{h,-}^{\alpha,\beta,\gamma} \lambda x_{k_i}|^{p_{k_i}} = |\lambda|^{p_{k_i}} \leq |\lambda|^{\frac{1}{i^2}} \rightarrow 1, \quad \text{as } i \rightarrow +\infty$$

Moreover,

$$(2.1) \quad \frac{g(x)}{2} < h(x) < 2g(x),$$

provided, the series for the operator $\Delta_{h,-}^{\alpha,\beta,\gamma}$ exists. This implies $g(\lambda\bar{x}) \geq \frac{1}{2}$, which is a contradiction. So, this proves the condition.

Secondly, our claim is now reduced to show the completeness property. So, let (x^i) be any arbitrary Cauchy sequence in $w(\Delta_h^{\alpha,\beta,\gamma}, p)$.

By definition, for each $\epsilon > 0$, there exists $I \in \mathbb{N}_0$ such that

$$g(x^i - x^j) < \epsilon, \quad \text{for all } i, j \geq I$$

implies $\sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma} (x_k^i - x_k^j) \right|^{p_k} \right)^{\frac{1}{M}} < \epsilon$, for all $i, j > I$. This implies that for each sufficiently large $k \in \mathbb{N}_0$ the sequence (x_k^i) , forming a Cauchy sequence in $\mathbb{C}$, which is complete and so convergent to l^i (say), i.e., $|x_k^i - l_k^i|^{p_k} < \frac{\epsilon}{F}$, where let $\left| \Delta_{h,-}^{\alpha,\beta,\gamma} x_k \right| = F$.

Now,

$$g(x^i - l^i) = \sup_n \left(\frac{1}{n+1} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma} (x_k^i - l_k^i) \right|^{p_k} \right)^{\frac{1}{M}} \leq \frac{\epsilon}{F} \sum_{k=0}^n \left| \Delta_{h,-}^{\alpha,\beta,\gamma} \right| < \left(\frac{\epsilon F}{F} \right)^{\frac{1}{M}} < \epsilon,$$

for all $i > I$, for all k . Now (x^i) is the Cauchy sequence with strong limit l^i ultimately gives $l^i \in w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ ([24], page 320). $\square$

Theorem 2.4. *The space $w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ is not separable in general.*

Proof. The proof follows from [14, Theorem 3.7]. Let us take the sequence $x = (x_k)$, such that for each $\varphi > 0 \in \mathbb{R}$

$$x_k = k^\varphi + r, \quad \text{where } r \in \mathbb{R}.$$

Obviously, for each $p \in \mathbb{R}$, $(x_k) \in w(\Delta_{1,-}^{\alpha,\beta,\beta}, p)$, (for $\alpha > \varphi$, $\beta = \gamma$).

Now consider $p = (p_k) = (e)$ and the set, $A = \{k^\varphi + r, k^\varphi + s, k^\varphi + t, \dots\} = \{x^r, x^s, x^t, \dots\}$, where $|r - s| > \frac{1}{3}$.

$$\text{Now, } g(x^r - x^s) = \sup_n \frac{1}{n+1} \sum_{k=0}^n |\Delta_{1,-}^{\alpha,\beta,\beta} (x_k^r - x_k^s)| = |r - s| > \frac{1}{3}.$$

Clearly, A is uncountable. Again, consider D be any arbitrary dense subset of $w(\Delta_{1,-}^{\alpha,\beta,\beta}, p)$.

Moreover, for each $x^r = (x_k^r)_{k \in \mathbb{N}_0} \in w(\Delta_{1,-}^{\alpha,\beta,\beta}, p)$, we can find $y^r = (y_k^r)_{k \in \mathbb{N}_0} \in D$ such that $g(x^r - y^r) < \frac{1}{6}$.

Now, let us define the transformation f as $f : A \rightarrow D$, $x^r = (x_k^r) \mapsto f(x^r) = (y_k^r)$. Again $g(x^r - y^s) = g(x^r - x^s + x^s - y^s) \geq g(x^r - y^s) - g(y^s - x^s) > \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$.

This inequality represents for $x^r \neq x^s$ implies that $y^r \neq y^s$ with respect to the paranorm. This shows that f is one-to-one. Furthermore, $f(A) \subset D$ and $f(A)$ is hence uncountable. Since, D is any arbitrary dense set so, the space $w(\Delta_{1,-}^{\alpha,\beta,\beta}, p)$ is not dense in general. $\square$

Theorem 2.5. *The space $w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$ forms a solid space. But, $w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ does not form solid in general.*

Proof. Suppose (λ_k) is any sequence of scalar such that $|\lambda_k| \leq 1$, for all $k \in \mathbb{N}_0$.

Now, the sequence $(\lambda_k x_k)$, for $(x_k) \in w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$. So,

$$\begin{aligned} \sup_n \frac{1}{n+1} \sum_{k=0}^n |\Delta_{h,-}^{\alpha,\beta,\gamma} \lambda_k x_k|^{p_k} &= \sup_n \frac{1}{n+1} \sum_{k=0}^n |\lambda_k|^{p_k} |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} \\ &\leq \sup_n \frac{1}{n+1} \sum_{k=0}^n |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} < +\infty. \end{aligned}$$

This implies that $(\lambda_k x_k) \in w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ and makes the space $w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ as a solid space. But the space $w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ does not form solid in general, which can be proved by following counter example.

Let $(p_k) = (1, 1, 1, \dots)$, $\beta = \gamma$, $h = 1$ and the sequence of scalar $\lambda = (\lambda_k)$ be

$$\lambda_k = \begin{cases} \frac{1}{2}, & k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $|\lambda_k| < 1$, for all $k \in \mathbb{N}_0$. Suppose that, $x_k = c$ is any constant sequence, which gives us $x \in w(\Delta_{1,-}^{\alpha,\beta,\beta}, p)$. But,

$$y_k = \lambda_k x_k = \begin{cases} \frac{c}{2}, & k \text{ is even,} \\ 0, & k \text{ is odd,} \end{cases} \quad \text{and} \quad \Delta_{1,-}^{\alpha,\beta,\beta} y_k = \begin{cases} c 2^{\alpha-2}, & k \text{ is even,} \\ -c 2^{\alpha-2}, & k \text{ is odd,} \end{cases}$$

i.e., we can not get an unique $l \in \mathbb{C}$ such that $(y_k) \notin w(\Delta_{1,-}^{\alpha,\beta,\beta}, p)$. $\square$

Theorem 2.6. *The space $w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ does not form a sequence algebra, in general.*

Proof. Let $x = (x_k)$ and $y = (y_k)$ be two sequences such that, $x_k = k$ and $y_k = k^2$, for all $k \in \mathbb{N}_0$ with $(p_k) = e$. Clearly, $x = (x_k), y = (y_k) \in w_\infty(\Delta_{1,-}^{2,\beta,\beta}, p)$. But, for $z_k = x_k y_k = k^3$, for all $k \in \mathbb{N}_0$ and $\Delta_{1,-}^{2,\beta,\beta} z_k = 6k - 2$.

Again, $\sup_n \left(\frac{1}{n+1} \sum_{k=0}^n (6k - 2) \right) = +\infty$, which consequently, gives us $z = (z_k) \notin w_\infty(\Delta_{1,-}^{2,\beta,\beta}, p)$ and completes the proof. $\square$

Theorem 2.7. (a) If $\alpha + \beta > \gamma$, then $c \subset w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$.

(b) If $\Delta_{h,-}^{\alpha,\beta,\gamma}(e)$ exists, then $\ell_\infty \subset w_\infty(\Delta_h^{\alpha,\beta,\gamma}, p)$.

(c) $w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) \subset w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) \subset w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$.

(d) $c(\Delta_{h,-}^{\alpha,\beta,\gamma}) \subset w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$, for $p_k < M$, for all $k \in \mathbb{N}_0$, where $c(\Delta_{h,-}^{\alpha,\beta,\gamma}) = \{x \in w : (\Delta_{1,-}^{\alpha,\beta,\gamma} x_k) \in c\}$.

Proof. (a) Let $x \in c$ be any convergent sequence, with $\alpha + \beta > \gamma$. This implies, $(\Delta_{h,-}^{\alpha,\beta,\gamma} x_k)$ is convergent. Ultimately, we obtain

$$\lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k - l|^{p_k} = 0, \quad \text{for some } l \in \mathbb{C},$$

i.e.,

$$x \in w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) \implies c \subset w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p).$$

Furthermore, this inclusion is strict by the following example.

Let $x_k = k^a$, for all $k \in \mathbb{N}_0$, $a > 1$, being a fixed real number. If $\alpha + \beta - \gamma > a$, then $x = (x_k) \in w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$. But, (x_k) is a divergent sequence. The proofs of (b) and (c) are similar as above.

(d) Let $x \in c(\Delta_{h,-}^{\alpha,\beta,\gamma})$, which implies $|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k - l| < \epsilon^{\frac{1}{M}} < \epsilon$, for some $l \in \mathbb{C}$.

Clearly,

$$\lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k - l|^{p_k} < (\epsilon^{\frac{1}{M}})^M = \epsilon,$$

i.e., $x \in w(\Delta_h^{\alpha,\beta,\gamma}, p)$.

This inclusion being strict by taking $(x_k) = (1, 3, 1, 3, 1, \dots)$ (see [14, Theorem 3.3]), such that $w(\Delta_{1,-}^{1,\beta,\beta}, e) \not\subset c(\Delta_{1,-}^{1,\beta,\beta})$.

This completes the proof. $\square$

We obtain our next results by following to Maddox [22]. Hence consider the set $w_0(\Delta_h^{\alpha,\beta,\gamma}, p)$ denoting the set such that,

$$(2.2) \quad 2^{-r} \left| \sum_r \Delta_{h,-}^{\alpha,\beta,\gamma} x_k \right|^{p_k} \rightarrow 0,$$

where $\sum_r$ is the sum over the k for $k \in [2^r, 2^{r+1})$ and r being any integer.

Theorem 2.8. ([22, Theorem 7]) $w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) \subset w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, e)$ if and only if

(a) there exists an integer $N > 1$ such that

$$B_r = \max_r M^{-\frac{1}{p_k}} 2^{-r+\frac{r}{p_k}} = \mathcal{O}(1),$$

(b) $\inf_{s>1} \limsup_{r \rightarrow \infty} 2^{-r} M_r(s) = 0$, where $M_r(s)$ is the number of k in $[2^r, 2^{r+1})$, such that $p_k \geq s$.

Proof. Let $x \in w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$, i.e., there exists $N \in \mathbb{N}_0$ for sufficiently large r , such that,

$$\frac{1}{2^r} \sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} < \frac{1}{M}.$$

Now, we may subdivide the sum as

$$2^{-r} \sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = 2^{-r} \sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} + 2^{-r} \sum_2 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k},$$

representing the sum $\sum_1$ and $\sum_2$ over $p_k < 1$ and $p_k \geq 1$. Let us consider $p_k < 1$ and $q_k = \frac{1}{p_k}$. We get the following inequalities

$$M^q |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} 2^{-rq} \leq M |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} 2^{-r} \leq M 2^{-r} \sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k}.$$

Again,

$$2^{-r} \sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \leq 2^{-r} M^{-q_k} \sum_1 2^{rq_k} M^{q_k} |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|.$$

This implies,

$$\begin{aligned} |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| &\leq M^{1-q_k} |x_k|^{p_k} 2^{-r+rq}, \\ 2^{-r} \sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| &\leq 2^{-2r} M \sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} 2^{rq_k} M^{-q_k} \\ &\leq 2^{-r} M B_r \sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = \mathcal{O}(1) \cdot o(1) = o(1), \end{aligned}$$

where the notation $\mathcal{O}$ is defined earlier and $o(f_2(y)) = f_1(y)$, for every positive constant M and there exists a constant y_0 , such that $0 \leq f_1(y) < M f_2(y)$, for all $y \geq y_0$.

Our claim is now reduced to prove that, $2^{-r} \sum_2 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| = o(1)$.

Let $\epsilon > 0$ be arbitrary. Then, there exists $s > 1$ such that $2^{-r} N_r(s) < \epsilon$, for all sufficiently large r . We may write the sum as

$$\sum_2 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| = \sum_3 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} + \sum_4 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|,$$

over the sum $|\Delta_{h,-}^{\alpha,\beta,\gamma}| \leq 1$ and $|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| > 1$, respectively. Again, $p_k \geq 1$, resulting

$$2^{-r} \sum_4 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| = 2^{-r} \sum_4 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = o(1).$$

Let us subdivide the sum in two parts, i.e., $1 \leq p_k \leq s$ and $p_k > 1$, respectively, as $\sum_5, \sum_6$.

Moreover, we found

$$\sum_6 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \leq \sum_6 1 = M_r(s).$$

This implies $2^{-r} \sum_6 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| < \epsilon$, for sufficiently large r . Now, only we want to show it for $\sum_5$.

Letting r so large that

$$2^{-r} \sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} < \min\{\epsilon, \epsilon^s\}.$$

From [22] we get that for sufficiently large r that,

$$2^{-r} \sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \leq 2^{-r} \sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} + \left(2^{-r} \sum_6 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} \right)^{\frac{1}{s}} < 2\epsilon.$$

So, all the inequalities obtained above ultimately resulting our claim that

$$2^{-r} \sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \rightarrow 0, \quad \text{as } r \rightarrow +\infty,$$

i.e.,

$$x \in w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, e) \implies w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) \subset w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, e).$$

This completes the proof. $\square$

3. DUAL SPACES

Suppose X and Y are two non-empty sequence spaces, then the set

$$S(X, Y) = \left\{ z \in w : xz = (x_k z_k) \in Y, \text{ for every } x \in X \right\},$$

is called the multiplier space of X and Y . The special multiplier space that are $S(X, l_1)$, $S(X, cs)$ and $S(X, bs)$, respectively, called as α -, β - and γ -duals of X (cf. [12]). In order to remove ambiguity, let us take $X^{\alpha*} = S(X, l_1)$, $X^{\beta*} = S(X, cs)$, $X^{\gamma*} = S(X, bs)$.

Theorem 3.1. (a) $\{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*} = D_1$, where

$$D_1 = \bigcap_{M>1} \left\{ a = (a_n) \in w : \sup_k \sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{\frac{1}{p_k}} \right| < +\infty \right\}.$$

(b) $\{w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*} = D_2$, where

$$D_2 = \bigcup_{M>1} \left\{ a = (a_n) \in w : \sup_k \sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{-\frac{1}{p_k}} \right| < +\infty \right\}.$$

(c) $\{w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*} = D_3$, where

$$D_3 = D_2 \cap \bigcup_{M>1} \left\{ a = (a_n) \in w : \sup_k \sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n \right| < +\infty \right\}.$$

Proof. The proof follows from Theorem 8 of [8]. Suppose $a \in D_1$ and $x \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$. Now,

$$\begin{aligned} \sum_n |a_n x_n| &= \sum_n |a_n| \cdot |x_n| \\ &= \sum_n |a_n| M^{\frac{1}{p_k}} M^{-\frac{1}{p_k}} \sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) \right| \\ &\quad \times \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) \right|^{-1} |x_n| \end{aligned}$$

$$\leq \sup_n \left| \frac{1}{k+1} \sum_{j=0}^{k-n} \frac{(-\alpha)_j}{j!} \right| \cdot \left| M^{-\frac{1}{p_k}} |x_n| \sum_n \left| \sum_k \right| (-1)^{n-k} \right. \\ \left. \times \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n \right| M^{\frac{1}{p_k}} \Big| < +\infty, \quad \text{for all } k \in \mathbb{N}_0.$$

So,

$$a \in \{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*} \implies D_1 \subset \{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*}.$$

Conversely, suppose $a \notin D_1$, Then, exists $M > 1$ such that,

$$\sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{\frac{1}{p_k}} \right| = +\infty.$$

Thus, we can find strictly increasing sequence of integer $n(s)$ such that,

$$n(1) = 1 < n(2) < n(3) < \dots < n(s) < n(s+1) \\ < \dots < \sum_{n=n(s)}^{n(s+1)} \sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) M^{-\frac{1}{p_k}} \right| > 1.$$

Now, let us define a sequence $x = (x_n)$ by

$$x_n = \begin{cases} 0, & \text{if } 1 < n < n(s), \\ M^{-\frac{1}{p_k}} \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) \operatorname{sgn}(a_n) \right|, & n(s) < n < n(s+1). \end{cases}$$

Clearly, $x = (x_n) \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$.

Moreover,

$$\sum_{n=0}^{+\infty} |a_n x_n| = \sum_{n=n(1)}^{n(2)} |a_n y_n| + \sum_{n=n(2)}^{n(3)} |a_n y_n| + \dots \\ = \sum_{n=n(1)}^{n(2)} |a_n| \cdot \left| M^{-\frac{1}{p_k}} \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) \right| \\ + \sum_{n=n(2)}^{n(3)} |a_n| \cdot \left| M^{-\frac{1}{p_k}} \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) \right| + \dots \\ \geq \sum_{n=1}^{+\infty} 1 > +\infty.$$

i.e., $a = (a_n) \notin \{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*}$. By contra positively, we get $w_\infty\{(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*} \subset D_1$, which consequently results $\{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha*} = D_1$.

So, it completes the proof. Similarly, we may prove the other two results. $\square$

Theorem 3.2. (a) $\{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta*} = D_4$, where

$$D_4 = \bigcap_{M>1} \left\{ a = (a_n) \in w : \lim_{n \rightarrow +\infty} \sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{\frac{1}{p_k}} \right| \text{ converges} \right\}.$$

(b) $\{w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta*} = D_5$, where

$$D_5 = \bigcup_{M>1} \left\{ a = (a_n) \in w : \sup_n \sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{-\frac{1}{p_k}} \right| < \infty \right\}.$$

(c) $\{w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta*} = D_6 \cap D_7$, where

$$D_6 = \bigcup_{M>1} \left\{ a = (a_n) \in w : \sup_n \sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{-\frac{1}{p_k}} - b_k \right| = 0, \text{ for all } b_k \in \mathbb{C} \right\}$$

and

$$D_7 = \left\{ a = (a_n) \in w : \lim_{n \rightarrow +\infty} \sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n - b_k \right| = 0, \text{ for all } b_k \in \mathbb{C} \right\}.$$

Proof. On contrary, let $a = (a_n)$ be a sequence such that $a \notin \{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta*}$. Then, there exists $x = (x_n) \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$ such that $\sum_n a_n x_n$ does not converge.

Consequently, this leads that $\sum_n |a_n x_n|$ diverges to ∞ . Furthermore,

$$\begin{aligned} \sum_n |a_n x_n| &= \sum_n \left| a_n M^{\frac{1}{p_k}} M^{-\frac{1}{p_k}} \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) \frac{1}{k+1} \sum_{j=0}^{k-n} \frac{(-\alpha)_j}{j!} x_n \right| \\ &\leq \sup_n \sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{\frac{1}{p_k}} \right. \\ &\quad \times \left. \left| \frac{1}{k+1} \sum_{j=0}^{k-n} \frac{(-\alpha)_j}{j!} M^{-\frac{1}{p_k}} x_n \right| \right| \rightarrow \infty, \quad \text{for larger } n. \end{aligned}$$

Since, $x \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$, implies

$$\sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_n M^{\frac{1}{p_k}} \right| \rightarrow +\infty,$$

which is a contradiction to the fact that $a \in D_4$. So, we get $a \notin \{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta*}$ and $D_4 \subset \{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta*}$.

Conversely, suppose that $a \in \{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta*}$. By definition $\sum_n a_n x_n$ converges, which necessarily gives $\lim_{n \rightarrow +\infty} a_n x_n = 0$. So, for each $\epsilon > 0$, there exists $N \in \mathbb{N}_0$ such that $|a_n x_n| < \epsilon$, for all $n > N$. Again,

$$|a_n x_n| = \left| a_n \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) M^{\frac{1}{p_k}} M^{-\frac{1}{p_k}} \frac{1}{k+1} \sum_{j=0}^{k-n} \frac{(-\alpha)_{k-n}}{(k-n)!} x_n \right|$$

$$\leq \sup_{n,k > N} \left| \frac{1}{k+1} \sum_{j=0}^{k-n} \frac{(-\alpha)_j}{j!} M^{-\frac{1}{p_k}} x_n \right| \cdot \left| \sum_k (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) \right. \\ \left. \times M^{\frac{1}{p_k}} a_n \right| < \epsilon,$$

for some $M > 1$. For $x \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$, it brings

$$\sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) M^{\frac{1}{p_k}} a_n \right| < \epsilon,$$

i.e., $a \in D_4$, which implies $\{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\beta^*} = D_4$.

Otherwise, for any larger $k \in \mathbb{N}$ by Theorem 5.1 in [18], there exists $b = (b_k)$ be any sequence of scalar in $\mathbb{C}$, such that

$$\lim_{n \rightarrow +\infty} \sum_k \left| (-1)^{n-k} \prod_{l=k}^n (l+1) \bar{D}_{n-k}^k (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) M^{\frac{1}{p_k}} a_n - b_k \right| = 0,$$

i.e., converges. So, it completes the proof. Similarly, we may obtain the other two results. $\square$

4. MATRIX TRANSFORMATIONS

In this section, we characterize some matrix transformations among newly constructed sequence spaces as defined earlier. We characterize matrix transformations among spaces $w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$, $w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$ and $w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$ with classical sequence spaces $c(q)$, $c_0(q)$ and $\ell_\infty(q)$.

In brief, for A , be an infinite matrix from X to Y , i.e., $A : X \rightarrow Y$, where for all $x = (x_n) \in X$, implies that

$$\{(Ax)_n\} = \left\{ \sum_{k \in \mathbb{N}_0} a_{nk} x_k \right\}_{n \in \mathbb{N}_0} \in Y.$$

Suppose that (X, Y) denote the set of all matrix transformations from X to Y . Before proceeding further. We need the following propositions from [18], for $(q_k)_{k \in \mathbb{N}_0}$, non decreasing bounded sequence of positive real numbers. For more convenience, we can replace the $\sum_{k=0}^\infty$ by $\sum_k$.

(a) $A \in (\ell_\infty(p), \ell_\infty(q))$ if and only if for all M , such that

$$(4.1) \quad \sup_n \left(\sum_k |a_{nk}| M^{\frac{1}{p_k}} \right)^{q_n} < +\infty.$$

(b) $A \in (\ell_\infty(p), c(q))$ if and only if exists η_k , such that

$$(4.2) \quad \sup_n \sum_k |a_{nk}| M^{\frac{1}{p_k}} < +\infty, \quad \text{for all integers } M > 1,$$

and

$$(4.3) \quad \lim_n \left(\sum_k |a_{nk} - \eta_k| M^{-\frac{1}{p_k}} \right)^{q_n} = 0, \quad \text{for all } M > 1.$$

(c) $A \in (c_0(p), \ell_\infty(q))$ if and only if exists M , such that

$$(4.4) \quad \sup_n \left(\sum_k |a_{nk}| M^{-\frac{1}{p_k}} \right)^{q_n} < +\infty.$$

(d) $A \in (c_0(p), c(q))$ if and only if (4.4) holds for all L exist M , and $\eta_k \in \mathbb{R}$ such that

$$(4.5) \quad \limsup_M \sup_n \sum_k |a_{nk}| M^{-\frac{1}{p_k}} < +\infty,$$

$$(4.6) \quad \sup_n L^{\frac{1}{q_n}} \sum_k |a_{nk} - \eta_k| M^{-\frac{1}{p_k}} < +\infty,$$

$$(4.7) \quad \lim_{n \rightarrow +\infty} |a_{nk} - \eta_k|^{q_n} = 0.$$

(e) $A \in (c_0(p), c_0(q))$ if and only if for all L , exists M , such that

$$(4.8) \quad \sup_n L^{\frac{1}{q_n}} \sum_k |a_{nk}| M^{-\frac{1}{p_k}} < +\infty.$$

(f) $A \in (c(p), \ell_\infty(q))$ if and only if (4.2) holds

$$(4.9) \quad \sup_n \left| \sum_k a_{nk} \right|^{q_n} < +\infty.$$

(g) $A \in (\ell_\infty(p), c_0(q))$ if and only if for all M ,

$$(4.10) \quad \lim_n \left(\sum_k |a_{nk} - \eta_k| M^{-\frac{1}{p_k}} \right)^{q_n} = 0 < +\infty.$$

(h) $A \in (c(p), c(q))$ if and only if (4.2), (4.5), (4.6) hold, exists $\alpha \in \mathbb{R}$, such that

$$(4.11) \quad \lim_n \left| \sum_k a_{nk} - \eta \right|^{q_n} = 0.$$

(i) $A \in (c_0(p), \ell(q))$ if and only if exists $M \in \mathbb{N}$

$$(4.12) \quad \sup_k \sum_n \left| \sum_{k \in K} a_{nk} M^{-\frac{1}{p_k}} \right|^{q_n} < +\infty.$$

(j) $A \in (c(p), \ell(q))$ if and only if (4.11) holds and

$$(4.13) \quad \sum_n \left| \sum_k a_{nk} \right|^{q_n} < +\infty.$$

Consider the infinite matrix $\hat{A} = (\hat{a}_{nk})$ via the matrix $A = (a_{nk})$ as

$$\hat{a}_{nk} = \sum_{j=k}^{+\infty} (-1)^{k-i} \prod_{l=i}^k (l+1) \bar{D}_{k-i}^i (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}) a_{nj},$$

where $\bar{D}_{k-i}^i (C_1 \Delta_{1,-}^{\alpha,\beta,\gamma})$ is defined as in equation (1.5) and $\hat{A}$ is called the associated matrix of A .

Theorem 4.1. (a) $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$ if and only if (4.1) holds, with $\hat{a}_{nk}$ instead of a_{nk} .

(b) $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c(q))$ if and only if (4.2) and (4.3) hold, with $\hat{a}_{nk}$ instead of a_{nk} .

(c) $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c_0(q))$ if and only if (4.9) holds and (4.2), (4.3) hold, with $\hat{a}_{nk}$ instead of a_{nk} , with $\eta_k = 0$.

Theorem 4.2. (a) $A \in (w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$ if and only if (4.4), (4.8) hold, with $\hat{a}_{nk}$ instead of a_{nk} .

(b) $A \in (w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c(q))$ if and only if (4.5), (4.6) and (4.11) hold, with $\hat{a}_{nk}$ instead of a_{nk} .

(c) $A \in (w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c_0(q))$ if and only if (4.6), (4.7) hold, with $\hat{a}_{nk}$ instead of a_{nk} with $\eta_k = 0$.

Theorem 4.3. (a) $A \in (w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$ if and only if (4.4) holds, with $\hat{a}_{nk}$ instead of a_{nk} .

(b) $A \in (w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c(q))$ if and only if (4.5), (4.6), (4.7) hold, with $\hat{a}_{nk}$ instead of a_{nk} .

(c) $A \in (w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c_0(q))$ if and only if (4.1), (4.3) holds and (4.6), (4.7) hold, with $\hat{a}_{nk}$ instead of a_{nk} with $\eta_k = 0$.

Proof. Suppose that $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$. Then, by the definition $\sum_{k=0}^{+\infty} a_{nk}y_k \in \ell_\infty(q)$, for every $y = (y_k) \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$. Again, $y = (y_k) \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)$, if and only if $\bar{y} = C_1\Delta_{1,-}^{\alpha,\beta,\gamma}(y) \in \ell_\infty(p)$. From Lemma 4.1 of [28], we get $\sum_{k=0}^{+\infty} \hat{a}_{nk}\bar{y}_k \in \ell_\infty(q)$, and the matrix $\hat{A} \in (\ell_\infty(p), \ell_\infty(q))$. Using the equation (4.1), $\hat{A} \in (\ell_\infty(p), \ell_\infty(q))$ if and only if $\sup_n \left(\sum_k |\hat{a}_{nk}| M^{\frac{1}{p_k}} \right)^{q_n} < +\infty$. This completes the proof of the first bit. We can prove the remaining parts of the theorem and the next theorems using similar argument. $\square$

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SOME CLASSES OF SIMULTANEOUS COSPECTRAL GRAPHS FOR ADJACENCY, LAPLACIAN AND NORMALIZED LAPLACIAN MATRICES

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ABSTRACT. In this paper we construct several classes of non-regular graphs which are co-spectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian, and hence we answer a question asked by S. Butler. We make these constructions starting with two pairs (G_1, H_1) and (G_2, H_2) of A -cospectral regular graphs, taking their R -graph $\mathcal{R}(G_i)$, $\mathcal{R}(H_i)$, $i = 1, 2$, and finally making some kind of partial joins between $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$; and $\mathcal{R}(H_1)$ and $\mathcal{R}(H_2)$. Moreover, we determine the number of spanning trees and the Kirchhoff index of the newly constructed graphs.

1. INTRODUCTION

In recent years, construction of cospectral graphs for different matrices is one of the interesting research problem in the area of spectral graph theory. Here we construct some graphs which give an answer to the question “Is there an example of two non-regular graphs which are cospectral with respect to the adjacency, combinatorial Laplacian and normalized Laplacian at the same time?” asked by Butler [2]. To present the results of the paper we need some definitions and terminology as follow. All graphs considered in the paper are simple and undirected. For any graph G , we take $V(G)$ and $E(G)$ as the vertex set and edge set of G , respectively. The *adjacency matrix* of graph G , denoted by $A(G)$, is a square matrix whose rows and columns are indexed by vertices of graph G , and the $(u, v)^{\text{th}}$ entry is 1 if and only if vertex u is adjacent to vertex v and 0 otherwise. If $D(G)$ is the diagonal matrix of vertex degrees in G , then

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the *Laplacian matrix* $L(G)$ is defined as $L(G) = D(G) - A(G)$ and the *normalized Laplacian matrix* $\mathcal{L}(G)$ of G is defined as $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$ with the convention that $D(G)^{-1}(u, u) = 0$ if degree of u is zero. For a given square matrix M of size n , we denote the characteristic polynomial $\det(xI_n - M)$ by $f_M(x)$. The eigenvalues of $A(G)$, $L(G)$ and $\mathcal{L}(G)$ are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, and $0 = \delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G) \leq 2$, respectively, where n is the number of vertices of G . The multiset of eigenvalues of $A(G)$ (respectively, $L(G)$, $\mathcal{L}(G)$) is called the *adjacency* (respectively, *Laplacian*, *normalized Laplacian*) *spectrum* of G , and denoted by A -spectrum (respectively, L -spectrum, $\mathcal{L}$ -spectrum). Two graphs are said to be A -cospectral (respectively, L -cospectral, $\mathcal{L}$ -cospectral) if they have the same A -spectrum (respectively, L -spectrum, $\mathcal{L}$ -spectrum).

The R -graph $\mathcal{R}(G)$ [6] of a graph G is the graph obtained from G by introducing a new vertex u_e for each $e \in E(G)$ and making u_e adjacent to both the end vertices of e . The set of such new vertices is denoted by $I(G)$, i.e., $I(G) = V(\mathcal{R}(G)) \setminus V(G)$. The partial joins of R -graphs which are considered in the paper are given in the definition below.

Definition 1.1. Let G_1 and G_2 be two vertex-disjoint graphs with number of vertices n_1 and n_2 , and edges m_1 and m_2 , respectively. Then the following hold.

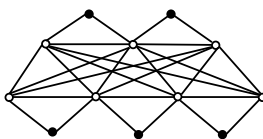
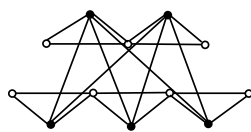
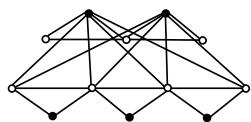
- (i) The R -vertex-vertex join of G_1 and G_2 , denoted by $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$, is the graph obtained from $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$. The graph $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $3m_1 + n_1n_2 + 3m_2$ edges.
- (ii) The R -edge-edge join of G_1 and G_2 , denoted by $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$, is the graph obtained from $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $I(G_1)$ with every vertex of $I(G_2)$. The graph $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1(3 + m_2) + 3m_2$ edges.
- (iii) The R -edge-vertex join of G_1 and G_2 , denoted by $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$, is the graph obtained from $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$. The graph $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1(3 + n_2) + 3m_2$ edges. (We note that R -vertex-edge join of G_1 and G_2 is isomorphic to the R -edge-vertex join of G_2 and G_1 .)

Example 1.1. Let us consider two graphs $G_1 = P_3$ and $G_2 = P_4$ (see Figures 1, 2 and 3). The set of dark vertices of G_1 and G_2 are $I(G_1)$ and $I(G_2)$, respectively.

In the following lemma we find the degrees of vertices in the above constructed graphs.

Lemma 1.1. (i) The degree of any vertex v in $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$ is given by

$$d_{\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)}(v) = \begin{cases} n_2 + 2d_{G_1}(v), & \text{if } v \in V(G_1), \\ 2, & \text{if } v \in I(G_1) \cup I(G_2), \\ n_1 + 2d_{G_2}(v), & \text{if } v \in V(G_2). \end{cases}$$

FIGURE 1. R -vertex-vertex join of P_3 and P_4 FIGURE 2. R -edge-edge join of P_3 and P_4 FIGURE 3. R -edge-vertex join of P_3 and P_4

(ii) The degree of any vertex v in $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ is given by

$$d_{\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)}(v) = \begin{cases} 2d_{G_1}(v), & \text{if } v \in V(G_1), \\ 2 + m_2, & \text{if } v \in I(G_1), \\ 2d_{G_2}(v), & \text{if } v \in V(G_2), \\ 2 + m_1, & \text{if } v \in I(G_2). \end{cases}$$

(iii) The degree of any vertex v in $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ is given by

$$d_{\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)}(v) = \begin{cases} 2d_{G_1}(v), & \text{if } v \in V(G_1), \\ 2 + n_2, & \text{if } v \in I(G_1), \\ 2d_{G_2}(v) + m_1 & \text{if } v \in V(G_2), \\ 2, & \text{if } v \in I(G_2). \end{cases}$$

For two matrices A and B , of same size $m \times n$, the *Hadamard product* $A \bullet B$ of A and B is a matrix of the same size $m \times n$ with entries given by $(A \bullet B)_{ij} = (A)_{ij} \cdot (B)_{ij}$ (that is entrywise multiplication). Hadamard product is commutative, that is $A \bullet B = B \bullet A$.

Notation. Throughout the paper, for any positive integers k , n_1 and n_2 , I_k denotes the identity matrix of size k , $J_{n_1 \times n_2}$ denotes $n_1 \times n_2$ matrix whose all entries are 1, $\mathbf{1}_n$ stands for the column vector of size n with all entries equal to 1, $K_{n \times n}$ denotes an $n \times n$ matrix whose all entries are the same. In other words, $K_{n \times n} = \alpha J_{n \times n}$, for a real number α . For any positive integers s and t , $O_{s \times t}$ denotes the zero matrix of size $s \times t$.

To prove our results we need some basics as given below.

Lemma 1.2 (Schur complement [7]). *Suppose that the order of all four matrices M , N , P and Q satisfy the rules of operations on matrices. Then we have*

$$\begin{aligned} \begin{vmatrix} M & N \\ P & Q \end{vmatrix} &= |Q| \cdot |M - NQ^{-1}P| \quad (\text{if } Q \text{ is a non-singular square matrix}) \\ &= |M| \cdot |Q - PM^{-1}N| \quad (\text{if } M \text{ is a non-singular square matrix}). \end{aligned}$$

Lemma 1.3 ([7]). *For a square matrix A of size n and a scalar α , $\det(A + \alpha J_{n \times n}) = \det(A) + \alpha \mathbf{1}_n^T \text{adj}(A) \mathbf{1}_n$, where $\text{adj}(A)$ is the adjugate matrix of A .*

Lemma 1.4. *For any real numbers $c, d > 0$, we have*

$$(cI_n - dJ_{n \times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c - nd)}J_{n \times n}.$$

Proof. We have

$$\begin{aligned} (cI_n - dJ_{n \times n})^{-1} &= \frac{\text{adj}(cI_n - dJ_{n \times n})}{\det(cI_n - dJ_{n \times n})} = \frac{c^{n-2}(c - nd)I_n + c^{n-2}dJ_{n \times n}}{c^{n-1}(c - nd)} \\ &= \frac{1}{c}I_n + \frac{d}{c(c - nd)}J_{n \times n}. \quad \square \end{aligned}$$

For a graph G on n vertices and m edges, the *vertex-edge incidence matrix* $R(G)$ of G is a matrix of size $n \times m$, with entry $r_{ij} = 1$ if the i^{th} vertex is incident to the j^{th} edge, and 0 otherwise. In particular, if G is an r -regular graph then $R(G)R(G)^T = A(G) + rI_n = 2rI_n - L(G) = r(2I_n - \mathcal{L}(G))$.

Notation. The M -coronal of an $n \times n$ matrix M , denoted by $\Gamma_M(x)$, is defined [5,10] as the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is, $\Gamma_M(x) = \mathbf{1}_n^T(xI_n - M)^{-1}\mathbf{1}_n$.

Lemma 1.5 ([5]). *If M is an $n \times n$ matrix with each row sum equal to a constant t , then $\Gamma_M(x) = \frac{n}{x-t}$.*

Butler [2] constructed non-regular bipartite graphs which are cospectral with respect to both the adjacency and normalized Laplacian matrices, and then asked for existence of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian. In this paper we construct several classes of such graphs taking help of the operations R -vertex-vertex join, R -edge-edge join, and R -edge-vertex join. We also find the number of spanning trees and Kirchhoff index for all the partial join of R -graphs constructed here.

2. ADJACENCY, LAPLACIAN AND NORMALIZED LAPLACIAN SPECTRA OF THE GRAPHS

In this section we consider regular graphs G_i on n_i vertices, m_i edges, and with degree of regularity r_i , $i = 1, 2$. To obtain the required matrices we label the vertices of the graphs in the following way. Let $V(G_1) = \{v_1, \dots, v_{n_1}\}$, $I(G_1) = \{e_1, \dots, e_{m_1}\}$, $V(G_2) = \{u_1, \dots, u_{n_2}\}$, $I(G_2) = \{f_1, \dots, f_{m_2}\}$. Then, $V(G_1) \cup I(G_1) \cup V(G_2) \cup I(G_2)$ is a partition for all $V(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2))$, $V(\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2))$ and $V(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2))$.

Lemma 2.1. For $i = 1, 2$, let G_i be a graph with n_i vertices and m_i edges. Then, we have the following:

$$\begin{aligned}
 & \text{(i)} \\
 & A(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} A(G_1) & R(G_1) & J_{n_1 \times n_2} & O_{n_1 \times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1 \times n_2} & O_{m_1 \times m_2} \\ J_{n_2 \times n_1} & O_{n_2 \times m_1} & A(G_2) & R(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix}; \\
 & \text{(ii)} \\
 & A(\mathcal{R}(G_1)\overline{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} A(G_1) & R(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1 \times n_2} & J_{m_1 \times m_2} \\ O_{n_2 \times n_1} & O_{n_2 \times m_1} & A(G_2) & R(G_2) \\ O_{m_2 \times n_1} & J_{m_2 \times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix}; \\
 & \text{(iii)} \\
 & A(\mathcal{R}(G_1)\overline{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} A(G_1) & R(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ R(G_1)^T & O_{m_1} & J_{m_1 \times n_2} & O_{m_1 \times m_2} \\ O_{n_2 \times n_1} & J_{n_2 \times m_1} & A(G_2) & R(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix}.
 \end{aligned}$$

Theorem 2.1. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the adjacency spectrum of $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$ consists of:

- (i) two roots of the equation $x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)$, for every eigenvalue $\lambda_i(G_1)$, $i = 2, 3, \dots, n_1$, of $A(G_1)$;
- (ii) two roots of the equation $x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2)$, for every eigenvalue $\lambda_j(G_2)$, $j = 2, 3, \dots, n_2$, of $A(G_2)$,
- (iii) the eigenvalue 0 with multiplicity $m_1 + m_2 - n_1 - n_2$;
- (iv) four roots of the equation $x^4 - (r_1 + r_2)x^3 - (2r_1 + n_1n_2 + 2r_2 - r_1r_2)x^2 + 4r_1r_2x + 4r_1r_2 = 0$.

Proof. The adjacency characteristic polynomial of $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$ is

$$\begin{aligned}
 f_{A(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2))}(x) &= \det \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & -J_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1 \times n_2} & O_{m_1 \times m_2} \\ -J_{n_2 \times n_1} & O_{n_2 \times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix} \\
 &= x^{m_2} \det(S),
 \end{aligned}$$

where

$$\begin{aligned}
 S &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & -J_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1 \times n_2} \\ -J_{n_2 \times n_1} & O_{n_2 \times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\
 &\quad - \begin{pmatrix} O_{n_1 \times m_2} \\ O_{m_1 \times m_2} \\ -R(G_2) \end{pmatrix} \frac{1}{x} \begin{pmatrix} O_{m_2 \times n_1} & O_{m_2 \times m_1} & -R(G_2)^T \end{pmatrix}
 \end{aligned}$$

$$= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & -J_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1 \times n_2} \\ -J_{n_2 \times n_1} & O_{n_2 \times m_1} & xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T \end{pmatrix}.$$

Hence,

$$\begin{aligned} \det(S) &= \det\left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T\right) \det(W) \\ &= \prod_{j=1}^{n_2} \left\{ x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x} \right\} \det(W), \end{aligned}$$

where

$$\begin{aligned} W &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix} \\ &\quad - \begin{pmatrix} -J_{n_1 \times n_2} \\ O_{m_1 \times n_2} \end{pmatrix} \left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T \right)^{-1} \begin{pmatrix} -J_{n_2 \times n_1} & O_{n_2 \times m_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A(G_1) - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x)J_{n_1 \times n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{aligned} \det(W) &= x^{m_1} \det \left(xI_{n_1} - A(G_1) - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x)J_{n_1 \times n_1} - \frac{1}{x}R(G_1)R(G_1)^T \right) \\ &= x^{m_1} \left[\det \left(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T \right) \right. \\ &\quad \left. - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \mathbf{1}_{n_1}^T \operatorname{adj} \left(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T \right) \mathbf{1}_{n_1} \right] \\ &= x^{m_1} \det \left(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T \right) \\ &\quad \times \left[1 - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \mathbf{1}_{n_1}^T \left(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T \right)^{-1} \mathbf{1}_{n_1} \right] \\ &= x^{m_1} \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x} \right\} \\ &\quad \times \left[1 - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \Gamma_{A(G_1) + \frac{1}{x}R(G_1)R(G_1)^T}(x) \right] \\ &= x^{m_1} \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x} \right\} \left[1 - \frac{n_2}{x - r_2 - \frac{2r_2}{x}} \cdot \frac{n_1}{x - r_1 - \frac{2r_1}{x}} \right]. \end{aligned}$$

Therefore,

$$f_{A(\mathcal{R}(G_1))\check{\vee}\mathcal{R}(G_2)}(x) = x^{m_1} x^{m_2} \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x} \right\}$$

$$\begin{aligned}
& \times \prod_{j=1}^{n_2} \left\{ x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x} \right\} \\
& \times \left[1 - \frac{n_2}{x - r_2 - \frac{2r_2}{x}} \cdot \frac{n_1}{x - r_1 - \frac{2r_1}{x}} \right] \\
& = x^{m_1 - n_1} x^{m_2 - n_2} \prod_{i=2}^{n_1} \left\{ x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1) \right\} \\
& \times \prod_{j=2}^{n_2} \left\{ x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2) \right\} \left[x^4 - (r_1 + r_2)x^3 \right. \\
& \quad \left. - (2r_1 + n_1n_2 + 2r_2 - r_1r_2)x^2 + 4r_1r_2x + 4r_1r_2 \right],
\end{aligned}$$

and the result follows immediately. $\square$

Theorem 2.2. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the adjacency spectrum of $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ consists of:

- (i) two roots of the equation $x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)$, for every eigenvalue $\lambda_i(G_1)$, $i = 2, 3, \dots, n_1$, of $A(G_1)$;
- (ii) two roots of the equation $x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2)$, for every eigenvalue $\lambda_j(G_2)$, $j = 2, 3, \dots, n_2$, of $A(G_2)$;
- (iii) the eigenvalue 0 with multiplicity $m_1 + m_2 - n_1 - n_2$;
- (iv) four roots of the equation $x^4 - (r_1 + r_2)x^3 - (2r_1 + m_1m_2 + 2r_2 - r_1r_2)x^2 + (4r_1r_2 + m_1m_2r_1 + m_1m_2r_2)x + 4r_1r_2 - m_1m_2r_1r_2 = 0$.

Proof. The adjacency characteristic polynomial of $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ is

$$\begin{aligned}
f_{A(\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2))}(x) &= \det \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1 \times n_2} & -J_{m_1 \times m_2} \\ O_{n_2 \times n_1} & O_{n_2 \times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ O_{m_2 \times n_1} & -J_{m_2 \times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix} \\
&= x^{m_2} \det(S),
\end{aligned}$$

where

$$\begin{aligned}
S &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1 \times n_2} \\ O_{n_2 \times n_1} & O_{n_2 \times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\
&\quad - \begin{pmatrix} O_{n_1 \times m_1} \\ -J_{m_1 \times m_2} \\ -R(G_2) \end{pmatrix} \frac{1}{x} \begin{pmatrix} O_{m_2 \times n_1} & -J_{m_2 \times m_1} & -R(G_2)^T \end{pmatrix} \\
&= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} - \frac{m_2}{x} J_{m_1 \times m_1} & -\frac{1}{x} J_{m_1 \times m_2} R(G_2)^T \\ O_{n_2 \times n_1} & -\frac{1}{x} R(G_2) J_{m_2 \times m_1} & xI_{n_2} - A(G_2) - \frac{1}{x} R(G_2) R(G_2)^T \end{pmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}\det(S) &= \det \left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T \right) \det(W) \\ &= \prod_{j=1}^{n_2} \left\{ x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x} \right\} \det(W),\end{aligned}$$

where

$$\begin{aligned}W &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - \frac{m_2}{x}J_{m_1 \times m_1} \end{pmatrix} - \begin{pmatrix} O_{n_1 \times n_2} \\ -\frac{1}{x}J_{m_1 \times m_2}R(G_2)^T \end{pmatrix} \\ &\quad \times \left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T \right)^{-1} \begin{pmatrix} O_{n_2 \times n_1} & -\frac{1}{x}R(G_2)J_{m_2 \times m_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - \frac{m_2}{x}J_{m_1 \times m_1} - \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)}J_{m_1 \times m_1} \end{pmatrix}.\end{aligned}$$

Then,

$$\begin{aligned}&\det(W) \\ &= \det \left(xI_{m_1} - \frac{m_2}{x}J_{m_1 \times m_1} - \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)}J_{m_1 \times m_1} \right) \\ &\quad \times \det \left(xI_{n_1} - A(G_1) - R(G_1) \right. \\ &\quad \times \left. \left(xI_{m_1} - \frac{m_2}{x}J_{m_1 \times m_1} - \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)}J_{m_1 \times m_1} \right)^{-1} R(G_1)^T \right) \\ &= x^{m_1} \left\{ 1 - \left(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)} \right) \frac{m_1}{x} \right\} \det \left[xI_{n_1} - A(G_1) \right. \\ &\quad \left. - R(G_1) \left\{ \frac{1}{x}I_{m_1} + \frac{\left(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)} \right) J_{m_1 \times m_1}}{x(x - m_1(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)))} \right\} R(G_1)^T \right] \\ &= x^{m_1} \left\{ 1 - \left(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)} \right) \frac{m_1}{x} \right\} \\ &\quad \times \det \left(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T \right. \\ &\quad \left. - \frac{\left(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)} \right) R(G_1)J_{m_1 \times m_1}R(G_1)^T}{x(x - m_1(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)))} \right) \\ &= x^{m_1} \left\{ 1 - \left(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T(x)} \right) \frac{m_1}{x} \right\}\end{aligned}$$

$$\begin{aligned}
& \times \det \left(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T \right. \\
& \quad \left. - r_1^2 \frac{(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x))}{x(x - m_1(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)))} J_{n_1 \times n_1} \right) \\
& = x^{m_1} \left\{ 1 - \left(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x) \right) \frac{m_1}{x} \right\} \\
& \quad \times \det \left(xI_{n_1} - A(G_1) - \frac{1}{x}(r_1I_{n_1} + A(G_1)) \right) \\
& \quad \times \left[1 - \frac{r_1^2(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x))\Gamma_{A(G_1)+\frac{1}{x}R(G_1)R(G_1)^T}(x)}{x(x - m_1(\frac{m_2}{x} + \frac{r_2^2}{x^2}\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)))} \right] \\
& = x^{m_1} \left\{ 1 - \left(\frac{m_2}{x} + \frac{r_2^2 n_2}{x^2(x - r_2 - \frac{2r_2}{x})} \right) \frac{m_1}{x} \right\} \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{1}{x}(r_1 + \lambda_i(G_1)) \right\} \\
& \quad \times \left[1 - \frac{r_1^2 \left(\frac{m_2}{x} + \frac{r_2^2 n_2}{x^2(x - r_2 - \frac{2r_2}{x})} \right) n_1}{x \left\{ x - m_1 \left(\frac{m_2}{x} + \frac{r_2^2 n_2}{x^2(x - r_2 - \frac{2r_2}{x})} \right) \right\} (x - r_1 - \frac{2r_1}{x})} \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{A(\mathcal{R}(G_1)\overline{\vee}\mathcal{R}(G_2))}(x) & = x^{m_1} x^{m_2} \left\{ 1 - \left(\frac{m_2}{x} + \frac{r_2^2 n_2}{x^2(x - r_2 - \frac{2r_2}{x})} \right) \frac{m_1}{x} \right\} \\
& \quad \times \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x} \right\} \\
& \quad \times \prod_{j=1}^{n_2} \left\{ x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x} \right\} \\
& \quad \times \left[1 - \frac{r_1^2 \left(\frac{m_2}{x} + \frac{r_2^2 n_2}{x^2(x - r_2 - \frac{2r_2}{x})} \right) n_1}{x \left\{ x - m_1 \left(\frac{m_2}{x} + \frac{r_2^2 n_2}{x^2(x - r_2 - \frac{2r_2}{x})} \right) \right\} (x - r_1 - \frac{2r_1}{x})} \right] \\
& = x^{m_1-n_1} x^{m_2-n_2} \prod_{i=2}^{n_1} \{x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)\} \\
& \quad \times \prod_{j=2}^{n_2} \{x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2)\} \\
& \quad \times [x^4 - (r_1 + r_2)x^3 - (2r_1 + m_1m_2 + 2r_2 - r_1r_2)x^2 \\
& \quad + (4r_1r_2 + m_1m_2r_1 + m_1m_2r_2)x + 4r_1r_2 - m_1m_2r_1r_2],
\end{aligned}$$

and hence the result follows. $\square$

Theorem 2.3. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the adjacency spectrum of $\mathcal{R}(G_1) \bar{\vee} \mathcal{R}(G_2)$ consists of:

- (i) two roots of the equation $x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)$, for every eigenvalue $\lambda_i(G_1)$, $i = 2, 3, \dots, n_1$, of $A(G_1)$;
- (ii) two roots of the equation $x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2)$, for every eigenvalue $\lambda_j(G_2)$, $j = 2, 3, \dots, n_2$, of $A(G_2)$;
- (iii) the eigenvalue 0 with multiplicity $m_1 + m_2 - n_1 - n_2$;
- (iv) four roots of the equation $x^4 - (r_1 + r_2)x^3 - (2r_1 + m_1n_2 + 2r_2 - r_1r_2)x^2 + (4r_1r_2 + r_1m_1n_2)x + 4r_1r_2 = 0$.

Proof. The adjacency characteristic polynomial of $\mathcal{R}(G_1) \bar{\vee} \mathcal{R}(G_2)$ is

$$f_{A(\mathcal{R}(G_1) \bar{\vee} \mathcal{R}(G_2))}(x) = \det \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -R(G_1)^T & xI_{m_1} & -J_{m_1 \times n_2} & O_{m_1 \times m_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix} \\ = x^{m_2} \det(S),$$

where

$$S = \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} & -J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ - \begin{pmatrix} O_{n_1 \times m_2} \\ O_{m_1 \times m_2} \\ -R(G_2) \end{pmatrix} \frac{1}{x} \begin{pmatrix} O_{m_2 \times n_1} & O_{m_2 \times m_1} & -R(G_2)^T \end{pmatrix} \\ = \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} & -J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T \end{pmatrix}.$$

Hence,

$$\det(S) = \det\left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T\right) \det(W) \\ = \prod_{j=1}^{n_2} \left\{x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x}\right\} \det(W),$$

where

$$W = \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix} \\ - \begin{pmatrix} O_{n_1 \times n_2} \\ -J_{m_1 \times n_2} \end{pmatrix} \left(xI_{n_2} - A(G_2) - \frac{1}{x}R(G_2)R(G_2)^T\right)^{-1} \begin{pmatrix} O_{n_2 \times n_1} & -J_{n_2 \times m_1} \end{pmatrix} \\ = \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T} J_{m_1 \times m_1} \end{pmatrix}.$$

Then,

$$\begin{aligned}
\det(W) &= \det \left(xI_{m_1} - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T} J_{m_1 \times m_1} \right) \det \left(xI_{n_1} - A(G_1) \right. \\
&\quad \left. - R(G_1)(xI_{m_1} - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T} J_{m_1 \times m_1})^{-1} R(G_1)^T \right) \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \frac{m_1}{x} \right) \det \left[xI_{n_1} - A(G_1) \right. \\
&\quad \left. - R(G_1) \left\{ \frac{1}{x}I_{m_1} + \frac{\Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x)}{x(x - m_1\Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x))} J_{m_1 \times m_1} \right\} R(G_1)^T \right] \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \frac{m_1}{x} \right) \det \left(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T \right. \\
&\quad \left. - \frac{\Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x)}{x(x - m_1\Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x))} R(G_1)J_{m_1 \times m_1}R(G_1)^T \right) \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \frac{m_1}{x} \right) \det \left(xI_{n_1} - A(G_1) \right. \\
&\quad \left. - \frac{1}{x}R(G_1)R(G_1)^T - r_1^2 \frac{\Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x)}{x(x - m_1\Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x))} J_{n_1 \times n_1} \right) \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \frac{m_1}{x} \right) \\
&\quad \det \left(xI_{n_1} - A(G_1) - \frac{1}{x}(r_1 I_{n_1} + A(G_1)) \right) \\
&\quad \times \left[1 - \frac{r_1^2 \Gamma_{A(G_1) + \frac{1}{x}R(G_1)R(G_1)^T}(x) \Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x)}{x \left(x - m_1\Gamma_{A(G_2) + \frac{1}{x}R(G_2)R(G_2)^T}(x) \right)} \right] \\
&= x^{m_1} \left(1 - \frac{m_1 n_2}{x(x - r_2 - \frac{2r_2}{x})} \right) \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{1}{x}(r_1 + \lambda_i(G_1)) \right\} \\
&\quad \times \left[1 - \frac{r_1^2 n_1 n_2}{x(x - r_1 - \frac{2r_1}{x}) \left(x - \frac{m_1 n_2}{x - r_2 - \frac{2r_2}{x}} \right) (x - r_2 - \frac{2r_2}{x})} \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{A(\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2))}(x) &= x^{m_1} x^{m_2} \left(1 - \frac{m_1 n_2}{x \left(x - r_2 - \frac{2r_2}{x} \right)} \right) \\
&\quad \times \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{r_1}{x} - \frac{\lambda_i(G_1)}{x} \right\} \\
&\quad \times \prod_{j=1}^{n_2} \left\{ x - \lambda_j(G_2) - \frac{r_2}{x} - \frac{\lambda_j(G_2)}{x} \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \left[1 - \frac{r_1^2 n_1 n_2}{x \left(x - r_1 - \frac{2r_1}{x} \right) \left(x - \frac{m_1 n_2}{x - r_2 - \frac{2r_2}{x}} \right) \left(x - r_2 - \frac{2r_2}{x} \right)} \right] \\
& = x^{m_1 - n_1} x^{m_2 - n_2} \prod_{i=2}^{n_1} \{x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)\} \\
& \quad \times \prod_{j=2}^{n_2} \{x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2)\} \left[x^4 - (r_1 + r_2)x^3 \right. \\
& \quad \left. - (2r_1 + m_1 n_2 + 2r_2 - r_1 r_2)x^2 + (4r_1 r_2 + r_1 m_1 n_2)x + 4r_1 r_2 \right],
\end{aligned}$$

and the result follows. $\square$

In the similar way we can obtain Laplacian and normalized Laplacian spectra of the partial join of R -graphs, which are given below.

Lemma 2.2. *We have the following Laplacian matrices:*

$$\begin{aligned}
& \text{(i)} \\
& L(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} (r_1 + n_2)I_{n_1} + L(G_1) & -R(G_1) & -J_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -R(G_1)^T & 2I_{m_1} & O_{m_1 \times n_2} & O_{m_1 \times m_2} \\ -J_{n_2 \times n_1} & O_{n_2 \times m_1} & (r_2 + n_1)I_{n_2} + L(G_2) & -R(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & -R(G_2)^T & 2I_{m_2} \end{pmatrix}; \\
& \text{(ii)} \\
& L(\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} r_1 I_{n_1} + L(G_1) & -R(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -R(G_1)^T & (2 + m_2)I_{m_1} & O_{m_1 \times n_2} & -J_{m_1 \times m_2} \\ O_{n_2 \times n_1} & O_{n_2 \times m_1} & r_2 I_{n_2} + L(G_2) & -R(G_2) \\ O_{m_2 \times n_1} & -J_{m_2 \times m_1} & -R(G_2)^T & (2 + m_1)I_{m_2} \end{pmatrix}; \\
& \text{(iii)} \\
& L(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} r_1 I_{n_1} + L(G_1) & -R(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -R(G_1)^T & (2 + n_2)I_{m_1} & -J_{m_1 \times n_2} & O_{m_1 \times m_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & (r_2 + m_1)I_{n_2} + L(G_2) & -R(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & -R(G_2)^T & 2I_{m_2} \end{pmatrix}.
\end{aligned}$$

Theorem 2.4. *For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the Laplacian spectrum of $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$ consists of:*

- (i) roots of the equation $x^2 - (2 + r_1 + n_2 + \mu_i(G_1))x + 2n_2 + 3\mu_i(G_1) = 0$, for every eigenvalue $\mu_i(G_1)$, $i = 2, 3, \dots, n_1$, of $L(G_1)$;
- (ii) roots of the equation $x^2 - (2 + r_2 + n_1 + \mu_j(G_2))x + 2n_1 + 3\mu_j(G_2) = 0$, for every eigenvalue $\mu_j(G_2)$, $j = 2, 3, \dots, n_2$, of $L(G_2)$;
- (iii) the eigenvalue 2 with multiplicity $m_1 + m_2 - n_1 - n_2$;
- (iv) four roots of the equation $x^4 - (4 + r_1 + r_2 + n_1 + n_2)x^3 + (4 + 4n_1 + 4n_2 + 2r_1 + 2r_2 + r_1 r_2 + r_1 n_1 + r_2 n_2)x^2 - 2(2n_1 + 2n_2 + r_1 n_1 + r_2 n_2)x = 0$.

Theorem 2.5. *For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the Laplacian spectrum of $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ consists of:*

- (i) roots of the equation $x^2 - (2 + r_1 + m_2 + \mu_i(G_1))x + r_1m_2 + 3\mu_i(G_1) + m_2\mu_i(G_1) = 0$, for every eigenvalue $\mu_i(G_1)$, $i = 2, 3, \dots, n_1$, of $L(G_1)$;
- (ii) roots of the equation $x^2 - (2 + r_2 + m_1 + \mu_j(G_2))x + r_2m_1 + 3\mu_j(G_2) + m_1\mu_j(G_2) = 0$, for every eigenvalue $\mu_j(G_2)$, $j = 2, 3, \dots, n_2$, of $L(G_2)$;
- (iii) the eigenvalue $2 + m_2$ with multiplicity $m_1 - n_1$;
- (iv) the eigenvalue $2 + m_1$ with multiplicity $m_2 - n_2$;
- (v) four roots of the equation $x^4 - (4 + r_1 + r_2 + m_1 + m_2)x^3 + (4 + 2r_1 + 2r_2 + r_1r_2 + r_1m_1 + r_2m_2 + 2m_1 + 2m_2 + r_1m_2 + r_2m_1)x^2 - (2r_1m_2 + 2r_2m_1 + r_1r_2m_1 + r_1r_2m_2)x = 0$.

Theorem 2.6. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the Laplacian spectrum of $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ consists of:

- (i) roots of the equation $x^2 - (2 + r_1 + n_2 + \mu_i(G_1))x + r_1n_2 + 3\mu_i(G_1) + n_2\mu_i(G_1) = 0$, for every eigenvalue $\mu_i(G_1)$, $i = 2, 3, \dots, n_1$, of $L(G_1)$;
- (ii) roots of the equation $x^2 - (2 + r_2 + m_1 + \mu_j(G_2))x + 2m_1 + 3\mu_j(G_2) = 0$, for every eigenvalue $\mu_j(G_2)$, $j = 2, 3, \dots, n_2$, of $L(G_2)$;
- (iii) the eigenvalue $2 + n_2$ with multiplicity $m_1 - n_1$;
- (iv) the eigenvalue 2 with multiplicity $m_2 - n_2$;
- (v) four roots of the equation $x^4 - (4 + r_1 + r_2 + m_1 + n_2)x^3 + (4 + 2r_1 + 2r_2 + 4m_1 + 2n_2 + r_1r_2 + r_1m_1 + r_1n_2 + r_2n_2)x^2 - (4m_1 + 2r_1m_1 + 2r_1n_2 + r_1r_2n_2)x = 0$.

Lemma 2.3. We have the following normalized Laplacian matrices:

(i)

$$\mathcal{L}(\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -cR(G_1) & -K_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -cR(G_1)^T & I_{m_1} & O_{m_1 \times n_2} & O_{m_1 \times m_2} \\ -K_{n_2 \times n_1} & O_{n_2 \times m_1} & \mathcal{L}(G_2) \bullet B(G_2) & -dR(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

where $K_{n_1 \times n_2}$ is the matrix of size $n_1 \times n_2$ with all entries equal to $\frac{1}{\sqrt{(2r_1+n_2)(2r_2+n_1)}}$, $B(G_1)$ is the $n_1 \times n_1$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_1}{2r_1+n_2}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{2r_2+n_1}$, c is the constant whose value is $\frac{1}{\sqrt{2(2r_1+n_2)}}$, d is the constant whose value is $\frac{1}{\sqrt{2(2r_2+n_1)}}$;

(ii)

$$\mathcal{L}(\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -cR(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -cR(G_1)^T & I_{m_1} & O_{m_1 \times n_2} & -K_{m_1 \times m_2} \\ O_{n_2 \times n_1} & O_{n_2 \times m_1} & \mathcal{L}(G_2) \bullet B(G_2) & -dR(G_2) \\ O_{m_2 \times n_1} & -K_{m_2 \times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

where $K_{m_1 \times m_2}$ is the matrix of size $m_1 \times m_2$ with all entries equal to $\frac{1}{\sqrt{(2+m_2)(2+m_1)}}$, $B(G_1)$ is the $n_1 \times n_1$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_1}{2r_1}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal

entries are $\frac{r_2}{2r_2}$, c is the constant whose value is $\frac{1}{\sqrt{2r_1(2+m_2)}}$, d is the constant whose value is $\frac{1}{\sqrt{2r_2(2+m_1)}}$;

(iii)

$$\mathcal{L}(\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)) = \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -cR(G_1) & O_{n_1 \times n_2} & O_{n_1 \times m_2} \\ -cR(G_1)^T & I_{m_1} & -K_{m_1 \times n_2} & O_{m_1 \times m_2} \\ O_{n_2 \times n_1} & -K_{n_2 \times m_1} & \mathcal{L}(G_2) \bullet B(G_2) & -dR(G_2) \\ O_{m_2 \times n_1} & O_{m_2 \times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

where $K_{m_1 \times n_2}$ is the matrix of size $m_1 \times n_2$ with all entries equal to $\frac{1}{\sqrt{(2+n_2)(2r_2+m_1)}}$, $B(G_1)$ is the $n_1 \times n_1$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_1}{2r_1}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{2r_2}$, c is the constant whose value is $\frac{1}{\sqrt{2r_1(2+n_2)}}$, d is the constant whose value is $\frac{1}{\sqrt{2(2r_2+m_1)}}$.

Theorem 2.7. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the normalized Laplacian spectrum of $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ consists of:

- (i) roots of the equation $2(2r_1+n_2)x^2 - 2(3r_1+2n_2+r_2\delta_i(G_1))x + 2n_2+3r_1\delta_i(G_1) = 0$, for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \dots, n_1$, of $\mathcal{L}(G_1)$;
- (ii) roots of the equation $2(2r_2+n_1)x^2 - 2(3r_2+2n_1+r_2\delta_j(G_2))x + 2n_1+3r_2\delta_j(G_2) = 0$, for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \dots, n_2$, of $\mathcal{L}(G_2)$;
- (iii) the eigenvalue 1 with multiplicity $m_1 + m_2 - n_1 - n_2$;
- (iv) four roots of the equation $(4r_1r_2 + 2r_1n_1 + 2r_2n_2 + n_1n_2)x^4 - (12r_1r_2 + 7r_1n_1 + 7r_2n_2 + 4n_1n_2)x^3 + (9r_1r_2 + 8r_1n_1 + 8r_2n_2 + 5n_1n_2)x^2 - (3r_1n_1 + 3r_2n_2 + 2n_1n_2)x = 0$.

Theorem 2.8. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the normalized Laplacian spectrum of $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ consists of:

- (i) roots of the equation $2(2+m_2)x^2 - (6+3m_2+2\delta_i(G_1)+m_2\delta_i(G_1))x + m_2+3\delta_i(G_1)+m_2\delta_i(G_1) = 0$, for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \dots, n_1$, of $\mathcal{L}(G_1)$;
- (ii) roots of the equation $2(2+m_1)x^2 - (6+3m_1+2\delta_j(G_2)+m_1\delta_j(G_2))x + m_1+3\delta_j(G_2)+m_1\delta_j(G_2) = 0$, for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \dots, n_2$, of $\mathcal{L}(G_2)$;
- (iii) the eigenvalue 1 with multiplicity $m_1 + m_2 - n_1 - n_2$;
- (iv) four roots of the equation $4(4+2m_1+2m_2+m_1m_2)x^4 - 12(4+2m_1+2m_2+m_1m_2)x^3 + (36+22m_1+22m_2+9m_1m_2)x^2 - 2(3m_1+3m_2+m_1m_2)x = 0$.

Theorem 2.9. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then, the normalized Laplacian spectrum of $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$ consists of:

- (i) roots of the equation $2(2+n_2)x^2 - (6+3n_2+2\delta_i(G_1)+n_2\delta_i(G_1))x + n_2+3\delta_i(G_1)+n_2\delta_i(G_1) = 0$, for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \dots, n_1$, of $\mathcal{L}(G_1)$;
- (ii) roots of the equation $2(2r_2+m_1)x^2 - 2(3r_2+2m_1+r_2\delta_j(G_2))x + 2m_1+3r_2\delta_j(G_2) = 0$, for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \dots, n_2$, of $\mathcal{L}(G_2)$;
- (iii) the eigenvalue 1 with multiplicity $m_1 + m_2 - n_1 - n_2$;

- (iv) *four roots of the equation $(8r_2 + 4r_2n_2 + 4m_1 + 2m_1n_2)x^4 - (24r_2 + 12r_2n_2 + 14m_1 + 7m_1n_2)x^3 + (18r_2 + 11r_2n_2 + 16m_1 + 7m_1n_2)x^2 - (3r_2n_2 + 6m_1 + 2m_1n_2)x = 0$.*

3. SIMULTANEOUS COSPECTRAL GRAPHS

In this section we present the main result of the paper. We construct several classes of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian. For the construction of these graphs we consider two pairs of A -cospectral regular graphs, which are readily available in the literature, for example see [11]. Then we take partial join of R -graphs belong to different pairs.

The following lemma is immediate from the definition of Laplacian and normalized Laplacian matrices.

Lemma 3.1. (i) *If G is an r -regular graph, then $L(G) = rI_n - A(G)$ and $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$.*

(ii) *If G_1 and G_2 are A -cospectral regular graphs, then they are also cospectral with respect to the Laplacian and normalized Laplacian matrices.*

Observation. From all the theorems given in the previous section we observe that the adjacency, Laplacian and normalized Laplacian spectra of all the partial join graphs $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$, $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$, and $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$, depend only on the number of vertices, number of edges, degree of regularities, and the corresponding spectrum of G_1 and G_2 . Furthermore, we note that, although G_1 and G_2 are regular graphs, $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$, $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$, and $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$ are non-regular graphs.

The following theorem is the main result of the paper.

Theorem 3.1. *Let $G_i, H_i, i = 1, 2$, be regular graphs, where G_1 need not be different from H_1 . If G_1 and H_1 are A -cospectral, and G_2 and H_2 are A -cospectral then $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$ (respectively, $\mathcal{R}(G_1)\bar{\vee}\mathcal{R}(G_2)$, $\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)$) and $\mathcal{R}(H_1)\check{\vee}\mathcal{R}(H_2)$ (respectively, $\mathcal{R}(H_1)\bar{\vee}\mathcal{R}(H_2)$, $\mathcal{R}(H_1)\check{\vee}\mathcal{R}(H_2)$) are simultaneously A -cospectral, L -cospectral and $\mathcal{L}$ -cospectral.*

Proof. Follows from Lemma 3.1 and the above observation. $\square$

4. SPANNING TREES AND KIRCHHOFF INDICES

Applying the results on Laplacian spectra, we find the number of spanning trees and Kirchhoff index of all the partial join graphs constructed in the paper.

Let $t(G)$ denote the number of *spanning trees* of G . It is well known [6] that if G is a connected graph on n vertices with Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \leq \dots \leq \mu_n(G)$, then $t(G) = \frac{\mu_2(G) \cdots \mu_n(G)}{n}$.

The *Kirchhoff index* of a graph G , denoted by $Kf(G)$, is defined as the sum of resistances between all pairs of vertices [1, 9] in G . For a connected graph G on n vertices, the Kirchhoff index [8] can be expressed as $Kf(G) = n \sum_{i=2}^n \frac{1}{\mu_i(G)}$.

Theorem 4.1. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then,

(i)

$$\begin{aligned} & t(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)) \\ &= \frac{2^{m_1+m_2-n_1-n_2} \cdot 2(2n_1+2n_2+r_1n_1+r_2n_2) \cdot \prod_{i=2}^{n_1} (2n_2+3\mu_i(G_1)) \cdot \prod_{j=2}^{n_2} (2n_1+3\mu_j(G_2))}{n_1+n_2+m_1+m_2}; \end{aligned}$$

(ii)

$$\begin{aligned} & t(\mathcal{R}(G_1)\overline{\check{\vee}}\mathcal{R}(G_2)) \\ &= (2+m_2)^{m_1-n_1} \cdot (2+m_1)^{m_2-n_2} \\ & \times \frac{(2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2) \cdot \prod_{i=2}^{n_1} (r_1m_2+3\mu_i(G_1)+m_2\mu_i(G_1)) \cdot \prod_{j=2}^{n_2} (r_2m_1+3\mu_j(G_2)+m_1\mu_j(G_2))}{n_1+n_2+m_1+m_2}; \end{aligned}$$

(iii)

$$\begin{aligned} & t(\mathcal{R}(G_1)\overline{\check{\vee}}\mathcal{R}(G_2)) \\ &= (2+n_2)^{m_1-n_1} \cdot 2^{m_2-n_2} \cdot (4m_1+2r_1m_1+2r_1n_2+r_1r_2n_2) \\ & \times \frac{\prod_{i=2}^{n_1} (r_1n_2+3\mu_i(G_1)+n_2\mu_i(G_1)) \cdot \prod_{j=2}^{n_2} (2m_1+3\mu_j(G_2))}{n_1+n_2+m_1+m_2}. \end{aligned}$$

Theorem 4.2. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then,

(i)

$$\begin{aligned} & Kf(\mathcal{R}(G_1)\check{\vee}\mathcal{R}(G_2)) \\ &= (n_1+n_2+m_1+m_2) \left(\frac{m_1+m_2-n_1-n_2}{2} \right. \\ & \quad + \frac{4+4n_1+4n_2+2r_1+2r_2+r_1r_2+r_1n_1+r_2n_2}{2(2n_1+2n_2+r_1n_1+r_2n_2)} + \sum_{i=2}^{n_1} \frac{2+r_1+n_2+\mu_i(G_1)}{2n_2+3\mu_i(G_1)} \\ & \quad \left. + \sum_{j=2}^{n_2} \frac{2+r_2+n_1+\mu_j(G_2)}{2n_1+3\mu_j(G_2)} \right); \end{aligned}$$

(ii)

$$\begin{aligned} & Kf(\mathcal{R}(G_1)\overline{\check{\vee}}\mathcal{R}(G_2)) \\ &= (n_1+n_2+m_1+m_2) \left(\frac{m_1-n_1}{2+m_2} + \frac{m_2-n_2}{2+m_1} + \sum_{i=2}^{n_1} \frac{2+r_1+m_2+\mu_i(G_1)}{r_1m_2+3\mu_i(G_1)+m_2\mu_i(G_1)} \right. \\ & \quad \left. + \frac{4+2r_1+2r_2+r_1r_2+r_1m_1+r_2m_2+2m_1+2m_2+r_1m_2+r_2m_1}{2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2} \right) \end{aligned}$$

$$(iii) \quad + \sum_{j=2}^{n_2} \frac{2 + r_2 + m_1 + \mu_j(G_2)}{r_2 m_1 + 3\mu_j(G_2) + m_1 \mu_j(G_2)} \Bigg);$$

$$\begin{aligned} & Kf(\mathcal{R}(G_1) \bar{\vee} \mathcal{R}(G_2)) \\ &= (n_1 + n_2 + m_1 + m_2) \left(\frac{m_1 - n_1}{2 + n_2} + \frac{m_2 - n_2}{2} \right. \\ &\quad + \frac{4 + 2r_1 + 2r_2 + 4m_1 + 2n_2 + r_1 r_2 + r_1 m_1 + r_1 n_2 + r_2 n_2}{4m_1 + 2r_1 m_1 + 2r_1 n_2 + r_1 r_2 n_2} \\ &\quad \left. + \sum_{i=2}^{n_1} \frac{2 + r_1 + n_2 + \mu_i(G_1)}{r_1 n_2 + 3\mu_i(G_1) + n_2 \mu_i(G_1)} + \sum_{j=2}^{n_2} \frac{2 + r_2 + m_1 + \mu_j(G_2)}{2m_1 + 3\mu_j(G_2)} \right). \end{aligned}$$

5. CONCLUDING REMARKS

The main result of the paper is based on regular A -cospectral graphs and certain operations on a pair of these graphs so that the operated (or resultant) graphs are non-regular and their adjacency, Laplacian and normalized Laplacian spectra depend on only the order, size, degree of regularity and spectra of the original graphs. Thus following the technique of this paper one may construct some more graphs like here.

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ON THE BOUNDEDNESS OF q -HAUSDORFF OPERATORS ON q -HARDY SPACES

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ABSTRACT. E. Liflyand and F. Móricz proved that the Hausdorff operator generated by a function $\varphi \in L^1(\mathbb{R})$ is a linear operator bounded on the real Hardy space $H^1(\mathbb{R})$ by using the classical Fourier transform and the Hilbert transform, they also proved that this operator commutes with Hilbert transform. In this work, we extend these results to the context of q -harmonic analysis associated with the q -Rubin's operator, we introduce the q -Hilbert transform on the real line, we study some of its main properties. Next, we define the q -Hardy spaces $\mathbb{H}_q^1(\mathbb{R}_q)$ by means of the q -Hilbert transforms, we finally study the q -Hausdorff operator and we prove the boundedness property and the commuting relation of this operator and q -Hilbert transform in q -Hardy spaces.

1. INTRODUCTION AND PRELIMINARIES

The Hausdorff operator is one of the important operators in harmonic analysis, it has a deep root in the study of the Fourier analysis. Particularly, a one-dimensional Hausdorff operator in Euclidean space is closely related to the summability of the classical Fourier series. The modern theory of Hausdorff operator started with the work of Siskakis in the complex analysis setting and with the work of Liflyand-Móricz in the Fourier transform setting [20], for more information about the background and the development of the Hausdorff operator one can refer to [7, 13, 15, 18, 19, 21] and references therein. For a locally integrable function φ on $(0, +\infty)$, the one-dimensional

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Hausdorff operator is defined in the integral form by

$$h_\varphi(f)(x) = \int_0^{+\infty} \frac{\varphi(t)}{t} f\left(\frac{x}{t}\right) dt.$$

It is worth pointing out that if the kernel function φ is taken appropriately, then the Hausdorff operator reduces to many classical operators in analysis such as the Hardy operator, the Cesàro operator, the Riemann-Liouville fractional integral operator and the Hardy-Littlewood average operator (see, e.g. [7, 13] and references therein).

Let $L^1(\mathbb{R})$, be the space of all Lebesgue measurable and integrable complex-valued functions defined on $\mathbb{R}$ endowed with the usual norm

$$\|f\|_1 := \int_{\mathbb{R}} |f(t)| dt < +\infty.$$

Then, we define the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R})$ by the following equality

$$\widehat{f}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(t) e^{-itx} dt, \quad \text{for all } x \in \mathbb{R}.$$

We also consider $\mathcal{H}(f)$ the Hilbert transform of a function $f \in L^1(\mathbb{R})$ in the form

$$(1.1) \quad \mathcal{H}(f)(x) := \frac{1}{\pi} \text{P. V.} \int_{\mathbb{R}} \frac{f(x-y)}{y} dy = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{+\infty} \frac{f(x-y) - f(x+y)}{y} dy,$$

for all $x \in \mathbb{R}$. We recall the definition of the Hardy space $H^1(\mathbb{R})$ as follows (see [20])

$$H^1(\mathbb{R}) := \left\{ f \in L^1(\mathbb{R}) : \mathcal{H}(f) \in L^1(\mathbb{R}) \right\},$$

endowed with the norm $\|f\|_{H^1} := \|f\|_{L^1} + \|\mathcal{H}(f)\|_{L^1}$. It has been proved in [2, 24] that $\mathcal{H}$ is a linear multiplier operator given by

$$\widehat{\mathcal{H}(f)}(t) = -i \operatorname{sgn}(t) \widehat{f}(t), \quad \text{for all } t \in \mathbb{R},$$

where $\operatorname{sgn}(t)$ equals $+1$ for $t > 0$, -1 for $t < 0$ and cancels at 0. This familiar identity is valid for all $f \in H^1(\mathbb{R})$ and plays a crucial role in this work.

In [20], E. Liflyand and F. Móricz studied the Hausdorff operators related to the Fourier transform on the real Hardy spaces $H^1(\mathbb{R})$ and proved the boundedness property and the commuting relations of these operators and Hilbert transforms. These results were recently obtained for the Dunkl-Hausdorff operator in [8]. For further details, the reader is invited to consult the following references [1, 3, 7, 12, 20, 26].

In the present paper, by using some elements of the q -harmonic analysis associated to the q -Rubin's operator introduced in [22, 23], we attempt to explore the validity of these results in case of the q -Fourier transform. To describe our results, we introduce some necessary fundamental concepts of quantum analysis which will be used in this paper. For this purpose, we refer the reader to the book by G. Gasper and M. Rahman [10], for the definitions, notations and properties of the q -shifted factorials and the q -hypergeometric functions. We can also see the following references [4, 9–11, 25].

We mention that, in this paper, we will follow all the so-mentioned works which, for a convergence argument, they imposed that the parameter $0 < q < 1$ satisfies the condition

$$\frac{\log(1-q)}{\log(q)} \in 2\mathbb{Z}.$$

Throughout this paper, we assume $0 < q < 1$ and we denote

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \widehat{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}.$$

For $a \in \mathbb{C}$, the q -shifted factorials are defined by:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{l=0}^{n-1} (1 - aq^l), \quad n = 1, 2, \dots, \quad (a; q)_\infty = \prod_{l=0}^{+\infty} (1 - aq^l).$$

We also denote for all $x \in \mathbb{C}$ and $n \in \mathbb{N}$

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [n]_q! = [1]_q \times [2]_q \times \dots \times [n]_q = \frac{(q; q)_n}{(1 - q)^n}, \quad [0]_q! = 1.$$

The q -Gamma function is given by (see [16])

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad x \neq 0, -1, -2, \dots$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x), \quad \operatorname{Re}(x) > 0,$$

and tends to the classical Gamma function Γ when q tends to 1^- .

The q -derivative of a function f is given by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0,$$

where $\mathcal{D}_q f(0) = f'(0)$ provided $f'(0)$ exists. We recall the following definition of the q -trigonometric functions. We define the q -cosine by

$$\cos(x; q^2) = \sum_{n=0}^{+\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!},$$

and the q -sine by

$$\sin(x; q^2) = \sum_{n=0}^{+\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}.$$

So, we deduce the q -analogue of the classical exponential function by

$$(1.2) \quad e(ix; q^2) = \cos(x; q^2) + i \sin(x; q^2).$$

These three functions are entire on $\mathbb{C}$ and when q tends to 1^- , they tend to the corresponding classical ones pointwise and uniformly on compacts.

The following estimates are available for all $x \in \mathbb{R}_q$ (see [22])

$$|\cos(x; q^2)| \leq \frac{1}{(q; q)_\infty}, \quad |\sin(x; q^2)| \leq \frac{1}{(q; q)_\infty}.$$

So,

$$(1.3) \quad |e(ix; q^2)| \leq \frac{2}{(q; q)_\infty}.$$

The Rubin's q -differential operator is defined in [22, 23] by

$$\partial_q(f)(x) = \begin{cases} \frac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2(1-q)x}, & \text{if } x \neq 0, \\ \lim_{x \rightarrow 0} \partial_q(f)(x) \text{ in } \mathbb{R}_q, & \text{if } x = 0. \end{cases}$$

Remark that if f is differentiable at x , then $\lim_{q \rightarrow 1^-} \partial_q(f)(x) = f'(x)$.

The following equality for the q -exponential function (see [23])

$$(1.4) \quad \partial_q e(itx; q^2) = ite(itx; q^2), \quad x \in \mathbb{R}_q.$$

holds, for every $t \in \mathbb{C}$.

The q -Jackson integrals are defined by (see [16, Sections 1.10 and 1.11])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{+\infty} q^n f(aq^n), \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

$$\int_0^{+\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

$$\int_{-\infty}^{+\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} q^n [f(q^n) + f(-q^n)],$$

provided the sums converge absolutely. In particular, for $a \in \mathbb{R}_q^+$,

$$\int_a^{+\infty} f(x) d_q x = (1-q)a \sum_{n=-\infty}^{-1} q^n f(aq^n).$$

Lemma 1.1. *The q -analogue of the integration theorem by a change of variable can be stated when $u(x) = \alpha x^\beta$, $\alpha \in \mathbb{C}$ as follows:*

$$\int_{u(a)}^{u(b)} f(u) d_q u = \int_a^b f(u(t)) \mathcal{D}_{q^{1/\beta}} u(t) d_{q^{1/\beta}} t.$$

Proof. See [17]. □

The proof of Lemma 1.2 is straightforward: just use the definition of q -Jackson integrals.

Lemma 1.2. *If the stated integrals exist, then for all $a \in \mathbb{R}_q$, we have the following.*

- If f is odd, then

$$\int_{-a}^a f(t) d_q t = 0.$$

- If f is even, then

$$\int_{-a}^a f(t) d_q t = 2 \int_0^a f(t) d_q t.$$

In what follows, let us fix some notations.

- $\mathcal{C}_q^p(\mathbb{R}_q)$ is the space of functions f , p times q -differentiable on $\widehat{\mathbb{R}}_q$ such that for all $0 \leq n \leq p$, $\partial_q^n f$ is continuous on $\widehat{\mathbb{R}}_q$.
- $\mathcal{S}_q(\mathbb{R}_q)$ is the space of functions f defined on $\mathbb{R}_q$ satisfying

$$P_{n,m,q}(f) = \sup_{x \in \mathbb{R}_q} |x^m \partial_q^n f(x)| < +\infty, \quad \text{for all } n, m \in \mathbb{N},$$

and

$$\lim_{x \rightarrow 0} \partial_q^n(f)(x) \quad (\text{in } \mathbb{R}_q) \quad \text{exists.}$$

- $L_q^p(\mathbb{R}_q)$, $p \in [1, +\infty]$, is the set of all functions defined on $\mathbb{R}_q$ such that

$$\|f\|_{q,p} = \begin{cases} \left(\int_{-\infty}^{+\infty} |f(x)|^p d_q x \right)^{1/p} < +\infty, & \text{if } 1 \leq p < +\infty, \\ \sup_{x \in \mathbb{R}_q} |f(x)| < +\infty, & \text{if } p = +\infty. \end{cases}$$

Definition 1.1. The q -Fourier transform $\widehat{f}(\cdot; q^2)$ associated with the q -Rubin operator ∂_q is defined for every function f in $L_q^1(\mathbb{R}_q)$ by

$$(1.5) \quad \widehat{f}(x; q^2) = K \int_{-\infty}^{+\infty} f(t) e(-itx; q^2) d_q t,$$

for all $x \in \widehat{\mathbb{R}}_q$ where $K = (1+q)^{1/2}/2\Gamma_{q^2}(1/2)$.

It was shown in [23] that the q -Fourier transform $\widehat{f}(\cdot; q^2)$ verifies the following properties.

- If $f, \partial_q f \in L_q^1(\mathbb{R}_q)$, then

$$\widehat{\partial_q f}(x; q^2) = ix \widehat{f}(x; q^2), \quad \text{for all } x \in \widehat{\mathbb{R}}_q.$$

- If $f \in L_q^1(\mathbb{R}_q)$, then $\widehat{f}(\cdot; q^2) \in \mathcal{C}_q^0(\mathbb{R}_q)$ and we have

$$(1.6) \quad \|\widehat{f}(\cdot; q^2)\|_{q,\infty} \leq \frac{2K}{(q; q)_\infty} \|f\|_{q,1}.$$

Theorem 1.1. (i) The q -Fourier transform $\widehat{f}(\cdot; q^2)$ is an isomorphism from $L_q^2(\mathbb{R}_q)$ into itself and satisfies the following q -Plancherel formula:

$$(1.7) \quad \|\widehat{f}(\cdot; q^2)\|_{q,2} = \|f\|_{q,2}, \quad \text{for all } f \in L_q^2(\mathbb{R}_q).$$

(ii) If $f \in L_q^1(\mathbb{R}_q)$ such that $\widehat{f}(\cdot; q^2) \in L_q^1(\mathbb{R}_q)$, then the q -inversion formula holds and we have

$$f(x) = K \int_{-\infty}^{+\infty} \widehat{f}(t; q^2) e(itx; q^2) d_q t, \quad \text{for all } x \in \widehat{\mathbb{R}}_q.$$

The q -translation operator $\mathcal{T}_{q,x}$, $x \in \widehat{\mathbb{R}}_q$ is defined on $\mathcal{S}_q(\mathbb{R}_q)$ by (see [5, 22])

$$\begin{aligned} \mathcal{T}_{q,x}(f)(y) &= K \int_{-\infty}^{+\infty} \widehat{f}(t; q^2) e(itx; q^2) e(ity; q^2) d_q t, \\ \mathcal{T}_{q,0}(f)(y) &= f(y). \end{aligned}$$

In particular the product formula

$$(1.8) \quad \mathcal{T}_{q,x}(e(it.; q^2))(y) = e(itx; q^2)e(ity; q^2), \quad x, y, t \in \mathbb{R}_q,$$

holds. It was shown in [8] that the q -translation operator can be also defined on $L_q^2(\mathbb{R}_q)$ and we have the following result.

Proposition 1.1. *For all $f \in L_q^2(\mathbb{R}_q)$, we have $\mathcal{T}_{q,x}f \in L_q^2(\mathbb{R}_q)$ and*

$$\|\mathcal{T}_{q,x}(f)\|_{q,2} \leq \frac{2}{(q; q)_\infty} \|f\|_{q,2}, \quad \text{for all } x \in \widehat{\mathbb{R}}_q.$$

Furthermore, the q -translation operator verifies the following properties.

Proposition 1.2. *For all $f \in L_q^1(\mathbb{R}_q)$, $t, x \in \mathbb{R}_q$, we have*

$$(1.9) \quad \begin{aligned} \mathcal{T}_{q,x}(f)(t) &= \mathcal{T}_{q,t}(f)(x), \\ \int_{-\infty}^{+\infty} \mathcal{T}_{q,x}(f)(t) d_q t &= \int_{-\infty}^{+\infty} f(t) d_q t, \\ \widehat{\mathcal{T}_{q,x}(f)}(t; q^2) &= e(itx; q^2) \widehat{f}(t; q^2). \end{aligned}$$

2. q -HILBERT TRANSFORM

In this section, we introduce the q -analogue of Hilbert transform and we study some of its main properties by using the q -harmonic analysis associated with the q -Rubin operator ∂_q .

For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we define the q -analogue of Hilbert transform under the q -Rubin operator ∂_q , also called the q -Hilbert transform operator $\mathcal{H}_q$ by the principal value of the singular integral

$$(2.1) \quad \mathcal{H}_q(f)(x) := a_q \text{ P.V. } \int_{-\infty}^{+\infty} \frac{\mathcal{T}_{q,x}(f)(-t)}{t} d_q t,$$

where $a_q = (1+q)/2q\Gamma_{q^2}^2(1/2)$ and the principal value integral is defined to be $\lim_{\varepsilon \rightarrow 0^+} \int_{|t| > \varepsilon}$.

Remark 2.1. (1) Letting q tends to 1^- subject to the formula (2.1), gives, at least formally, the classical Hilbert transform given by the formula (1.1).

(2) In view of formula (1.9), the function $\mathcal{T}_{q,x}(f) \in \mathcal{S}_q(\mathbb{R}_q)$ and one can write

$$(2.2) \quad \mathcal{H}_q(f)(x) = a_q \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{+\infty} \frac{\mathcal{T}_{q,x}(f)(-t) - \mathcal{T}_{q,x}(f)(t)}{t} d_q t.$$

Let now us calculate the q -Hilbert transform of some basic functions, for this, we need the following proposition.

Proposition 2.1. *The following identities are valid*

$$(2.3) \quad \int_{-\infty}^0 \frac{\sin(t; q^2)}{t} d_q t = \int_0^{+\infty} \frac{\sin(t; q^2)}{t} d_q t = \frac{\Gamma_{q^2}^2(1/2)}{1+q^{-1}}.$$

Proof. See [14, Proposition 5.3]. □

Proposition 2.2. *The following identities are fulfilled for all $x \in \mathbb{R}_q$:*

$$(2.4) \quad \mathcal{H}_q(\cos(\cdot; q^2))(x) = \sin(x; q^2),$$

$$(2.5) \quad \mathcal{H}_q(\sin(\cdot; q^2))(x) = -\cos(x; q^2).$$

Proof. Using the linearity of the q -translation and (1.2), we conclude that

$$(2.6) \quad \mathcal{T}_{q,x}(e(i\cdot; q^2))(t) = \mathcal{T}_{q,x}(\cos(\cdot; q^2))(t) + i\mathcal{T}_{q,x}(\sin(\cdot; q^2))(t).$$

On other hands, it follows from (1.2) again that

$$(2.7) \quad \begin{aligned} e(ix; q^2) \cdot e(it; q^2) &= \cos(x; q^2) \cos(t; q^2) - \sin(x; q^2) \sin(t; q^2) \\ &+ i \left[\cos(x; q^2) \sin(t; q^2) + \sin(x; q^2) \cos(t; q^2) \right]. \end{aligned}$$

Using the product formula (1.8) and combining the relations (2.6) and (2.7), we have

$$\mathcal{T}_{q,x}(\cos(\cdot; q^2))(t) = \cos(x; q^2) \cos(t; q^2) - \sin(x; q^2) \sin(t; q^2),$$

$$\mathcal{T}_{q,x}(\sin(\cdot; q^2))(t) = \cos(x; q^2) \sin(t; q^2) + \sin(x; q^2) \cos(t; q^2).$$

Therefore, by using the fact that $\cos(\cdot; q^2)$ is even and $\sin(\cdot; q^2)$ is odd, it follows from the definition of q -Hilbert transform (2.2) and Proposition 2.1 that

$$\begin{aligned} \mathcal{H}_q(\cos(\cdot; q^2))(x) &= a_q \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{+\infty} \frac{\mathcal{T}_{q,x}(\cos(\cdot; q^2))(-t) - \mathcal{T}_{q,x}(\cos(\cdot; q^2))(t)}{t} d_q t \\ &= 2a_q \sin(x; q^2) \int_0^{+\infty} \frac{\sin(t; q^2)}{t} d_q t \\ &= \sin(x; q^2). \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathcal{H}_q(\sin(\cdot; q^2))(x) &= a_q \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{+\infty} \frac{\mathcal{T}_{q,x}(\sin(\cdot; q^2))(-t) - \mathcal{T}_{q,x}(\sin(\cdot; q^2))(t)}{t} d_q t \\ &= -2a_q \cos(x; q^2) \int_0^{+\infty} \frac{\sin(t; q^2)}{t} d_q t \\ &= -\cos(x; q^2). \end{aligned}$$

Thus, Proposition 2.2 is proved. □

Remark 2.2. From (2.4) and (2.5), we have

$$\mathcal{H}_q(e(i\cdot; q^2))(x) = -ie(ix; q^2), \quad x \in \mathbb{R}_q.$$

Now, we are giving a very important property for the q -Hilbert transform under the q -Rubin operator, it has been proved in [24] and also in [2] that the Hilbert transform is a multiplier operator. Now, in the following theorem, we give a q -Fourier version of this result.

Theorem 2.1. *Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. Then, the q -Hilbert transform is a multiplier operator with*

$$(2.8) \quad \widehat{\mathcal{H}_q(f)}(t; q^2) = -i \operatorname{sgn}(t) \widehat{f}(t; q^2),$$

for all $t \in \mathbb{R}_q$.

Proof. For $\xi \in \mathbb{R}_q$, we put

$$\mathcal{G}_{q,\xi}(\varepsilon, A) = \int_{\varepsilon}^A \frac{e(-it\xi; q^2) - e(it\xi, q^2)}{t} d_q t, \quad (\varepsilon, A) \in \mathbb{R}_q^+ \times \mathbb{R}_q^+.$$

In view of (1.2), we get

$$\begin{aligned} \mathcal{G}_{q,\xi}(\varepsilon, A) &= \int_{\varepsilon}^A \frac{e(-it\xi; q^2) - e(it\xi; q^2)}{t} d_q t = -2i \int_{\varepsilon}^A \frac{\sin(t\xi; q^2)}{t} d_q t \\ &= -2i \operatorname{sgn}(\xi) \int_{\varepsilon\xi}^{A\xi} \frac{\sin(t; q^2)}{t} d_q t. \end{aligned}$$

Consequently, by (2.3), we have

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ A \rightarrow +\infty}} \mathcal{G}_{q,\xi}(\varepsilon, A) = -2i \operatorname{sgn}(\xi) \frac{\Gamma_{q^2}^2(1/2)}{1 + q^{-1}} = \frac{-i \operatorname{sgn}(\xi)}{a_q}.$$

Furthermore, since

$$\mathcal{T}_{q,x}(f)(t) = \int_{-\infty}^{+\infty} \widehat{\mathcal{T}_{q,x}(f)}(\xi; q^2) e(it\xi; q^2) d_q \xi,$$

then, using Fubini's theorem and dominated convergence theorem, we get

$$\begin{aligned} \mathcal{H}_q(f)(x) &= a_q \int_0^{+\infty} \int_{-\infty}^{+\infty} \widehat{\mathcal{T}_{q,x}(f)}(\xi; q^2) \left(\frac{e(-it\xi; q^2) - e(it\xi; q^2)}{t} d_q \xi \right) d_q t \\ &= a_q \lim_{\substack{\varepsilon \rightarrow 0 \\ A \rightarrow +\infty}} \int_{-\infty}^{+\infty} \widehat{\mathcal{T}_{q,x}(f)}(\xi; q^2) \mathcal{G}_{q,\xi}(\varepsilon, A) d_q \xi \\ &= a_q \lim_{\substack{\varepsilon \rightarrow 0 \\ A \rightarrow +\infty}} \int_{-\infty}^{+\infty} e(ix\xi; q^2) \widehat{f}(\xi; q^2) \mathcal{G}_{q,\xi}(\varepsilon, A) d_q \xi \\ &= -i \int_{-\infty}^{+\infty} \operatorname{sgn}(\xi) \widehat{f}(\xi; q^2) e(ix\xi; q^2) d_q \xi. \end{aligned}$$

By the uniqueness of q -Fourier transform, we get exactly (2.8). $\square$

Some direct application of the last theorem include the following.

Proposition 2.3. *Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. Then, the following equality is valid for the q -Hilbert transform $\mathcal{H}_q$:*

$$(2.9) \quad \mathcal{H}_q(\mathcal{H}_q f) = -f.$$

Proof. Let $t \in \mathbb{R}_q$, then by using (2.8), we get

$$\widehat{\mathcal{H}_q(\mathcal{H}_q f)}(t; q^2) = -i \operatorname{sgn}(t) \widehat{\mathcal{H}_q f}(t; q^2) = -i \operatorname{sgn}(t) \times -i \operatorname{sgn}(t) \widehat{f}(t; q^2) = -\widehat{f}(t; q^2).$$

Hence, by uniqueness of q -Fourier transform, we have the result. $\square$

Note that, from (2.9), we can see that $\mathcal{H}_{q,\alpha}$ is a unitary operator in $L_q^2(\mathbb{R}_q)$ and we have $\mathcal{H}_q^2 = -\mathcal{I}$, where $\mathcal{I}$ denotes the identity operator in $L_q^2(\mathbb{R}_q)$. From this, the inverse q -Hilbert transform operator can be written symbolically as $\mathcal{H}_q^{-1} = -\mathcal{H}_q$, and so,

$$f(x) = -a_q \text{ P. V. } \int_{-\infty}^{+\infty} \frac{\mathcal{T}_{q,x}(\mathcal{H}_q f)(-t)}{t} d_q t.$$

Theorem 2.2. (i) *The q -Hilbert transform $\mathcal{H}_q$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have the following isometric property*

$$(2.10) \quad \|\mathcal{H}_q(f)\|_{q,2} = \|f\|_{q,2}.$$

(ii) *The q -Hilbert transform can be uniquely extended to an isometric isomorphism on $L_q^2(\mathbb{R}_q)$.*

Proof. Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. From (1.3) and (1.4), we can easily see that the q -Fourier transform leaves $\mathcal{S}_q(\mathbb{R}_q)$ invariant and by taking into account the relation (1.9), we conclude that $\mathcal{T}_{q,x}(f)$ leaves $\mathcal{S}_q(\mathbb{R}_q)$ invariant, too. Therefore, $\mathcal{H}_q(f) \in \mathcal{S}_q(\mathbb{R}_q)$ and the application $\mathcal{H}_q : \mathcal{S}_q(\mathbb{R}_q) \rightarrow \mathcal{S}_q(\mathbb{R}_q)$ is well defined. The bijectivity of $\mathcal{H}_q$ follows immediately from relation (2.9). On the other hand, in view of Theorem 2.1 and q -Plancherel formula (1.7), we have

$$\begin{aligned} \|\mathcal{H}_q(f)\|_{q,2}^2 &= \|\widehat{\mathcal{H}_q(f)}(\cdot; q^2)\|_{q,2}^2 = \int_{-\infty}^{+\infty} |\widehat{\mathcal{H}_q f}(t; q^2)|^2 d_q t \\ &= \int_{-\infty}^{+\infty} |-i \operatorname{sgn}(t) \widehat{f}(t; q^2)|^2 d_q t = \int_{-\infty}^{+\infty} |\widehat{f}(t; q^2)|^2 d_q t \\ &= \|\widehat{f}(\cdot; q^2)\|_{q,2}^2 = \|f\|_{q,2}^2. \end{aligned}$$

Then we have (2.10). The result of (ii) follows from (i) and the density of $\mathcal{S}_q(\mathbb{R}_q)$ in $L_q^2(\mathbb{R}_q)$ (see [4]). $\square$

Let $\mathcal{D}_s$, $s > 0$, $s \in \mathbb{R}_q$, be the dilation operator defined by $\mathcal{D}_s f(x) = f(sx)$ and $\mathcal{R}$ be the reflection operator $\mathcal{R}f(x) = f(-x)$. Then, the following properties are fulfilled for the q -Hilbert transform $\mathcal{H}_q$.

Proposition 2.4. *The q -Hilbert transform $\mathcal{H}_q$ commutes with q -translations $\mathcal{T}_{q,x}$, $x \in \mathbb{R}_q$, positive dilations $\mathcal{D}_s$ and anticommutes with the reflection $\mathcal{R}$, that means*

$$(2.11) \quad \mathcal{H}_q(\mathcal{T}_{q,x} f) = \mathcal{T}_{q,x}(\mathcal{H}_q f),$$

$$(2.12) \quad \mathcal{H}_q(\mathcal{D}_s f) = \mathcal{D}_s(\mathcal{H}_q f),$$

$$(2.13) \quad \mathcal{H}_q(\mathcal{R}f) = -\mathcal{R}(\mathcal{H}_q f),$$

for all $f \in L_q^2(\mathbb{R}_q)$.

Proof. Let $t \in \mathbb{R}_q$, according to the formula (1.9) and Theorem 2.1, we have

$$\begin{aligned}\widehat{\mathcal{T}_{q,x}\mathcal{H}_q f}(t; q^2) &= e(itx; q^2)\widehat{\mathcal{H}_q f}(t; q^2) = -i \operatorname{sgn}(t)e(itx; q^2)\widehat{f}(t; q^2) \\ &= -i \operatorname{sgn}(t)\widehat{\mathcal{T}_{q,x} f}(t; q^2) = \widehat{\mathcal{H}_q \mathcal{T}_{q,x} f}(t; q^2).\end{aligned}$$

By injectivity of q -Fourier transform, we get (2.11). To prove (2.12), let $f \in L_q^2(\mathbb{R}_q)$, then $\mathcal{D}_s f \in L_q^2(\mathbb{R}_q)$ and by a simple change of variable, we can see that

$$\widehat{\mathcal{D}_s f}(t; q^2) = s^{-1}\widehat{f}(t/s; q^2), \quad t, s \in \mathbb{R}_q, s > 0.$$

From this and (2.8), we have

$$\begin{aligned}\widehat{\mathcal{H}_q \mathcal{D}_s f}(t; q^2) &= -i \operatorname{sgn}(t)\widehat{\mathcal{D}_s f}(t; q^2) = -i \operatorname{sgn}(t)s^{-1}\widehat{f}(t/s; q^2) \\ &= s^{-1}(-i \operatorname{sgn}(t/s)\widehat{f}(t/s; q^2)) = s^{-1}\widehat{\mathcal{H}_q f}(t/s; q^2) = \widehat{\mathcal{D}_s \mathcal{H}_q f}(t; q^2).\end{aligned}$$

By uniqueness of q -Fourier transform, we get (2.12) and it is equally obvious that

$$\mathcal{H}_q(\mathcal{D}_s f) = -\mathcal{D}_s(\mathcal{H}_q f), \quad \text{for } s < 0.$$

From this, (2.13) is proved in the case $s = -1$. □

3. THE q -HAUSDORFF OPERATORS IN q -HARDY SPACES

To give the main results of this work, we first introduce the q -analogue of Hardy spaces using the q Hilbert transform on the image under the q -Fourier transform. Next, we prove the boundedness of q -Hausdorff operator by using the closed graph theorem and the fact that if $f \in L_q^1(\mathbb{R}_q)$ satisfies

$$\widehat{f}(t; q^2) = 0, \quad \text{for } t < 0 \text{ (or } t > 0), t \in \mathbb{R}_q,$$

then $f \in \mathbb{H}_q^1(\mathbb{R}_q)$.

Let $f \in L_q^1(\mathbb{R}_q)$ and $x \in \widehat{\mathbb{R}_q}$. Since for all $z, t \in \mathbb{R}_q$, $|e(-itz; q^2)| \leq \frac{2}{(q; q)_\infty}$, and $t \mapsto e(itx; q^2)$ is in $L_q^1(\mathbb{R}_q)$, then

$$\begin{aligned}& \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(z)e(itx; q^2)e(-itz; q^2)| d_q t d_q z \\ & \leq \frac{2}{(q; q)_\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(z)| |e(itx; q^2)| d_q t d_q z \\ & = \frac{2}{(q; q)_\infty} \|f\|_{q,1} \|e(ix; q^2)\|_{q,1} < +\infty.\end{aligned}$$

Therefore, we can define the q -Hilbert transform on $L_q^1(\mathbb{R}_q)$ as

$$\mathcal{H}_q(f)(x) = -i \int_{-\infty}^{+\infty} \operatorname{sgn}(t)\widehat{f}(t; q^2)e(itx; q^2) d_q t.$$

So, the q -Hardy space $\mathbb{H}_q^1(\mathbb{R}_q)$ is an important subspace of $L_q^1(\mathbb{R}_q)$ defined by

$$\mathbb{H}_q^1(\mathbb{R}_q) := \left\{ f \in L_q^1(\mathbb{R}_q) : \mathcal{H}_q(f) \in L_q^1(\mathbb{R}_q) \right\},$$

endowed with the norm

$$(3.1) \quad \|f\|_{\mathbb{H}_q^1} := \|f\|_{q,1} + \|\mathcal{H}_q(f)\|_{q,1}.$$

This space is a Banach algebra under pointwise addition, scalar multiplication and convolution. It is much better adapted to problems arising in the theory of q -harmonic analysis.

Furthermore, it follows from (2.8) that, if $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, then

$$(3.2) \quad \widehat{\mathcal{H}_q f}(t; q^2) = -i \operatorname{sgn}(t) \widehat{f}(t; q^2)$$

and

$$(3.3) \quad \mathcal{H}_q(\mathcal{H}_q f)(t) = -f(t), \quad t \in \mathbb{R}_q.$$

From (3.3), we can easily see that $\mathcal{H}_q$ is a bounded operator from $\mathbb{H}_q^1(\mathbb{R}_q)$ into itself. In particular, if $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, then $\mathcal{H}_q(f) \in \mathbb{H}_q^1(\mathbb{R}_q)$ and we have

$$\|\mathcal{H}_q(f)\|_{\mathbb{H}_q^1} = \|f\|_{\mathbb{H}_q^1}.$$

In what follows, we give some auxiliary results interesting in themselves.

Proposition 3.1. *If f belongs to $\mathbb{H}_q^1(\mathbb{R}_q)$, then*

$$(3.4) \quad \int_{-\infty}^{+\infty} f(t) d_q t = 0.$$

Proof. Since $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, then by using the definition of q -Hardy space, we have $f \in L_q^1(\mathbb{R}_q)$ and $\mathcal{H}_q(f) \in L_q^1(\mathbb{R}_q)$, by using the relation (1.6) and the fact that

$$\begin{aligned} \|\widehat{f}(\cdot; q^2)\|_{q,\infty} &\leq \frac{4K}{(q, q)_\infty} \|f\|_{q,1} < +\infty, \\ \|\widehat{\mathcal{H}_q f}(\cdot; q^2)\|_{q,\infty} &\leq \frac{4K}{(q, q)_\infty} \|\mathcal{H}_q(f)\|_{q,1} < +\infty, \end{aligned}$$

we have $\widehat{f}(\cdot; q^2), \widehat{\mathcal{H}_q f}(\cdot; q^2) \in \mathcal{C}_q^0(\mathbb{R}_q)$. From (3.2), it follows that

$$\widehat{\mathcal{H}_q f}(0; q^2) = -i \widehat{f}(0; q^2) = i \widehat{f}(0; q^2).$$

So, $\widehat{f}(0; q^2) = 0$, which gives (3.4). □

Proposition 3.2. *If f belong to $L_q^1(\mathbb{R}_q)$ and satisfies:*

$$(3.5) \quad \widehat{f}(t; q^2) = 0, \quad \text{for } t > 0 \text{ and } t \in \mathbb{R}_q,$$

or

$$(3.6) \quad \widehat{f}(t; q^2) = 0, \quad \text{for } t > 0 \text{ and } t \in \mathbb{R}_q,$$

then $f \in \mathbb{H}_q^1(\mathbb{R}_q)$.

Proof. It follows from (2.8) and (3.5) that

$$\widehat{\mathcal{H}_q f}(t; q^2) = -i \operatorname{sgn}(t) \widehat{f}(t; q^2) = i \widehat{f}(t; q^2).$$

By uniqueness of q -Fourier transform, we get

$$(3.7) \quad \mathcal{H}_q(f)(t) = if(t),$$

for almost all $t \in \mathbb{R}_q$, therefore the result follows from (3.7). A similar proof is obtained from (3.6). $\square$

Proposition 3.3. *If $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, then there are two functions f_1 and f_2 both in $\mathbb{H}_q^1(\mathbb{R}_q)$ such that $f = f_1 + f_2$ and*

$$\begin{aligned} \widehat{f_1}(t; q^2) &= 0, & \text{for } t > 0 \text{ and } t \in \mathbb{R}_q, \\ \widehat{f_2}(t; q^2) &= 0, & \text{for } t > 0 \text{ and } t \in \mathbb{R}_q. \end{aligned}$$

Proof. We set

$$f_1(t) = \frac{f(t) + i\mathcal{H}_q(f)(t)}{2} \quad \text{and} \quad f_2(t) = \frac{f(t) - i\mathcal{H}_q(f)(t)}{2}.$$

By (3.3), we have $f_1, f_2 \in \mathbb{H}_q^1(\mathbb{R}_q)$. Furthermore, assume that $t < 0$ with $t \in \mathbb{R}_q$ and according to the formula (3.2), we get

$$\begin{aligned} 2\widehat{f_1}(t; q^2) &= \widehat{f}(t; q^2) + i\widehat{\mathcal{H}_q f}(t; q^2) = \widehat{f}(t; q^2) + i(-i \operatorname{sgn}(t) \widehat{f}(t; q^2)) \\ &= (1 + \operatorname{sgn}(t)) \widehat{f}(t; q^2) = 0, \end{aligned}$$

and analogously, we have

$$2\widehat{f_2}(t; q^2) = (1 - \operatorname{sgn}(t)) \widehat{f}(t; q^2) = 0, \quad \text{for } t > 0, t \in \mathbb{R}_q,$$

which completes the proof. $\square$

In the following, we introduce the q -analogue of Hausdorff operator in q -Fourier analysis, also known as the q -Hausdorff operator. We define the q -Hausdorff operator $h_{q,\psi}$ of a function $f \in L_q^1(\mathbb{R}_q)$ generated by a function ψ belonging also to $L_q^1(\mathbb{R}_q)$ as follows:

$$(3.8) \quad \widehat{h_{q,\psi} f}(t; q^2) = \int_{-\infty}^{+\infty} \widehat{f}(tx; q^2) \psi(x) d_q x, \quad t \in \mathbb{R}_q,$$

provided the integral is convergent, where

$$(3.9) \quad h_{q,\psi} f(x) = \int_{-\infty}^{+\infty} \frac{\psi(t)}{|t|} f\left(\frac{x}{t}\right) d_q t, \quad x \in \mathbb{R}_q.$$

For $x > 0$, $x \in \mathbb{R}_q$, by a change of variables, one has

$$h_{q,\psi} f(x) = \int_{-\infty}^{+\infty} \frac{f(t)}{|t|} \psi\left(\frac{x}{t}\right) d_q t.$$

The existence of such a function $h_{q,\psi} f$ in $L_q^1(\mathbb{R}_q)$ will be clear from the proof of the following theorem.

Theorem 3.1. *Let f, ψ belong to $L_q^1(\mathbb{R}_q)$. Then, the q -Hausdorff operator $h_{q,\psi} : L_q^1(\mathbb{R}_q) \rightarrow L_q^1(\mathbb{R}_q)$ is a bounded operator with*

$$\|h_{q,\psi}f\|_{q,1} \leq \|f\|_{q,1}\|\psi\|_{q,1},$$

and the formula (3.8) holds.

Proof. Let $f, \psi \in L_q^1(\mathbb{R}_q)$, from (3.9), we have

$$|h_{q,\psi}f(x)| \leq \int_{-\infty}^{+\infty} \left| \frac{\psi(t)}{t} \right| \cdot \left| f\left(\frac{x}{t}\right) \right| d_q t.$$

Putting $\xi = x/t$ and by using Fubini's theorem, we have

$$\begin{aligned} \|h_{q,\psi}f\|_{q,1} &= \int_{-\infty}^{+\infty} |h_{q,\psi}f(x)| d_q x \\ &= \int_{-\infty}^{+\infty} \left| \frac{\psi(t)}{t} \right| d_q t \int_{-\infty}^{+\infty} \left| f\left(\frac{x}{t}\right) \right| d_q x \\ &= \int_0^{+\infty} \frac{|\psi(t)|}{t} d_q t \int_{-\infty}^{+\infty} \left| f\left(\frac{x}{t}\right) \right| d_q x - \int_{-\infty}^0 \frac{|\psi(t)|}{t} d_q t \int_{-\infty}^{+\infty} \left| f\left(\frac{x}{t}\right) \right| d_q x \\ &= \int_0^{+\infty} |\psi(t)| d_q t \int_{-\infty}^{+\infty} |f(\xi)| d_q \xi + \int_{-\infty}^0 |\psi(t)| d_q t \int_{-\infty}^{+\infty} |f(\xi)| d_q \xi \\ &= \int_{-\infty}^{+\infty} |\psi(t)| d_q t \int_{-\infty}^{+\infty} |f(\xi)| d_q \xi \\ &= \|f\|_{q,1} \|\psi\|_{q,1}, \end{aligned}$$

which completes the proof of boundedness of $h_{q,\psi}$ on $L_q^1(\mathbb{R}_q)$. Finally, to get (3.8), it follows from (1.5) and (3.9) that

$$\begin{aligned} \widehat{h_{q,\psi}f}(t; q^2) &= K \int_{-\infty}^{+\infty} e(-itx; q^2) h_{q,\psi}f(x) d_q x \\ &= K \int_{-\infty}^{+\infty} e(-itx; q^2) d_q x \int_{-\infty}^{+\infty} \frac{\psi(\xi)}{|\xi|} f\left(\frac{x}{\xi}\right) d_q \xi \\ &= \int_0^{+\infty} \frac{\psi(\xi)}{\xi} d_q \xi K \int_{-\infty}^{+\infty} f\left(\frac{x}{\xi}\right) e(-itx; q^2) d_q x \\ &\quad - \int_{-\infty}^0 \frac{\psi(\xi)}{\xi} d_q \xi K \int_{-\infty}^{+\infty} f\left(\frac{x}{\xi}\right) e(-itx; q^2) d_q x \\ &= \int_0^{+\infty} \psi(\xi) d_q \xi K \int_{-\infty}^{+\infty} f(u) e(-it\xi u; q^2) d_q u \\ &\quad + \int_{-\infty}^0 \psi(\xi) d_q \xi K \int_{-\infty}^{+\infty} f(u) e(-it\xi u; q^2) d_q u \\ &= \int_{-\infty}^{+\infty} \psi(\xi) d_q \xi K \int_{-\infty}^{+\infty} f(u) e(-it\xi u; q^2) d_q u \\ &= \int_{-\infty}^{+\infty} \widehat{f}(t\xi; q^2) \psi(\xi) d_q \xi. \end{aligned}$$

Then, Theorem 3.1 is proved. □

Theorem 3.2. *Let ψ belongs to $L_q^1(\mathbb{R}_q)$, such that*

$$(3.10) \quad \psi(t) = 0, \quad \text{for } t < 0, t \in \mathbb{R}_q.$$

Then, $h_{q,\psi} : \mathbb{H}_q^1(\mathbb{R}_q) \rightarrow \mathbb{H}_q^1(\mathbb{R}_q)$ is a bounded operator.

Proof. Assume that $\psi \in L_q^1(\mathbb{R}_q)$ such that condition (3.10) is satisfied. Let $f \in \mathbb{H}_q^1(\mathbb{R}_q)$. By Proposition 3.3, there are two functions f_1 and f_2 both in $\mathbb{H}_q^1(\mathbb{R}_q)$ such that $f = f_1 + f_2$. The q -Hausdorff operator is clearly linear, hence

$$(3.11) \quad h_{q,\psi}f = h_{q,\psi}f_1 + h_{q,\psi}f_2.$$

Furthermore, from (3.8) and by a change of variable, we get

$$\begin{aligned} \widehat{h_{q,\psi}f}(t; q^2) &= \int_{-\infty}^{+\infty} \widehat{f}(tx; q^2) \psi(x) d_q x = \frac{1}{t} \int_{-\infty}^{+\infty} \text{sgn}(t) \widehat{f}(\xi; q^2) \psi(\xi/t) d_q \xi \\ &= \frac{1}{|t|} \int_{-\infty}^{+\infty} \widehat{f}(\xi; q^2) \psi(\xi/t) d_q \xi. \end{aligned}$$

According to Proposition 3.3, we have

$$(3.12) \quad \widehat{f_1}(\xi; q^2) = 0, \quad \text{for } \xi < 0,$$

and the condition (3.10) yields to

$$(3.13) \quad \psi(\xi/t) = 0, \quad \text{for } t < 0 \text{ and } \xi > 0.$$

Therefore, from (3.12) and (3.13), we conclude that

$$(3.14) \quad \widehat{h_{q,\psi}f_1}(t; q^2) = \frac{1}{|t|} \int_0^{+\infty} \widehat{f_1}(\xi; q^2) \psi(\xi/t) d_q \xi = 0, \quad \text{for } t < 0.$$

In the same way, we establish that

$$(3.15) \quad \widehat{h_{q,\psi}f_2}(t; q^2) = \frac{1}{|t|} \int_0^{+\infty} \widehat{f_2}(\xi; q^2) \psi(\xi/t) d_q \xi = 0, \quad \text{for } t > 0.$$

Finally, by Theorem 3.1, Proposition 3.2 and combining the formulas (3.14) and (3.15), both $h_{q,\psi}f_1$ and $h_{q,\psi}f_2$ belongs to $\mathbb{H}_q^1(\mathbb{R}_q)$. From (3.11) it follows that $h_{q,\psi}f \in \mathbb{H}_q^1(\mathbb{R}_q)$, which completes the proof. □

Remark 3.1. In the general case, we decompose $\psi \in L_q^1(\mathbb{R}_q)$ in the trivial way:

$$\psi(x) = \psi_1(x) + \psi_2(x),$$

where

$$\psi_1(x) = \psi(x) \chi_{(0, +\infty)}(x) \quad \text{and} \quad \psi_2(x) = \psi(x) - \psi_1(x),$$

$\chi_{(0, +\infty)}(x)$ being the indicator function on $(0, +\infty)$. Clearly, we have

$$h_{q,\psi}f = h_{q,\psi_1}f + h_{q,\psi_2}f, \quad f \in \mathbb{H}_q^1(\mathbb{R}_q),$$

and the special cases in Theorem 3.2 apply separately to $h_{q,\psi_1}f$ and $h_{q,\psi_2}f$, respectively. Therefore, we have $h_{q,\psi}f \in \mathbb{H}_q^1(\mathbb{R}_q)$ whenever $f \in \mathbb{H}_q^1(\mathbb{R}_q)$.

Theorem 3.3. *If ψ belongs to $L_q^1(\mathbb{R}_q)$, then the q -Hausdorff operator $h_{q,\psi} : \mathbb{H}_q^1(\mathbb{R}_q) \rightarrow \mathbb{H}_q^1(\mathbb{R}_q)$ is a bounded operator.*

Proof. In this proof, we shall apply the closed graph theorem, which says that a linear operator mapping a Banach space into another Banach space is bounded if and only if it is closed. Linearity of the q -Hausdorff operator $h_{q,\psi}$ is obvious. We have to show that the graph of $h_{q,\psi}$ is closed. To this effect, assume that a sequence of functions $\{f_n : n = 1, 2, \dots\}$, belonging to $\mathbb{H}_q^1(\mathbb{R}_q)$ exists, such that

$$f_n \rightarrow f \quad \text{and} \quad h_{q,\psi} f_n \rightarrow g, \quad \text{as } n \rightarrow +\infty,$$

with $f, g \in \mathbb{H}_q^1(\mathbb{R}_q)$. This implies that

$$\|f_n - f\|_{\mathbb{H}_q^1} \rightarrow 0 \quad \text{and} \quad \|h_{q,\psi} f_n - g\|_{\mathbb{H}_q^1} \rightarrow 0.$$

From (3.1), the above expression yields

$$\|f_n - f\|_{q,1} + \|\mathcal{H}_q(f_n - f)\|_{q,1} \rightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

and

$$\|h_{q,\psi} f_n - g\|_{q,1} + \|\mathcal{H}_q(h_{q,\psi} f_n - g)\|_{q,1} \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Hence,

$$\|f_n - f\|_{q,1} \rightarrow 0 \quad \text{and} \quad \|h_{q,\psi} f_n - g\|_{q,1} \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

By Theorem 3.1, we have $h_{q,\psi} : L_q^1(\mathbb{R}_q) \rightarrow L_q^1(\mathbb{R}_q)$ is a bounded operator and hence by the closed graph theorem its graph must be closed.

Since

$$(f_n, h_{q,\psi} f_n) \rightarrow (f, g), \quad \text{as } n \rightarrow +\infty.$$

Hence,

$$(f, g) \in \text{graph}(h_{q,\psi}),$$

whence we conclude that $h_{q,\psi} f = g$. This implies that the graph of $h_{q,\psi}$ is closed in $\mathbb{H}_q^1(\mathbb{R}_q)$. $\square$

Theorem 3.4. *If ψ belongs to $L_q^1(\mathbb{R}_q)$, then for all $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, we have*

$$\mathcal{H}_q(h_{q,\psi} f) = h_{q,\text{sgn}(\cdot)\psi(\cdot)}(\mathcal{H}_q(f)).$$

Proof. Let us put

$$\tilde{\psi}(t) = \text{sgn}(t)\psi(t), \quad t \in \mathbb{R}_q,$$

and let $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, then by Theorem 3.3, we have $h_{q,\psi} f \in \mathbb{H}_q^1(\mathbb{R}_q)$, thus $\mathcal{H}_q(h_{q,\psi} f) \in L_q^1(\mathbb{R}_q)$, then in view of formula (3.2) and the fact that the function signum verify

$$\text{sgn}(t) \times \text{sgn}(x) = \text{sgn}(tx), \quad \text{for all } t, x \in \mathbb{R}_q,$$

we have conclude that

$$\begin{aligned}\widehat{\mathcal{H}_q(h_{q,\psi}f)}(t; q^2) &= -i \operatorname{sgn}(t) \widehat{h_{q,\psi}f}(t; q^2) = -i \operatorname{sgn}(t) \int_{-\infty}^{+\infty} \widehat{f}(tx; q^2) \psi(x) d_q x \\ &= -i \operatorname{sgn}(t) \int_{-\infty}^{+\infty} \operatorname{sgn}^2(x) \widehat{f}(tx; q^2) \psi(x) d_q x \\ &= \int_{-\infty}^{+\infty} -i \operatorname{sgn}(tx) \widehat{f}(tx; q^2) \widetilde{\psi}(x) d_q x \\ &= \int_{-\infty}^{+\infty} \widehat{\mathcal{H}_q f}(tx; q^2) \widetilde{\psi}(x) d_q x = \widehat{h_{q,\widetilde{\psi}}(\mathcal{H}_q f)}(t; q^2).\end{aligned}$$

Then, by uniqueness of the q -Fourier transform, we have the desired result. $\square$

A consequence of the last theorem is to give another version of the proof of Theorem 3.3.

Theorem 3.5. *If ψ belongs to $L_q^1(\mathbb{R}_q)$, then for all $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, we have $h_{q,\psi}f \in \mathbb{H}_q^1(\mathbb{R}_q)$ and*

$$\|h_{q,\psi}f\|_{\mathbb{H}_q^1} \leq \|f\|_{\mathbb{H}_q^1} \|\psi\|_{q,1}.$$

Proof. Assume that $f \in \mathbb{H}_q^1(\mathbb{R}_q)$. Then, $\mathcal{H}_q(f) \in L_q^1(\mathbb{R}_q)$. Thus, in view of (3.1) and Theorem 3.1, we get

$$\begin{aligned}\|h_{q,\psi}f\|_{\mathbb{H}_q^1} &= \|h_{q,\psi}f\|_{q,1} + \|\mathcal{H}_q(h_{q,\psi}f)\|_{q,1} = \|h_{q,\psi}f\|_{q,1} + \|h_{q,\widetilde{\psi}}(\mathcal{H}_q f)\|_{q,1} \\ &\leq \|f\|_{q,1} \|\psi\|_{q,1} + \|\mathcal{H}_q f\|_{q,1} \|\widetilde{\psi}\|_{q,1} \\ &= \|f\|_{q,1} \|\psi\|_{q,1} + \|\mathcal{H}_q f\|_{q,1} \|\psi\|_{q,1} = (\|f\|_{q,1} + \|\mathcal{H}_q f\|_{q,1}) \|\psi\|_{q,1} \\ &= \|f\|_{\mathbb{H}_q^1} \|\psi\|_{q,1}.\end{aligned}\quad \square$$

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BI-UNIVALENT FUNCTION SUBCLASSES WITH
 (p, q) -DERIVATIVE OPERATOR LINKED TO HORADAM
POLYNOMIALS

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ABSTRACT. In the open unit disk $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$, two subclasses of bi-univalent functions related to Horadam polynomials are presented and examined in this paper. For functions belonging to the recently established classes, we obtain the estimates of the first two coefficients. Furthermore, an estimate of the Fekete-Szegő problem is provided for functions in these classes. We also provide some observations and draw relevant connections to earlier research.

1. PRELIMINARIES

The quantum calculus is very important because it is applied in numerous subfields within computer science, mathematics, physics, and other relevant disciplines. The significance of the q -derivative operator is demonstrated by its ability to be applied to various subclasses of holomorphic functions. Jackson [27] first investigated the q -analogue of the derivative and integral operator in 1908, along with some of its uses. Later, in [26], the concept of q -extension of the family of starlike functions was introduced. Subsequently, q -calculus was studied in the context of theory of univalent functions by Srivastava [38] and several mathematicians: the q -analogue of the Ruscheweyh operator was presented in [29]; authors looked at some of its applications for multivalent functions in [7, 8]; the convolution concept was used in [56] to establish the q -starlike functions linked to the generalized conic domain. Many authors have recently published several articles on sbclases of q -starlike functions

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and other related classes (see [2, 17, 19, 31, 42–44, 47]). Scholars investigating these topics might find Srivastava's expository review paper [39] to be beneficial.

Furthermore, the extension of the q -calculus to the (p, q) -calculus, was taken into consideration by the researchers. The (p, q) -calculus, which includes the (p, q) -number, is first examined around the same time (1991) and subsequently on its own by [6, 11, 12, 54]. In [12], the (p, q) -number was explored as a way to unify or generalize various forms of q -oscillator algebras. The investigation of the (p, q) -number in [11] allows for the construction of a (p, q) -Harmonic oscillator. The (p, q) -numbers are investigated in [54] in order to calculate the (p, q) -Stirling numbers. Fibonacci oscillators were studied with the presentation of the (p, q) -number in [6]. Consequently, many mathematical, physical, and chemical problems require knowledge of (p, q) -calculus. Expanding upon the previously mentioned papers, numerous scientist have studied the (p, q) -calculus in a variety of research fields since 1991. A syntax for embedding the q -series into a (p, q) -series was given by the results in [28]. Additionally, they looked into (p, q) -hypergeometric series and discovered some outcomes that matched (p, q) -extensions of the well-known q -identities. The q -identities are extended correspondingly to yield the (p, q) -series (see, e.g., [5]). We provide some basic definitions of the (p, q) -calculus concepts used in this paper. The (p, q) -bracket number is given by $[j]_{p,q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + \cdots + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p - q}$, $p \neq q$, which is an extension of q -number (see [27]), that is $[j]_q = \frac{1 - q^j}{1 - q}$, $q \neq 1$. Note that $[j]_{p,q}$ is symmetric and if $p = 1$, then $[j]_{p,q} = [j]_q$.

Let $\{\zeta \in \mathbb{C} : |\zeta| < 1\} = \mathfrak{D}$ be the open unit disk, and let the set of complex numbers be $\mathbb{C}$. Let $\mathbb{N}$ represent the natural number set and $\mathbb{R}$ the real number set.

Definition 1.1 ([53]). Let $0 < q < p \leq 1$ and g be a function defined on $\mathbb{C}$. Then the (p, q) -derivative of g is defined by

$$D_{p,q}g(\zeta) = \frac{g(p\zeta) - g(q\zeta)}{(p - q)\zeta}, \quad \zeta \neq 0,$$

and $D_{p,q}g(0) = g'(0)$, provided $g'(0)$ exists.

We note that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}$ and $D_{p,q}\ln(\zeta) = \frac{\ln(p/q)}{(p-q)\zeta}$. Also, we observe that $[j]_{p,q} \rightarrow j$ if $p = 1$ and $q \rightarrow 1^-$. Therefore, $D_{p,q}g(\zeta) \rightarrow g'(\zeta)$ as $p = 1$ and $q \rightarrow 1^-$. Any function's (p, q) -derivative is a linear operator. More accurately $D_{p,q}(ag(\zeta) + bh(\zeta)) = aD_{p,q}g(\zeta) + bD_{p,q}h(\zeta)$, for any constants a and b . The product rule and quotient rule are satisfied by the (p, q) -derivative (see [35]). The exponential functions are used to define the (p, q) -analogues of many functions, including sine, cosine, and tangent, in the same way as their well-known Euler expressions. In addition, the (p, q) -derivatives of these functions have been examined by Duran et al. [15].

The set of functions g that are regular in $\mathfrak{D}$ and have the following form is represented by $\mathcal{A}$:

$$(1.1) \quad g(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j, \quad \zeta \in \mathfrak{D},$$

with $g'(0) - 1 = 0 = g(0)$. $\mathcal{S}$ represents a sub-set of $\mathcal{A}$ that consists of univalent functions in $\mathfrak{D}$. If $g \in \mathcal{A}$ is of the form (1.1), then

$$(1.2) \quad D_{p,q}g(\zeta) = 1 + \sum_{j=2}^{\infty} [j]_{p,q} d_j \zeta^{j-1}, \quad \zeta \in \mathfrak{D}.$$

We say that $\mathbf{g}_1$ is subordinate to $\mathbf{g}_2$ for $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{A}$ regular in $\mathfrak{D}$, if there is a Schwarz function $\psi(\zeta)$ that is regular in $\mathfrak{D}$ with $|\psi(\zeta)| < 1$, $\psi(0) = 0$, such that $\mathbf{g}_1(\zeta) = \mathbf{g}_2(\psi(\zeta))$ (see [16]). The notation $\mathbf{g}_1 \prec \mathbf{g}_2$ or $\mathbf{g}_1(\zeta) \prec \mathbf{g}_2(\zeta)$, $\zeta \in \mathfrak{D}$, denotes this subordination. Specifically, when $\mathbf{g}_2 \in \mathcal{S}$, we have

$$\mathbf{g}_1(\zeta) \prec \mathbf{g}_2(\zeta) \Leftrightarrow \mathbf{g}_1(0) = \mathbf{g}_2(0) \text{ and } \mathbf{g}_1(\mathfrak{D}) \subset \mathbf{g}_2(\mathfrak{D}).$$

The Koebe theorem (see [16]) states that the inverse of each function g in $\mathcal{S}$ is given by

$$(1.3) \quad g^{-1}(w) = w - d_2 w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2 d_3 + d_4)w^4 + \cdots = f(w)$$

satisfying $\zeta = g^{-1}(g(\zeta))$, $g(g^{-1}(w)) = w$, $|w| < r_0(g)$ and $1/4 \leq r_0(g)$, $\zeta, w \in \mathfrak{D}$. In $\mathfrak{D}$, a member g of $\mathcal{A}$ given by (1.1) is called bi-univalent if $g \in \mathcal{S}$ and $g^{-1} \in \mathcal{S}$. The set of such functions in $\mathfrak{D}$ is represented by σ . $\frac{1}{2} \log \left(\frac{1+\zeta}{1-\zeta} \right)$, $-\log(1-\zeta)$ and $\frac{\zeta}{1-\zeta}$ are some of the functions in the σ family. Nevertheless, despite being in $\mathcal{S}$, $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function do not belong to σ . For a concise analysis and to discover some of the remarkable characteristics of the family σ , see [9, 10, 30, 51] and the citation provided in these papers. Similar to the well-known subclasses of the family $\mathcal{S}$, Srivastava et al. [45] have introduced a number of subclasses of the family σ . In reality, many writers have since investigated a variety of alternative subfamilies of σ as follow-ups to the aforementioned subfamilies (see, for example [14, 20, 21, 41, 52]). The majority of these publications focus on the analysis of the Fekete-Szegő problem of functions in distinct σ subclasses.

The (p, q) -calculus was used previously to study several subclasses of the class $\mathcal{S}$ and the class σ . In [46], the subordination principle is used to define the classes of (p, q) -starlike and (p, q) -convex functions. Novel subclasses of the class σ associated with (p, q) -differential operators have also been presented and examined in a number of studies (refer to [3, 4, 13, 23, 32, 33]).

The Horadam polynomials $\mathcal{H}_j(\varkappa, u, v; r, s)$ (or $\mathcal{H}_j(\varkappa)$) have recently been studied and quantified using the recurrence relation given below by Horzum and Koçer [24] (see also [25]):

$$(1.4) \quad \mathcal{H}_j(\varkappa) = r\varkappa\mathcal{H}_{j-1}(\varkappa) + s\mathcal{H}_{j-2}(\varkappa),$$

with $\mathcal{H}_1(\varkappa) = u$, $\mathcal{H}_2(\varkappa) = v\varkappa$, where $\varkappa, r, s, u, v \in \mathbb{R}$, and $j \in \mathbb{N} \setminus \{1, 2\}$. It is evident from (1.4) that

$$(1.5) \quad \mathcal{H}_3(\varkappa) = rv\varkappa^2 + su = Z.$$

For $j \in \mathbb{N}$, the sequence $\mathcal{H}_j(\varkappa)$, has the following generating function (see [24]):

$$(1.6) \quad \mathbf{H}(\varkappa, \zeta) := \sum_{j=1}^{\infty} \mathcal{H}_j(\varkappa) \zeta^{j-1} = \frac{(v - ur)\varkappa\zeta + u}{1 - r\varkappa\zeta - s\zeta^2},$$

where $\varkappa \in \mathbb{R}$, $\zeta \in \mathbb{C}$ with $\varkappa \neq \Re(\zeta)$.

For specific selections of u, v, r and s , Horadam polynomials $\mathcal{H}_j(\varkappa, u, v, ;, r, s)$ leads to various known polynomials (see [40]). Interesting findings regarding coefficient estimates and Fekete-Szegő functional [18] have been found in [22, 34, 36, 48, 49, 55] for members of certain subclasses of σ associated with Horadam polynomials.

We present two new subclasses of σ subordinate to polynomials $\mathcal{H}_j(\varkappa)$ as in (1.4) and its generating function (1.6). The Fekete-Szegő functional on specific subclasses of σ and the patterns in the citations discussed above on coefficient-related problems serve as the inspiration for these families. $\mathbf{H}(\varkappa, \zeta)$ is as in (1.6), $g^{-1}(w) = f(w)$ as in (1.3), $\varkappa \in \mathbb{R}$, $\zeta \in \mathfrak{D}$ and $w \in \mathfrak{D}$ are assumed throughout this paper, unless otherwise mentioned.

Definition 1.2. A function g in σ that possesses the series (1.1) is said to be a part of the class $\mathfrak{Y}_{\sigma, p, q}^{\tau, \delta}(\nu, \varkappa)$, $\nu \geq 1$, $0 < \delta \leq 1$ and $\tau \geq 1$, if

$$\frac{1}{2} \left\{ \frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^{\tau} + (1 - \nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^{\tau} + (1 - \nu)}{D_{p,q}g(\zeta)} \right)^{\frac{1}{\delta}} \right\} \\ \prec 1 - u + \mathbf{H}(\varkappa, \zeta)$$

and

$$\frac{1}{2} \left\{ \frac{\nu[D_{p,q}(w D_{p,q}f(w))]^{\tau} + (1 - \nu)}{D_{p,q}f(w)} + \left(\frac{\nu[D_{p,q}(w D_{p,q}f(w))]^{\tau} + (1 - \nu)}{D_{p,q}f(w)} \right)^{\frac{1}{\delta}} \right\} \\ \prec 1 - u + \mathbf{H}(\varkappa, w).$$

For particular selections of p, q, ν and τ , the family $\mathfrak{Y}_{\sigma, p, q}^{\tau, \delta}(\nu, \varkappa)$ includes many new and existing subfamilies of σ . This is shown below.

1. $\mathfrak{H}_{\sigma, p, q}^{\delta}(\nu, \varkappa) \equiv \mathfrak{Y}_{\sigma, p, q}^{1, \delta}(\nu, \varkappa)$, $\nu \geq 1$, $0 < \delta \leq 1$, is the set of members g of σ that satisfy

$$\frac{1}{2} \left\{ \frac{\nu D_{p,q}(\zeta D_{p,q}g(\zeta)) + (1 - \nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu D_{p,q}(\zeta D_{p,q}g(\zeta)) + (1 - \nu)}{D_{p,q}g(\zeta)} \right)^{\frac{1}{\delta}} \right\} \\ \prec 1 - u + \mathbf{H}(\varkappa, \zeta)$$

and

$$\frac{1}{2} \left\{ \frac{\nu D_{p,q}(w D_{p,q} f(w)) + (1 - \nu)}{D_{p,q} f(w)} + \left(\frac{\nu D_{p,q}(w D_{p,q} f(w)) + (1 - \nu)}{D_{p,q} f(w)} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathcal{K}, w).$$

2. $\mathfrak{I}_{\sigma,p,q}^{\tau,\delta}(\mathcal{K}) \equiv \mathfrak{Y}_{\sigma,p,q}^{\tau,\delta}(1, \mathcal{K})$, $0 < \delta \leq 1$ and $\tau \geq 1$, is the set of elements g of σ that satisfy

$$\frac{1}{2} \left\{ \frac{[D_{p,q}(\zeta D_{p,q} g(\zeta))]^\tau}{D_{p,q} g(\zeta)} + \left(\frac{[D_{p,q}(\zeta D_{p,q} g(\zeta))]^\tau}{D_{p,q} g(\zeta)} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathcal{K}, \zeta)$$

and

$$\frac{1}{2} \left\{ \frac{[D_{p,q}(w D_{p,q} f(w))]^\tau}{D_{p,q} f(w)} + \left(\frac{[D_{p,q}(w D_{p,q} f(w))]^\tau}{D_{p,q} f(w)} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathcal{K}, w).$$

3. If $p = 1$ and $q \rightarrow 1^-$ in the set $\mathfrak{Y}_{\sigma,p,q}^{\tau,\delta}(\nu, \mathcal{K})$, then we obtain a subset $\Upsilon_{\sigma}^{\tau,\delta}(\nu, \mathcal{K})$, $\tau \geq 1$, $0 < \delta \leq 1$, $\nu \geq 1$, which is the collection of members g of σ that satisfy

$$\frac{1}{2} \left\{ \frac{\nu[(\zeta g'(\zeta))']^\tau + (1 - \nu)}{g'(\zeta)} + \left(\frac{\nu[(\zeta g'(\zeta))']^\tau + (1 - \nu)}{g'(\zeta)} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathcal{K}, \zeta)$$

and

$$\frac{1}{2} \left\{ \frac{\nu[(w f'(w))']^\tau + (1 - \nu)}{f'(w)} + \left(\frac{\nu[(w f'(w))']^\tau + (1 - \nu)}{f'(w)} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathcal{K}, w).$$

4. $\mathfrak{Z}_{\sigma,p,q}^{\tau}(\nu, \mathcal{K}) \equiv \mathfrak{Y}_{\sigma,p,q}^{\tau,1}(\nu, \mathcal{K})$, $\nu \geq 1$ and $\tau \geq 1$ is the set of elements g of σ that satisfy

$$\left\{ \frac{\nu[D_{p,q}(\zeta D_{p,q} g(\zeta))]^\tau + (1 - \nu)}{D_{p,q} g(\zeta)} \right\} \prec 1 - u + H(\mathcal{K}, \zeta)$$

and

$$\left\{ \frac{\nu[D_{p,q}(w D_{p,q} f(w))]^\tau + (1 - \nu)}{D_{p,q} f(w)} \right\} \prec 1 - u + H(\mathcal{K}, w).$$

Remark 1.1. i) $\mathfrak{H}_{\sigma,p,q}^{\delta}(1, \mathcal{K}) \equiv \mathfrak{I}_{\sigma,p,q}^{1,\delta}(\mathcal{K})$;

ii) $\mathfrak{H}_{\sigma,p,q}^1(\nu, \mathcal{K}) \equiv \mathfrak{Z}_{\sigma,p,q}^1(\nu, \mathcal{K})$;

iii) $\mathfrak{Z}_{\sigma,p,q}^{\tau}(1, \mathcal{K}) \equiv \mathfrak{I}_{\sigma,p,q}^{\tau,1}(\mathcal{K})$.

Definition 1.3. A function g in σ that possesses the series (1.1) is said to be a part of the class $\mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma, \mathcal{K})$, $0 \leq \gamma \leq 1$, $0 < \delta \leq 1$ and $\tau \geq 1$, if

$$\frac{1}{2} \left\{ \frac{\zeta(D_{p,q} g(\zeta))^\tau}{\gamma g(\zeta) + (1 - \gamma)\zeta} + \left(\frac{\zeta(D_{p,q} g(\zeta))^\tau}{\gamma g(\zeta) + (1 - \gamma)\zeta} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathcal{K}, \zeta)$$

and

$$\frac{1}{2} \left\{ \frac{w(D_{p,q}f(w))^\tau}{\gamma f(w) + (1-\gamma)w} + \left(\frac{w(D_{p,q}f(w))^\tau}{\gamma f(w) + (1-\gamma)w} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathfrak{x}, w).$$

For particular selections of γ , τ , and δ , the family $\mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma, \mathfrak{x})$ includes many new and preexisting subfamilies of σ , as shown below.

1. $\mathfrak{C}_{\sigma,p,q}^{\tau,\delta}(\mathfrak{x}) \equiv \mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(0, \mathfrak{x})$, $0 < \delta \leq 1$ and $\tau \geq 1$, is the set of members g of σ that satisfy

$$\frac{1}{2} \left((D_{p,q}g(\zeta))^\tau + (D_{p,q}g(\zeta))^{\frac{\tau}{\delta}} \right) \prec 1 - u + H(\mathfrak{x}, \zeta)$$

and

$$\frac{1}{2} \left((D_{p,q}f(w))^\tau + D_{p,q}f(w)^{\frac{\tau}{\delta}} \right) \prec 1 - u + H(\mathfrak{x}, w).$$

2. $\mathfrak{D}_{\sigma,p,q}^{\tau,\delta}(x) \equiv \mathfrak{T}_{\Sigma,p,q}^{\tau,\delta}(1, \mathfrak{x})$, $0 < \delta \leq 1$ and $\tau \geq 1$, is the family of elements g of σ that satisfy

$$\frac{1}{2} \left\{ \frac{\zeta(D_{p,q}g(\zeta))^\tau}{g(\zeta)} + \left(\frac{\zeta(D_{p,q}g(\zeta))^\tau}{g(\zeta)} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathfrak{x}, \zeta)$$

and

$$\frac{1}{2} \left\{ \frac{w(D_{p,q}f(w))^\tau}{f(w)} + \left(\frac{w(D_{p,q}f(w))^\tau}{f(w)} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathfrak{x}, w).$$

3. If $p = 1$ and $q \rightarrow 1^-$ in the class $\mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\nu, \mathfrak{x})$, then we have a subset $\Gamma_\sigma^{\tau,\delta}(\nu, \mathfrak{x})$, $\tau \geq 1$, $0 < \delta \leq 1$, $0 \leq \gamma \leq 1$ of $g \in \sigma$ that satisfy

$$\frac{1}{2} \left\{ \frac{\zeta(g'(\zeta))^\tau}{\gamma g(\zeta) + (1-\gamma)\zeta} + \left(\frac{\zeta(g'(\zeta))^\tau}{\gamma g(\zeta) + (1-\gamma)\zeta} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathfrak{x}, \zeta)$$

and

$$\frac{1}{2} \left\{ \frac{w(f'(w))^\tau}{\gamma f(w) + (1-\gamma)w} + \left(\frac{w(f'(w))^\tau}{\gamma f(w) + (1-\gamma)w} \right)^{\frac{1}{\delta}} \right\} \prec 1 - u + H(\mathfrak{x}, w).$$

4. $\mathfrak{D}_{\sigma,p,q}^\tau(\gamma, \mathfrak{x}) \equiv \mathfrak{T}_{\sigma,p,q}^{\tau,1}(\gamma, \mathfrak{x})$, $0 \leq \gamma \leq 1$ and $\tau \geq 1$, is the group of $g \in \sigma$ that satisfy

$$\frac{\zeta(D_{p,q}g(\zeta))^\tau}{\gamma g(\zeta) + (1-\gamma)\zeta} \prec 1 - u + H(\mathfrak{x}, \zeta)$$

and

$$\frac{w(D_{p,q}f(w))^\tau}{\gamma f(w) + (1-\gamma)w} \prec 1 - u + H(\mathfrak{x}, w).$$

Remark 1.2. i) $\mathfrak{D}_{\sigma,p,q}^{\tau,1}(x) \equiv \mathfrak{D}_{\sigma,p,q}^\tau(1, \mathfrak{x})$;

ii) $\Gamma_\sigma^{\tau,1}(\nu, \mathfrak{x}) \equiv \mathfrak{D}_{\sigma,p=1,q \rightarrow 1^-}^\tau(\gamma, \mathfrak{x})$.

In Section 2, we find estimates for $|d_2|$, $|d_3|$ and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, for functions in $\mathfrak{Y}_{\sigma,p,q}^{\tau,\delta}(\nu, \varkappa)$. In Section 3, we find estimates for $|d_2|$, $|d_3|$ and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, for functions in $\mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma, \varkappa)$. Interesting results are also presented along with relevant connections to the published research.

2. RESULTS FOR THE CLASS $\mathfrak{Y}_{\sigma,p,q}^{\tau,\delta}(\nu, \varkappa)$

First, we determine the coefficient estimates for any function $g \in \mathfrak{Y}_{\sigma,p,q}^{\tau,\delta}(\nu, \varkappa)$, the class as defined in Definition 1.2.

Theorem 2.1. *Let $0 < \delta \leq 1$, $\nu \geq 1$ and $\tau \geq 1$. If $g \in \mathfrak{Y}_{\sigma,p,q}^{\tau,\delta}(\nu, \varkappa)$, then*

$$(2.1) \quad |d_2| \leq \frac{2\delta|\nu\varkappa|\sqrt{|v\varkappa|}}{\sqrt{|(2\delta(\delta+1)X + (1-\delta)W^2[2]_{p,q}^2)(v\varkappa)^2 - (\delta+1)^2W^2[2]_{p,q}^2Z|}}},$$

$$(2.2) \quad |d_3| \leq \frac{4\delta^2(v\varkappa)^2}{(\delta+1)^2W^2[2]_{p,q}^2} + \frac{2\delta|\nu\varkappa|}{(\delta+1)U[3]_{p,q}},$$

and for $\mu \in \mathbb{R}$

$$(2.3) \quad |d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta|\nu\varkappa|}{(\delta+1)U[3]_{p,q}}, & |1-\mu| \leq J, \\ \frac{4\delta^2|\nu\varkappa|^3|1-\mu|}{|(2\delta(\delta+1)X + (1-\delta)W^2[2]_{p,q}^2)(v\varkappa)^2 - (\delta+1)^2W^2[2]_{p,q}^2Z|}, & |1-\mu| \geq J, \end{cases}$$

where

$$(2.4) \quad J = \left| \frac{(2\delta(\delta+1)X + (1-\delta)W^2[2]_{p,q}^2)v^2\varkappa^2 - (\delta+1)^2W^2[2]_{p,q}^2Z}{2\delta(1+\delta)U[3]_{p,q}v^2\varkappa^2} \right|,$$

$$(2.5) \quad X = U[3]_{p,q} + V[2]_{p,q}^2,$$

$$(2.6) \quad U = \nu\tau[3]_{p,q} - 1,$$

$$(2.7) \quad V = 1 - \nu\tau[2]_{p,q} + \frac{\nu\tau(\tau-1)[2]_{p,q}^2}{2},$$

$$(2.8) \quad W = \nu\tau[2]_{p,q} - 1,$$

and Z is as in (1.5).

Proof. Let $g \in \mathfrak{Y}_{\sigma,p,q}^{\tau,\delta}(\gamma, \varkappa)$. Then, because of Definition 1.3, we obtain

$$(2.9) \quad \frac{1}{2} \left\{ \frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} \right)^{\frac{1}{\delta}} \right\} \\ = 1 - u + H(\varkappa, \mathbf{m}(\zeta))$$

and

$$(2.10) \quad \frac{1}{2} \left\{ \frac{\nu[D_{p,q}(w D_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} + \left(\frac{\nu[D_{p,q}(w D_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} \right)^{\frac{1}{\delta}} \right\} \\ = 1 - u + H(\varkappa, \mathbf{n}(w)),$$

where

(2.11) $\mathbf{m}(\zeta) = m_1\zeta + m_2\zeta^2 + m_3\zeta^3 + \dots$ and $\mathbf{n}(w) = n_1w + n_2w^2 + n_3w^3 + \dots$,
are some functions holomorphic in $\mathfrak{D}$ with $|\mathbf{m}(\zeta)| < 1$, $|\mathbf{n}(w)| < 1$ and is known that

$$(2.12) \quad |\mathbf{m}_i| \leq 1 \quad \text{and} \quad |\mathbf{n}_i| \leq 1, \quad i \in \mathbb{N}.$$

From (2.9)–(2.11), it follows that

$$(2.13) \quad \frac{1}{2} \left\{ \frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} \right)^{\frac{1}{\delta}} \right\} \\ = 1 - u + \mathcal{H}_1(\mathcal{K}) + \mathcal{H}_2(\mathcal{K})\mathbf{m}(\zeta) + \mathcal{H}_3(\mathcal{K})\mathbf{m}^2(\zeta) + \dots$$

and

$$(2.14) \quad \frac{1}{2} \left\{ \frac{\nu[D_{p,q}(w D_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} + \left(\frac{\nu[D_{p,q}(w D_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} \right)^{\frac{1}{\delta}} \right\} \\ = 1 - u + \mathcal{H}_1(\mathcal{K}) + \mathcal{H}_2(\mathcal{K})\mathbf{n}(w) + \mathcal{H}_3(\mathcal{K})\mathbf{n}^2(w) + \dots$$

In the light of (1.4), we determine from (2.13) and (2.14) that

$$(2.15) \quad \frac{1}{2} \left\{ \frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} \right)^{\frac{1}{\delta}} \right\} \\ = 1 + \mathcal{H}_2(\mathcal{K})\mathbf{m}_1\zeta + [\mathcal{H}_2(\mathcal{K})\mathbf{m}_2 + \mathcal{H}_3(\mathcal{K})\mathbf{m}_1^2]\zeta^2 + \dots$$

and

$$(2.16) \quad \frac{1}{2} \left\{ \frac{\nu[D_{p,q}(w D_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} + \left(\frac{\nu[D_{p,q}(w D_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} \right)^{\frac{1}{\delta}} \right\} \\ = 1 + \mathcal{H}_2(\mathcal{K})\mathbf{n}_1w + [\mathcal{H}_2(\mathcal{K})\mathbf{n}_2 + \mathcal{H}_3(\mathcal{K})\mathbf{n}_1^2]w^2 + \dots$$

Using (1.2) and comparing (2.15) and (2.16), we have

$$(2.17) \quad \frac{(\delta+1)W[2]_{p,q}}{2\delta} d_2 = \mathcal{H}_2(\mathcal{K})\mathbf{m}_1,$$

$$(2.18) \quad \left(\frac{\delta+1}{2\delta} \right) (U[3]_{p,q}d_3 + V[2]_{p,q}^2d_2^2) + \left(\frac{1-\delta}{4\delta^2} \right) W^2[2]_{p,q}^2d_2^2 = \mathcal{H}_2(\mathcal{K})\mathbf{m}_2 + \mathcal{H}_3(\mathcal{K})\mathbf{m}_1^2,$$

$$(2.19) \quad -\frac{(\delta+1)W[2]_{p,q}}{2\delta} d_2 = \mathcal{H}_2(\mathcal{K})\mathbf{n}_1$$

and

$$(2.20) \quad \left(\frac{\delta+1}{2\delta} \right) (U[3]_{p,q}(2d_2^2 - d_3) + V[2]_{p,q}^2d_2^2) + \left(\frac{1-\delta}{4\delta^2} \right) W^2[2]_{p,q}^2d_2^2 = \mathcal{H}_2(\mathcal{K})\mathbf{n}_2 + \mathcal{H}_3(\mathcal{K})\mathbf{n}_1^2,$$

where U , V and W are as mentioned in (2.6), (2.7) and (2.8), respectively. From (2.17) and (2.19), we easily obtain

$$(2.21) \quad \mathbf{m}_1 = -\mathbf{n}_1,$$

and also

$$(2.22) \quad \frac{(\delta + 1)^2 W^2 [2]_{p,q}^2}{2\delta^2} d_2^2 = (\mathbf{m}_1^2 + \mathbf{n}_1^2) (\mathcal{H}_2(\varkappa))^2.$$

The bound on $|d_2|$ is obtained by adding (2.18) and (2.20)

$$(2.23) \quad \left[\left(\frac{\delta + 1}{\delta} \right) X + \left(\frac{1 - \delta}{2\delta^2} \right) W^2 [2]_{p,q}^2 \right] d_2^2 = \mathcal{H}_2(\varkappa) (\mathbf{m}_2 + \mathbf{n}_2) + \mathcal{H}_3(\varkappa) (\mathbf{m}_1^2 + \mathbf{n}_1^2),$$

where X is as in (2.5). The value of $\mathbf{m}_1^2 + \mathbf{n}_1^2$ from (2.22) is substituted in (2.23), yielding

$$(2.24) \quad d_2^2 = \frac{2\delta^2 \mathcal{H}_2^3(\varkappa) (\mathbf{m}_2 + \mathbf{n}_2)}{(2\delta(\delta + 1)X + (1 - \delta)W^2 [2]_{p,q}^2) \mathcal{H}_2^2(\varkappa) - (\delta + 1)^2 W^2 [2]_{p,q}^2 \mathcal{H}_3(\varkappa)}.$$

Using (1.5) and applying (2.12) to the coefficients $\mathbf{m}_2$ and $\mathbf{n}_2$ yields (2.1).

We deduct (2.20) from (2.18) to get the bound on $|d_3|$:

$$(2.25) \quad d_3 = d_2^2 + \frac{\mathcal{H}_2(\varkappa) (\mathbf{m}_2 - \mathbf{n}_2)}{\left(\frac{\delta + 1}{\delta} \right) U[3]_{p,q}}.$$

Then in view of (2.21) and (2.22), (2.25) becomes

$$d_3 = \frac{2\delta^2 \mathcal{H}_2^2(\varkappa) (\mathbf{m}_1^2 + \mathbf{n}_1^2)}{(\delta + 1)^2 W^2 [2]_{p,q}^2} + \frac{\delta \mathcal{H}_2(\varkappa) (\mathbf{m}_2 - \mathbf{n}_2)}{(\delta + 1) U[3]_{p,q}},$$

and applying (2.12) for the coefficients $\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}_1$ and $\mathbf{n}_2$ we get (2.2).

From (2.24) and (2.25), for $\mu \in \mathbb{R}$, we get in view of (1.4) that

$$\begin{aligned} & |d_3 - \mu d_2^2| \\ &= |\mathcal{H}_2(\varkappa)| \left| \left(\mathfrak{B}_2(\mu, \varkappa) + \frac{\delta}{(\delta + 1) U[3]_{p,q}} \right) \mathbf{m}_2 + \left(\mathfrak{B}_2(\mu, \varkappa) - \frac{\delta}{(\delta + 1) U[3]_{p,q}} \right) \mathbf{n}_2 \right|, \end{aligned}$$

where

$$\mathfrak{B}_2(\mu, \varkappa) = \frac{2\delta^2 (1 - \mu) \mathcal{H}_2^2(\varkappa)}{(2\delta(\delta + 1)X + (1 - \delta)W^2 [2]_{p,q}^2) \mathcal{H}_2^2(\varkappa) - (\delta + 1)^2 W^2 [2]_{p,q}^2 \mathcal{H}_3(\varkappa)}.$$

Clearly

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta |\mathcal{H}_2(\varkappa)|}{(\delta + 1) U[3]_{p,q}}, & 0 \leq |\mathfrak{B}_2(\mu, \varkappa)| \leq \frac{\delta}{(\delta + 1) U[3]_{p,q}}, \\ 2|\mathcal{H}_2(\varkappa)| |\mathfrak{B}_2(\mu, \varkappa)|, & |\mathfrak{B}_2(\mu, \varkappa)| \geq \frac{\delta}{(\delta + 1) U[3]_{p,q}}, \end{cases}$$

which results in (2.3) with J as in (2.4) and Z as in (1.5). $\square$

Corollary 2.1. *Let us assume that $\tau = 1$ in Theorem 2.1. Then, for any function $g \in \mathfrak{H}_{\sigma,p,q}^{\delta}(\nu, \varkappa)$ the upper bounds of $|d_2|$, $|d_3|$ and $|d_3 - \mu d_2^2|$, $\mu \in \mathfrak{R}$, are given by (2.1), (2.2) and (2.3), respectively, with $U = U_1 = \nu[3]_{p,q} - 1$, $V = V_1 = 1 - \nu[2]_{p,q}$, $W = W_1 = -V_1$ and $X = X_1 = U_1[3]_{p,q} + V_1[2]_{p,q}^2$. To change U_1 , V_1 , W_1 and X_1 for U , V , W and X , respectively, for J in (2.4) and Z , as stated in (1.5).*

Corollary 2.2. *Let us assume that $\nu = 1$ in Theorem 2.1. Then for any function $g \in \mathfrak{H}_{\sigma,p,q}^{\tau,\delta}(\varkappa)$ the upper bounds of $|d_2|$, $|d_3|$ and $|d_3 - \mu d_2^2|$, $\mu \in \mathfrak{R}$, are given by (2.1), (2.2) and (2.3), respectively, with $U = U_2 = \tau[3]_{p,q} - 1$, $V = V_2 = 1 - \tau[2]_{p,q} + \frac{\tau(\tau-1)[2]_{p,q}^2}{2}$, $W = W_2 = \tau[2]_{p,q} - 1$ and $X = X_2 = U_2[3]_{p,q} + V_2[2]_{p,q}^2$. For J in (2.4), U_2 , V_2 , W_2 and X_2 should be used in place of U , V , W and X , respectively, and Z , as stated in (1.5).*

Taking $p = 1$ and $q \rightarrow 1^-$ in the Theorem 2.1, we get the following.

Corollary 2.3. *Let $0 < \delta \leq 1$, $\tau \geq 1$ and $\nu \geq 1$. If $g \in \Upsilon_{\sigma}^{\tau,\delta}(\nu, \varkappa)$, then*

$$|d_2| \leq \frac{\delta|\nu\varkappa|\sqrt{2|\nu\varkappa|}}{\sqrt{[(\delta(\delta+1)Y + 2(1-\delta)(2\nu\tau-1)^2)(\nu\varkappa)^2 - 2(\delta+1)^2(2\nu\tau-1)^2Z]}},$$

$$|d_3| \leq \left(\frac{\delta\nu\varkappa}{(\delta+1)(2\nu\tau-1)} \right)^2 + \frac{2\delta|\nu\varkappa|}{3(\delta+1)(3\nu\tau-1)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta|\nu\varkappa|}{3(\delta+1)(3\nu\tau-1)}, & |1-\mu| \leq J_1, \\ \frac{2\delta^2|\nu\varkappa|^3|1-\mu|}{[(\delta(\delta+1)Y + 2(1-\delta)(2\nu\tau-1)^2)(\nu\varkappa)^2 - 2(\delta+1)^2(2\nu\tau-1)^2Z]}, & |1-\mu| \geq J_1, \end{cases}$$

where

$$J_1 = \left| \frac{(\delta(\delta+1)Y + 2(1-\delta)(2\nu\tau-1)^2)(\nu\varkappa)^2 - 2(\delta+1)^2(2\nu\tau-1)^2Z}{3\delta(\delta+1)(3\nu\tau-1)\nu^2\varkappa^2} \right|,$$

$Y = 8\nu\tau^2 - 7\nu\tau + 1$ and Z is as in (1.5).

Remark 2.1. i) In Corollary 2.3, $\delta = 1$ yields Theorem 2.2 of Swamy and Sailaja [50]. Moreover, we obtain Corollaries 2.3 and 2.4 of Swamy and Sailaja [50], for $\tau = 1$ and $\nu = 1$, respectively.

ii) Using $\delta = \tau = \nu = 1$ in Corollary 2.3 we derive Corollary 1 of Horhan et al. [34].

Taking $\delta = 1$ in the above theorem, we get the following.

Corollary 2.4. *If $g \in \mathfrak{H}_{\sigma,p,q}^{\tau}(\nu, \varkappa)$, $\nu \geq 1$ and $\tau \geq 1$, then*

$$|d_2| \leq \frac{|\nu\varkappa|\sqrt{|\nu\varkappa|}}{\sqrt{|X(\nu\varkappa)^2 - W^2[2]_{p,q}^2Z|}}, \quad |d_3| \leq \frac{(\nu\varkappa)^2}{W^2[2]_{p,q}^2} + \frac{|\nu\varkappa|}{U[3]_{p,q}},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|v\kappa|}{U[3]_{p,q}}, & |1 - \mu| \leq \left| \frac{Xv^2\kappa^2 - W^2[2]^2Z}{U[3]_{p,q}v^2\kappa^2} \right|, \\ \frac{|v\kappa|^3|1-\mu|}{|X(v\kappa)^2 - W^2[2]_{p,q}^2Z|}, & |1 - \mu| \geq \left| \frac{Xv^2\kappa^2 - W^2[2]^2Z}{U[3]_{p,q}v^2\kappa^2} \right|, \end{cases}$$

where X , U , V and W are as detailed in (2.5), (2.6), (2.7) and (2.8), respectively.

Remark 2.2. i) In Corollary 2.4, $q \rightarrow 1^-$ and $p = 1$ yields Theorem 2.2 of Swamy and Sailaja [50]. Moreover, we obtain Corollaries 2.3 and 2.4 of Swamy and Sailaja [50], for $\tau = 1$ and $\nu = 1$, respectively.

ii) Using $\nu = \tau = 1$, $q \rightarrow 1^-$ and $p = 1$ in Corollary 2.4 we derive Corollary 1 of Horhan et al. [34].

3. RESULTS FOR THE CLASS $\mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma, \kappa)$

First, we calculate the coefficient estimates for any function $g \in \mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma, \kappa)$, the class as defined in Definition 1.3.

Theorem 3.1. Let $0 < \delta \leq 1$, $\tau \geq 1$ and $0 \leq \nu \leq 1$. If $g \in \mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma, \kappa)$, then

$$(3.1) \quad |d_2| \leq \frac{2\delta|v\kappa|\sqrt{|v\kappa|}}{\sqrt{|(2\delta(\delta+1)(A+S) + (1-\delta)B^2)(v\kappa)^2 - (\delta+1)^2B^2Z|}},$$

$$(3.2) \quad |d_3| \leq \frac{4\delta^2(v\kappa)^2}{(\delta+1)^2B^2} + \frac{2\delta|v\kappa|}{(\delta+1)A},$$

and for $\mu \in \mathbb{R}$

$$(3.3) \quad |d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta|v\kappa|}{(\delta+1)A}, & |1 - \mu| \leq \mathbb{Q}, \\ \frac{4\delta^2|v\kappa|^3|1-\mu|}{|(2\delta(\delta+1)(A+S) + (1-\delta)B^2)(v\kappa)^2 - (\delta+1)^2B^2Z|}, & |1 - \mu| \geq \mathbb{Q}, \end{cases}$$

where

$$(3.4) \quad \mathbb{Q} = \left| \frac{(2\delta(\delta+1)(A+S) + (1-\delta)B^2)v^2\kappa^2 - (\delta+1)^2B^2Z}{2\delta(1+\delta)Av^2\kappa^2} \right|,$$

$$(3.5) \quad A = \tau[3]_{p,q} - \gamma,$$

$$(3.6) \quad S = \frac{\tau(\tau-1)[2]_{p,q}^2}{2} - \gamma\tau[2]_{p,q} + \gamma^2,$$

$$(3.7) \quad B = \tau[2]_{p,q} - \gamma,$$

and Z is as in (1.5).

Proof. Let $g \in \mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma, \kappa)$. Following that, due to Definition 1.3, we obtain

$$(3.8) \quad \frac{1}{2} \left\{ \frac{\zeta(D_{p,q}g(\zeta))^\tau}{\gamma g(\zeta) + (1-\gamma)\zeta} + \left(\frac{\zeta(D_{p,q}g(\zeta))^\tau}{\gamma g(\zeta) + (1-\gamma)\zeta} \right)^{\frac{1}{\delta}} \right\} = 1 - u + H(\kappa, \mathbf{m}(\zeta))$$

and

$$(3.9) \quad \frac{1}{2} \left\{ \frac{w(D_{p,q}f(w))^\tau}{\gamma f(w) + (1-\gamma)w} + \left(\frac{w(D_{p,q}f(w))^\tau}{\gamma f(w) + (1-\gamma)w} \right)^{\frac{1}{\delta}} \right\} = 1 - u + H(\varkappa, \mathbf{n}(w)),$$

where $\mathbf{m}(\zeta)$ and $\mathbf{n}(w)$ are regular functions as given in (2.11) satisfying (2.12).

By going through the steps in the proof of Theorem 2.1 to obtain (2.13), (2.14), (2.15), and (2.16), one can easily get the following in view (3.8) and (3.9).

$$(3.10) \quad \frac{(\delta+1)B}{2\delta} d_2 = \mathcal{H}_2(\varkappa) \mathbf{m}_1,$$

$$(3.11) \quad \left(\frac{\delta+1}{2\delta} \right) (Ad_3 + Sd_2^2) + \left(\frac{1-\delta}{4\delta^2} \right) B^2 d_2^2 = \mathcal{H}_2(\varkappa) \mathbf{m}_2 + \mathcal{H}_3(\varkappa) \mathbf{m}_1^2,$$

$$(3.12) \quad -\frac{(\delta+1)B}{2\delta} d_2 = \mathcal{H}_2(\varkappa) \mathbf{n}_1$$

and

$$(3.13) \quad \left(\frac{\delta+1}{2\delta} \right) (A(2d_2^2 - d_3) + Sd_2^2) + \left(\frac{1-\delta}{4\delta^2} \right) B^2 d_2^2 = \mathcal{H}_2(\varkappa) \mathbf{n}_2 + \mathcal{H}_3(\varkappa) \mathbf{n}_1^2,$$

where A , S and B are as mentioned in (3.5), (3.6) and (3.7), respectively. From (3.10) and (3.12), we easily obtain

$$(3.14) \quad \mathbf{m}_1 = -\mathbf{n}_1,$$

and also

$$(3.15) \quad \frac{(\delta+1)^2 B^2}{2\delta^2} d_2^2 = (\mathbf{m}_1^2 + \mathbf{n}_1^2) (\mathcal{H}_2(\varkappa))^2.$$

We add (3.11) and (3.13) to obtain the bound on $|d_2|$:

$$(3.16) \quad \left(\left(\frac{\delta+1}{\delta} \right) (A+S) + \left(\frac{1-\delta}{2\delta^2} \right) B^2 \right) d_2^2 = \mathcal{H}_2(\varkappa) (\mathbf{m}_2 + \mathbf{n}_2) + \mathcal{H}_3(\varkappa) (\mathbf{m}_1^2 + \mathbf{n}_1^2).$$

The value of $\mathbf{m}_1^2 + \mathbf{n}_1^2$ from (3.15) is substituted in (3.16) to obtain

$$(3.17) \quad d_2^2 = \frac{2\delta^2 \mathcal{H}_2^3(\varkappa) (\mathbf{m}_2 + \mathbf{n}_2)}{(2\delta(\delta+1)(A+S) + (1-\delta)B^2) \mathcal{H}_2^2(\varkappa) - (\delta+1)^2 B^2 \mathcal{H}_3(\varkappa)}.$$

Using (1.5) and applying (2.12) for the coefficients $\mathbf{m}_2$ and $\mathbf{n}_2$, we obtain (3.1).

We subtract (3.13) from (3.11) to get the bound on $|d_3|$:

$$(3.18) \quad d_3 = d_2^2 + \frac{\mathcal{H}_2(\varkappa) (\mathbf{m}_2 - \mathbf{n}_2)}{\left(\frac{\delta+1}{\delta} \right) A}.$$

Then in view of (3.14) and (3.15), (3.18) becomes

$$d_3 = \frac{2\delta^2 \mathcal{H}_2^2(\varkappa) (\mathbf{m}_1^2 + \mathbf{n}_1^2)}{(\delta+1)^2 B^2} + \frac{\delta \mathcal{H}_2(\varkappa) (\mathbf{m}_2 - \mathbf{n}_2)}{(\delta+1)A},$$

and applying (2.12) for the coefficients $\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}_1$ and $\mathbf{n}_2$ we get (3.2).

From (3.17) and (3.18), for $\mu \in \mathbb{R}$, we get in view of (1.4) that

$$|d_3 - \mu d_2^2| = \frac{|\mathcal{H}_2(\varkappa)|}{2} \left| \left(\mathfrak{B}_1(\mu, \varkappa) + \frac{\delta}{(\delta+1)A} \right) \mathfrak{m}_2 + \left(\mathfrak{B}_1(\mu, \varkappa) - \frac{\delta}{(\delta+1)A} \right) \mathfrak{n}_2 \right|,$$

where

$$\mathfrak{B}_1(\mu, \varkappa) = \frac{2\delta^2(1-\mu)\mathcal{H}_2^2(\varkappa)}{(2\delta(\delta+1)(A+S) + (1-\delta)B^2)\mathcal{H}_2^2(\varkappa) - (\delta+1)^2B^2\mathcal{H}_3(\varkappa)}.$$

Clearly,

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|\mathcal{H}_2(\varkappa)|}{(\delta+1)A}, & 0 \leq |\mathfrak{B}_1(\mu, \varkappa)| \leq \frac{\delta}{(\delta+1)A}, \\ |\mathcal{H}_2(\varkappa)| \cdot |\mathfrak{B}_1(\mu, \varkappa)|, & |\mathfrak{B}_1(\mu, \varkappa)| \geq \frac{\delta}{(\delta+1)A}, \end{cases}$$

from which we conclude (3.3) with $\mathbb{Q}$ as in (3.4) and Z as in (1.5). $\square$

Corollary 3.1. *Let $\gamma = 0$ in the above theorem. Then, the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, for any function $g \in \mathfrak{H}_{\sigma,p,q}^\delta(\varkappa)$ are given by (3.1), (3.2) and (3.3), respectively, with $A = A_1 = \tau[3]_{p,q}$, $S = S_1 = \frac{\tau(\tau-1)[2]_{p,q}^2}{2}$ and $B = B_1 = \tau[2]_{p,q}$. A_1 , S_1 and B_1 should be used in place of A , S and B , respectively, for $\mathbb{Q}$ in (3.4), and Z is as stated in (1.5).*

Corollary 3.2. *Let $\gamma = 1$ in the above theorem. Then, the upper bounds of $|d_2|$, $|d_3|$ and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, for any function $g \in \mathfrak{J}_{\sigma,p,q}^{\tau,\delta}(\varkappa)$ are given by (3.1), (3.2) and (3.3), respectively, with $A = A_2 = \tau[3]_{p,q} - 1$, $S = S_2 = 1 + \frac{\tau(\tau-1)[2]_{p,q}^2}{2} - \tau[2]_{p,q}$ and $B = B_2 = \tau[2]_{p,q} - 1$. A_2 , S_2 and B_2 should be used in place of A , S and B , respectively, for $\mathbb{Q}$ in (3.4), and Z is as stated in (1.5).*

Taking $p = 1$ and $q \rightarrow 1^-$ in the Theorem 3.1, we get the following.

Corollary 3.3. *Let $0 < \delta \leq 1$, $\tau \geq 1$ and $0 \leq \gamma \leq 1$. If $g \in \Gamma_{\sigma}^{\tau,\delta}(\gamma, \varkappa)$, then*

$$|d_2| \leq \frac{2\delta|v\varkappa|\sqrt{2|v\varkappa|}}{\sqrt{|2\delta(\delta+1)F(v\varkappa)^2 - (\delta+1)^2(2\tau-\gamma)^2Z|}},$$

$$|d_3| \leq \left(\frac{2\delta v\varkappa}{(\delta+1)(2\tau-\gamma)} \right)^2 + \frac{2\delta|v\varkappa|}{(\delta+1)(3\tau-\gamma)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta|v\varkappa|}{(\delta+1)(3\tau-\gamma)}, & |1-\mu| \leq \mathbb{Q}_1, \\ \frac{2\delta^2|v\varkappa|^3|1-\mu|}{|2\delta(\delta+1)F(v\varkappa)^2 - (\delta+1)^2(2\tau-\gamma)^2Z|}, & |1-\mu| \geq \mathbb{Q}_1, \end{cases}$$

where

$$\mathbb{Q}_1 = \left| \frac{2\delta(\delta+1)F(v\varkappa)^2 - (\delta+1)^2(2\tau-\gamma)^2Z}{\delta(\delta+1)(3\tau-\gamma)v^2\varkappa^2} \right|,$$

$$F = (2\tau+1)(\tau-\gamma) + \gamma^2 + (1-\delta)(2\tau-\gamma)^2,$$

and Z is as given in (1.5).

Remark 3.1. i) In Corollary 3.3, $\gamma = 1$ and $\tau = 1$ yield Theorem 2 of Srivastava et al. [40].

ii) Corollary 3.3, where $\delta = 1$, yields Corollary 3.3 in [49]. Furthermore, Theorem 2.1 of [1] is obtained when $\gamma = 1$.

iii) In Corollary 3.3, $\delta = 1, \tau = 1$ and $\gamma = 0$ yield Corollary 2 of Horan et al. [34].

Corollary 3.4. *If $g \in \mathfrak{D}_{\sigma,p,q}^{\tau}(\gamma, \varkappa)$, $0 \leq \gamma \leq 1$ and $\tau \geq 1$, then*

$$|d_2| \leq \frac{|v\varkappa|\sqrt{|v\varkappa|}}{\sqrt{|(A+S)(v\varkappa)^2 - B^2Z|}}, \quad |d_3| \leq \frac{(v\varkappa)^2}{B^2} + \frac{|v\varkappa|}{A},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|v\varkappa|}{A}, & |1 - \mu| \leq \left| \frac{(A+S)v^2\varkappa^2 - B^2Z}{Av^2\varkappa^2} \right|, \\ \frac{|v\varkappa|^3|1-\mu|}{|(A+S)(v\varkappa)^2 - B^2Z|}, & |1 - \mu| \geq \left| \frac{(A+S)v^2\varkappa^2 - B^2Z}{Av^2\varkappa^2} \right|, \end{cases}$$

where Z , A , S and B are as mentioned in (1.5), (3.5), (3.6) and (3.7), respectively.

Remark 3.2. i) From Corollary 3.4, letting $p = 1$ and $q \rightarrow 1^-$, we obtain Corollary 3.3, which is demonstrated in [49]. Moreover, Theorem 2.1 of [1] is obtained when $\gamma = 1$.

ii) We obtain Corollary 2 of Horan et al. [34], if we take $p = 1$, $q \rightarrow 1^-$, $\gamma = 0$ and $\tau = 1$ in Corollary 3.4.

4. CONCLUSION

This study establishes upper bounds on $|d_2|$ and $|d_3|$ for functions in two subfamilies of σ related to Horadam polynomials. Moreover, the Fekete-Szegő functional $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$ has been identified for functions in these subfamilies. By varying the parameters in Theorem 2.1 and Theorem 3.1, we have been able to highlight a number of implications. Additionally, pertinent links to the ongoing research are found. Nevertheless, this paper does not address all of the significant subclasses of σ that exist in the literature. For example, authors [32, 37] have examined various subclasses involving operators introduced in (p, q) -calculus. It is recommended that the interested reader review these papers and the associated references.

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BRYANT-SCHNEIDER GROUP OF BASARAB LOOP

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ABSTRACT. A loop $(Q, \circ)$ is called Basarab loop if it is both a left and a right Basarab loop; $(x \circ yx^\rho) \circ xz = x \circ yz$ and $yx \circ (x^\lambda z \circ x) = yz \circ x$ hold for all $x, y, z \in Q$ respectively. In this paper, the characterizations of the Bryant-Schneider group of a Basarab loop are studied using the left and right Basarab loop identities. It is shown that the element, $x^\lambda (x^\rho)$ is in the left (right) nucleus if and only if the middle inner map T_x (inverse T_x^{-1}) is an automorphism. It is revealed that every crypto-automorphism of a Basarab loop is an element of the Bryant-Schneider group. Some related algebraic properties were also characterized. Furthermore, elements of the Bryant-Schneider group of a Basarab loop in terms of pseudo-automorphism and automorphism are also characterized. A subgroup of the Bryant-Schneider group, characterized by the Basarab loop, is established. Finally, a right pseudo-automorphic characterization of the isotopy-isomorphism of a Basarab loop is carried out.

1. INTRODUCTION

Let Q be a non-empty set. We can define a binary operation, denoted by ‘ $\circ$ ’, on this set. If the result of $s \circ t$ is always an element of Q for any s and t in Q , then the pair $(Q, \circ)$ is called a groupoid. Moreover, if the equations $k \circ s = l$ and $t \circ k = l$ have unique solutions s and t for all k and l in Q , then $(Q, \circ)$ is called a quasigroup.

Furthermore, if $(Q, \circ)$ is a quasigroup and there exists a unique element, denoted as the identity $e \in Q$, with the property that for any $s \in Q$ we have $s \circ e = e \circ s = s$,

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then $(Q, \circ)$ is known as a loop. We use st to represent $s \circ t$, with the understanding that ' $\circ$ ' has lower priority than juxtaposition among factors to be multiplied.

Additionally, if $(Q, \circ)$ is a groupoid and k is a fixed element in Q , then the left and right translations of k , denoted by L_k and R_k respectively, are defined as $sL_k = k \circ s$ and $sR_k = s \circ k$ for all s in Q . It is evident that $(Q, \circ)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, the inverse mappings L_k^{-1} and R_k^{-1} exist.

Let

$$k \backslash l = lL_k^{-1} = kM_l \quad \text{and} \quad k/l = kR_l^{-1} = lM_k^{-1},$$

and note that

$$k \backslash l = m \Leftrightarrow k \circ m = l \quad \text{and} \quad k/l = m \Leftrightarrow m \circ l = k.$$

Thus, for any quasigroup $(Q, \circ)$, we have two new binary operations; right division ($/$) and left division ($\backslash$). M_k is the middle translation for any fixed $k \in Q$. Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. Using the operations ($\backslash$) and ($/$), the definition of a loop can be restated as follows.

Definition 1.1. A loop $(Q, \circ, /, \backslash, e)$ is a set Q together with three binary operations ($\circ$), ($/$), ($\backslash$) and one nullary operation e such that

- (i) $k \circ (k \backslash l) = l$, $(l/k) \circ k = l$ for all $k, l \in Q$;
- (ii) $k \backslash k = l/l$ or $e \circ k = k \circ e = k$ for all $k, l \in Q$.

We also stipulate that ($/$) and ($\backslash$) have higher priority than ($\circ$) among factors to be multiplied. For instance, $k \circ l/m$ and $k \circ l \backslash m$ stand for $k(l/m)$ and $a(b \backslash c)$, respectively.

In a loop $(Q, \circ)$ with identity element e , the *left inverse element* of $k \in Q$ is the element $kJ_\lambda = k^\lambda \in Q$ such that

$$k^\lambda \circ k = e,$$

while the *right inverse element* of $k \in Q$ is the element $kJ_\rho = k^\rho \in Q$ such that

$$k \circ k^\rho = e.$$

If $x^\rho = x^\lambda$, then we shall write $J_\lambda = J_\rho = J$ where $xJ = x^{-1}$.

For more study on quasigroup and loop theories, readers can check [10, 11, 18, 20–23, 25, 26, 29, 30].

The study of a Bryant-Schneider group on an arbitrary loop was first introduced in the work of Robinson [28] in 1980. The concept has been extended to different loop structures, e.g., the study of the isotopy-isomorphism of Bol loops, Moufang loops, and Osborn loops. In 2003 Adeniran [1] studied some properties of Bryant-Schneider groups of certain Bol loops. Jaiyéqlá [13], and Jaiyéqlá et al. [14, 15] extended the concept of the Bryant-Schneider group to the study of Smarandache loop, Osborn loop, and their universality. A characterization of isostrophy Bryant-Schneider group-invariant of Bol loops was studied in 2023 by Jaiyéqlá et al. [12]. The isotopic and pseudo-automorphic characterizations of a loop were presented by (Capodaglio [3], 1993), where the crypto-automorphism of a loop was defined as a generalization

of pseudo-automorphism. Further study of crypto-automorphism was carried out by (Oyebo et al. [24], 2024). The work revealed that crypto-automorphisms of a quasigroup with left and right identity elements forms a group.

In this study, we shall study the properties of the crypto-automorphism group and Bryant-Schneider group of left (right) Basarab loop.

2. PRELIMINARIES

Definition 2.1. The set of all permutations on a non-empty set Q forms a group called the symmetric group of Q , denoted as $SYM(Q)$. Let $(Q, \circ)$ be a loop, and let A, B , and C be elements of $SYM(Q)$. If $xA \circ yB = (x \circ y)C$ for all $x, y \in Q$, then the triple (A, B, C) is termed an autotopism, and these triples form a group known as the autotopism group of $(Q, \circ)$, denoted as $AUT(Q, \circ)$. If $A = B = C$, then A is called an automorphism of $(Q, \circ)$, and the set of all such automorphisms forms a group called the automorphism group of $(Q, \circ)$, denoted as $AUM(Q, \circ)$. See [26].

Definition 2.2. Let $(Q, \circ)$ be a loop.

- (a) $\phi \in SYM(Q)$ is called a left pseudo-automorphism with companion $a \in Q$ if $(\phi L_a, \phi, \phi L_a) \in AUT(Q, \circ)$. The set of such maps forms a group (see [26]).
- (b) $\phi \in SYM(Q)$ is called a right pseudo-automorphism with companion $a \in Q$ if $(\phi, \phi R_a, \phi R_a) \in AUT(Q, \circ)$. The set of such maps forms a group (see [26]).
- (c) $\phi \in SYM(Q)$ is called a crypto-automorphism with companions $a, b \in Q$ if $(R_a \phi, L_b \phi, \phi) \in AUT(Q, \circ)$. The set of such maps forms a group (see [24]).
- (d) A mapping $\phi \in SYM(Q)$ such that $(\phi R_g^{-1}, \phi L_f^{-1}, \phi) \in AUT(Q, \circ)$ for some $f, g \in G$ is called a Bryant-Schneider map of $(Q, \circ)$. The set of such maps forms a group called the Bryant-Schneider group $BS(Q, \circ)$ of $(Q, \circ)$ (see [28]).

From Definition 2.2, it is clearly seen that

$$(\phi R_g^{-1}, \phi L_f^{-1}, \phi) = (\phi, \phi, \phi)(R_g^{-1}, L_f^{-1}, I),$$

which implies that ϕ is an isomorphism of $(G, \circ)$ onto some f, g -isotope of it.

Theorem 2.1 ([28]). *Let the set $BS(Q, \circ) = \{\phi \in SYM(Q) : \text{exists } f, g \in Q \ni (\phi R_g^{-1}, \phi L_f^{-1}, \phi) \in AUT(Q, \circ)\}$, then $BS(Q, \circ) \leq SYM(Q)$.*

Theorem 2.1 is associated with Theorem 2.2.

Theorem 2.2 ([26]). *Let $(Q, \circ)$ and $(H, \circ)$ be two isotopic loops. For some $f, g \in Q$, there exists an f, g -principal isotope $(Q, *)$ of $(Q, \circ)$ such that $(H, \circ) \cong (Q, *)$.*

Definition 2.3 ([26]). Let $(Q, \circ)$ be quasigroup with fixed elements $a, b \in Q$. The isotope of this form (R_a^{-1}, L_b^{-1}, I) is called LP-isotope.

Definition 2.4 ([26]). Let $(Q, \circ)$ be a quasigroup. Then, the following hold.

- (a) Right inverse property (RIP) holds if there is a mapping $J_\rho : x \rightarrow x^\rho$ such that $(y \circ x) \circ x^\rho = y$ for all $x, y \in Q$.

- (b) Left inverse property (LIP) holds if there is a mapping $J_\lambda : x \rightarrow x^\lambda$ such that $x^\lambda \circ (x \circ y) = y$ for all $x, y \in Q$.
- (c) Inverse property (IP) if (a) and (b) hold.
- (d) Right alternative property (RAP) if $y \circ xx = yx \circ x$ for all $x, y \in Q$.
- (e) Left alternative property (LAP) if $x \circ xy = xx \circ y$ for all $x, y \in Q$.
- (f) Flexible or elastic property if $(x \circ y) \circ x = x \circ (y \circ x)$ holds for all $x, y \in Q$.
- (g) Cross inverse property (CIP) if there exist mapping $J_\lambda : x \rightarrow x^\lambda$ or $J_\rho : x \rightarrow x^\rho$ such that $xy \circ x^\rho = y$ or $x \circ yx^\rho = y$ or $x^\lambda \circ yx = y$ or $x^\lambda y \circ x = y$ for all $x, y \in Q$.

Definition 2.5. A loop $(Q, \circ)$ is said to be

- (a) an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$;
- (b) an anti-automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$;
- (c) a power associative loop if $\langle x \rangle$ is a subgroup for all $x \in Q$ and a diassociative loop if $\langle x, y \rangle$ is a subgroup for all $x, y \in Q$.

Definition 2.6. Let $(Q, \circ)$ be a loop.

- (a) $N_\lambda = \{v \in Q : (v \circ x) \circ y = v \circ (x \circ y) \text{ for all } x, y \in Q\}$ is called the left nucleus of Q .
- (b) $N_\rho = \{v \in Q : y \circ (x \circ v) = (y \circ x) \circ v \text{ for all } x, y \in Q\}$ is called the right nucleus of Q .
- (c) $N_\mu = \{v \in Q : (y \circ v) \circ x = y \circ (v \circ x) \text{ for all } x, y \in Q\}$ is called the middle nucleus of Q .

Definition 2.7 ([3]). In a loop $(Q, \circ)$, a permutation ϕ is called a crypto-automorphism if there exists $a, b \in Q$ called the companions of ϕ such that for every $x, y \in Q$

$$(x \circ a)\phi \circ (b \circ y)\phi = (x \circ y)\phi.$$

Hence, ϕ is called a crypto-automorphism with companion (a, b) .

Definition 2.8. Let $(Q, \circ)$ be a quasigroup. A mapping $\phi \in SYM(Q)$ will be defined as a two-middle pseudo-automorphism if there exists elements a and b in Q such that $(\phi R_a^{-1}, \phi L_b^{-1}, \phi) \in AUT(Q, \circ)$.

Remark 2.1. The collection of two-middle pseudo-automorphisms will be denoted as $PS_{\mu 2}(Q, \circ)$. It is worth noting that $PS_\mu(Q, \circ) \subseteq PS_{\mu 2}(Q, \circ)$.

Theorem 2.3 ([24]). *The set of crypto-automorphisms $CAUM(Q, \circ)$ of a quasigroup $(Q, \circ)$ with right and left identity elements forms a group.*

Theorem 2.4 ([24]). *Let $(Q, \circ)$ be a loop. $BS(Q, \circ) = CAUM(Q, \circ)$.*

Theorem 2.5 ([26]). *Let $(Q, \circ)$ be an inverse property loop. Then, for any $a \in Q$:*

- (a) $J_\lambda R_a J_\rho = L_{a^\lambda}$;

- (b) $J_\rho R_a J_\rho = L_{a^\rho}$;
- (c) $J_\rho L_a J_\rho = R_{a^\rho}$;
- (d) $J_\lambda L_a J_\rho = R_{a^\lambda}$.

Theorem 2.6 ([26]). *Let $P = (A, B, C) \in \text{AUT}(Q, \circ)$ of a loop $(Q, \circ)$.*

- (a) *If $(Q, \circ)$ is a left inverse property loop (LIPL), then $P_\lambda = (JAJ, C, B) \in \text{AUT}(Q, \circ)$.*
- (b) *If $(Q, \circ)$ is a right inverse property loop (RIPL), then $P_\rho = (C, JBJ, A) \in \text{AUT}(Q, \circ)$.*

Theorem 2.7 ([26, 31]). *Let $(Q, \circ)$ be a RIP or LIP or AAIP loop. Then, $J_\rho = J_\lambda = J$, i.e., $a^\rho = a^\lambda = a^{-1}$ for all $a \in Q$.*

Definition 2.9. A loop $(Q, \circ)$ is called

- (a) left Basarab loop (LBaL) if it satisfies the identity $(x \circ yx^\rho) \circ xz = x \circ yz$ for all $x, y, z \in Q$;
- (b) right Basarab loop (RBaL) if it satisfies the identity $yx \circ (x^\lambda z \circ x) = yz \circ x$ for all $x, y, z \in Q$.

A Basarab loop was studied in Basarab's work before Cote et al. [2] in 2010 worked on classification of loops of generalised Bol-Moufang type. Basarab published two prominent papers [4, 5] in 1992, focusing on IK-loops in [6] and [7]. Some recent studies of Basarab loop have been carried out in various fashions. The properties of Basarab loop with invariants of inverse properties were examined in [16]. This study revealed that a Basarab loop exhibits a cross inverse property if and only if it is an abelian group, or if all left (right) translations of the loop are right (left) regular. The same authors also investigated the Basarab loop and the generators of its total inner mapping group [17]. In 2021, the characterization of subloops of a Basarab loop was explored [8], and a recent announcement was made by Effiong et al. [9] regarding the holomorphic characterization of a Basarab loop.

Theorem 2.8 ([16]). (a) *In a left (right) Basarab loop, the following are equivalent:*

- i flexibility;
- ii RIP (LIP);
- iii RAP (LAP);
- iv AAIP.

(b) *In a Basarab loop, the following are equivalent:*

- i flexibility;
- ii right inverse property;
- iii left inverse property;
- iv inverse property;
- v right alternative property;
- vi left alternative property;
- vii alternative property.

3. MAIN RESULTS

Lemma 3.1. *Let $(Q, \circ)$ be a loop. Then, $(Q, \circ)$ is a*

- (a) *left Basarab loop if and only if any of the following hold for all $a, b \in Q$:*
 - i $(R_{a^\rho}L_a, L_a, L_a) \in \text{AUT}(Q, \circ)$;
 - ii $R_{a^\rho}L_aR_{ab} = R_bL_a$;
 - iii $R_{a^\rho}L_a = T_a^{-1} = R_bL_aR_{ab}^{-1}$;
 - iv $L_aL_{a \circ b a^\rho} = L_bL_a$;
 - v $L_{a \circ b a^\rho} = L_a^{-1}L_bL_a$;
- (b) *right Basarab loop if and only if any of the following hold for all $a, b \in Q$:*
 - i $(R_a, L_{a^\lambda}R_a, R_a) \in \text{AUT}(Q, \circ)$;
 - ii $R_aL_{a^\lambda b \circ a} = R_bR_a$;
 - iii $L_{a^\lambda b \circ a} = R_a^{-1}R_bR_a$;
 - iv $L_{a^\lambda}R_aL_{ba} = L_bR_a$;
 - v $L_{a^\lambda}R_a = T_a = L_bR_aL_{ba}^{-1}$.

Proof. Use Definition 2.9 with the aid of autotopisms and translations. □

Theorem 3.1. *Let $(Q, \circ)$ be a Basarab loop.*

- (a) $a^\rho \in \mathbf{N}_\rho$ *if and only if* $T_a^{-1} \in \text{AUM}(Q, \circ)$.
- (b) $a^\lambda \in \mathbf{N}_\lambda$ *if and only if* $T_a \in \text{AUM}(Q, \circ)$.

Proof. (a) If $(Q, \circ)$ is a left Basarab loop, then from Lemma 3.1, $(R_{a^\rho}L_a, L_a, L_a) \in \text{AUT}(Q, \circ)$ for all $a \in Q$. If $a^\rho \in \mathbf{N}_\rho$, then $(I, R_{a^\rho}, R_{a^\rho})$ is an autotopism of $(Q, \circ)$ for all $a \in Q$. The product

$$(I, R_{a^\rho}, R_{a^\rho})(R_{a^\rho}L_a, L_a, L_a) = (R_{a^\rho}L_a, R_{a^\rho}L_a, R_{a^\rho}L_a)$$

is equivalent to $(T_a^{-1}, T_a^{-1}, T_a^{-1})$ is an autotopism of $(Q, \circ)$ for all $a \in Q$. Thus, T_a^{-1} is automorphism of $(Q, \circ)$.

For the converse, we reverse the procedures to obtain $a^\rho \in \mathbf{N}_\rho$ if and only if $(R_{a^\rho}L_a, R_{a^\rho}L_a, R_{a^\rho}L_a) \in \text{AUT}(Q, \circ)$ is equivalent to $T_a^{-1} \in \text{AUM}(Q, \circ)$.

(b) If $(Q, \circ)$ is a right Basarab loop, in Lemma 3.1, $(R_a, L_{a^\lambda}R_a, R_a) \in \text{AUT}(Q, \circ)$ for all $a \in Q$. If $a^\lambda \in \mathbf{N}_\lambda$, then $(L_{a^\lambda}, I, L_{a^\lambda})$ is an autotopism of $(Q, \circ)$ for all $a \in Q$. The product

$$(L_{a^\lambda}, I, L_{a^\lambda})(R_a, L_{a^\rho}R_a, R_a) = (L_{a^\lambda}R_a, L_{a^\rho}R_a, L_{a^\lambda}R_a)$$

is equivalent to $(T_a, T_a, T_a) \in \text{AUT}(Q, \circ)$. Hence, T_a is an automorphism of $(Q, \circ)$. For the converse, we simply do the reverse procedure. □

Theorem 3.2. *Let U be a crypto-automorphism of LBaL $(Q, \circ)$ with companion (a, b) . Then, $L_xU \in \text{BS}(Q, \circ)$ for all $x \in Q$.*

Proof. If $(Q, \circ)$ is a LBaL then $P = (R_{x^\rho}L_x, L_x, L_x)$ is an autotopism of $(Q, \circ)$ for all $x \in Q$. If U be a crypto-automorphism of $(Q, \circ)$, then $T = (R_aU, L_bU, U)$ is an

autotopism of $(Q, \circ)$ with a twin (a, b) . The product

$$(3.1) \quad PT = (R_{x^\rho}L_x, L_x, L_x)(R_aU, L_bU, U) = (R_{x^\rho}L_xR_aU, L_xL_bU, L_xU) \in AUT(Q, \circ).$$

Set $L_xU = \omega$ in (3.1) to obtain

$$(3.2) \quad (R_{x^\rho}L_xR_aU, L_xL_bU, \omega) \in AUT(Q, \circ).$$

Writing this in an equivalent form, for all $y, z \in Q$, we have

$$(3.3) \quad yR_{x^\rho}L_xR_aU \circ zL_xL_bU = (y \circ z)\omega.$$

Put $y = e$ in (3.3) to obtain

$$(3.4) \quad \begin{aligned} eR_{x^\rho}L_xR_aU \circ zL_xL_bU &= (e \circ z)\omega \Rightarrow (x \circ ex^\rho)R_aU \circ zL_xL_bU = z\omega \\ &\Rightarrow zL_xL_bUL_aU = z\omega \\ &\Rightarrow zL_xL_bU = z\omega L_{aU}^{-1} \Rightarrow L_xL_bU = \omega L_{aU}^{-1}. \end{aligned}$$

Also, put $z = e$ in (3.3) to get

$$(3.5) \quad \begin{aligned} yR_{x^\rho}L_xR_aU \circ eL_xL_bU &= y\omega \Rightarrow yR_{x^\rho}L_xR_aU \circ (ex)L_bU = y\omega \\ &\Rightarrow yR_{x^\rho}L_xR_aUR_{(bx)U} = y\omega \\ &\Rightarrow yR_{x^\rho}L_xR_aU = y\omega R_{(bx)U}^{-1} \\ (3.6) \quad &\Rightarrow R_{x^\rho}L_xR_aU = \omega R_{(bx)U}^{-1}. \end{aligned}$$

Using (3.4) and (3.6) in (3.2), we have $(\omega R_{(bx)U}^{-1}, \omega L_{aU}^{-1}, \omega) \in AUT(Q, \circ)$. Thus, $\omega \in BS(Q, \circ)$ for all $a, b \in Q$. $\square$

Corollary 3.1. *Let U be a crypto-automorphism of $LBaL(Q, \circ)$ with companion (a, b) . Then, $L_aU \in PS_{\mu_2}(Q, \circ)$ with companion $((bx)U, (aU)^\rho)$.*

Proof. This follows as a consequence of Theorem 3.2. $\square$

Theorem 3.3. *Let U be a crypto-automorphism of $RBaL(Q, \circ)$ with companion (a, b) . Then,*

- (a) $R_xU \in BS(Q, \circ)$ for all $x \in Q$;
- (b) $R_xU \in PS_{\mu_2}(Q, \circ)$ with companion $(bU, ((xa)U)^\rho)$ for all $x \in Q$

Proof. Follow similar steps as in Theorem 3.2. $\square$

Theorem 3.4. *Let $(Q, \circ)$ be a $LBaL$ with RIP . If the map $\theta \in BS(Q, \circ)$ such that $\theta = \omega L_x^{-1}$ where $\omega : e \rightarrow e$, then ω is a unique right pseudo-automorphism with companion $(xg)^{-1}$ for all $x \in Q$ and for some $g \in Q$.*

Proof. Let $\theta \in BS(Q, \circ)$. Then, $M = (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q, \circ)$ for some $f, g \in Q$. Since $(Q, \circ)$ is a $LBaL$, $N = (R_{x^\rho}L_x, L_x, L_x) \in AUT(Q, \circ)$ for all $x \in Q$. The product

$$(3.7) \quad MN = (\theta R_g^{-1}, \theta L_f^{-1}, \theta)(R_{x^\rho}L_x, L_x, L_x) = (\theta R_g^{-1}R_{x^\rho}L_x, \theta L_f^{-1}L_x, \theta L_x) \in AUT(Q, \circ),$$

for some $f, g \in Q$ and for all $x \in Q$.

Let $(Q, \circ)$ be a RIPL, applying Theorem 2.6 to the autotopism in (3.7), we get

$$(3.8) \quad (\theta L_x, J\theta L_f^{-1} L_x J, \theta R_g^{-1} R_{x\rho} L_x) \in AUT(Q, \circ),$$

for some $f, g \in Q$ and for all $x \in Q$.

Let $\alpha = J\theta L_f^{-1} L_x J$ in (3.8). Then,

$$(3.9) \quad (\theta L_x, \alpha, \theta R_g^{-1} R_{x\rho} L_x) \in AUT(Q, \circ),$$

for some $g \in Q$ and for all $x \in Q$. For all $y, z \in Q$, we have

$$y\theta L_x \circ z\alpha = (y \circ z)\theta R_g^{-1} R_{x\rho} L_x.$$

If $\theta = \omega L_x^{-1}$, then $\theta L_x = \omega$, so, $\alpha = J\omega L_x^{-1} L_f^{-1} L_x J$. Next, we have

$$(3.10) \quad y\omega \circ z\alpha = (y \circ z)\omega L_x^{-1} R_g^{-1} R_{x\rho} L_x.$$

Let $y = e$ in (3.10). Then, we have

$$(3.11) \quad e\omega \circ z\alpha = z\omega L_x^{-1} R_g^{-1} R_{x\rho} L_x \Rightarrow z\alpha = z\omega L_x^{-1} R_g^{-1} R_{x\rho} L_x \Rightarrow \alpha = \omega L_x^{-1} R_g^{-1} R_{x\rho} L_x.$$

Using (3.10) and (3.11), (3.9) becomes

$$(\omega, \omega L_x^{-1} R_g^{-1} R_{x\rho} L_x, \omega L_x^{-1} R_g^{-1} R_{x\rho} L_x) \in AUT(Q, \circ),$$

for some $g \in Q$ and for all $x \in Q$.

Since, $(Q, \circ)$ is a left Basarab loop, $R_{x\rho} L_x = R_g L_x R_{xg}^{-1}$. Hence,

$$\begin{aligned} & (\omega, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}) \\ &= (\omega, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}) = (\omega, \omega R_{xg}^{-1}, \omega R_{xg}^{-1}) \end{aligned}$$

is an autotopism of $(Q, \circ)$ for some $g \in Q$ and for all $x \in Q$. Thus, ω is a right pseudo-automorphism with companion $(xg)^{-1}$.

Let $\omega_1 L_{x_1}^{-1} = \omega_2 L_{x_2}^{-1}$, where $\omega_1, \omega_2 : e \rightarrow e$, for all $x_1, x_2 \in Q$. Then, $L_{x_1}^{-1} L_{x_2} = \omega_1^{-1} \omega_2$. So, $e L_{x_1}^{-1} L_{x_2} = e \omega_1^{-1} \omega_2$ implies $L_{x_1}^{-1} L_{x_2} = I$, implies $L_{x_1} = L_{x_2}$. Hence, $x_1 = x_2$ and so $\omega_1 = \omega_2$. This implies that for all $x \in Q$, there exists a unique ω such that $\theta = \omega L_x^{-1}$. Hence, $\theta = \omega L_x^{-1}$ where ω is a unique right pseudo-automorphism with companion $(xg)^{-1}$ for all $x \in Q$ and for some $g \in Q$. $\square$

Corollary 3.2. *Let $(Q, \circ)$ be a LBaL with RIP. For some $\theta \in BS(Q, \circ)$ such that $\omega L_x^{-1} = \theta$, where $\omega : e \rightarrow e$, $\omega \in AUM(Q, \circ)$ is unique for all $x \in Q$.*

Proof. Set $x = g^{-1}$ in Theorem 3.4. $\square$

Theorem 3.5. *Let $(Q, \circ)$ be a RBaL with LIP. If the map $\theta \in BS(Q, \circ)$ such that $\theta = \omega R_x^{-1}$, where $\omega : e \rightarrow e$, then ω is a unique left pseudo-automorphism with companion $(fx)^{-1}$ for all $x \in Q$ and for some $f \in Q$.*

Proof. Let $\theta \in BS(Q, \circ)$ then $M = (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q, \circ)$ for some $f, g \in Q$. Since $(Q, \circ)$ is a RBaL, $N = (R_x, L_{x^\lambda} R_x, R_x) \in AUT(Q, \circ)$ for all $x \in Q$. The product

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta)(R_x, L_{x^\lambda} R_x, R_x) = (\theta R_g^{-1} R_x, \theta L_f^{-1} L_{x^\lambda} R_x, \theta R_x) \in AUT(Q, \circ),$$

for some $f, g \in Q$ and all $x \in Q$. If $(Q, \circ)$ is LIPL, by Theorem 2.6,

$$(3.12) \quad (J\theta R_g^{-1} R_x J, \theta R_x, \theta L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q, \circ),$$

for some $f, g \in Q$ and all $x \in Q$. Let $\alpha = J\theta R_g^{-1} R_x J$ in (3.12), to get

$$(\alpha, \theta R_x, \theta L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q, \circ),$$

for all $x \in Q$ and some $f \in Q$.

If $\theta = \omega R_x^{-1}$, then $\omega = \theta R_x$. Thus, $\alpha = J\omega R_x^{-1} R_g^{-1} R_x J$. Here, (3.12) becomes

$$(3.13) \quad (\alpha, \omega, \omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q, \circ),$$

for all $x \in Q$ and some $f \in Q$. For all $y, z \in Q$, we have $y\alpha \circ z\omega = (y \circ z)\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x$. Put $z = e$. Then,

$$(3.14) \quad y\alpha \circ e\omega = y\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x \Rightarrow y\alpha = y\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x \Rightarrow \alpha = \omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x.$$

Now, using (3.14) in (3.13), we get

$$(3.15) \quad (\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x, \omega, \omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q, \circ),$$

for all $x \in Q$ and some $f \in Q$. Since $(Q, \circ)$ is a RBaL, by Lemma 3.1, $L_{x^\lambda} R_x = L_f R_x L_{fx}^{-1}$ for all $x, f \in Q$. Thus, we have the last autotopism

$$(\omega R_x^{-1} L_f^{-1} L_f R_x L_{fx}^{-1}, \omega, \omega R_x^{-1} L_f^{-1} L_f R_x L_{fx}^{-1}) = (\omega L_{fx}^{-1}, \omega, \omega L_{fx}^{-1}) \in AUT(Q, \circ),$$

for all $x \in Q$ and some $f \in Q$. Thus, ω is a left pseudo-automorphism with companion $(fx)^{-1}$ for all $x \in Q$ and for some $f \in Q$. The proof of the uniqueness of ω is similar to the one in Theorem 3.4. $\square$

Corollary 3.3. *Let $(Q, \circ)$ be a RBaL with LIP. For some $\theta \in BS(Q, \circ)$ such that $\theta = \omega R_x^{-1}$, where $\omega : e \rightarrow e$, $\omega \in AUM(Q, \circ)$ is unique for all $x \in Q$ and for some $f \in Q$.*

Proof. Set $x = f^{-1}$ in Theorem 3.5. $\square$

Theorem 3.6. *Let $(Q, \circ)$ be a LBaL and let $\omega = \omega(f, g) \in BS(Q, \circ)$ such that $\omega : e \rightarrow e$. Then, $\omega = \omega(f, g) \equiv \omega(f, f^\rho)$ and $\omega \equiv \omega(f, g) = \omega(g^\lambda, g)$.*

Proof. Let $\omega \in BS(Q, \circ)$, then $(\omega R_g^{-1}, \omega L_f^{-1}, \omega) \in AUT(Q, \circ)$ for some $f, g \in Q$. Since $(Q, \circ)$ is a LBaL, $(R_{x^\rho} L_x, L_x, L_x) \in AUT(Q, \circ)$ for all $x \in Q$. The product

$$(3.16) \quad (\omega R_g^{-1}, \omega L_f^{-1}, \omega)(R_{x^\rho} L_x, L_x, L_x) = (\omega R_g^{-1} R_{x^\rho} L_x, \omega L_f^{-1} L_x, \omega L_x)$$

is an autotopism of $(Q, \circ)$ for some $f, g \in Q$ and for all $x \in Q$. For all $y, z \in Q$ we have

$$(3.17) \quad y\omega R_g^{-1} R_{x^\rho} L_x \circ z\omega L_f^{-1} L_x = (y \circ z)\omega L_x.$$

Put $z = e$ in (3.17) to obtain

$$\begin{aligned} y\omega R_g^{-1}R_{x^\rho}L_x \circ e\omega L_f^{-1}L_x &= y\omega L_x \Rightarrow y\omega R_g^{-1}R_{x^\rho}L_x \circ eL_f^{-1}L_x = y\omega L_x \\ &\Rightarrow y\omega R_g^{-1}R_{x^\rho}L_x R_{xf^\rho} = y\omega L_x \\ &\Rightarrow R_g^{-1}R_{x^\rho}L_x R_{xf^\rho} = L_x \\ &\Rightarrow R_g^{-1} = L_x R_{xf^\rho}^{-1} L_x^{-1} R_{x^\rho}^{-1}, \end{aligned}$$

for some $f \in Q$ and for all $x \in Q$. Set $x = f$ in the last equality to get

$$R_g^{-1} = L_f L_f^{-1} R_{f^\rho}^{-1} = R_{f^\rho}^{-1}.$$

Hence, $g = f^\rho$. On the other hand, set $y = e$ in (3.17) to get

$$\begin{aligned} e\omega R_g^{-1}R_{x^\rho}L_x \circ z\omega L_f^{-1}L_x &= z\omega L_x \Rightarrow eR_g^{-1}R_{x^\rho}L_x \circ z\omega L_f^{-1}L_x = z\omega L_x \\ &\Rightarrow (x \circ g^{-1}x^\rho) (z\omega L_f^{-1}L_x) = z\omega L_x \\ &\Rightarrow L_f^{-1} = L_x L_{(x \circ g^{-1}x^\rho)}^{-1} L_x^{-1}. \end{aligned}$$

Set $x = g^\lambda$, to get

$$L_f^{-1} = L_{g^\lambda} L_{(g^\lambda \circ g^{-1}(g)^\lambda)^\rho}^{-1} L_{g^\lambda}^{-1} = L_{g^\lambda}^{-1}.$$

Thus, $L_f^{-1} = L_{g^\lambda}^{-1}$ implies $f = g^\lambda$. □

Corollary 3.4. *Let $(Q, \circ)$ be a LBaL. If $\omega \equiv \omega(f, g) \in BS(Q, \circ)$, such that $\omega : e \rightarrow e$, for some $f, g \in Q$, the following hold:*

- (a) ω is a right pseudo-automorphism with companion f ;
- (b) $|f| = 2$ or $|g| = 2$.

Proof. Putting $R_g^{-1} = R_{f^\rho}^{-1}$ and $L_f^{-1} = L_{g^\lambda}^{-1}$ in (3.16), we get the autotopism

$$(\omega R_{f^\rho}^{-1} R_{x^\rho} L_x, \omega L_{g^\lambda}^{-1} L_x, \omega L_x)$$

of $(Q, \circ)$ for some $f, g \in Q$ and for all $x \in Q$. Set $x = f$, to get $(\omega L_f, \omega L_{g^\lambda}^{-1} L_f, \omega L_f) \in AUT(Q, \circ)$ for some $f, g \in Q$. Let $g = f$. Then, $(\omega L_f, \omega L_{f^\lambda}^{-1} L_f, \omega L_f) \in AUT(Q, \circ)$ for some $f \in Q$. For all $y, z \in Q$, we have $y\omega L_f \circ z\omega L_{f^\lambda}^{-1} L_f = (y \circ z)\omega L_f$. Setting $z = e$, we get

$$y\omega L_f \circ e\omega L_{f^\lambda}^{-1} L_f = y\omega L_f \Rightarrow y\omega L_f \circ f^2 = y\omega L_f \Rightarrow y\omega L_f R_{f^2} = y\omega L_f \Rightarrow |f| = 2.$$

Thus, $(\omega L_f, \omega, \omega L_f) \in AUT(Q, \circ)$ for some $f \in Q$. Therefore, ω is a pseudo-automorphism with companion f for some $f \in Q$. □

Theorem 3.7. *Let $(Q, \circ)$ be a LBaL with AAIP. Then,*

$$\begin{aligned} BS'(Q, \circ) &= \left\{ \omega \in BS(Q, \circ) \mid \omega : e \rightarrow e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \right\} \\ &= \left\{ \omega \in SYM(Q) \mid \text{exists } f \in Q \ni (\omega R_{f^{-1}}^{-1}, \omega L_f^{-1}, \omega) \in AUT(Q), \text{ } e\omega = e \right. \\ &\quad \left. \text{and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q \right\} \\ &= \left\{ \omega \in SYM(Q) \mid \text{exists } g \in Q \ni (\omega R_g^{-1}, \omega L_{g^{-1}}^{-1}, \omega) \in AUT(Q), \right. \\ &\quad \left. e\omega = e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q \right\} \\ &\leq BS(Q, \circ). \end{aligned}$$

Proof. Let

$$BS'(Q, \circ) = \left\{ \omega \in BS(Q, \circ) \mid \omega : e \rightarrow e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \right\} \subseteq BS(Q, \circ).$$

Using the connection from Theorem 3.6,

$$\begin{aligned} BS'(Q, \circ) &= \left\{ \omega \in BS(Q, \circ) \mid \omega : e \rightarrow e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \right\} \\ &= \left\{ \omega \in SYM(Q) \mid \text{exists } f \in Q \ni (\omega R_{f^{-1}}^{-1}, \omega L_f^{-1}, \omega) \in AUT(Q), \right. \\ &\quad \left. e\omega = e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q \right\} \\ &= \left\{ \omega \in SYM(Q) \mid \text{exists } g \in Q \ni (\omega R_g^{-1}, \omega L_{g^{-1}}^{-1}, \omega) \in AUT(Q), \right. \\ &\quad \left. e\omega = e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q \right\}. \end{aligned}$$

Note that $eI = e$ and $(gI)^{-1} = (g^{-1})I$ for all $g \in Q$ and $(IR_e^{-1}, IL_e^{-1}, I) = (I, I, I) \in AUT(Q, \circ)$. Then, $I \in BS'(Q, \circ)$. Hence, $BS'(Q, \circ)$ contains the identity map, thus $BS'(Q, \circ)$ is not empty.

Let $\alpha, \sigma \in BS'(Q, \circ)$. Then, $\alpha, \sigma \in BS(Q, \circ)$ and $e\alpha = e$ and $(x\alpha)^{-1} = (x^{-1})\alpha$, $e\sigma = e$ and $(x\sigma)^{-1} = (x^{-1})\sigma$ for all $x \in Q$.

In addition, there exist $f_1, g_1, f_{11}, g_{11} \in Q$ with $g_1 = f_1^{-1}, f_{11} = g_{11}^{-1}$ such that

$$\begin{aligned} P &= (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), \quad T = (\sigma R_{g_{11}}^{-1}, \sigma L_{f_{11}}^{-1}, \sigma), \\ T^{-1} &= (R_{g_{11}}\sigma^{-1}, L_{f_{11}}\sigma^{-1}, \sigma^{-1}) \in AUT(Q, \circ), \\ PT^{-1} &= (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_{11}}\sigma^{-1}, L_{f_{11}}\sigma^{-1}, \sigma^{-1}) \\ &= (\alpha R_{g_1}^{-1} R_{g_{11}}\sigma^{-1}, \alpha L_{f_1}^{-1} L_{f_{11}}\sigma^{-1}, \alpha\sigma^{-1}) \in AUT(Q, \circ). \end{aligned}$$

Let $\varrho = \sigma R_{g_1}^{-1} R_{g_{11}}\sigma^{-1}$ and $\gamma = \sigma L_{f_1}^{-1} L_{f_{11}}\sigma^{-1}$, such that $(\alpha\sigma^{-1}\varrho, \alpha\sigma^{-1}\gamma, \alpha\sigma^{-1}) \in AUT(Q, \circ)$ if and only if for all $a, b \in Q$

$$(3.18) \quad a\alpha\sigma^{-1}\varrho \circ b\alpha\sigma^{-1}\gamma = (a \circ b)\alpha\sigma^{-1}.$$

Setting $a = e$ in Q and replacing b by $b\sigma\alpha^{-1}$ in (3.18), we have

$$(e\alpha\sigma^{-1}\varrho) \circ (b\gamma) = b \Rightarrow b\gamma L_{(e\alpha\sigma^{-1}\varrho)} = b \Rightarrow \gamma = L_{(e\alpha\sigma^{-1}\varrho)}^{-1}.$$

Analogously, setting $b = e$ in Q , the identity element and replacing a by $a\sigma\alpha^{-1}$ in (3.18), we have

$$(a\varrho) \circ (e\alpha\sigma^{-1}\gamma) = a \Rightarrow a\varrho R_{(e\alpha\sigma^{-1}\gamma)} = a \Rightarrow \varrho = R_{(e\alpha\sigma^{-1}\gamma)}^{-1}.$$

Thus, $g = e\alpha\sigma^{-1}\gamma = e\gamma = e\sigma L_{f_1}^{-1} L_{f_{11}}\sigma^{-1} = [f_{11} \circ (f_1 \setminus e)]\sigma^{-1} = [f_{11} \circ f_1^{-1}]\sigma^{-1}$ and $f = e\alpha\sigma^{-1}\varrho = e\varrho = e\sigma R_{f_1}^{-1} R_{f_{11}}\sigma^{-1} = eR_{f_1}^{-1} R_{f_{11}}\sigma^{-1} = [(e/f_1^{-1}) \circ f_{11}^{-1}]\sigma^{-1} = (f_1 \circ f_{11}^{-1})\sigma^{-1}$. Then, using Theorem 2.7, $f^{-1} = [(f_1 \circ f_{11}^{-1})\sigma^{-1}]^{-1} = (f_1 \circ f_{11}^{-1})^{-1}\sigma^{-1} = (f_{11} \circ f_1^{-1})\sigma^{-1} = g$. Hence, $PT^{-1} = (\alpha\sigma^{-1}\varrho, \alpha\sigma^{-1}\gamma, \alpha\sigma^{-1}) = (\alpha\sigma^{-1}R_{f_1}^{-1}, \alpha\sigma^{-1}L_{f_1}^{-1}, \alpha\sigma^{-1})$ is an autotopism of $(Q, \circ)$, $e\alpha\sigma^{-1} = e$ and $(x^{-1})\alpha\sigma^{-1} = (x\alpha\sigma^{-1})^{-1}$ for all $x \in Q$. So, $\alpha\sigma^{-1} \in BS'(Q, \circ)$. Also,

$$PT^{-1} = (\alpha\sigma^{-1}R_g^{-1}, \alpha\sigma^{-1}L_{g^{-1}}^{-1}, \alpha\sigma^{-1})$$

is an autotopism of $(Q, \circ)$ and $e\alpha\sigma^{-1} = e$ and $(x^{-1})\alpha\sigma^{-1} = (x\alpha\sigma^{-1})^{-1}$ for all $x \in Q$. So, $\alpha\sigma^{-1} \in BS'(Q, \circ)$. Hence, $BS'(Q, \circ) \leq BS(Q, \circ)$. $\square$

Corollary 3.5. *Let $(Q, \circ)$ be a LBaL with AAIP. Then, $AUM(Q, \circ) \leq BS'(Q, \circ) \leq BS(Q, \circ)$.*

Proof. This follows as a consequence of Theorem 3.7. $\square$

Theorem 3.8. *Let $(Q, \circ)$ be a LBaL with a RIP and let $H = (A, B, C) \in AUT(Q, \circ)$. Then, there exists a right pseudo-automorphism ω with companion yx , where $eA = x$ and $eB = y$ such that*

$$(A, B, C) = (\omega, \omega R_{yx}, \omega R_{yx})(L_{x^{-1}}, JL_{x^{-1}}J, R_{x\rho}^{-1}L_{x^{-1}})^{-1}.$$

Proof. Let $(Q, \circ)$ be a LBaL, then $(R_{x\rho}^{-1}L_{x^{-1}}, L_{x^{-1}}, L_{x^{-1}}) \in AUT(Q, \circ)$ for all $x \in Q$. Applying Theorem 2.6, $(L_{x^{-1}}, JL_{x^{-1}}J, R_{x\rho}^{-1}L_{x^{-1}}) \in AUT(Q, \circ)$ for all $x \in Q$. Suppose that $K = (A, B, C)$ is an autotopism of $(Q, \circ)$, then the product

$$\begin{aligned} & (A, B, C)(L_{x^{-1}}, JL_{x^{-1}}J, R_{x\rho}^{-1}L_{x^{-1}}) \\ &= (AL_{x^{-1}}, BJL_{x^{-1}}J, CR_{x\rho}^{-1}L_{x^{-1}}) \in AUT(Q, \circ), \quad \text{for all } x \in Q. \end{aligned}$$

Replace $AL_{x^{-1}}$ with ω and note that $e\omega = eAL_{x^{-1}} = x^{-1} \circ x = e$,

$$(3.19) \quad t\omega \circ zBJL_{x^{-1}}J = (t \circ z)CR_{x\rho}^{-1}L_{x^{-1}}$$

for all $t, z \in Q$. Set $t = e$ in (3.19), we obtain

$$\begin{aligned} e\omega \circ zBJL_{x^{-1}}J &= zCR_{x\rho}^{-1}L_{x^{-1}} \Rightarrow zBJL_{x^{-1}}J = zCR_{x\rho}^{-1}L_{x^{-1}} \\ &\Rightarrow BJL_{x^{-1}}J = CR_{x\rho}^{-1}L_{x^{-1}}, \end{aligned}$$

for all $x \in Q$. So, (3.19) becomes

$$(3.20) \quad t\omega \circ zBJL_{x^{-1}}J = (t \circ z)BJL_{x^{-1}}J,$$

for all $t, z \in Q$. Let $z = e$, then it follows from (3.20) that $y\omega \circ eB J L_{x^{-1}} J = yB J L_{x^{-1}} J$ implies $\omega y J L_{x^{-1}} J = B J L_{x^{-1}} J$, implies $\omega R_{yx} = B J L_{x^{-1}} J$. Using this information in (3.20), we get $(\omega, \omega R_{yx}, \omega R_{yx}) \in \text{AUT}(Q, \circ)$. Hence, ω is a right pseudo-automorphism with companion (yx) . So,

$$\begin{aligned} (A, B, C)(L_{x^{-1}}, J L_{x^{-1}} J, R_{x^\rho}^{-1} L_{x^{-1}}) &= (\omega, \omega R_{yx}, \omega R_{yx}) \\ \Rightarrow (A, B, C) &= (\omega, \omega R_{yx}, \omega R_{yx})(L_{x^{-1}}, J L_{x^{-1}} J, R_{x^\rho}^{-1} L_{x^{-1}})^{-1}. \quad \square \end{aligned}$$

Theorem 3.9. *Let $(Q, \circ)$ be a LBaL with a RIP such that $|xz| = 2$ for all $z \in Q$, and let $a \oplus b = aR_x^{-1} \circ bL_y^{-1}$, for any arbitrarily fixed $a, b \in Q$ and for all $x, y \in Q$. Then, $(Q, \oplus) \cong (Q, \circ)$ if and only if there exists right pseudo-automorphism ω of $(Q, \circ)$ with companion yx .*

Proof. Let C be isomorphism between $(Q, \oplus)$ and $(Q, \circ)$. Then,

$$(a \circ b)C = aC \oplus bC = aC R_x^{-1} \circ bC L_y^{-1} = aB \circ bA,$$

for all $a, b \in Q$, where $B = C R_x^{-1}$ and $A = C L_y^{-1}$. So, $(B, A, C) \in \text{AUT}(Q, \circ)$.

Let e and e^* denote the identity elements in $(Q, \circ)$ and $(Q, \oplus)$, respectively. So, since C is an isomorphism, we have $eC = e^*$, where $e^* = y \circ x$, and

$$(3.21) \quad eA = eC R_x^{-1} = e^* R_x^{-1} = (y \circ x) R_x^{-1} = y,$$

$$(3.22) \quad eB = eC L_y^{-1} = e^* L_y^{-1} = (y \circ x) L_y^{-1} = x.$$

By Theorem 3.8, and going by (3.21), and (3.22), we have

$$(B, A, C) = (\omega, \omega R_{yx}, \omega R_{yx})(L_{x^{-1}}, J L_{x^{-1}} J, R_{x^\rho}^{-1} L_{x^{-1}})^{-1},$$

where ω is a right pseudo-automorphism of $(Q, \circ)$ with companion yx .

Conversely, if ω is right pseudo-automorphism with companion yx , then $(\omega, \omega R_{yx}, \omega R_{yx})$ is a autotopism of $(Q, \circ)$. On the other hand, since $(Q, \circ)$ is LBaL, we have that $(L_{y^{-1}}, J L_{y^{-1}} J, R_{y^\rho}^{-1} L_{y^{-1}})^{-1}$ is an autotopism of $(Q, \circ)$. Now,

$$\begin{aligned} (B, A, C) &= (\omega, \omega R_{yx}, \omega R_{yx})(L_{y^{-1}}, J L_{y^{-1}} J, R_{y^\rho}^{-1} L_{y^{-1}})^{-1} \\ &= (\omega, \omega R_{yx}, \omega R_{yx})(L_{y^{-1}}^{-1}, J L_{y^{-1}}^{-1} J, L_{y^{-1}}^{-1} R_{y^\rho}) \\ &= (\omega L_{y^{-1}}^{-1}, \omega R_{yx} J L_{y^{-1}}^{-1} J, \omega R_{yx} L_{y^{-1}}^{-1} R_{y^\rho}) \in \text{AUT}(Q, \circ). \end{aligned}$$

Writing it in an equivalent relation form, for all $a, b \in Q$, we have

$$(3.23) \quad a\omega L_{y^{-1}}^{-1} \circ b\omega R_{yx} J L_{y^{-1}}^{-1} J = (a \circ b)\omega R_{yx} L_{y^{-1}}^{-1} R_{y^\rho} \Rightarrow (a \circ b)C = aB \circ bA,$$

where $C = \omega R_{yx} L_{y^{-1}}^{-1} R_{y^\rho}$, $B = \omega L_{y^{-1}}^{-1}$ and $A = \omega R_{yx} J L_{y^{-1}}^{-1} J$. So,

$$\begin{aligned} eB &= e\omega L_{y^{-1}}^{-1} = eL_{y^{-1}}^{-1} = y^{-1} \setminus e = y, \\ eA &= e\omega R_{yx} J L_{y^{-1}}^{-1} J = eR_{yx} J L_{y^{-1}}^{-1} J = [y^{-1} \setminus (yx)^{-1}]^{-1} \\ (3.24) \quad \Rightarrow eA &= [y \setminus (yx)]^{-1} = x^{-1} = x. \end{aligned}$$

Note that we used the assumptions that $|z| = 2$ for all $z \in Q$ and Q has RIP, which by Theorem 2.8 is equivalent to flexibility in a left Basarab loop.

Now, setting $b = e$ in (3.23), to get

$$(3.25) \quad aC = aB \circ eA \Rightarrow aC = aB \circ x \Rightarrow aC = aBR_x \Rightarrow B = CR_x^{-1}.$$

Put $a = e$ in (3.23) to obtain

$$(3.26) \quad bC = eB \circ bA \Rightarrow bC = y \circ bA \Rightarrow bC = bAL_y \Rightarrow A = CL_y^{-1}.$$

So, using (3.23), (3.25) and (3.26), we obtain

$$(a \circ b)C = aB \circ bA = aCR_x^{-1} \circ bCL_y^{-1} = aC \oplus bC,$$

for all $a, b \in Q$. Hence, $(Q, \oplus) \cong (Q, \circ)$. □

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S SPECTRAL THEORY OF MULTIVALUED LINEAR OPERATOR IN BANACH SPACES

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ABSTRACT. In this paper, we begin with the definition of the S -resolvent set of a linear relation. Throughout this paper, X will denote a normed linear space over the complex field $\mathbb{C}$. Operator S plays the role of a transition multivalued linear operator from X . It is the main goal of the present note to study the basic spectral properties of T linked to the transition multivalued linear operator S .

1. INTRODUCTION

Let $T, S : X \rightarrow Y$. For all $\lambda \in \mathbb{C}$, the map $P(\lambda) := T + \lambda S$ is called a linear bundle. It is known that many problems of mathematical physics (for example, quantum theory...) are reduced to some study of certain reversibility conditions of operators $P(\lambda)$. In light of this, this paper shares the same concern of these studies. In fact, it highlights some recent mathematical developments, including the spectral theory of multivalued linear operator bundle. The study of the spectral theory of linear operators, including the study of S -essential spectrum, has recently been the core subject of several works.

Most importantly, A. Jeribi [13] has treated the notion of S -essential spectra in a comprehensive account of the known definitions of essential spectra (see Chapter 9). In [2] T. Alvarez, A. Ammar and A. Jeribi have characterized some S -essential spectra of a closed linear relation in terms of certain linear relations of semi-Fredholm type. In [3], they studied the decomposition of Frobenius-Schur in order to determine the essential spectrum of a matrix multivalued linear operator.

Most importantly in [4] A. Ammar, A. Jeribi and B. Saadaoui considered a 2×2 block multivalued linear operator matrices and described its essential pseudospectrum.

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Among the most important works on the spectral theory of block linear relation matrix, we mention [5, 6] where the authors proposed the development of the essential spectra of a 2×2 block linear relation matrix. Recently, B. Saadaoui [15] continued this study to investigate the (P,Q)-Outer Generalized inverses and their stability of pseudo spectrum. In [7, 16] B. Saadaoui generalized some known results in the condition S-spectrum of a compact operator in a right quaternionic Hilbert space. Several problems can be described by systems of mixed order linear differential equations in mathematical physics. The localization of the essential spectra gave substantial physical information. The findings acquired in this memorandum provides through the conformable ones in [8], and they are robustly affined to notions from different spectral problems in applied sciences (for concerning works see, for sample, [11, 12, 17]). In specific, the research of different sort of degenerate equations on Banach spaces could be done using the connotations and outcomes gained in the current note, cf. [11]. Paradigms to uncover the applicability of our theoretical processing will be extended in [5]. More punctually, the main outcomes of this matter will be exercised to research different perturbations of multivalued linear operator in Banach spaces in the spirit of the results obtained in [9, 14].

The purpose of this paper is to extend the results of [10] to the case of multivalued linear operators. It consists of three sections. In Section 2, we present some basic notations and results connected to the main body of the work. In Section 3 we give a characterization of the S -resolvent set of a closed multivalued linear operator T .

2. PRELIMINARY RESULTS

This section contains some definitions and auxiliary results which will be needed in the rest of this paper. We adhere with the notation and the terminology of the book [10]. Let X, Y and Z be infinite dimensional vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. T multivalued linear operator or simply a linear relation $T : X \rightarrow Y$ is a mapping from a subspace $\mathcal{D}(T)$ of X , called the domain of T , into $P(Y) \setminus \{\emptyset\}$ (the collection of non-empty subsets of Y) such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all non-zero scalars $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator, which is equivalent to $T(0) = \{0\}$. We denote by $\mathcal{LR}(X, Y)$ the class of linear relations everywhere defined and we write $\mathcal{LR}(X) := \mathcal{LR}(X, X)$. $T \in \mathcal{LR}(X, Y)$ is uniquely determined by its graph $G(T)$, which is defined by:

$$G(T) := \{(x, y) \in X \times Y : x \in \mathcal{D}(T), y \in Tx\},$$

so that we can identify T with $G(T)$. The closure of T , denoted by $\overline{T}$, is the linear relation defined by

$$G(\overline{T}) := \overline{G(T)}.$$

We denote by $\mathcal{CR}(X, Y)$ the class of all closed linear relations from X to Y . If $X = Y$, we take $\mathcal{CR}(X, X) := \mathcal{CR}(X)$.

The inverse of T is a linear relation T^{-1} given by:

$$G(T^{-1}) := \{(y, x) \in X \times Y : (x, y) \in G(T)\}.$$

If $G(T)$ is closed, then T is said to be closed. We design by $R(T) = T(\mathcal{D}(T))$ the range of T . T is called surjective if $R(T) = Y$. The subspace $N(T) := T^{-1}(0)$ is called the null space of T . T is called injective if $N(T) = \{0\}$, that is, if T^{-1} is a single valued linear operator.

Notice that when $x \in \mathcal{D}(T)$, $y \in Tx$ if and only if $Tx = y + T(0)$.

For $T, S \in \mathcal{LR}(X, Y)$, the notation $T \subset S$ means that $G(T) \subset G(S)$. The linear relation $T + S$ is defined by:

$$G(T + S) := \{(x, y) \in X \times Y : y = u + v \text{ with } (x, u) \in G(T), (x, v) \in G(S)\}.$$

Let $T \in \mathcal{LR}(X, Y)$ and $S \in \mathcal{LR}(Y, Z)$ where $R(T) \cap \mathcal{D}(S) \neq \emptyset$. The product of ST is defined by:

$$G(ST) := \{(x, z) \in X \times Z : (x, u) \in G(T) \text{ and } (u, z) \in G(S) \text{ for some } u \in Y\}.$$

Let Q_T denote the quotient map from X onto $X/\overline{T(0)}$. We shall denote $Q_{\overline{T(0)}}$ by Q_T . Clearly, $Q_T T$ is a single valued operator and the norm of T is defined by $\|T\| := \|Q_T T\|$. We say that T is continuous if for each neighborhood V in $R(T)$, $T^{-1}(V)$ is a neighborhood in $\mathcal{D}(T)$ (equivalently $\|T\| < +\infty$); bounded if it is continuous with $\mathcal{D}(T) = X$; open if T^{-1} is continuous equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) := \sup \left\{ \lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\| \text{ for } x \in \mathcal{D}(T) \right\},$$

where $d(x, N(T))$ is the distance between x and $N(T)$. Continuous linear relations defined everywhere on X are referred to as bounded linear relations. The class of such relations is denoted by $\mathcal{BR}(X, Y)$. If $X = Y$, we take $\mathcal{BR}(X, X) := \mathcal{BR}(X)$.

If M and N are subspaces of X and of the dual space X' , respectively, then

$$M^\perp := \{x' \in X' : x'(x) = 0 \text{ for all } x \in M\}$$

and

$$N^\top := \{x \in X : x'(x) = 0 \text{ for all } x' \in N\}.$$

The conjugate of $T \in \mathcal{LR}(X, Y)$ is the linear relation T' defined by

$$G(T') := G(-T^{-1})^\perp \subset Y' \times X',$$

so that $(y', x') \in G(T')$ if and only if $y'(y) = x'(x)$ for all $(x, y) \in G(T)$.

Lemma 2.1 ([10]). *Let X and Y be two vector spaces and let $T \in \mathcal{LR}(X, Y)$. Then,*

- (a) $\mathcal{D}(T^{-1}) = R(T)$; $\mathcal{D}(T) = R(T^{-1})$;
- (b) T is injective if and only if $T^{-1}T = I_{\mathcal{D}(T)}$;
- (c) T is single valued if and only if $T(0) = \{0\}$;
- (d) $TT^{-1}y = y + T(0)$, $y \in R(T)$, and $T^{-1}Tx = x + T^{-1}(0)$.

Proposition 2.1 ([10]). *Let $R, S, T \in \mathcal{LR}(X)$. Then,*

- (a) *$(R + S)T \subset RT + ST$ with equality if T is single valued;*
- (b) *$T(R + S)$ is an extension of $TR + TS$ and $TR + TS = T(R + S)$ if $\mathcal{D}(T)$ is the whole space.*

Lemma 2.2. *Let X and Y be two vector spaces. Let $T, S \in \mathcal{LR}(X, Y)$.*

- (a) ([10, Exercise I.2.14 (b)]) *If $\mathcal{D}(T) = \mathcal{D}(S)$ and $T(0) = S(0)$, then $T = S$ or the graphs of T and S are incomparable.*
- (b) ([10, Definition II.5.1 (2) and, Proposition II.5.3]) *T^{-1} is closed if, and only if, T is closed if, and only if, $Q_T T$ is closed single valued and $T(0)$ is a closed space.*
- (c) ([10, Definition II.5.1 (6)]) *If T is continuous, $\mathcal{D}(T)$ and $T(0)$ are closed, then T is closed.*

Lemma 2.3. ([2, Lemma 2.2 (ii)] and [1, Lemma 3.1])

- (a) *Let $S, T \in \mathcal{LR}(X, Y)$. If $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$, then $Q_{T+S} = Q_T$ and $T - S + S = T$.*
- (b) *If $T \in \mathcal{LR}(X, Y)$ and $S \in \mathcal{LR}(Y, Z)$ are closed with $\alpha(S) < +\infty$ and $R(S)$ closed, then $ST \in \mathcal{CR}(X, Z)$.*
- (c) *If $T(0) \subset N(S)$ or $T(0) \subset N(R)$, then $(R + S)T = RT + ST$.*

Lemma 2.4. ([10, Exercise II.5.18]) *Let $T \in \mathcal{BR}(X, Y)$ be single valued and $S \in \mathcal{CR}(Y, Z)$. Then, $ST \in \mathcal{CR}(X, Z)$.*

Lemma 2.5. ([10, Theorem II.3.11 and Corollary III.7.7]) *Let $S, T \in \mathcal{LR}(Y, Z)$.*

- (a) *If $N(S) \subset R(T)$, then $\gamma(ST) \geq \gamma(S)\gamma(T)$.*
- (b) *Let T be open and injective with dense range. Then, for any relation S such that $S(0) \subset \overline{T(0)}$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| < \gamma(T)$, we have $T + S$ is open and injective with dense range.*

Proposition 2.2. ([10, Proposition VI.5.2]) *Let X be complete, and let $S, T \in \mathcal{CR}(X)$ be bijective. Then, ST has the same properties.*

Theorem 2.1. ([10, Theorem III.4.2]) *Let $T \in \mathcal{LR}(X, Y)$ be closed. Then,*

- (a) *T is continuous if and only if $\mathcal{D}(T)$ is closed;*
- (b) *T is open if and only if $R(T)$ is closed.*

3. S-SPECTRAL THEORY OF MULTIVALUED LINEAR OPERATORS IN BANACH SPACES

Definition 3.1. Let $S, T \in \mathcal{LR}(X)$ with $\|S\| \neq 0$. We define the S-resolvent set of T by:

$$\rho_S(T) := \{\lambda \in \mathbb{C} : \lambda S - T \text{ is injective, open with dense range}\}.$$

The spectra of T is the set $\sigma_S(T) := \mathbb{C} \setminus \rho_S(T)$.

Lemma 3.1. *Let $S \in \mathcal{LR}(X)$ be continuous. For $T \in \mathcal{CR}(X)$ such that $\|S\| \neq 0$, $S(0) \subset T(0)$ and $\overline{D(T)} \subset \mathcal{D}(S)$, we have that instead of we define the S -resolvent set of T by:*

$$\rho_S(T) := \left\{ \lambda \in \mathbb{C} : (\lambda S - T)^{-1} \text{ is single valued and everywhere defined} \right\}.$$

Proof. We remark that if X is Banach, then the Lemma 3.1 is true.

Indeed, let $\lambda \in \rho_S(T)$. Then, $\lambda S - T$ is closed. Indeed, $\lambda S - T$ is closed if, and only if, $Q_{S-T}(\lambda S - T)$ is closed and $(\lambda S - T)(0)$ is closed [2, Proposition II.5.3]. But, $(\lambda S - T)(0) = \lambda S(0) - T(0) = T(0)$ closed (as $S(0) \subset T(0)$ and T closed) and $Q_{S-T}(\lambda S - T) = \lambda Q_T S - Q_T T = \lambda Q_A Q_S S - Q_T T$ (where $A := T(0)/\overline{S(0)}$ by virtue of [2, Lemma IV.5.2]) which is closed by [2, Exercise II.5.16]. Now, the result follows immediately from Theorem 2.1 and Lemma 2.3. Let $\lambda \in \rho_S(T)$, then $\lambda S - T$ is injective, open with dense range. Furthermore, $N(\lambda S - T) = (\lambda S - T)^{-1}(0) = \{0\}$, then $(\lambda S - T)^{-1}$ is single valued.

On the other hand, $\lambda S - T$ is open, then applying Theorem 2.1 we obtain $R(\lambda S - T)$ is closed. In short, $\overline{R(\lambda S - T)} = R(\lambda S - T) = X$. $\square$

Example 3.1. We see the following examples, where $\sigma_S(T)$ can be discrete or the whole complex plane.

- (a) Let $T = \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then, $\sigma_S(T) = \{3\}$.
- (b) Let $T = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Then, $\sigma_S(T) = \mathbb{C}$.
- (c) Let $T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, $\sigma_S(T) = \emptyset$.

Remark 3.1. (a) It follows immediately from Definition 3.1 that, for $S = I$ we have

$$\rho_I(T) := \left\{ \lambda \in \mathbb{C} : \lambda - T \text{ is injective, open with dense range} \right\}.$$

(b) If X is a finite dimension space and S is an invertible operator such that $\sigma(S) \setminus \{1\}$ is not empty, then $\sigma_S(S) = \{1\}$, which implies that $\sigma(S) \neq \sigma_S(S)$.

Theorem 3.1. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. Let $\lambda \in \rho_S(T)$, and let $\mu \in \mathbb{C}$ such that*

$$|\mu - \lambda| < \frac{\gamma(\lambda S - T)}{\|S\|}.$$

Then, $\mu \in \rho_S(T)$ and

$$\|(\mu S - T)^{-1}\| \leq \frac{\|(\lambda S - T)^{-1}\|}{1 - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\|}.$$

In particular, $\rho_S(T)$ is open.

Proof. Let $\lambda \in \rho_S(T)$, and let $(x, y) \in G(T)$. Since $(\lambda S - T)^{-1}$ is a bounded operator, it follows from the identity $(\lambda S - T)^{-1}(\lambda S - T)x = x$ that

$$\|(\lambda S - T)^{-1}\| \cdot \|(\lambda S - T)x\| \geq \|x\|.$$

For each $\mu \in \mathbb{C}$ one has

$$\begin{aligned} \|(\mu S - T)x\| &= \|(\lambda S - T)x - (\mu - \lambda)Sx\| \\ &\geq \|(\lambda S - T)x\| - |\mu - \lambda| \cdot \|Sx\| \\ &\geq \|(\lambda S - T)x\| - |\mu - \lambda| \cdot \|S\| \cdot \|x\|, \end{aligned}$$

which leads to

$$\begin{aligned} \|(\lambda S - T)^{-1}\| \cdot \|(\mu S - T)x\| &\geq \|(\lambda S - T)^{-1}\| \cdot \|(\lambda S - T)x\| \\ &\quad - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\| \cdot \|x\| \\ &\geq \|x\| - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\| \cdot \|x\| \\ &= (1 - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\|) \cdot \|x\|. \end{aligned}$$

This inequality shows that $(\mu S - T)^{-1}$ is a bounded operator. This leads to

$$\|(\mu S - T)^{-1}\| \leq \frac{\|(\lambda S - T)^{-1}\|}{1 - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\|}.$$

□

Lemma 3.2. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. If $\lambda \in \rho_S(T)$ and $|\mu - \lambda| < \frac{\gamma(\lambda S - T)}{\|S\|}$, then $\rho_S(T)$ is an open set of $\mathbb{C}$.*

Proof. Since $\lambda \in \rho_S(T)$, one has $\mu \in \rho_S(T)$ and $R(\lambda S - T) = X$, then $\gamma(\lambda S - T) > 0$. Let $|\mu - \lambda| < \frac{\gamma(\lambda S - T)}{\|S\|}$. Then by Lemma 2.3 1. and Lemma 2.5 2., we have $\mu S - T = (\mu - \lambda)S + (\lambda S - T)$ is injective and open with dense range. Therefore, $\mu \in \rho_S(T)$. In particular, $\rho_S(T)$ is open. □

Theorem 3.2. *Let X be a Banach space. Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{BR}(X)$ with $\mathcal{D}(T) = X$ such that $S(0) \subset T(0)$ and $\dim S(0) < +\infty$. If $0 \in \rho_I(S)$, then*

$$\rho_S(T) = \rho_I(S^{-1}T) \cap \rho_I(TS^{-1}).$$

Proof. First of all, it should be mentioned that $\lambda I - S^{-1}T$ is closed. In fact, $\mathcal{D}(S) = X$ is closed, $S(0)$ is closed (since $\dim S(0) < +\infty$) and S is continuous so that by Lemma 2.2 (c) S is closed. Then by Lemma 2.2 (b), S^{-1} is closed. Further $R(S^{-1}) = \mathcal{D}(S) = X$, $\dim N(S^{-1}) = \dim S(0) < +\infty$ and T is closed and thus by Lemma 2.3 (b), $S^{-1}T$ is closed. Now, applying Lemma 2.3, we get $\lambda I - S^{-1}T$ is closed.

Moreover, $\lambda I - TS^{-1}$ is closed. Indeed, since $0 \in \rho_I(S)$, then S^{-1} is single valued (since S is injective). We get S^{-1} is closed, yet $\mathcal{D}(S^{-1}) = R(S) = X$, then S^{-1} is a bounded single valued. Thus, by Lemma 2.4, we have TS^{-1} is closed, which implies by Lemma 2.3, that $\lambda I - TS^{-1}$ is closed.

We can say that $S(0) \subset ST^{-1}(0) = (TS^{-1})^{-1}(0) = N(TS^{-1})$. So, using Lemma 2.3, we infer that $\lambda S - T = (\lambda I - TS^{-1})S$. Let $x \in \mathcal{D}(S^{-1}) = R(S) = X$, then there is $a \in X$ such that $(x, a) \in G(S^{-1})$. This is equivalent to $(a, x) \in G(S)$. Therefore, $(x, x) \in G(SS^{-1})$. We have $I \subset SS^{-1}$ and consequently $\lambda S - T \subset SS^{-1}(\lambda S - T) = S(\lambda I - S^{-1}T)$. So, we get $\lambda S - T \subset S(\lambda I - S^{-1}T)$. It is easy to notice that

$$S(\lambda I - S^{-1}T)(0) = S(\lambda(0) - S^{-1}T(0)) = SS^{-1}T(0).$$

By using Lemma 2.1 (d) we get $SS^{-1}T(0) = T(0) + S(0) = (\lambda S - T)(0)$, additionally $\mathcal{D}(\lambda S - T) = \mathcal{D}(\lambda S) \cap \mathcal{D}(T) = \mathcal{D}(T)$ and

$$\mathcal{D}(S(\lambda I - S^{-1}T)) := \left\{ x \in \mathcal{D}(\lambda I - S^{-1}T) : \begin{array}{l} (\lambda I - S^{-1}T)x \cap \mathcal{D}(S) \\ = (\lambda I - S^{-1}T)x \cap X \neq \emptyset \end{array} \right\}.$$

Equivalently,

$$\begin{aligned} \mathcal{D}(\lambda I - S^{-1}T) &= \mathcal{D}(\lambda) \cap \mathcal{D}(S^{-1}T) = \mathcal{D}(S^{-1}T) \\ &= \{x \in \mathcal{D}(T) : Tx \cap \mathcal{D}(S^{-1}) = Tx \cap R(S) \neq \emptyset\} \\ &= \mathcal{D}(T) = \mathcal{D}(\lambda S - T). \end{aligned}$$

Then, by Lemma 2.2 (a), we infer that $\lambda S - T = S(\lambda I - S^{-1}T)$. Let $\lambda \in \rho_S(T)$ such that $\lambda S - T$ is closed and bijective. First of all, it should be mentioned that $\lambda I - S^{-1}T$ is bijective. Indeed,

$$R(\lambda I - S^{-1}T) = R(S^{-1}(\lambda S - T)) = R(S^{-1}) = \mathcal{D}(S) = X,$$

and let $x \in N(\lambda I - S^{-1}T)$ if and only if

$$x \in \mathcal{D}(S^{-1}T) \text{ and } (\lambda I - S^{-1}T)x = (\lambda I - S^{-1}T)(0)$$

if and only if $S(\lambda I - S^{-1}T)x = S(\lambda I - S^{-1}T)(0)$ if and only if $(\lambda S - T)x = (\lambda S - T)(0)$ if and only if $x = \{0\}$ (since $\lambda S - T$ is injective).

Then, $\lambda I - S^{-1}T$ is bijective. In the same manner we can prove that $\lambda I - TS^{-1}$ is bijective. Accordingly, $\lambda I - TS^{-1}$ and $\lambda I - S^{-1}T$ are bijective and closed, then

$$(3.1) \quad \rho_S(T) \subset \rho(TS^{-1}) \cap \rho(S^{-1}T).$$

Conversely, let $\lambda \in \rho_I(S^{-1}T) \cap \rho_I(TS^{-1})$. Then, $\lambda I - S^{-1}T$ and $\lambda I - TS^{-1}$ are bijective. Hence, S is surjective, then $R(\lambda S - T) = SR(\lambda I - S^{-1}T) = R(S) = X$ and $N(\lambda S - T) \subset N(S^{-1}(\lambda S - T)) = N(\lambda I - S^{-1}T) = \{0\}$. Then,

$$(3.2) \quad \rho(TS^{-1}) \cap \rho(S^{-1}T) \subset \rho_S(T).$$

By (3.1) and (3.2) we conclude that $\rho(TS^{-1}) \cap \rho(S^{-1}T) = \rho_S(T)$. □

Remark 3.2. If T and S are single valued with S is invertible, then

$$\rho_S(T) = \rho(TS^{-1}) = \rho(S^{-1}T).$$

Proposition 3.1. *Let X be a Banach space. Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{BR}(X)$ be single valued with $\mathcal{D}(T) = X$. If $0 \in \rho_I(S)$, then*

$$\rho_S(T) \setminus \{0\} = \rho_I(S^{-1}T) \setminus \{0\} = \rho_I(TS^{-1}) \setminus \{0\}.$$

Proof. Let $0 \in \rho_I(S^{-1})$, then $N(S^{-1}) = S(0) = \{0\}$ and $R(S^{-1}) = \mathcal{D}(S) = X$. Consequently, S^{-1} is single valued, hence $S^{-1}(0) = N(S) = \{0\}$. Since S^{-1} is closed (since S is closed), $\mathcal{D}(S^{-1}) = R(S) = X$, and S^{-1} is single valued, then S^{-1} is bounded.

Therefore, by Proposition 3.2 we get $\rho_S(T) \setminus \{0\} = \rho_I(S^{-1}T) \setminus \{0\} = \rho_I(TS^{-1}) \setminus \{0\}$. $\square$

Definition 3.2. Let $S, T \in \mathcal{LR}(X)$ and $\lambda \in \mathbb{C}$. We call the S-resolvent of T in λ the operator defined by:

$$R_S(\lambda, T) := (\lambda S - T)^{-1}.$$

Theorem 3.3. Let $T \in \mathcal{CR}(X)$ and let $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. Let $\lambda, \mu \in \rho_S(T)$. Then, the following hold.

- (a) $R_S(\mu, T) - R_S(\lambda, T) = (\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T)$.
- (b) If $\mu \in \rho_S(T)$ and $|\lambda - \mu| \cdot \|S\| \leq \gamma(\mu S - T)$, then

$$R_S(\lambda, T) = \sum_{i=0}^{+\infty} (\lambda - \mu)^i R_S(\mu, T)(SR_S(\mu, T))^i.$$

Proof. (a) Let $(x, y) \in G(R_S(\mu, T) - R_S(\lambda, T))$, so that $(x, y_1) \in G(R_S(\mu, T))$ and $(x, y_2) \in G(R_S(\lambda, T))$ with $y = y_1 + y_2$. One has $(y_1, x) \in G(\mu S - T)$ and $(y_2, x) \in G(\lambda S - T) = G(\mu S - T + (\lambda - \mu)S)$. Then, $x \in (\mu S - T + (\lambda - \mu)S)y_2 = (\mu S - T)y_2 + (\lambda - \mu)Sy_2$. By Lemma 2.3 we get $x - (\lambda - \mu)Sy_2 \in (\mu S - T)y_2$. Hence, $(y_2, x - (\lambda - \mu)Sy_2) \in G(\mu S - T)$, so that

$$(y, (\lambda - \mu)Sy_2) = (y_1, x) - (y_2, x - (\lambda - \mu)Sy_2) \in G(\lambda S - T).$$

Implies that $((\lambda - \mu)Sy_2, y) \in G(R_S(\lambda, T))$. By Lemma 2.3 which shows that $(y_2, y) \in (\lambda - \mu)R_S(\lambda, T)S$. Then,

$$(x, y) \in G((\lambda - \mu)R_S(\lambda, T)SR_S(\mu, T)),$$

which leads to $R_S(\mu, T) - R_S(\lambda, T) \subseteq (\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T)$.

Conversely, let $(x, y) \in G((\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T))$. Then, $(x, z) \in G(R_S(\lambda, T))$, $(z, w) \in G((\lambda - \mu)S)$ and $(w, y) \in G(R_S(\mu, T))$ for some $y, z \in X$. It follows from $(x, z) \in G(R_S(\lambda, T))$ that $(z, x) \in G(\lambda S - T)$. So that,

$$x \in (\lambda S - T)z = (\mu S - T)z + (\lambda - \mu)Sz.$$

By using Lemma 2.3 we get $x + (\mu - \lambda)Sz \in (\mu S - T)z$. Hence,

$$(z, x + (\mu - \lambda)Sz) \in G(\mu S - T).$$

Since $w \in (\mu - \lambda)Sz$, then $(z, x - w) \in (z, x + (\mu - \lambda)Sz) \in G(\mu S - T)$. Consequently,

$$(x, z + y) = (x - w, z) + (w, y) \in G(R_S(\mu, T)).$$

Finally,

$$(x, y) = (x, z + y) - (x, z) \in G(R_S(\mu, T)) - G(R_S(\lambda, T)) \subset G(R_S(\mu, T) - R_S(\lambda, T)).$$

We deduce that,

$$(\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T) \subseteq R_S(\mu, T) - R_S(\lambda, T).$$

So, $(\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T) = R_S(\mu, T) - R_S(\lambda, T)$.

(b) Since $(\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T) = R_S(\mu, T) - R_S(\lambda, T)$, then

$$R_S(\lambda, T) = \sum_{i=0}^n (\lambda - \mu)^i R_S(\mu, T)(SR_S(\mu, T))^i + (\lambda - \mu)^{n+1} R_S(\lambda, T)(SR_S(\mu, T))^{n+1}.$$

From the estimation

$$\|(\lambda - \mu)^{n+1} R_S(\lambda, T)(SR_S(\mu, T))^{n+1}\| \leq \|R_S(\lambda, T)\|(|\lambda - \mu| \cdot \|R_S(\mu, T)\| \cdot \|S\|)^{n+1},$$

and the inequality $|\lambda - \mu| \cdot \|R_S(\mu, T)\| \cdot \|S\| < 1$, it follows that the rest term

$$(\lambda - \mu)^{n+1} R_S(\lambda, T)(SR_S(\mu, T))^{n+1}$$

tends to 0 as $n \rightarrow +\infty$. This completes the proof. $\square$

Theorem 3.4. Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$, which is continuous, satisfy $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. For all $\lambda \in \rho_S(T)$, we have $R_S(\lambda, T)$ commutes with S . Then,

$$\lim_{\mu \rightarrow \lambda} (\mu - \lambda)^{-1} (R_S(\mu, T) - R_S(\lambda, T)) = -R_S(\lambda, T)SR_S(\lambda, T).$$

Proof. Let $\lambda \in \rho_S(T)$. We have from Theorem 3.3 that

$$\begin{aligned} & (\mu - \lambda)^{-1} (R_S(\mu, T) - R_S(\lambda, T)) + R_S(\lambda, T)SR_S(\lambda, T) \\ &= -R_S(\mu, T)SR_S(\lambda, T) + R_S(\lambda, T)SR_S(\lambda, T) \\ &= (R_S(\lambda, T)S - R_S(\mu, T)S)R_S(\lambda, T). \end{aligned}$$

Hence, we can write

$$\begin{aligned} & \|(\mu - \lambda)^{-1} (R_S(\mu, T) - R_S(\lambda, T)) + R_S(\lambda, T)SR_S(\lambda, T)\| \\ & \leq \|R_S(\lambda, T) - R_S(\mu, T)\| \cdot \|S\| \cdot \|R_S(\lambda, T)\|. \end{aligned}$$

Now we take $|\mu - \lambda| \cdot \|S\| < \|R_S(\lambda, T)\|^{-1}$. Then,

$$\sum_{n \geq 0} |\mu - \lambda|^n \cdot \|S\|^n \cdot \|R_S(\lambda, T)\|^n < +\infty,$$

with $(I - (\lambda - \mu)SR_S(\lambda, T))^{-1} = \sum_{n \geq 0} (\mu - \lambda)^n S^n R_S(\lambda, T)^n$. Since $R_S(\lambda, T)$ and S commute, then for $x \in \mathcal{D}(T)$ we have

$$(\lambda - \mu)R_S(\lambda, T)^{-1}SR_S(\lambda, T)x = (\lambda - \mu)R_S(\lambda, T)^{-1}R_S(\lambda, T)Sx.$$

By Lemme 2.1 4., we get

$$\begin{aligned} (\lambda - \mu)R_S(\lambda, T)^{-1}SR_S(\lambda, T)x &= (\lambda - \mu)Sx + T(0) \\ &= R_S(\lambda, T)^{-1}x - R_S(\mu, T)^{-1}x. \end{aligned}$$

From the above we find

$$\begin{aligned} & R_S(\lambda, T)^{-1}(I - (\lambda - \mu)SR_S(\lambda, T))x \\ &= R_S(\lambda, T)^{-1}x + (\mu - \lambda)R_S(\lambda, T)^{-1}SR_S(\lambda, T))x \\ &= R_S(\lambda, T)^{-1}x - R_S(\lambda, T)^{-1}x + R_S(\mu, T)^{-1}x \\ &= R_S(\lambda, T)^{-1}(0) + R_S(\mu, T)^{-1}x. \end{aligned}$$

Since $R_S(\lambda, T)^{-1}(0) = (\lambda S - T)(0) = T(0)$ and $R_S(\mu, T)^{-1}x = (\lambda S - T)x$, then

$$R_S(\lambda, T)^{-1}(I - (\lambda - \mu)SR_S(\lambda, T))x = R_S(\mu, T)^{-1}x.$$

This implies that $R_S(\mu, T) = (I - (\lambda - \mu)SR_S(\lambda, T))^{-1}R_S(\lambda, T)$. So, in the first place, we have

$$R_S(\mu, T) = \sum_{n \geq 0} (\lambda - \mu)^n S^n R_S(\lambda, T)^{n+1}.$$

In the second place, we have

$$\begin{aligned} R_S(\mu, T) - R_S(\lambda, T) &= \sum_{n \geq 0} (\lambda - \mu)^n S^n R_S(\lambda, T)^{n+1} - R_S(\lambda, T) \\ &= R_S(\lambda, T) \sum_{n \geq 1} (\lambda - \mu)^n S^n R_S(\lambda, T)^n, \end{aligned}$$

and we conclude

$$\|R_S(\mu, T) - R_S(\lambda, T)\| \leq \|R_S(\lambda, T)\| \sum_{n \geq 1} |(\lambda - \mu)|^n \cdot \|S\|^n \cdot \|R_S(\lambda, T)\|^n.$$

Arguing as above we conclude that

$$\lim_{\mu \rightarrow \lambda} (\mu - \lambda)^{-1}(R_S(\mu, T) - R_S(\lambda, T)) = -R_S(\lambda, T)SR_S(\lambda, T).$$

□

Corollary 3.1. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$, which is continuous, satisfy $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. For all $\lambda \in \rho_S(T)$, we have $R_S(\lambda, T)$ commutes with S .*

The function $\varphi : \lambda \rightarrow R_S(\lambda, T)$ is holomorphic for all $\lambda \in \rho_S(T)$.

Proof. If λ and $\mu \in \rho_S(T)$, then

$$\lim_{\mu \rightarrow \lambda} \frac{\varphi(\mu) - \varphi(\lambda)}{\mu - \lambda} = -R_S(\lambda, T)SR_S(\lambda, T).$$

□

Proposition 3.2. *Let T be open and injective with dense range. Then, for any relation $\|S\| \neq 0$ such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$ we have*

$$\sigma_S(T) \subset \left\{ \lambda \in \mathbb{C} : |\lambda| < \frac{\gamma(T)}{\|S\|} \right\}.$$

Proof. Choose λ such that $0 < |\lambda| < \frac{\gamma(T)}{\|S\|}$. Then, by Lemma 2.5 2., we have $\lambda S - T$ is open, injective and has dense range. Thus, $\lambda \in \rho_S(T)$. $\square$

Proposition 3.3. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$, which is continuous, satisfy $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. For all $\lambda \in \rho_S(T)$, we have $R_S(\lambda, T)$ commutes with S , and $\rho_S(T)$ is nonempty and unbounded. Then,*

$$\lim_{|\lambda| \rightarrow +\infty} \|R_S(\lambda, T)\| = 0, \quad \text{for all } \lambda \in \rho_S(T).$$

Proof. Let $\lambda, \mu \in \rho_S(T)$ and μ be fixed,

$$(R_S(\lambda, T) - R_S(\mu, T))(\mu S - T) = (\mu - \lambda)R_S(\lambda, T)SR_S(\mu, T)(\mu S - T).$$

This is equivalent to $R_S(\lambda, T)(\mu S - T) - I_{\mathcal{D}(T)} = (\mu - \lambda)R_S(\lambda, T)S$. It follows that, $|\mu - \lambda| \cdot \|R_S(\lambda, T)\| \cdot \|S\| \leq 1 + \|\mu S - T\| \cdot \|R_S(\lambda, T)\|$. Thus, since $\|\mu S - T\| < +\infty$,

$$(|\mu - \lambda| \cdot \|S\| - \|\mu S - T\|) \|R_S(\lambda, T)\| < 1.$$

Then, $\|R_S(\lambda, T)\| < \frac{1}{|\mu - \lambda| \cdot \|S\| - \|\mu S - T\|}$. Consequently, $\lim_{|\lambda| \rightarrow +\infty} \|R_S(\lambda, T)\| = 0$. $\square$

Theorem 3.5. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$, which is continuous, satisfy $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. For all $\lambda \in \rho_S(T)$, we have $R_S(\lambda, T)$ commutes with S , and $\rho_S(T)$ is non-empty and unbounded. Then, the S -spectrum of T is non-empty.*

Proof. Suppose that $\rho_S(T) = \mathbb{C}$. We have for $x \in X$ and $x' \in X'$,

$$\lim_{\mu \rightarrow \lambda} \frac{x'R_S(\lambda, T)x - x'R_S(\mu, T)x}{\lambda - \mu} = x'R_S(\lambda, T)SR_S(\lambda, T)x.$$

Thus, the (single valued) function $f(\lambda) = x'R_S(\lambda, T)SR_S(\lambda, T)x$ is an entire analytic function.

Moreover, $f(\lambda) \leq \|x'\| \cdot \|R_S(\lambda, T)\| \cdot \|S\| \cdot \|R_S(\lambda, T)\| \cdot \|x\|$. By Proposition 3.3, it is clear that $\lim_{|\lambda| \rightarrow \infty} \|R_S(\lambda, T)\| = 0$, then we have $f(\lambda) = 0$ for all λ . Since x' is arbitrary, we have $R_S(\lambda, T)x = 0$ for all $x \in X$. Thus, $X = N(R_S(\lambda, T)) = (\lambda S - T)(0) = T(0)$, and hence,

$$0 = R_S(\lambda, T)(\lambda S - T)(0) = R_S(\lambda, T)(\lambda S - T)x = x, \quad \text{for all } x \in X$$

(since $R_S(\lambda, T)$ is injective), which contradicts with our assumption that X is non-trivial. $\square$

Definition 3.3. The augmented S -spectrum of T is the set

$$\bar{\sigma}_S(T) = \begin{cases} \sigma_S(T) \cup \{+\infty\}, & \text{if } 0 \in \sigma_S(T^{-1}), \\ \sigma_S(T), & \text{otherwise.} \end{cases}$$

The augmented S -spectrum is non-empty (if $X \neq \{0\}$) since if $\{\infty\} \not\subset \bar{\sigma}_S(T)$ then $\bar{\sigma}_S(T) = \sigma_S(T)$ is non-empty by Theorem 3.5.

The Möbius transformation $\eta(\lambda) = (\mu - \lambda)^{-1}$, where μ is a fixed point of $\mathbb{C}$, and a topological homeomorphism from $\mathbb{C}_\infty$ onto itself.

Theorem 3.6. *Let $S \in \mathcal{LR}(X, Y)$ be continuous and $T \in \mathcal{CR}(X, Y)$ such that $S(0) \subset \overline{T(0)}$ and $\{0\} \neq \mathcal{D}(T) \supset \mathcal{D}(S)$ with $\mu \in \rho_S(T)$. Then,*

$$\eta(\overline{\sigma_S}(T)) = \sigma(SR_S(\mu, T)).$$

Proof. Without loss of generality, we assume that X is complete, T is closed and S is continuous. Let $\lambda \in \mathbb{C}$, $\lambda \neq \mu$, and let $A := (\mu - \lambda)((\mu - \lambda)^{-1} - SR_S(\mu, T))$. Then,

$$\begin{aligned} \lambda S - T &= (\mu S - T) - (\mu - \lambda)S \quad (\text{since } S(0) \subset T(0)) \\ &= (I - (\mu - \lambda)SR_S(\lambda, T))(\mu S - T) \\ &= A(\mu S - T). \end{aligned}$$

We shall verify that A is injective. Suppose that $\lambda \in \rho_S(T)$. For $x \in X$ we have $A(x) = A(0) = S(0)$, then $(\mu - \lambda)((\mu - \lambda)^{-1} - SR_S(\mu, T))x = S(0) \subset (\lambda S - T)(0)$, which give $R_S(\lambda, T)(\mu - \lambda)((\mu - \lambda)^{-1} - SR_S(\mu, T))x \subset R_S(\lambda, T)(\lambda S - T)(0) = 0$. This implies that $(R_S(\lambda, T) - R_S(\lambda, T) + R_S(\mu, T))x = 0$, equivalent to $R_S(\mu, T)x = 0$. Therefore, $(\mu S - T)R_S(\mu, T)x = (\mu S - T)(0) = T(0)$. By using Lemma 2.1 (d) we get $x + (\mu S - T)(0) = x + T(0) = T(0)$, so we conclude that $x = 0$. Then, A is injective. Next we have $X = R(\lambda S - T) = R(A(\mu S - T)) \subset R(A)$. Hence, S is surjective. As required, since S is both bijective and open, it follows that $(\mu - \lambda)^{-1} \in \rho_I(SR_S(\lambda, T))$.

Conversely, let $(\mu - \lambda)^{-1} \in \rho_I(SR_S(\lambda, T))$. For $x \in \mathcal{D}(T)$, we have

$$\|(\lambda S - T)x\| = \|A(\mu S - T)x\| \geq \gamma(A(\mu S - T))d(x, R_S(\lambda, T)A^{-1}(0)).$$

Since S is injective, then $R_S(\lambda, T)A^{-1}(0) = R_S(\lambda, T)(0) = 0$. Therefore,

$$\|(\lambda S - T)x\| \geq \gamma(A(\mu S - T))\|x\|.$$

By Theorem 2.5 (a) as S is injective we have $\gamma(A(\mu S - T)) \geq \gamma(A)\gamma(\mu S - T)$. Hence, $\gamma(A) > 0$ from the hypothesis, and $\gamma(\mu S - T) > 0$ since $\mu \in \rho_S(T)$. Hence, it is easy to verify that $\lambda S - T$ is injective and open, and we have $\lambda S - T$ is surjective. We deduce that $\lambda \in \rho_S(T)$. $\square$

Proposition 3.4. *Let the relations S and T commute such that $\overline{\mathcal{D}(T)} \subset \mathcal{D}(S)$. Then, $(\lambda S - T)$ and $(\mu S - T)$ have the same properties.*

Proof. It is clear that the domains of $(\mu S - T)(\lambda S - T)$ and $(\lambda S - T)(\mu S - T)$ are each equal to $\mathcal{D}(T^2)$. We have for $x \in \mathcal{D}(T^2)$

$$\begin{aligned} (\lambda S - T)(\mu S - T)x &= (\lambda S - T)(\mu Sx - Tx) \\ &= (\lambda S - T)\{\mu y_1 - y_2 : y_1 \in Sx \text{ and } y_2 \in Tx, y_1, y_2 \in \mathcal{D}(T)\} \\ &= \{\lambda \mu S y_1 - \lambda S y_2 - \mu T y_1 + T y_2 : y_1 \in Sx \text{ and } y_2 \in Tx, \\ &\quad y_1, y_2 \in \mathcal{D}(T)\} \\ &= \lambda \mu S^2 x - \lambda S T x - \mu T S x + T^2 x. \end{aligned}$$

Since T and S commute, then

$$\begin{aligned} (\lambda S - T)(\mu S - T)x &= \lambda\mu S^2x - \lambda T Sx - \mu S T x + T^2x \\ &= (\mu S - T)(\lambda S - T)x. \end{aligned}$$

□

Proposition 3.5. *Let $\mu \in \rho_S(T)$ such that $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$ be continuous with $S(0) \subset T(0)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$ and let $\lambda \neq \mu$. Then,*

$$N(\lambda S - T) = N((\mu - \lambda)^{-1} - (\mu S - T)^{-1}S).$$

Proof. Let $\mu \in \rho_S(T)$ and $\lambda \in \mathbb{C}$ such that $\mu \neq \lambda$.

(a) $T(0) + (\mu - \lambda)Sx = (\mu S - T)x$. Indeed, let $(x, y) \in G(T - T + (\mu - \lambda)S)$, then $x \in \mathcal{D}(T)$ and $y \in (T - T + (\mu - \lambda)S)x$, so that, $y \in T(0) + (\mu - \lambda)Sx$.

From another angle, $T(0) = S(0) + T(0) \subset (\lambda S - T)x$, we obtain that

$$\begin{aligned} y \in T(0) + (\mu - \lambda)Sx &\subset (\lambda S - T)x + (\mu - \lambda)Sx \\ &= (\mu S - T)x \quad (\text{by Lemma 2.3 1.}). \end{aligned}$$

We infer that $G(T - T + (\mu - \lambda)S) \subset G(\mu S - T)$. Furthermore, $(T - T + (\mu - \lambda)S)(0) = T(0) = (\mu S - T)(0)$ and clearly $\mathcal{D}(T - T + (\mu - \lambda)S) = \mathcal{D}(T) \cap \mathcal{D}(S) = \mathcal{D}(T) = \mathcal{D}(\mu S - T)$. So, by Lemma 2.2 (a) and Lemma 2.3 (a), we have

$$T(0) + (\mu - \lambda)Sx = (\lambda S - T)x + (\mu - \lambda)Sx = (\mu S - T)x.$$

(b) $N(\lambda S - T) = N((\mu - \lambda)^{-1} - (\mu S - T)^{-1}S)$. Indeed, we first note that $x \in N((\mu - \lambda)^{-1} - (\mu S - T)^{-1}S)$ if, and only if, $x = (\mu - \lambda)(\mu S - T)^{-1}Sx$, because

$$\begin{aligned} (I - (\mu - \lambda)(\mu S - T)^{-1}S)(0) &= (\mu - \lambda)(\mu S - T)^{-1}(\mu S - T)(0) \\ &= (\mu S - T)^{-1}(0) = 0. \end{aligned}$$

The latter implies that $x \in R((\mu S - T)^{-1}) = \mathcal{D}(\mu S - T)$ and, hence, it suffices to consider the case $x \in \mathcal{D}(T)$, $x \neq 0$: $x \in N(\lambda S - T)$ if and only if

$$x \in \mathcal{D}(\lambda S - T) = \mathcal{D}(T) \quad \text{and} \quad (\lambda S - T)x = (\lambda S - T)(0) = T(0)$$

if and only if

$$(\mu S - T)x = (\lambda S - T)x + (\mu - \lambda)Sx = T(0) - \lambda Sx + \mu Sx \quad (\text{by (a)})$$

if and only if

$$(\mu S - T)^{-1}(\mu S - T)x = (\mu S - T)^{-1}((\mu - \lambda)Sx + T(0))$$

if and only if

$$x + (\mu S - T)^{-1}(0) = (\mu - \lambda)(\mu S - T)^{-1}Sx + (\mu S - T)^{-1}(\mu S - T)(0)$$

if and only if

$$0 = (I - (\mu - \lambda)(\mu S - T)^{-1}S)x.$$

Then, $N(\lambda S - T) = N((\mu - \lambda)^{-1} - (\mu S - T)^{-1}S)$. □

Theorem 3.7. *Let $T \in \mathcal{CR}(X)$ and let $S \in \mathcal{BR}(X)$ be single valued such that S and T commute, with $\mathcal{D}(T) = X$ and $0 \in \rho(S)$. Then, for any complex polynomial P , we have*

$$\sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1})) = P(\sigma_S(T)).$$

Proof. Let $\lambda \in \mathbb{C}$, and let

$$\mu I - P(\lambda) := c \prod_{j=1}^n S(\alpha_j - \lambda).$$

Then,

$$\mu I - P(S^{-1}T) := c \prod_{j=1}^n S(\alpha_j I - S^{-1}T)$$

and

$$\mu I - P(TS^{-1}) := c \prod_{j=1}^n S(\alpha_j I - TS^{-1}).$$

Without loss of generality, we assume that X is complete with $S^{-1}T$ and TS^{-1} closed.

Let $\mu \in \sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1}))$. If $\alpha_j \in \rho_S(T) = \rho(TS^{-1}) \cap \rho(TS^{-1})$ for all $1 \leq j \leq n$, then $(\alpha_j I - TS^{-1})$ and $(\alpha_j I - S^{-1}T)$ would be bijective for all $1 \leq j \leq n$. Since $0 \in \rho(S)$, then S is bijective. Hence, by Proposition 2.2, $\mu I - P(S^{-1}T)$ and $\mu I - P(TS^{-1})$ are bijective. Contradicting the assumption that $\mu \in \sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1}))$. Thus exists j , $1 \leq j \leq n$ such that $\alpha_j \in \sigma_S(T)$. Since $P(\alpha_j) = \mu$, it follows that $\mu \in P(\sigma_S(T))$.

Conversely, suppose that $\mu \in P(\sigma_S(T))$. Then, $\mu = P(\lambda)$ for some $\lambda \in \sigma_S(T)$. Thus, $\lambda S = \alpha_j S$ since S is single valued, then $\lambda = \lambda S S^{-1} = \alpha_j S S^{-1} = \alpha_j$ for some j such that $1 \leq j \leq n$. Since the factors commute (see Proposition 2.2), we may assume that $j = 1$. We notice the existence of two cases.

1st case. Suppose that $\alpha_1 S - T$ is injective. Then it cannot be surjective ($\alpha_1 = \lambda \in \sigma_S(T)$). Consequently, $\mu I - P(S^{-1}T)$ or $\mu I - P(TS^{-1})$ cannot be surjective. Thus, $\mu \in \sigma(P(S^{-1}T))$.

2nd case. $\alpha_j S - T$ is surjective for every j , $1 \leq j \leq n$. Since $\alpha_1 \in \sigma_S(T)$, $\alpha_1 S - T$ cannot be injective. Thus, $\mu I - P(S^{-1}T)$ or $\mu I - P(TS^{-1})$ is not injective. It follows that $\mu \in \sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1}))$. $\square$

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ON CENTRALLY-EXTENDED GENERALIZED JORDAN *-DERIVATIONS IN RINGS

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ABSTRACT. Let R be an associative ring with an involution $'$. In this article, we introduce the notions of *centrally-extended generalized Jordan *-derivation*, *centrally extended Jordan left *-centralizer* and characterize these mappings in involutive prime rings.

1. INTRODUCTION

Throughout this study, R is an associative ring with center $Z(R)$. R is called *prime*, if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$; and it is called *semiprime*, if $aRa = (0)$, implies $a = 0$. Clearly, every prime ring is semiprime ring but the converse need not be true, for instance $\mathbb{Z} \times \mathbb{Z}$. The symmetric ring of quotients of R is denoted by Q_s with center C , which is known as the extended centroid of R ; clearly $R \subseteq Q_s$ and $Z(R) \subseteq C$. It is well-known that if R is prime then Q_s is prime and C is a field. The central closure of R is denoted by $A(= RC + C)$; for more details of these objects, we refer the reader to [6]. For any $x, y \in R$, the commutator (resp. anti-commutator) of x, y is defined as $[x, y] = xy - yx$ (resp. $x \circ y = xy + yx$). It is established knowledge that R satisfies s_4 (the standard identity in four noncommuting variables), if for all $x_1, x_2, x_3, x_4 \in R$, the equation

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0,$$

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where S_4 is the symmetric group of degree 4 and $(-1)^\sigma$ is the sign of permutation $\sigma \in S_4$. For some interesting equivalent forms of s_4 , one can refer [11, Lemma 1].

For any $n \in \mathbb{Z}^+$, R is called an n -torsion free if for any $x \in R$, $nx = 0$ implies $x = 0$. An anti-automorphism $'*$ of R is called *involution* if $(x^*)^* = x$ for all $x \in R$. A ring equipped with an involution $'*$ is called **-ring* or *ring with involution* or *involutive ring*. An element x in a *-ring R is called *symmetric* if $x^* = x$, and it is called *skew-symmetric* if $x^* = -x$. The set of symmetric and skew symmetric elements of a ring R is denoted by $H(R)$ and $S(R)$, respectively. Moreover, if R is a prime ring endowed with the involution $'*$, then $'*$ can be uniquely extended to $Q_s(R)$ (see [15, page 4]).

Let R be a *-ring, an additive mapping $d : R \rightarrow R$ is called **-derivation* if $d(xy) = d(x)y^* + xd(y)$ for all $x, y \in R$ and it is called *Jordan *-derivation* if $d(x^2) = d(x)x^* + xd(x)$ for all $x \in R$. These notions are first mentioned in [12]. Note that the mapping $x \mapsto xa - ax^*$, where a is a fixed element of R , is an example of Jordan *-derivation, called *inner Jordan *-derivation*. Moreover, if $a \in Q_s$, then such a map is called *X-inner Jordan *-derivation*. The study of Jordan *-derivations has been originated from the problem of representability of quadratic forms by bilinear forms (see [28, 30]). Thereafter, some significant studies have taken place on the structure of Jordan *-derivations in rings (see [5, 14, 16, 21]).

In [2], Ali introduced the notion of *generalized *-derivation*, which is a self-map F of R associated with a *-derivation d satisfying $F(xy) = F(x)y^* + xd(y)$ for all $x, y \in R$. In addition to this, a self-map F of R is called a *generalized Jordan *-derivation* associated with a Jordan *-derivation d if $F(x^2) = F(x)x^* + xd(x)$ for all $x \in R$ (see [16], [19]). For any fixed $a, b \in R$, a map $x \mapsto ax^* + xb$ is a basic example of generalized Jordan *-derivation, which is called *generalized inner Jordan *-derivation*. Further, if a, b comes from Q_s , then this map is called *generalized X-inner Jordan *-derivation*. There has been an ongoing interest in the investigation of such mappings, for more details we refer the reader to [1, 4, 5, 16, 19] and references therein.

A mapping f of R is called *centralizing* (resp. *commuting*) on a subset S of R if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) for all $x \in S$. To our best knowledge, Divinsky [17] initiated the study of commuting and centralizing mappings in rings by proving a classical result which states that a simple Artinian ring is commutative if it admits a commuting nontrivial automorphism. Since then, many significant results on commuting and centralizing mappings have been established by Posner [27], Mayne [25], Bell and Martindale [8], Brešar [10]. Moreover, let R be a *-ring, a mapping $f : R \rightarrow R$ is called **-centralizing* (**-commuting*) on a subset S of R if $[f(x), x^*] \in Z(R)$ (resp. $[f(x), x^*] = 0$) for all $x \in S$ (see [3]).

Bell and Daif [7] introduced *centrally-extended derivations* and discussed their existence. Accordingly, a mapping $d : R \rightarrow R$ is called *centrally-extended derivation* if $d(x + y) - d(x) - d(y) \in Z(R)$ and $d(xy) - d(x)y - xd(y) \in Z(R)$ for all $x, y \in R$. Motivated by this, El-Deken and Nabel [31] introduced that a mapping d is called *centrally-extended *-derivation* if $d(x + y) - d(x) - d(y) \in Z(R)$ and $d(xy) -$

$d(x)y^* - xd(y) \in Z(R)$ for all $x, y \in R$. Furthermore, a mapping F is called *centrally-extended generalized *-derivation* associated with a centrally-extended *-derivation d if $F(x+y) - F(x) - F(y) \in Z(R)$ and $F(xy) - F(x)y^* - xd(y) \in Z(R)$ for all $x, y \in R$. In a recent paper [9], we introduced and studied the concept of *centrally-extended Jordan *-derivation*, which is a mapping $d : R \rightarrow R$ such that $d(x+y) - d(x) - d(y) \in Z(R)$ and $d(x \circ y) - d(x)y^* - d(y)x^* - xd(y) - yd(x) \in Z(R)$ for all $x, y \in R$. Nowadays, centrally-extended mappings are getting attention of many researchers, consequently there has been rising literature on these mappings in rings under different settings, for instance, see [7, 9, 18, 26, 31–34].

In this article, we shall introduce centrally-extended generalized Jordan *-derivation in rings and discuss their existence in noncommutative prime ring under suitable torsion conditions. We also investigate some specific functional identities involving centrally-extended generalized Jordan *-derivations. Precisely, in Section 4 of this article, we prove a structural result on centrally-extended generalized Jordan *-derivations which plays a key role in Sections 5 and 6, where we study centralizing and hypercommuting conditions involving centrally-extended generalized Jordan *-derivations, respectively.

2. PRELIMINARIES

The following lemmas constitute a set of results that will be instrumental in the development of the paper.

Lemma 2.1 (Brauer's Trick). *Let G be a group and H_1, H_2 be subgroups of G . If $G = H_1 \cup H_2$, then either $G = H_1$ or $G = H_2$.*

Lemma 2.2. *If R is a prime ring, then $Z(R)$ has no proper zero divisor.*

Lemma 2.3. *Let R is a prime for any $a \in Z(R)$. If there exists $b \in R$ such that $ab \in Z(R)$, then either $a = 0$ or $b \in Z(R)$.*

Lemma 2.4. ([1, Proposition 2.3]). *Let R be a 2-torsion free semiprime ring with involution $'*$ '. If $f : R \rightarrow R$ is an additive map such that $f(x^2) = f(x)x^*$ for all $x \in R$, then there exists $q \in Q_r(R)$ such that $f(x) = qx^*$ for all $x \in R$.*

Lemma 2.5. ([3, Lemma 2.2]). *Let R be a 2-torsion free semiprime ring with involution $'*$ '. If an additive mapping f of R into itself such that $[f(x), x^*] \in Z(R)$ for all $x \in R$, then $[f(x), x^*] = 0$ for all $x \in R$.*

Lemma 2.6. ([6, Theorem 6.4.6]). *Let R be a prime ring with extended centroid C , anti-automorphism g and maximal right ring of quotients $Q_{mr}(R) = Q$. If $0 \neq \phi = \phi(x_1, \dots, x_n, g(x_1), \dots, g(x_n)) \in Q_C < X \cup g(X) >$ is a g -identity on K ideal of R , then ϕ is a g -identity on $Q_s = Q_s(R)$.*

Lemma 2.7. ([9, Theorem 4.6]). *Let R be a 2-torsion free noncommutative prime ring. If R admits a non-zero centrally-extended Jordan derivation $\mathfrak{d} : R \rightarrow R$ such*

that $[\mathfrak{d}(x), x] \in Z(R)$ for all $x \in R$, then either $\mathfrak{d} = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.

Lemma 2.8. ([9, Theorem 4.7]). *Let R be a 2-torsion free noncommutative prime ring with involution $*$. If R admits a non-zero centrally-extended Jordan derivation $\mathfrak{d} : R \rightarrow R$ such that $[\mathfrak{d}(x), x^*] \in Z(R)$ for all $x \in R$, then either $\mathfrak{d} = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.*

Lemma 2.9. ([10, Proposition 3.1]). *Let R be a 2-torsion free semiprime ring and U be a Jordan subring of R . If an additive mapping F of R into itself is centralizing on U , then F is commuting on U .*

Lemma 2.10. ([10, Theorem 3.2]). *Let R be a prime ring. If an additive mapping $F : R \rightarrow R$ is commuting on R , then there exists $\lambda \in C$ and an additive $\xi : R \rightarrow C$, such that $F(x) = \lambda x + \xi(x)$ for all $x \in R$.*

Lemma 2.11. ([19, Theorem 2.2]). *Let R be a 2-torsion free prime ring with involution $'*$. Let $F : R \rightarrow R$ be a generalized Jordan $*$ -derivation associated with a Jordan $*$ -derivation d . Then, F is of the form $F(x) = qx^* + d(x)$ for all $x \in R$ and some $q \in Q_s(R)$.*

Lemma 2.12. ([20, Theorem 1]). *Let R be a prime ring with involution $'*$ and center $Z(R)$. If d is a non-zero derivation such that $[d(h), h] \in Z(R)$ for all $h \in H(R)$, then R satisfies s_4 .*

Lemma 2.13. ([20, Theorem 3]). *Let R be a prime ring with involution $'*$ and center $Z(R)$. If n be a fixed natural number such that $x^n \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4 identity.*

Lemma 2.14. ([20, Theorem 6]). *Let R be a prime ring with involution $'*$ and center $Z(R)$. If d is a non-zero derivation on R such that $d(x)x + xd(x) \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4 identity.*

Lemma 2.15. ([20, Theorem 7]). *Let R be a prime ring with involution $'*$ and center $Z(R)$. If d is a non-zero derivation on R such that $d(x)x + xd(x) \in Z(R)$ for all $x \in S(R)$, then R satisfies s_4 identity.*

Lemma 2.16. [21, Theorem 1.3]. *Let R be a 2-torsion free noncommutative prime ring with involution $'*$, then any Jordan $*$ -derivation on R is X -inner.*

Lemma 2.17. *Let R be a 2-torsion free prime ring. If $q_1 \in Q_s(R)$ such that $[q_1, h] \in C$ for all $h \in H(R)$, then R satisfy s_4 identity or $q_1 \in C$.*

Proof. Let us consider

$$(2.1) \quad [q_1, h] \in C, \quad \text{for all } h \in H(R).$$

Replacing h by h^2 , where $h \in H(R)$, we obtain $[q_1, h]h + h[q_1, h] \in C$, i.e., $d(h)h + hd(h) \in C$ for all $h \in H(R)$, where $d(x) = [q_1, x]$. If $d \neq 0$, we have the result by Lemma 2.14. If $d = 0$, then we conclude $q_1 \in C$, as desired. $\square$

Lemma 2.18. *Let R be a 2-torsion free prime ring with involution. If $[h, k] = 0$ for all $h \in H(R)$, $k \in S(R)$, then R satisfy s_4 identity.*

Proof. Suppose that R does not satisfy s_4 identity. By hypothesis, we have $[h, k] = 0 \in C$ for all $h \in H(R)$ and $k \in S(R)$. In view of Lemma 2.17, it follows that either R satisfies s_4 or $k \in Z(R)$ for all $k \in S(R)$. Under the given hypothesis, we left with $S(R) \subseteq Z(R)$. Clearly $h \circ k \in S(R)$ for all $h \in H(R)$ and $k \in S(R)$, therefore we have $[h \circ k, k] = 0$, i.e., $[h, k]k + k[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. For a fixed $h \in H(R)$, we have $d(k)k + kd(k) = 0$ for all $k \in S(R)$, where $d(x) = [x, h]$ for all $x \in R$. If $d \neq 0$, then we have the result by Lemma 2.15. In case $d = 0$, we conclude $H(R) \subseteq Z(R)$ and hence by Lemma 2.13, we have a contradiction. Hence, R must satisfies s_4 identity. $\square$

3. DEFINITIONS AND EXAMPLES

We begin our discussions with the definition of centrally-extended generalized Jordan *-derivations of rings with involution.

Definition 3.1. Let R be a ring with involution $'^*$. A mapping $F : R \rightarrow R$ is called *centrally-extended generalized Jordan *-derivation* associated with an centrally-extended Jordan *-derivation d , if

- (A) $F(x + y) - F(x) - F(y) \in Z(R)$,
 (B) $F(x \circ y) - F(x)y^* - F(y)x^* - xd(y) - yd(x) \in Z(R)$,

for all $x, y \in R$.

Remark 3.1. If R is 2-torsion free noncommutative prime ring with involution $'^*$, then for an additive mapping F to be a centrally-extended generalized Jordan *-derivation, it is sufficient to satisfy the condition $F(x^2) - F(x)x^* - xd(x) \in Z(R)$ for all $x \in R$.

Example 3.1. We now show the existence of centrally-extended generalized Jordan *-derivations in certain rings.

(I) Let $\mathbb{Z}$ be the ring of integers and $R = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} : x, y, z, t \in \mathbb{Z} \right\}$, a noncommutative prime ring. Then the mapping $*$: $R \rightarrow R$ such that $\begin{pmatrix} x & y \\ z & t \end{pmatrix}^* = \begin{pmatrix} t & -y \\ -z & x \end{pmatrix}$, $F : R \rightarrow R$ such that $F \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x - t & y \\ z & 0 \end{pmatrix}$ with associated mapping $d : R \rightarrow R$ defined as $d \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. One can easily verify that F is a centrally-extended generalized Jordan *-derivation with associated centrally-extended Jordan *-derivation d .

(II) Let R be a ring defined as $R := \mathfrak{R} \times \mathbb{Z}$, where $\mathfrak{R} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in \mathbb{Z}_2 \right\}$. For any $X = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \mathfrak{R}$, let us define $X^* = \begin{pmatrix} z & y \\ 0 & x \end{pmatrix}$ and hence $(X, k)^* = (X^*, k)$, which is an involution on R . Define $F : R \rightarrow R$ such that $F\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, k\right) = \left(\begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix}, 1\right)$ with associated mapping $d : R \rightarrow R$ defined as $d\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, k\right) = \left(\begin{pmatrix} z & y \\ 0 & x \end{pmatrix}, 1\right)$. We observe that F is a centrally-extended generalized Jordan $*$ -derivation with an associated centrally-extended Jordan $*$ -derivation.

Remark 3.2. It can be seen from the above example that every centrally-extended generalized Jordan $*$ -derivation is a centrally-extended Jordan $*$ -derivation, but the converse does not necessarily hold in general.

Definition 3.2. Let R be a ring with involution $'*$ '. A mapping $T : R \rightarrow R$ is called *centrally-extended Jordan $*$ -left centralizer* (resp. *centrally-extended Jordan $*$ -right centralizer*) if

- (A) $T(x + y) - T(x) - T(y) \in Z(R)$,
 (B) $T(x^2) - T(x)x^* \in Z(R)$ (resp. $T(x^2) - x^*T(x) \in Z(R)$),

for all $x, y \in R$. Moreover, T is called *centrally-extended Jordan $*$ -centralizer* if it is both a centrally-extended Jordan $*$ -left centralizer and a centrally-extended Jordan $*$ -right centralizer.

Example 3.2. Let R be a ring defined as $R := M_2(\mathbb{R}) \times \mathbb{C}$, where $M_2(\mathbb{R})$ denotes the ring of 2×2 matrices over real numbers. For any $r = (x, z_1), s = (y, z_2) \in R$, we define $r^* = (x^*, \bar{z}_1)$, where x^* is defined as in Example 3.1 (I) and $\bar{z}_1$ is the complex conjugate of z_1 . Define a mapping $T : R \rightarrow R$ such that $T(x, z) = (0, 1)$ for all $(x, z) \in R$. Then it can be easily verified that T is a centrally-extended Jordan $*$ -left centralizer of R .

4. AUXILIARY RESULTS

In this section, we shall mainly prove the following theorem which is crucial for the results proved in the subsequent sections.

Theorem 4.1. *Let R be a 2-torsion free noncommutative prime ring with involution $'*$ ' and F be a centrally-extended generalized Jordan $*$ -derivation of R associated with a centrally-extended Jordan $*$ -derivation $\mathfrak{d}$. Then, there exists $q \in Q_s(R)$ such that $F(x) = qx^* + \mathfrak{d}(x)$ for all $x \in R$.*

After proving few more facts in this regard, we shall return to the proof of Theorem 4.1.

Proposition 4.1. *Let R be a 2-torsion free prime ring with involution $'^*$ and $d : R \rightarrow R$ a Jordan $*$ -derivation. Then, d can be uniquely extended to A , unless R satisfies s_4 .*

Proof. Suppose that R does not satisfy s_4 identity. By Lemma 2.16, there exists $a \in Q_s(R)$ such that $d(x) = xa - ax^*$ for all $x \in R$. Let us define $\bar{d}(q) = qa - aq^*$ for all $q \in A$. Clearly, $\bar{d}$ is a well-defined map and it is an extension of d . Now, we claim that the extension of d is unique.

Let D be also an extension of d . Since A is a noncommutative 2-torsion free prime ring, D is X -inner on A , i.e., there exists $b \in A$ such that $D(u) = ub - bu^*$ for all $u \in A$. As D is an extension of d , so we have

$$(4.1) \quad xb - bx^* = xa - ax^*, \quad \text{for all } x \in R.$$

In particular by taking h for x in (4.1), where $h \in H(R)$, we obtain $[h, b - a] = 0$ for all $h \in H(R)$. Using Lemma 2.17, we have $b - a \in C$. That means there exists $c \in C$ such that $b = a + c$. From (4.1), we have $xc - cx^* = 0$ for all $x \in R$. Replacing x by k in the last expression, where $k \in S(R)$, we get $2ck = 0$ for all $k \in S(R)$. If $c \neq 0$, then by using Lemma 2.2, we have $S(R) = (0)$, thence $H(R) = R$; which implies that for any $x, y \in R$, we have $xy = (xy)^* = y^*x^* = yx$, which is a contradiction. In case $c = 0$, we have $a = b$. It proves our claim. $\square$

Proposition 4.2. *Let R be a 2-torsion free prime ring with involution $'^*$ and $F : R \rightarrow R$ a generalized Jordan $*$ -derivation associated with d a Jordan $*$ -derivation. Then, F can be uniquely extended to A , unless R satisfies s_4 .*

Proof. By Lemma 2.11, there exists $a \in Q_s(R)$ such that $F(x) = ax^* + d(x)$ for all $x \in R$. We define $\tilde{F}(u) = au^* + d(u)$ for all $u \in A$. By Proposition 4.1, $\tilde{F}$ is a well defined map and also it is an extension of F . Now, we will show that extension of F is unique. Let G be also extension of F . Using Lemma 2.11, there exists $b \in Q_s$ and g a Jordan $*$ -derivation on A such that $G(u) = bu^* + g(u)$ for all $u \in A$. Since, G is an extension of F . Therefore

$$(4.2) \quad ax^* + d(x) = bx^* + g(x), \quad \text{for all } x \in R.$$

By Lemma 2.16, there exists $c, q \in Q_s$ such that $d(x) = xc - cx^*$ and $g(x) = xq - qx^*$ for all $x \in R$. From (4.2), we have $(a - c)x^* + xc = (b - q)x^* + xq$ for all $x \in R$. As the preceding equation is a g -identity on R , application of Lemma 2.6 yields $(a - c)x^* + xc = (b - q)x^* + xq$ for all $x \in Q_s(R)$. Replacing x by 1, we obtain $a = b$. From (4.2), we find $d(x) = g(x)$ for all $x \in R$. In fact, Proposition 4.1 gives $d(x) = g(x)$ for all $x \in A$. It completes the proof. $\square$

For the sake of brevity, we omit the proof of the following result, as it follows proceeding along the same lines as the proof of Lemma 4.4 of [9], with insignificant variations.

Proposition 4.3. *Let R be a ring with involution $*$ and with no non-zero central ideal. If F is a centrally-extended generalized Jordan $*$ -derivation associated with centrally-extended Jordan $*$ -derivation $\mathfrak{d}$ of R , then F is additive.*

Corollary 4.1. *Let R be a noncommutative prime ring with involution $'*$ '. If F is a centrally-extended generalized Jordan $*$ -derivation of R associated with centrally-extended Jordan $*$ -derivation $\mathfrak{d}$, then F is additive.*

Proposition 4.4. *Let R be a 2-torsion free noncommutative prime ring with involution $'*$ ' and $T : R \rightarrow R$ a centrally-extended Jordan $*$ -left centralizer. Then, there exists $q \in Q_s(R)$ such that $T(x) = qx^*$ for all $x \in R$.*

Proof. Note that T is additive by Proposition 4.1. If $Z(R) = (0)$, then $T(x^2) = T(x)x^*$ for all $x \in R$. Thus, we conclude the desired result by Lemma 2.4. In case $Z(R) \neq (0)$, we first claim that there exists $0 \neq z \in Z(R)$ such that $z^* = z$.

Let us suppose that $0 \neq z_c \in Z(R)$. If $z_c^* = z_c$, then we are done. If $z_c^* \neq z_c$, then we take $z_1 = z_c + z_c^*$, and observe that $z_1 = z_1^*$. Therefore, we can say that there exists $0 \neq z \in Z(R)$ such that $z^* = z$.

By the assumption, we have

$$(4.3) \quad T(x^2) - T(x)x^* \in Z(R), \quad \text{for all } x \in R.$$

Polarizing (4.3), we get

$$(4.4) \quad T(xy + yx) - T(x)y^* - T(y)x^* \in Z(R), \quad \text{for all } x, y \in R.$$

Replacing y by z^2 , where $z \in H(R) \cap Z(R)$ in (4.4) to get

$$T(xz^2 + z^2x) - T(x)z^2 - T(z^2)x^* \in Z(R), \quad \text{for all } x \in R.$$

It implies

$$T(xz^2 + z^2x) - T(x)z^2 - T(z)zx^* - c_1x^* \in Z(R), \quad \text{for all } x \in R,$$

where c_1 is the corresponding central element. It implies

$$(4.5) \quad T(4xz^2) - 2T(x)z^2 - 2T(z)zx^* - 2c_1x^* \in Z(R).$$

Also, $4xz^2 = z(xz + zx) + (xz + zx)z$. From (4.4), we have

$$T(z(xz + zx) + (xz + zx)z) - T(z)(xz + zx)^* - T(xz + zx)z \in Z(R).$$

Again using (4.4) in the last summand of the above relation, we get

$$(4.6) \quad T(4xz^2) - T(z)(x^*z + zx^*) - (T(x)z + T(z)x^* + c_2)z \in Z(R),$$

where c_2 is the corresponding central element. Comparing (4.5) and (4.6) to obtain

$$T(x)z^2 - T(z)x^*z - c_1x^* \in Z(R).$$

It implies

$$([T(x)z, x^*] - [T(z)x^*, x^*])z = 0.$$

Application of Lemma 2.2 yields

$$(4.7) \quad [T(x)z - T(z)x^*, x^*] = 0.$$

Define $\mathfrak{G}(x) = T(x)z - T(z)x^*$ for all $x \in R$. From (4.7), we find $\mathfrak{G}$ is an additive *-commuting map. Applying involution in (4.7) and using Lemma 2.10 in order to get $\mathfrak{G}(x)^* = \lambda x + \sigma(x)$ for all $x \in R$, for some $\lambda \in C$ and $\sigma : R \rightarrow C$. It implies $\mathfrak{G}(x) = T(x)z - T(z)x^* = \lambda^*x^* + \sigma(x)^*$. Therefore, $T(x)z = (T(z) + \lambda^*)x^* + \sigma(x)^*$ for all $x \in R$. It can also be written as $T(x) = z^{-1}(T(z) + \lambda^*)x^* + \sigma(x)^*z^{-1}$. Thus, $T(x) = qx^* + \sigma'(x)$ for all $x \in R$ where $q = z^{-1}(T(z) + \lambda^*)$ and $\sigma'(x) = z^{-1}\sigma(x)^*$. From (4.3), we have $q(x^2)^* + \sigma'(x^2) - (qx^* + \sigma'(x))x^* \in Z(R)$ for all $x \in R$. It implies

$$(4.8) \quad x^*\sigma'(x) \in C, \quad \text{for all } x \in R.$$

For any fixed $x \in R$, using Lemma 2.3 in (4.8), we have either $x \in Z(R)$ or $\sigma'(x) = 0$. As σ' is additive function, by application of Lemma 2.1, we find either $x \in Z(R)$ for all $x \in R$ or $\sigma'(x) = 0$ for all $x \in R$. Since R is noncommutative, we have $\sigma' = 0$. Thus, $T(x) = qx^*$ for all $x \in R$. $\square$

Proof of Theorem 4.1. Let $T(x) = (F - \mathfrak{d})(x)$ for all $x \in R$. Then for any $x \in R$, $T(x^2) = (F - \mathfrak{d})(x^2) = F(x)x^* + x\mathfrak{d}(x) + c_1 - \mathfrak{d}(x)x^* - x\mathfrak{d}(x) - c_2$ where c_1 and c_2 are corresponding central element. It turns out to be $T(x^2) - T(x)x^* = c_1 + c_2 \in Z(R)$ for all $x \in R$. Therefore T is a centrally-extended Jordan *-left centralizer. Invoking Proposition 4.4, we get $T(x) = qx^*$ for all $x \in R$, where $q \in Q_s(R)$. Thus $F(x) = qx^* + \mathfrak{d}(x)$ for all $x \in R$. $\square$

5. CENTRALIZING CONDITIONS

An astonishing result of Posner [27] states that a prime ring R is commutative if it possesses a non-zero derivation d which is centralizing on R (i.e. $[d(x), x] \in Z(R)$). Proceeding this investigation, Mayne [24, 25] studied automorphisms and derivations which are centralizing on appropriate subsets of a prime ring. Later on, Bell and Martindale [8] examined centralizing mappings of semiprime rings. Since then several authors have extended these results in different directions. In 2014, Ali and Dar [3] introduced the notion of *-centralizing mappings in prime rings with involution. Motivated by these studies, in this section, we are intended to describe the structure of centralizing and *-centralizing centrally-extended generalized Jordan *-derivations of a prime ring with involution $*$.

Theorem 5.1. *Let R be a 2-torsion free prime ring with involution $'*$ and $F : R \rightarrow R$ a centrally-extended generalized Jordan *-derivation associated with centrally-extended Jordan *-derivation $\mathfrak{d}$. If $[F(x), x] \in Z(R)$ for all $x \in R$, then there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, unless R satisfies s_4 .*

Proof. Suppose that R does not satisfies s_4 . By hypothesis, we have

$$[F(x), x] \in Z(R), \quad \text{for all } x \in R.$$

If $Z(R) = (0)$, then from Lemma 2.11, we find that there exists $a \in Q_s(R)$ such that

$$(5.1) \quad F(x) = ax^* + \mathfrak{d}(x), \quad \text{for all } x \in R.$$

Since F is a additive and centralizing map, by using Lemma 2.9 and Lemma 2.10, there exists $\lambda \in C$ and a map $\sigma : R \rightarrow C$ such that

$$(5.2) \quad F(x) = \lambda x + \sigma(x), \quad \text{for all } x \in R.$$

In view of Proposition 4.2, it follows from equations (5.1) and (5.2) that

$$(5.3) \quad \lambda x + \sigma(x) = ax^* + \mathfrak{d}(x), \quad \text{for all } x \in A.$$

By Lemma 2.16 and Proposition 4.1, there exists $b \in Q_s(R)$ such that $\mathfrak{d}(x) = xb - xb^*$ for all $x \in A$. Therefore, from (5.3), we have

$$(5.4) \quad \lambda x + \sigma(x) = ax^* + xb - bx^*, \quad \text{for all } x \in A.$$

Taking 1 instead of x in (5.4), we find

$$(5.5) \quad a \in C.$$

Replacing x by h in (5.4), we obtain

$$(5.6) \quad \lambda h + \sigma(h) = ah + d(h), \quad \text{for all } h \in H(A),$$

where $d(x) = [x, b]$. Since $a \in C$, from (5.6), we see that $[d(h), h] = 0$ for all $h \in H(A)$. If $d \neq 0$, then Lemma 2.12 leads us to a contradiction.

On the other hand, let $d = 0$, i.e., $b \in C$. Moreover, from (5.6), we get $(\lambda - a)h \in C$. Since $\lambda - a \in C$, in view of Lemma 2.3, we have either $\lambda = a$ or $H(R) \subseteq Z(R)$. In the latter case, we have a contradiction by Lemma 2.13, so we left with $\lambda = a$. With this, from (5.4) we obtain

$$(5.7) \quad 2\lambda k - 2bk \in C, \quad \text{for all } k \in S(R).$$

As $\lambda, b \in C$, it implies either $S(R) \subseteq Z(R)$ or $\lambda = b$. In the former case, we have the desired result by Lemma 2.18 and in the latter case, (5.1) yields $F(x) = \lambda x^* + x\lambda - \lambda x^*$ for all $x \in R$, i.e., $F(x) = \lambda x$ for all $x \in R$, as desired.

In case $Z(R) \neq \{0\}$, we have $0 \neq h_c \in Z(R)$ such that $h_c^* = h_c$. By the assumption $[F(x), x] \in Z(R)$ for all $x \in R$. With the aid of Lemma 2.9 and Proposition 4.3, we have

$$(5.8) \quad [F(x), x] = 0, \quad \text{for all } x \in R.$$

From Lemma 2.10 and (5.8), we have

$$(5.9) \quad F(x) = \lambda x + \sigma(x), \quad \text{for all } x \in R,$$

for some $\lambda \in C$ and a map $\sigma : R \rightarrow C$. Application of Proposition 4.1 in (5.9) yields

$$(5.10) \quad qx^* + \mathfrak{d}(x) = \lambda x + \sigma(x), \quad \text{for all } x \in R,$$

for some $q \in Q_s(R)$. From (B), we find

$$(5.11) \quad F(x \circ h_c) - F(x)h_c - F(h_c)x^* - x\mathfrak{d}(h_c) - h_c\mathfrak{d}(x) \in Z(R), \quad \text{for all } x \in R.$$

Using (5.9) in (5.11), we find

$$\lambda(x \circ h_c) - \lambda x h_c - \lambda h_c x^* - \sigma(h_c)x^* - x\mathfrak{d}(h_c) - h_c\mathfrak{d}(x) \in C, \quad \text{for all } x \in R.$$

It implies

$$(5.12) \quad \lambda(x - x^*)h_c - \sigma(h_c)x^* - x\mathfrak{d}(h_c) - h_c\mathfrak{d}(x) \in C, \quad \text{for all } x \in R.$$

Replacing x by h_c in (5.12), we conclude

$$(5.13) \quad \mathfrak{d}(h_c) \in Z(R).$$

Replacing x by h in (5.12), where $h \in H(R)$, we obtain

$$(5.14) \quad -\sigma(h_c)h - h\mathfrak{d}(h_c) - h_c\mathfrak{d}(h) \in C.$$

It implies

$$(5.15) \quad \mathfrak{d}(h_c)[h, x] + \sigma(h_c)[h, x] + h_c[\mathfrak{d}(h), x] = 0, \quad \text{for all } h \in H(R), x \in R.$$

Replacing h by h^2 in (5.15), we find

$$(5.16) \quad (\mathfrak{d}(h_c) + \sigma(h_c))([h, x]h + h[h, x]) + h_c[\mathfrak{d}(h)h + h\mathfrak{d}(h), x] = 0, \quad \text{for all } h \in H(R), x \in R.$$

Using (5.13) and (5.15) in (5.16), we obtain

$$h_c(\mathfrak{d}(h)[h, x] + [h, x]\mathfrak{d}(h)) = 0, \quad \text{for all } h \in H(R), x \in R.$$

It implies that

$$(5.17) \quad \mathfrak{d}(h)[h, x] + [h, x]\mathfrak{d}(h) = 0, \quad \text{for all } h \in H(R), x \in R.$$

Polarizing h in (5.17), we find

$$(5.18) \quad \mathfrak{d}(h_1)[h, x] + \mathfrak{d}(h)[h_1, x] + [h, x]\mathfrak{d}(h_1) + [h_1, x]\mathfrak{d}(h) = 0, \quad \text{for all } h, h_1 \in H(R), x \in R.$$

In particular, replacing h_1 by h_c in (5.18) and thereby using (5.13), we conclude

$$(5.19) \quad 2\mathfrak{d}(h_c)[h, x] = 0, \quad \text{for all } h \in H(R), x \in R.$$

If $\mathfrak{d}(h_c) \neq 0$, then Lemma 2.2 in (5.19) implies $H(R) \subseteq Z(R)$. With the aid of Lemma 2.13, we arrive at a contradiction.

In case $\mathfrak{d}(h_c) = 0$, we obtain from (5.12) that

$$2\lambda kh_c + \sigma(h_c)k - h_c\mathfrak{d}(k) \in C, \quad \text{for all } k \in S(R).$$

It implies $h_c[\mathfrak{d}(k), k] = 0$, for all $k \in S(R)$. From the fact $h_c \neq 0$ and Lemma 2.2, it follows that

$$(5.20) \quad [\mathfrak{d}(k), k] = 0, \quad \text{for all } k \in S(R).$$

By using $\mathfrak{d}(h_c) = 0$ and replacing h by k^2 in (5.14), where $k \in S(R)$, we have

$$-\sigma(h_c)k^2 - h_c\mathfrak{d}(k^2) \in C.$$

It implies

$$-\sigma(h_c)k^2 + h_c[\mathfrak{d}(k), k] \in C, \quad \text{for all } k \in S(R).$$

From (5.20), we find

$$(5.21) \quad \sigma(h_c)k^2 \in C, \quad \text{for all } k \in S(R).$$

If $\sigma(h_c) \neq 0$, then Lemma 2.3 in (5.21) implies $k^2 \in Z(R)$ for all $k \in S(R)$. For any fixed $x \in R$, we have $[k, x]k + k[k, x] = d(k)k + kd(k) = 0$ for all $k \in S(R)$, where $d(y) = [y, x]$ for all $y \in R$. If $d \neq 0$, then Lemma 2.15 yields a contradiction. Now, if $d = 0$, then we have $[x, y] = 0$ for all $x, y \in R$, hence R is commutative, which is a contradiction.

In case $\sigma(h_c) = 0$, using fact $\mathfrak{d}(h_c) = 0$, from (5.14), we have $h_c \mathfrak{d}(h) \in Z(R)$ for all $h \in H(R)$. Since $h_c \neq 0$, we have $\mathfrak{d}(h) \in Z(R)$ for all $h \in H(R)$ by Lemma 2.2. For any fixed $h \in H(R)$, using Lemma 2.2 in (5.17) we find that for each $h \in H(R)$, either $\mathfrak{d}(h) = 0$ or $h \in Z(R)$. Using Lemma 2.1, we have either $\mathfrak{d}(h) = 0$ for all $h \in H(R)$ or $H(R) \subseteq Z(R)$. In latter case, we have a contradiction by Lemma 2.13. In case $\mathfrak{d}(h) = 0$ for all $h \in H(R)$, we have

$$(5.22) \quad F(h^2) - F(h)h \in C, \quad \text{for all } h \in H(R).$$

Using (5.9) in (5.22), we conclude

$$(5.23) \quad \sigma(h)h \in C, \quad \text{for all } h \in H(R).$$

For any fixed $h \in H(R)$, using Lemma 2.3 in (5.23), we have either $\sigma(h) = 0$ or $h \in Z(R)$. Aid of Lemma 2.1, we have either $\sigma(h) = 0$ for all $h \in H(R)$ or $H(R) \subseteq Z(R)$. In the latter case, we have the desired result by using Lemma 2.13. In case $\sigma(h) = 0$, replacing x by h , where $h \in H(R)$ in (5.10), we get

$$(5.24) \quad qh = \lambda h \text{ for all } h \in H(R).$$

Replacing h by h_c in (5.24) and thereby using Lemma 2.2, we conclude $q = \lambda$. Replacing x by k in (5.10), where $k \in S(R)$, we find

$$(5.25) \quad \mathfrak{d}(k) = 2\lambda k + \sigma(k), \quad \text{for all } k \in S(R).$$

Using (B), we have

$$(5.26) \quad F(h \circ k) + F(h)k - F(k)h - h\mathfrak{d}(k) \in Z(R), \quad \text{for all } k \in S(R), h \in H(R).$$

Using (5.9) and (5.25) in (5.26), we conclude

$$\lambda(h \circ k) + \lambda hk - \lambda kh - \sigma(k)h - h(2\lambda k + \sigma(k)) \in C, \quad \text{for all } k \in S(R), h \in H(R).$$

It implies

$$(5.27) \quad \sigma(k)h \in C, \quad \text{for all } k \in S(R), h \in H(R).$$

If there exists $k \in S(R)$ such that $\sigma(k) \neq 0$, then using Lemma 2.3 in (5.27), we find $H(R) \subseteq Z(R)$. With the aid of Lemma 2.13, we get a contradiction. In case $\sigma(k) = 0$ for all $k \in S(R)$. From (5.9) and fact $\sigma(k) = \sigma(h) = 0$, we find $F(x) = F(h + k) = F(h) + F(k) = \lambda h + \lambda k = \lambda(h + k) = \lambda x$ for all $x \in R$. $\square$

Theorem 5.2. *Let R be a 2-torsion free prime ring with involution $'*$ ' and $F : R \rightarrow R$ a centrally-extended generalized Jordan $*$ -derivation with an associated centrally-extended Jordan $*$ -derivation $\mathfrak{d}$. If $[F(x), x^*] \in Z(R)$ for all $x \in R$, then there exists $\lambda \in C$ such that $F(x) = \lambda x^*$ for all $x \in R$, unless R satisfies s_4 .*

Proof. Suppose that R does not satisfies s_4 . If $Z(R) = (0)$, then by Lemma 2.11, we find

$$(5.28) \quad F(x) = ax^* + \mathfrak{d}(x), \quad \text{for all } x \in R,$$

for some $a \in Q_s(R)$. From Lemma 2.5 and assumption, we find $[F(x), x^*] = 0$ for all $x \in R$. Applying involution, we obtain $[F(x)^*, x] = 0$ for all $x \in R$. By Lemma 2.10, we have

$$F(x)^* = \lambda x + \sigma(x), \quad \text{for all } x \in R.$$

It implies

$$(5.29) \quad F(x) = \lambda^* x^* + \sigma(x)^*, \quad \text{for all } x \in R.$$

By Proposition 4.2, Eq. (5.28) and (5.29), we conclude

$$(5.30) \quad \lambda^* x^* + \sigma(x)^* = ax^* + \mathfrak{d}(x), \quad \text{for all } x \in A.$$

Replacing x by 1 in (5.30), we obtain

$$(5.31) \quad a \in C.$$

Using (5.31) in (5.30), we obtain

$$[\mathfrak{d}(x), x^*] = 0, \quad \text{for all } x \in R.$$

Application of Lemma 2.8 implies $\mathfrak{d} = 0$ or R satisfy s_4 identity. For non trivial solution, we have $\mathfrak{d} = 0$. Thus using it in (5.28), we obtain $F(x) = ax^*$ for all $x \in R$, where $a \in C$ as desired.

Let $Z(R) \neq \{0\}$. Then there exists $0 \neq h_c \in Z(R)$ such that $h_c^* = h_c$. By the assumption, we have $[F(x), x^*] \in Z(R)$ for all $x \in R$. With the aid of Lemma 2.5 and Lemma 4.3, we have

$$(5.32) \quad [F(x), x^*] = 0, \quad \text{for all } x \in R.$$

Applying involution both sides in (5.32), we find

$$(5.33) \quad [F(x)^*, x] = 0, \quad \text{for all } x \in R.$$

Application of Lemma 2.10 in (5.33) implies

$$(5.34) \quad F(x)^* = \lambda x + \sigma(x), \quad \text{for all } x \in R,$$

for some $\lambda \in C$ and a mapping $\sigma : R \rightarrow C$. Applying involution both sides in (5.34), we get

$$(5.35) \quad F(x) = \lambda^* x^* + \sigma(x)^*, \quad \text{for all } x \in R.$$

From (B), we have

$$(5.36) \quad F(x \circ y) - F(x)y^* - F(y)x^* - x\mathfrak{d}(y) - y\mathfrak{d}(x) \in Z(R), \quad \text{for all } x, y \in R.$$

Using (5.35) in (5.36), we find

$$\lambda^*(x \circ y)^* - \lambda^*(x)^*y^* - \sigma(x)^*y^* - \lambda^*y^*x^* - \sigma(y)^*x^* - x\mathfrak{d}(y) - y\mathfrak{d}(x) \in C, .$$

for all $x, y \in R$. It implies

$$(5.37) \quad -\sigma(x)^*y^* - \sigma(y)^*x^* - x\mathfrak{d}(y) - y\mathfrak{d}(x) \in C, \quad \text{for all } x, y \in R.$$

Replacing x and y by h_c in (5.37), we find $h_c\mathfrak{d}(h_c) \in Z(R)$. It forces

$$(5.38) \quad \mathfrak{d}(h_c) \in Z(R).$$

Replacing y by h_c in (5.37) and using (5.38), we find

$$(5.39) \quad -\sigma(h_c)^*x^* - x\mathfrak{d}(h_c) - h_c\mathfrak{d}(x) \in Z(R), \quad \text{for all } x \in R.$$

Replacing x by h in (5.39), where $h \in H(R)$, we have

$$-\sigma(h_c)^*h - h\mathfrak{d}(h_c) - h_c\mathfrak{d}(h) \in C.$$

Commuting with x and using (5.38), we obtain

$$(5.40) \quad \mathfrak{d}(h_c)[h, x] + \sigma(h_c)^*[h, x] + h_c[\mathfrak{d}(h), x] = 0, \quad \text{for all } x \in R, h \in H(R).$$

Replacing h by h^2 in (5.40) and using it, we conclude

$$(5.41) \quad \mathfrak{d}(h)[h, x] + [h, x]\mathfrak{d}(h) = 0.$$

Polarizing (5.41), we find

$$(5.42) \quad \mathfrak{d}(h_1)[h, x] + \mathfrak{d}(h)[h_1, x] + [h, x]\mathfrak{d}(h_1) + [h_1, x]\mathfrak{d}(h) = 0, \quad \text{for all } h, h_1 \in H(R).$$

In particular, replacing h_1 by h_c in (5.42) to obtain

$$(5.43) \quad 2\mathfrak{d}(h_c)[h, x] = 0, \quad \text{for all } h \in H(R).$$

If $\mathfrak{d}(h_c) \neq 0$, then using Lemma 2.2 and (5.38) in (5.43), we find $H(R) \subseteq Z(R)$. With the aid of Lemma 2.13, we have contradiction.

In case $\mathfrak{d}(h_c) = 0$, from (5.39), we find $[\mathfrak{d}(x), x^*] = 0$ for all $x \in R$. For non trivial solution, Lemma 2.8 yields $\mathfrak{d} = 0$. Using it in (5.37), we get

$$\sigma(y)^*[x^*, y^*] = 0, \quad \text{for all } x, y \in R.$$

For any fixed $y \in R$, we have either $\sigma(y) = 0$ or $[y^*, R] = 0$. Application of Lemma 2.1 implies that either $\sigma = 0$ or $[y, R] = (0)$ for all $y \in R$. But R is noncommutative, so we left with $\sigma = 0$. Thus, from (5.35) we have $F(x) = ax^*$, where $a = \lambda^* \in C$. It completes the proof. $\square$

6. HYPERCOMMUTING CONDITIONS

A pair of mappings f and g satisfying the condition $f(x)x - xg(x) = 0$ (resp. $f(x)x^* - x^*g(x) = 0$) on an appropriate subset K of a ring (resp. ring with involution) R is called hypercommuting (resp. $*$ -hypercommuting). Obviously, it is a more general concept than that of commuting ($*$ -commuting) mappings. In this section, we study a pair (F, G) of centrally-extended generalized Jordan $*$ -derivations which is hypercommuting or $*$ -hypercommuting.

Theorem 6.1. *Let R be a 2-torsion free prime ring with involution $'^*$ and $F, G : R \rightarrow R$ are centrally-extended generalized Jordan $*$ -derivations with an associated centrally-extended Jordan $*$ -derivations $\mathfrak{d}, \mathfrak{g}$, respectively. If $F(x)x - xG(x) = 0$ for all $x \in R$, then there exists $\lambda \in C$ such that $F(x) = G(x) = \lambda x$ for all $x \in R$, unless R satisfies s_4 .*

Proof. Assume that R does not satisfy s_4 . By the hypothesis, we have

$$F(x)x - xG(x) = 0, \quad \text{for all } x \in R.$$

Suppose that $Z(R) = (0)$. Clearly in this case F and G becomes generalized Jordan $*$ -derivations. Thus with the aid of Lemma 2.11, we have $F(x) = ax^* + \mathfrak{d}(x)$, $G(x) = bx^* + \mathfrak{g}(x)$ for all $x \in R$, where $a, b \in Q_s(R)$. Using it in our hypothesis, we find

$$(ax^* + \mathfrak{d}(x))x - x(bx^* + \mathfrak{g}(x)) = 0, \quad \text{for all } x \in R.$$

The fact of Lemma 2.16 yields $\mathfrak{d}(x) = xc - cx^*$, $\mathfrak{g}(x) = xd - dx^*$ for all $x \in R$ for some $c, d \in Q_s(R)$. In this view it follows that R satisfies the functional identity

$$(6.1) \quad (ax^* + xc - cx^*)x = x(bx^* + xd - dx^*), \quad \text{for all } x \in R.$$

Application of Lemma 2.6 in (6.1) yields

$$(6.2) \quad (ax^* + xc - cx^*)x = x(bx^* + xd - dx^*), \quad \text{for all } x \in A.$$

Replacing x by 1 in (6.2), we find $a = b$. Polarizing (6.2), we obtain $(ax^* + xc - cx^*)y + (ay^* + yc - cy^*)x = y(bx^* + xd - dx^*) + x(by^* + yd - dy^*)$ for all $x, y \in A$. Replacing y by 1, we get $ax^* + xc - cx^* + ax = bx^* + xd - dx^* + xb$ for all $x, y \in A$. By using the fact $a = b$ in preceding equation, we find

$$(6.3) \quad xc - cx^* - xd + dx^* + [a, x] = 0, \quad \text{for all } x \in A.$$

Replacing x by h in (6.3), where $h \in H(R)$, to obtain

$$(6.4) \quad -[c, h] + [d, h] + [a, h] = 0.$$

It implies $[d - c + a, h] = 0$. With the aid of Lemma 2.17, we have $d - c + a \in C$. Replacing x by k in (6.3), where $k \in S(R)$ and thereby using the fact $d - c + a \in C$, we find $k \circ (c - d) + [c - d, k] = 0$ for all $k \in S(R)$. It implies

$$(6.5) \quad (c - d)k = 0, \quad \text{for all } k \in S(R).$$

Replacing k by $h \circ k$ in (6.5), where $h \in H(R)$, $k \in S(R)$ and using (6.5), we find

$$(6.6) \quad (c - d)hk = 0, \quad \text{for all } h \in H(R), k \in S(R).$$

From (6.5) and (6.6), we get

$$(6.7) \quad (c - d)[k, h] = 0, \quad \text{for all } k \in S(R), h \in H(R).$$

Replacing k by $k \circ h_1$ in (6.7), we obtain

$$(6.8) \quad (c - d)h_1[k, h] + (c - d)[k, h]h_1 + (c - d)k[h, h_1] + (c - d)[h, h_1]k = 0.$$

Using (6.5) and (6.7) in (6.8), we find

$$(6.9) \quad (c - d)h_1[k, h] = 0, \quad \text{for all } k \in S(R), h, h_1 \in H(R).$$

Equation (6.5) also implies

$$(6.10) \quad (c - d)k_1[k, h] = 0, \quad \text{for all } k, k_1 \in S(R), h \in H(R).$$

From (6.9) and (6.10), we have

$$\begin{aligned} 2(c - d)x[k, h] &= (c - d)(2x)[k, h] \\ &= (c - d)(h_1 + k_1)[k, h] \\ &= (c - d)h_1[k, h] + (c - d)k_1[k, h] \\ &= 0, \quad \text{for all } x \in R, h \in H(R), k \in S(R). \end{aligned}$$

Thus, we have $(c - d)R[k, h] = (0)$ for all $k \in S(R)$ and $h \in H(R)$. In case $c \neq d$, primeness of R implies $[H(R), S(R)] = (0)$. Lemma 2.18 leads us to the contradiction. In case $c = d$, we find $\mathfrak{d} = \mathfrak{g}$. Using the fact $a = b$, we conclude $F(x) = G(x)$ for all $x \in R$. Thus, we have the desired result by Theorem 5.1.

Let $Z(R) \neq (0)$. Then there exists $0 \neq h_c \in Z(R)$ such that $h_c^* = h_c$. By the assumption, we have

$$(6.11) \quad F(x)x - xG(x) = 0, \quad \text{for all } x \in R.$$

Replacing x by h_c in (6.11), we find

$$(6.12) \quad F(h_c) - G(h_c) = 0.$$

Proposition 4.1 yields

$$(6.13) \quad F(x) = q_1x^* + \mathfrak{d}(x), \quad \text{for all } x \in R,$$

$$(6.14) \quad G(x) = q_2x^* + \mathfrak{g}(x), \quad \text{for all } x \in R,$$

for some $q_1, q_2 \in Q_s(R)$. Using (6.13) and (6.14) in (6.12), we obtain

$$(6.15) \quad (q_1 - q_2)(h_c) + (\mathfrak{d} - \mathfrak{g})(h_c) = 0.$$

Polarizing (6.11) to obtain

$$(6.16) \quad F(x)y - yG(x) + F(y)x - xG(y) = 0, \quad \text{for all } x, y \in R.$$

Replacing y by h_c in (6.16) to obtain

$$(6.17) \quad (F(x) - G(x))h_c + F(h_c)x - xG(h_c) = 0, \quad \text{for all } x \in R.$$

Using (6.12) in (6.17), we have

$$(6.18) \quad (F(x) - G(x))h_c + [F(h_c), x] = 0, \quad \text{for all } x \in R.$$

Replacing h_c by h_c^2 in (6.18) and using Lemma 2.2, we conclude

$$(6.19) \quad [\mathfrak{d}(h_c), x] = 0, \quad \text{for all } x \in R.$$

It implies $\mathfrak{d}(h_c) \in Z(R)$. In the same way, we compute $\mathfrak{g}(h_c) \in Z(R)$. Using (6.13) and (6.19) in (6.18), we obtain

$$(F(x) - G(x) + [q_1, x])h_c = 0, \quad \text{for all } x \in R.$$

It implies

$$(6.20) \quad F(x) - G(x) + [q_1, x] = 0, \quad \text{for all } x \in R.$$

Since F and G are centrally-extended generalized Jordan *-derivation, replacing x by $h \circ h_c$ in (6.20) to obtain

$$(6.21) \quad \begin{aligned} & (F(h) - G(h))h_c + (F(h_c) - G(h_c))h + h_c(\mathfrak{d}(h) - \mathfrak{g}(h)) \\ & + h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) + 2[q, h]h_c \in Z(R), \quad \text{for all } h \in H(R). \end{aligned}$$

Using (6.12) and (6.20) in (6.21), we see that

$$(6.22) \quad (\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q, h]h_c + h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R).$$

Replacing h by h^2 in (6.22), we obtain

$$(6.23) \quad \begin{aligned} & (\mathfrak{d}(h) - \mathfrak{g}(h))hh_c + h(\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q_1, h]hh_c + h[q_1, h]h_c \\ & + h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R). \end{aligned}$$

From (6.22) and (6.23), we conclude

$$(6.24) \quad h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R).$$

If $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) \neq 0$, then using Lemma 2.3 in (6.24), we have $h^2 \in Z(R)$ for all $h \in H(R)$. Thus, we have the result by Lemma 2.13. If $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) = 0$, then from (6.15), we obtain

$$(6.25) \quad (q_1 - q_2)h_c = 0.$$

Using Lemma 2.2 in (6.25), we have $q_1 = q_2$. With the aid of (6.13), (6.14) and the fact $q_1 = q_2$ in (6.20), we find

$$(6.26) \quad \mathfrak{d}(x) - \mathfrak{g}(x) + [q_1, x] = 0, \quad \text{for all } x \in R.$$

Replacing x by $h \circ k$ where $h \in H(R)$, $k \in S(R)$ in (6.26), we obtain

$$(6.27) \quad \begin{aligned} & (\mathfrak{d}(h) - \mathfrak{g}(h))(-k) + k(\mathfrak{d}(h) - \mathfrak{g}(h)) + (\mathfrak{d}(k) - \mathfrak{g}(k))h + h(\mathfrak{d}(k) - \mathfrak{g}(k)) \\ & + [q_1, h]k + k[q_1, h] + [q_1, k]h + h[q_1, k] \in Z(R). \end{aligned}$$

Using (6.26) in (6.27) to conclude

$$(6.28) \quad 2[q_1, h]k \in Z(R), \quad \text{for all } h \in H(R), k \in S(R).$$

We now split the proof into the following two parts.

Case 1. If mapping induced on centroid is non identity map, then there exists $0 \neq z \in C$ such that $z^* \neq z$. Replacing h by $x + x^*$ and k by $y - y^*$ in (6.28), where $x, y \in R$ in order to obtain

$$(6.29) \quad 2[q_1, x + x^*](y - y^*) \in Z(R), \quad \text{for all } x, y \in R.$$

With the aid of Lemma 2.6 in (6.29), we have

$$(6.30) \quad [q_1, x + x^*](y - y^*) \in Z(R), \quad \text{for all } x, y \in Q_s(R).$$

Replace y by z in (6.30), we have

$$(6.31) \quad [q_1, x + x^*](z - z^*) \in C, \quad \text{for all } x \in Q_s(R).$$

Using Lemma 2.3 in (6.31), we have

$$(6.32) \quad [q_1, x + x^*] \in Z(R), \quad \text{for all } x \in Q_s(R).$$

Replacing x by h , where $h \in H(R)$ in (6.32), we obtain $[q_1, h] \in Z(R)$. Using Lemma 2.17, we conclude $q_1 \in Z(R)$. From (6.20), we have $F(x) = G(x)$ for all $x \in R$. Hence, by Theorem 5.1, we get the desired result.

Case 2. If mapping induced on centroid is an identity map, then $c^* = c$ for all $c \in C$. From (6.28), we have

$$([q_1, h]k)^* = [q_1, h]k, \quad \text{for all } h \in H(R), k \in S(R).$$

It implies

$$(6.33) \quad [q_1, h]k - k[q_1^*, h] = 0, \quad \text{for all } h \in H(R), k \in S(R).$$

Replacing k by $k \circ h_1$ in (6.33), where $k \in S(R)$, $h_1 \in H(R)$, we obtain

$$(6.34) \quad [q_1, h]h_1k + [q_1, h]kh_1 - kh_1[q_1^*, h] - h_1k[q_1^*, h] = 0.$$

Using (6.28) and (6.33) in (6.34) to conclude

$$(6.35) \quad [q_1, h]h_1k - kh_1[q_1^*, h] = 0, \quad \text{for all } h, h_1 \in H(R), k \in S(R).$$

Replacing h_1 by k_1^2 in (6.35), where $k_1 \in S(R)$, we find

$$([q_1, h]k_1)k_1k - kk_1(k_1[q_1^*, h]) = 0.$$

It implies

$$(6.36) \quad ([q_1, h]k_1)k_1k - kk_1([q_1, h]k_1)^* = 0, \quad \text{for all } h \in H(R), k, k_1 \in S(R).$$

Using (6.28) and (6.33) in (6.36), we obtain

$$(6.37) \quad [q_1, h]k_1[k_1, k] = 0.$$

For fixed $k_1 \in S(R)$, from the fact $[q_1, h]k_1 \in C$, using Lemma 2.2 in (6.37), we have either $[q_1, h]k_1 = 0$ for all $h \in H(R)$ or $[k_1, k] = 0$ for all $k \in S(R)$. Invoking Lemma 2.1 yields that either $[q_1, h]k_1 = 0$ for all $h \in H(R)$, $k_1 \in S(R)$ or $[k_1, k] = 0$ for all $k, k_1 \in S(R)$. In the latter case, replace k_1 by $h \circ k$ to obtain $[h, k]k + k[h, k] = 0$ for all $h \in H(R)$, $k \in S(R)$. For any fixed $h \in H(R)$, we obtain $d(k)k + kd(k) = 0$, where $d(x) = [h, x]$ for all $k \in S(R)$. With the aid of Lemma 2.15, we find either R satisfies s_4 identity or $d = 0$. In case $d = 0$, we have $H(R) \subseteq Z(R)$. Application of Lemma 2.13 gives a contradiction. If $[q_1, h]k_1 = 0$, then using the similar arguments as in (6.5), we find $[q_1, H(R)] = (0)$. With the aid of Lemma 2.17, we have $q_1 \in C$. From (6.20), we have $F(x) = G(x)$ for all $x \in R$. Thus, we get the desired conclusion from Theorem 5.1. $\square$

Theorem 6.2. *Let R be a 2-torsion free prime ring with involution $'^*$ and $F, G : R \rightarrow R$ are centrally-extended generalized Jordan $*$ -derivations associated with centrally-extended Jordan $*$ -derivations $\mathfrak{d}, \mathfrak{g}$, respectively. If $F(x)x^* - x^*G(x) = 0$ for all $x \in R$, then there exists $\lambda \in C$ such that $F(x) = G(x) = \lambda x^*$, unless R satisfies s_4 .*

Proof. Suppose that R does not satisfies s_4 . By the hypothesis, we have $F(x)x^* - x^*G(x) = 0$ for all $x \in R$.

If $Z(R) = (0)$, then application of Lemma 2.11 yields $F(x) = ax^* + \mathfrak{d}(x)$, $G(x) = bx^* + \mathfrak{g}(x)$ for all $x \in R$, where $a, b \in Q_s(R)$. Using it in our hypothesis, we find

$$(6.38) \quad (ax^* + \mathfrak{d}(x))x^* - x^*(bx^* + \mathfrak{g}(x)) = 0, \quad \text{for all } x \in R.$$

With the aid of Lemma 2.16, we conclude $\mathfrak{d}(x) = xc - cx^*$, $\mathfrak{g}(x) = xd - dx^*$ for all $x \in R$ for some $c, d \in U(R)$. Using it in (6.38), we find

$$(6.39) \quad (ax^* + xc - cx^*)x^* = x^*(bx^* + xd - dx^*), \quad \text{for all } x \in R.$$

Using Lemma 2.6 in (6.39), we conclude

$$(6.40) \quad (ax^* + xc - cx^*)x^* = x^*(bx^* + xd - dx^*), \quad \text{for all } x \in A.$$

Replacing x by 1 in (6.40), we get $a = b$. Polarizing (6.40), we obtain

$$(6.41) \quad \begin{aligned} & (ax^* + xc - cx^*)y^* + (ay^* + yc - cy^*)x^* \\ &= y^*(bx^* + xd - dx^*) + x^*(by^* + yd - dy^*), \quad \text{for all } x, y \in R. \end{aligned}$$

Replacing y by 1 in (6.41), we find

$$(6.42) \quad ax^* + xc - cx^* + ax^* = bx^* + xd - dx^* + x^*b, \quad \text{for all } x, y \in R.$$

Using the fact $a = b$ in (6.42), we conclude

$$(6.43) \quad xc - cx^* - xd + dx^* + [a, x^*] = 0, \quad \text{for all } x \in R.$$

Replacing x by h in (6.43), where $h \in H(R)$, we obtain

$$(6.44) \quad -[c, h] + [d, h] + [a, h] = 0.$$

Since (6.44) is the same as (6.4), using similar arguments, we can reach our conclusion.

Let $Z(R) \neq (0)$. Then there exists $0 \neq h_c \in Z(R)$ such that $h_c^* = h_c$. By the assumption, we have

$$(6.45) \quad F(x)x^* - x^*G(x) = 0, \quad \text{for all } x \in R.$$

Replacing x by h_c in (6.45), we obtain

$$(6.46) \quad F(h_c) - G(h_c) = 0.$$

Invoking Proposition 4.1, we have

$$(6.47) \quad F(x) = q_1x^* + \mathfrak{d}(x), \quad \text{for all } x \in R,$$

$$(6.48) \quad G(x) = q_2x^* + \mathfrak{g}(x), \quad \text{for all } x \in R,$$

for some $q_1, q_2 \in Q_s(R)$. Using (6.47) and (6.48) in (6.46), we obtain

$$(6.49) \quad (q_1 - q_2)(h_c) + (\mathfrak{d} - \mathfrak{g})(h_c) = 0.$$

Polarizing (6.45) to get

$$(6.50) \quad F(x)y^* - y^*G(x) + F(y)x^* - x^*G(y) = 0, \quad \text{for all } x, y \in R.$$

Replacing y by h_c in (6.50), we see that

$$(6.51) \quad (F(x) - G(x))h_c + F(h_c)x^* - x^*G(h_c) = 0, \quad \text{for all } x \in R.$$

Using (6.46) in (6.51), we have

$$(6.52) \quad (F(x) - G(x))h_c + [F(h_c), x^*] = 0, \quad \text{for all } x \in R.$$

Replacing h_c by h_c^2 in (6.52), we obtain

$$(6.53) \quad [\mathfrak{d}(h_c), x^*] = 0, \quad \text{for all } x \in R.$$

It implies $\mathfrak{d}(h_c) \in Z(R)$. In the same way, one can easily observe that $\mathfrak{g}(h_c) \in Z(R)$.

With the aid of (6.47), (6.53) in (6.52), we have

$$(6.54) \quad (F(x) - G(x) + [q_1, x^*])h_c = 0, \quad \text{for all } x \in R.$$

It yields

$$(6.55) \quad F(x) - G(x) + [q_1, x^*] = 0, \quad \text{for all } x \in R.$$

Replacing x by $h \circ h_c$ in (6.55) where $h \in H(R)$, we obtain

$$(6.56) \quad \begin{aligned} & (F(h) - G(h))h_c + (F(h_c) - G(h_c))h + h_c(\mathfrak{d}(h) - \mathfrak{g}(h)) \\ & + h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) + 2[q, h]h_c \in Z(R), \quad \text{for all } h \in H(R). \end{aligned}$$

Using (6.46) and (6.55) in (6.56), we conclude

$$(6.57) \quad (\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q, h]h_c + h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R).$$

Replacing h by h^2 in (6.57), we obtain

$$(6.58) \quad \begin{aligned} & (\mathfrak{d}(h) - \mathfrak{g}(h))hh_c + h(\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q_1, h]hh_c + h[q_1, h]h_c \\ & + h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R). \end{aligned}$$

From (6.57) and (6.58), we conclude

$$(6.59) \quad h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R).$$

If $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) \neq 0$, then using Lemma 2.3 in (6.59), we have $h^2 \in Z(R)$ for all $h \in H(R)$. Thus, a contradiction follows from Lemma 2.13. In case $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) = 0$, from (6.49), we obtain

$$(q_1 - q_2)h_c = 0,$$

and it implies $q_1 = q_2$. Now from (6.47), (6.48) and (6.55) we find

$$(6.60) \quad \mathfrak{d}(x) - \mathfrak{g}(x) + [q_1, x^*] = 0, \quad \text{for all } x \in R.$$

Replacing x by $h \circ k$ in (6.60), where $h \in H(R)$, $k \in S(R)$, we find

$$(6.61) \quad \begin{aligned} & (\mathfrak{d}(h) - \mathfrak{g}(h))(-k) + k(\mathfrak{d}(h) - \mathfrak{g}(h)) + (\mathfrak{d}(k) - \mathfrak{g}(k))h + h(\mathfrak{d}(k) \\ & - \mathfrak{g}(k)) - [q_1, h]k - k[q_1, h] - [q_1, k]h - h[q_1, k] \in Z(R). \end{aligned}$$

Using (6.60) in (6.61), we conclude

$$(6.62) \quad -2k[q_1, h] \in Z(R), \quad \text{for all } h \in H(R), k \in S(R).$$

As (6.62) is same as (6.28), similar arguments are taking us to the desired conclusion. $\square$

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STUDY OF KIRCHHOFF CURVATURE PROBLEM WITH ψ -HILFER DERIVATIVE AND $p(\cdot)$ -LAPLACIAN OPERATOR

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ABSTRACT. The paper focuses on the existence and multiplicity of weak solutions to nonlinear Kirchhoff-type equations involving ψ -Hilfer derivatives with $p(\cdot)$ -Laplacian operators and Dirichlet boundary conditions. Through the application of a critical point approach, along with genus theory and variational techniques, we establish the existence and multiplicity results within appropriate fractional ψ -Hilfer derivative spaces. Our novel main results contribute to the advancement of the literature on differential equations involving fractional ψ -Hilfer generalized curvature phenomena.

1. INTRODUCTION

Fractional derivatives go beyond classical derivatives, extending the scope of traditional differentiation. Initially akin to ordinary differentiation, the concept of fractional differentiation has evolved in recent research, emphasizing its broader applicability. Within mathematical analysis, fractional analysis explores diverse methods of defining real number powers or complex powers of the differentiation operator and integration. The study extends to fractional order differential equations, which represent generalized and non-integer differential equations in both time and space domains. These equations are characterized by a power-law memory kernel, capturing nonlocal relationships, as highlighted by Kolma [29]. Several approaches exist for incorporating fractional integro-differentiation operations, including the Riemann-Liouville, Caputo and Grunwald-Letnikov methods, along with their modifications. It is crucial to note

Key words and phrases. Generalized ψ -Hilfer derivative, Kirchhoff problem, curvature phenomena, critical point theorem, genus theory, variational approach.

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that a significant portion of the literature on fractional differentiation primarily concentrates on Riemann-Liouville and Caputo fractional derivatives. However, there are alternative well-established definitions, such as the fractional derivative of Hadamard, the fractional derivative of Erdélyi-Kober, and others. Readers interested in delving further into fractional calculus are referred to additional resources like [47] for more comprehensive insights.

A fresh look at capillary phenomena in current literature highlights the need for progress in this area. Capillary action, the movement of liquids through narrow spaces without relying on external forces, is observed in various situations, like paint moving between bristles or liquids flowing through thin tubes. This action is powered by the forces between the liquid and solid surfaces. In small tubes, a combination of surface tension and adhesive forces moves the liquid. Studying phenomena like water rising in tubes or the formation of drops and bubbles involves using calculus variations, providing a unified way to address mathematical questions across different scenarios. Recently, interest in capillary phenomena has grown, driven by its relevance in fields like industry, healthcare, and micro-fluids. Given the breadth of the subject, we will touch on a few instances for those interested [8, 27, 32].

In the context of evolving trends in modern physics and mechanics, it becomes crucial to consider the dynamic landscape when formulating mathematical models. Drawing insights from pertinent research, it is essential to narrow our focus on specific domains in order to gain a deeper understanding of the underlying theory of the primary research problem. To address this, we employ the generalized ψ -Hilfer fractional derivative to analyze a nonlinear Kirchhoff equation featuring a positive parameter. This equation adheres to Dirichlet boundary conditions and is formulated as follows:

$$(1.1) \quad \begin{cases} -(\alpha + \beta \mathcal{L}(u)) \mathbf{L}_{\tau(x)}^{\gamma, \kappa; \psi}(u) = \xi |u|^{r(x)-2} u - h(x) |u|^{\tau(x)-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with

$$\mathcal{L}(u) := \int_{\Omega} \frac{1}{\tau(x)} \left(|D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)} + \sqrt{1 + |D_{0+}^{\gamma, \kappa; \psi} u|^{2\tau(x)}} \right) dx$$

and

$$\mathbf{L}_{\tau(x)}^{\gamma, \kappa; \psi}(u) := D_T^{\gamma, \kappa; \psi} \left(\left(1 + \frac{|D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)}}{\sqrt{1 + |D_{0+}^{\gamma, \kappa; \psi} u|^{2\tau(x)}}} \right) |D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u \right),$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N > 3$, with smooth boundary $\partial\Omega$, α, β are positive constants, $h \in C(\bar{\Omega}, \mathbb{R})$, ξ is a positive parameter, $\tau, r \in C^+(\bar{\Omega})$ such that:

$$(1.2) \quad 1 < r^- \leq r(x) \leq r^+ < \tau^- \leq \tau(x) \leq \tau^+ < 2\tau^- < \tau^*(x) = \frac{N\tau(x)}{N - \gamma\tau(x)},$$

for all $x \in \Omega$, h satisfies the following hypothesis:

$$(\mathbf{H}_0) \quad h : \Omega \rightarrow [0, \infty) \text{ such that } h \in L^\infty(\Omega),$$

$D_T^{\gamma,\kappa;\psi}$ and $D_{0+}^{\gamma,\kappa;\psi}$ are ψ -Hilfer fractional derivatives of order $\gamma \in (\frac{1}{\tau(x)}, 1)$ and type $\kappa \in [0, 1]$ defined later, this type of derivative is introduced by Vanterler et al. [39], by means of the Gronwall inequality. They explored specific cases of the generalized fractional partial derivative, such as the ψ -Riemann-Liouville and ψ -Caputo fractional partial derivatives, discussed in detail in [36]. It's worth noting that when κ tends toward 1 and $\psi(x) = x$, our problem (1.1) simplifies to the integer case, elucidated on page 6 of [38]. This implies that our problem extends the scope of numerous Kirchhoff-type papers in the literature, especially in the context of fractional integration. Recent studies have investigated Kirchhoff equations using different operators. For instance, in [6], the authors employed a variational approach with the mountain pass theorem to establish the existence and multiplicity of solutions for a specific $p(\cdot)$ -Kirchhoff type equation

$$(1.3) \quad \begin{cases} -M \left(\int_{\Omega} \frac{1}{p(x) |\nabla u|^{p(x)}} dx \right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $1 < p(\cdot) < N$.

Chung in [7] used the mountain pass theorem combined with the minimum principle, to obtained at least two non-negative, non-trivial weak solutions for the following $p(x)$ -Kirchhoff-type equations

$$(1.4) \quad \begin{cases} -M \left(\int_{\Omega} \frac{1}{p(x) |\nabla u|^{p(x)}} dx \right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ u \geq 0, & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $1 < p(\cdot) < N$ and λ is a positive real parameter and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ is a Caratheodory function that may change sign. In [25] Shapour et al., used variational methods and critical point theory to established the existence of multiple solutions for problem (1.3). For more examples of Kirchhoff-type problem, we refer the reader to [15–18, 46].

In the study of curvature problems related to the $p(\cdot)$ -Laplacian, an example can be found in the work of authors in [26]. They employed Ricceri's variational principle, as developed by Bonanno and Molica Bisci, to establish the existence of at least one weak solution and infinitely many weak solutions for the following Neumann problem. This problem, originating from capillary phenomena, is a notable illustration in the context of $p(\cdot)$ -Laplacian analysis

$$(1.5) \quad \begin{cases} -\operatorname{div} \left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right) + \alpha(x) |u|^{p(x)-2} u = \lambda f(x, u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^1 , ν is the outer unit normal to $\partial\Omega$, $\alpha \in L^\infty(\Omega)$, f is an L^1 -Caratheodory function. Furthermore, in [34], the authors studied the existence and multiplicity of solutions for the following nonlinear eigenvalue problems for $p(x)$ -Laplacian-like operators

$$(1.6) \quad \begin{cases} -\operatorname{div} \left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right) = \lambda f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\Omega)$ and $p(x) > 2$, for all $x \in \Omega$, $\lambda > 0$ and f satisfies some growth condition and Ambrosetti-Rabinowitz type condition (AR).

In the context of Kirchhoff-type problem involving tempered fractional derivatives, we refer to [33]. The authors employed variational methods to establish the existence of infinitely many solutions to the following:

$$\begin{cases} M \left(\int_{\mathbb{R}} |D_+^{\alpha, \lambda} u(t)|^2 dt \right) D_-^{\alpha, \lambda} (D_+^{\alpha, \lambda} u(t)) = f(t, u(t)), & t \in \mathbb{R}, \\ u \in W_{\lambda}^{\alpha, 2}(\mathbb{R}), \end{cases}$$

where $D_{\pm}^{\alpha, \lambda}(\cdot)$ denote the left and right tempered fractional derivatives of order $\alpha \in (1/2, 1]$, $\lambda > 0$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $M \in C(\mathbb{R}^+, \mathbb{R}^+)$. In [31], the authors studied the existence and multiplicity of solutions to the following Kirchhoff equation with singular nonlinearity and Riemann-Liouville fractional derivative:

$$\begin{cases} \left(a + b \int_0^T |{}_0D_t^{\alpha}(u(t))|^p dt \right) {}_tD_T^{\alpha}(\Phi({}_0D_t^{\alpha}(u(t)))) = \frac{\lambda g(t)}{u^{\gamma}(t)} + f(t, u(t)), & t \in (0, T), \\ u(0) = u(T) = 0, \end{cases}$$

where $a \geq 1$, $b, \lambda > 0$, $p > 1$ are constants, $\frac{1}{p} < \lambda \leq 1$, $0 < \gamma < 1$, $g \in C([0, 1])$ and $f \in C^1([0, T] \times \mathbb{R}, \mathbb{R})$. Under appropriate assumptions on the function f , they employed variational methods to show the existence and multiplicity of positive solutions of the above problem with respect to the parameter λ . All the problems discussed above are associated to the stationary version of the Kirchhoff problem

$$(1.7) \quad \varrho \frac{\partial^2 u}{\partial \gamma^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

presented by Kirchhoff in 1883 [28], which extends d'Alembert's wave equation. One notable feature of model (1.7) is that it contains a nonlocal term $\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2$. The parameters L, h, E, m, P_0 in model (1.7) represent different physical meanings, which we will not cover here. For an application related to Kirchhoff problems in capillary phenomena, we refer to [45], where the authors proposed a computational model to simulate the interaction between a complex fluid and a solid material. The complex fluid is represented using a diffuse-interface model, governed by the incompressible

Navier-Stokes-Cahn-Hilliard equations. Preferential-wetting boundary conditions are applied at the fluid-solid interface. The fluid-solid interaction involves fluid traction on the interface, incorporating a contribution from capillary stress. The dynamic interface condition considers the traction exerted by the non-uniform fluid-solid surface tension. The authors formulated the complex-fluid-solid interaction problem in a weak form, employing an Arbitrary-Lagrangian-Eulerian approach for the Navier-Stokes-Cahn-Hilliard equations. They also appropriately reformulated the complex-fluid traction and the fluid-solid surface tension. To validate their model, further details can be found in the cited paper [45].

Our approach to proving the existence and multiplicity results for problems (1.1) relies on the utilization of the critical points theorem together with genus theory and variational approach.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of variable exponent Lebesgue spaces and ψ -Hilfer fractional derivative spaces. Moving on to Section 3, we present the existing solutions to problems (1.1), along with their corresponding proofs.

2. PRELIMINARY

In this section we collect preliminary concepts of the theory of variable exponent Lebesgue space, classical and fractional ψ -Hilfer derivative space (see [2, 4, 5, 11–14, 19–24, 40–43]).

2.1. Variable exponent Lebesgue space. In the following, we define

$$C^+(\mathbb{R}^N) = \left\{ \tau \in C(\mathbb{R}^N) : 1 < \tau^- \leq \tau^+ < +\infty \right\},$$

where

$$\tau^- := \inf_{x \in \mathbb{R}^N} \tau(x) \quad \text{and} \quad \tau^+ := \sup_{x \in \mathbb{R}^N} \tau(x).$$

Denote by $\mathbf{U}(\mathbb{R}^N)$ the set of all measurable real-valued functions defined in $\mathbb{R}^N$. For any $s \in C^+(\mathbb{R}^N)$, we denote the variable exponent Lebesgue space by

$$L^{\tau(x)}(\mathbb{R}^N) = \left\{ u \in \mathbf{U}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{\tau(x)} dx < +\infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{\tau(x)}} = \inf \left\{ \nu > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\nu} \right|^{\tau(x)} dx \leq 1 \right\},$$

then, the variable exponent Lebesgue space $(L^{\tau(x)}(\mathbb{R}^N), \|\cdot\|_{L^{\tau(x)}})$ becomes a Banach space.

We have the following generalized Holder inequality

$$(2.1) \quad \left| \int_{\mathbb{R}^N} u(x)v(x)dx \right| \leq 2\|u\|_{L^{\tau(x)}}\|v\|_{L^{\tau'(x)}},$$

for $u \in L^{\tau(x)}(\mathbb{R}^N)$, $v \in L^{\tau'(x)}(\mathbb{R}^N)$ such that $\frac{1}{\tau(x)} + \frac{1}{\tau'(x)} = 1$.

At this point, let define the following map $\sigma_{\tau(x)} : L^{\tau(x)}(\mathbb{R}^N) \rightarrow \mathbb{R}$ by

$$\sigma_{\tau(x)}(u) = \int_{\mathbb{R}^N} |u(x)|^{\tau(x)} dx.$$

Then, we can see the important relationship between the norm $\|\cdot\|_{L^{\tau(x)}}$ and the corresponding modular function $\sigma_{\tau(x)}(\cdot)$ given in the next proposition.

Proposition 2.1. *If u and $(u_k)_{k \in \mathbb{N}} \in L^{\tau(x)}(\mathbb{R}^N)$, we have*

$$(2.2) \quad \|u\|_{L^{\tau(x)}} < 1 \text{ (} = 1, > 1 \text{) if and only if } \sigma_{\tau(x)}(u) < 1 \text{ (} = 1, > 1 \text{),}$$

$$(2.3) \quad \|u\|_{L^{\tau(x)}} > 1, \text{ then } \|u\|_{L^{\tau(x)}}^{\tau^-} \leq \sigma_{\tau(x)}(u) \leq \|u\|_{L^{\tau(x)}}^{\tau^+},$$

$$(2.4) \quad \|u\|_{L^{\tau(x)}} < 1, \text{ then } \|u\|_{L^{\tau(x)}}^{\tau^+} \leq \sigma_{\tau(x)}(u) \leq \|u\|_{L^{\tau(x)}}^{\tau^-},$$

$$(2.5) \quad \lim_{k \rightarrow +\infty} \|u_k - u\|_{L^{\tau(x)}} = 0 \text{ if and only if } \lim_{k \rightarrow +\infty} \sigma_{\tau(x)}(u_k - u) = 0.$$

Remark 2.1. Note that, by (2.3) and (2.4), we can derive the two subsequent inequalities:

$$(2.6) \quad \|u\|_{L^{\tau(x)}} \leq \sigma_{\tau(x)}(u) + 1,$$

$$(2.7) \quad \sigma_{\tau(x)}(u) \leq \|u\|_{L^{\tau(x)}}^{\tau^+} + \|u\|_{L^{\tau(x)}}^{\tau^-}.$$

2.2. ψ -Hilfer fractional derivative space. Let $A := [c, d]$, $-\infty \leq c < d \leq +\infty$, $n - 1 < \gamma < n$, $n \in \mathbb{N}$, $\mathbf{f}, \psi \in C^n(A, \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in A$. We recall the following definitions (see [39]).

- The left-sided fractional ψ -Hilfer integrals of a function $\mathbf{f}$ is given by

$$(2.8) \quad \mathbf{I}_{c^+}^{\gamma; \psi} \mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_c^x \psi'(y) (\psi(x) - \psi(y))^{\gamma-1} \mathbf{f}(y) dy.$$

- The right-sided fractional ψ -Hilfer integrals of a function $\mathbf{f}$ is given by

$$(2.9) \quad \mathbf{I}_{d^-}^{\gamma; \psi} \mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_x^d \psi'(y) (\psi(y) - \psi(x))^{\gamma-1} \mathbf{f}(y) dy.$$

- The left-sided ψ -Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \kappa \leq 1$ is defined by

$$\mathbf{D}_{c^+}^{\gamma, \kappa; \psi} \mathbf{f}(x) = \mathbf{I}_{c^+}^{\kappa(n-\gamma); \psi} \left(\frac{1}{\psi'(x)} \cdot \frac{d}{dx} \right)^n \mathbf{I}_{c^+}^{(1-\kappa)(n-\gamma); \psi} \mathbf{f}(x).$$

- The right-sided ψ -Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \kappa \leq 1$ is defined by

$$\mathbf{D}_{d^-}^{\gamma, \kappa; \psi} \mathbf{f}(x) = \mathbf{I}_{d^-}^{\kappa(n-\gamma); \psi} \left(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx} \right)^n \mathbf{I}_{d^-}^{(1-\kappa)(n-\gamma); \psi} \mathbf{f}(x).$$

Choosing $\kappa \rightarrow 1$, we obtain ψ -Caputo fractional derivatives left-sided and right-sided, given by

$$(2.10) \quad \mathbf{D}_{c^+}^{\gamma; \psi} \mathbf{f}(x) = \mathbf{I}_{c^+}^{(n-\gamma); \psi} \left(\frac{1}{\psi'(x)} \cdot \frac{d}{dx} \right)^n \mathbf{f}(x),$$

$$(2.11) \quad D_{d^-}^{\gamma;\psi} \mathbf{f}(x) = \mathbf{I}_{d^-}^{(n-\gamma);\psi} \left(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx} \right)^n \mathbf{f}(x).$$

Remark 2.2. The ψ -Hilfer fractional derivatives defined as above can be written in the following form

$$D_{c^+}^{\gamma,\kappa;\psi} \mathbf{f}(x) = \mathbf{I}_{c^+}^{\mu-\gamma;\psi} D_{c^+}^{\gamma;\psi} \mathbf{f}(x)$$

and

$$D_{d^-}^{\gamma,\kappa;\psi} \mathbf{f}(x) = \mathbf{I}_{d^-}^{\mu-\gamma;\psi} D_{d^-}^{\gamma;\psi} \mathbf{f}(x),$$

with $\mu = \gamma + \kappa(n - \gamma)$ and $\mathbf{I}_{c^+}^{\mu-\gamma;\psi}$, $\mathbf{I}_{d^-}^{\mu-\gamma;\psi}$, $D_{c^+}^{\gamma;\psi}$ and $D_{d^-}^{\gamma;\psi}$ as defined in (2.8), (2.9), (2.10) and (2.11).

In this paper we take $\Omega = A_1 \times \cdots \times A_N = [c_1, d_1] \times \cdots \times [c_N, d_N]$ where $0 < c_i < d_i$ for all $i \in \mathbb{N}$, $0 < \gamma_1, \dots, \gamma_N < 1$.

• The ψ -Riemann-Liouville fractional partial integral of order γ of N -variables $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_N)$ is defined by

$$\mathbf{I}_{c,x}^{\gamma;\psi} \mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_{A_1} \int_{A_2} \cdots \int_{A_N} \psi'(y) (\psi(x) - \psi(y))^{\gamma-1} \mathbf{f}(y) dy,$$

with $\psi'(y)(\psi(x) - \psi(y))^{\gamma-1} = \psi'(y_1)(\psi(x_1) - \psi(y_1))^{\gamma_1-1} \cdots \psi'(y_N)(\psi(x_N) - \psi(y_N))^{\gamma_N-1}$ and $\Gamma(\gamma) = \Gamma(\gamma_1)\Gamma(\gamma_2) \cdots \Gamma(\gamma_N)$, $x_i = x_1 x_2 \cdots x_N$ and $dy_i = dy_1 dy_2 \cdots dy_N$, for all $i \in \{1, 2, \dots, N\}$.

• $D_{c,x_i}^{\gamma,\kappa;\psi}$ is defined by

$$D_{c,x_i}^{\gamma,\kappa;\psi} \mathbf{f}(x_i) = \mathbf{I}_{c,x_i}^{\kappa(n-\gamma);\psi} \left(\frac{1}{\psi'(x_i)} \cdot \frac{\partial^N}{\partial x_i} \right) \mathbf{I}_{c,x_i}^{(1-\kappa)(n-\gamma);\psi} \mathbf{f}(x_i),$$

with $\partial x_i = \partial x_1, \partial x_2, \dots, \partial x_N$ and $\psi'(x_i) = \psi'(x_1)\psi'(x_2) \cdots \psi'(x_N)$ for all $i \in \{1, 2, \dots, N\}$. Analogously, it is defined $D_{d,x_i}^{\gamma,\kappa;\psi}(\cdot)$.

Now that we have all the necessary tools, we are ready to commence our study. To facilitate this, we define the ψ -Hilfer fractional derivative space $\mathcal{H}_{\tau(x)}^{\gamma,\kappa,\psi}(\Omega)$ as follow

$$\mathcal{H}_{\tau(x)}^{\gamma,\kappa,\psi}(\Omega) = \mathcal{H}_{\tau(x)}(\Omega) := \left\{ u \in L^{\tau(x)}(\Omega) : |D_{0^+}^{\gamma,\kappa;\psi} u| \in L^{\tau(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{\mathcal{H}_{\tau(x)}} = \|u\|_{L^{\tau(x)}} + \|D_{0^+}^{\gamma,\kappa;\psi} u\|_{L^{\tau(x)}}.$$

Proposition 2.2 ([44]). *Let $0 < \gamma \leq 1$, $0 \leq \kappa \leq 1$ and $1 < \tau(x)$. $\mathcal{H}_{\tau(x)}(\Omega)$ is a reflexive and separable Banach space.*

Remark 2.3. We can define $\mathcal{H}_{\tau(x),0}^{\gamma,\kappa,\psi}(\Omega) := \mathcal{H}_{\tau(x),0}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}_{\tau(x)}^{\gamma,\kappa,\psi}(\Omega)$ which can be renormed by the equivalent norm $\|u\| := \|D_{0^+}^{\gamma,\kappa,\psi} u\|_{L^{\tau(x)}}$. This space is a separable and reflexive Banach space (see [44]).

Proposition 2.3 ([30]). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$. Let $p \in C^0(\overline{\Omega})$. If $r : \overline{\Omega} \rightarrow (1, +\infty)$ such that*

$$1 \leq r(x) < \tau^*(x) := \begin{cases} \frac{N\tau(x)}{N-\gamma\tau(x)}, & \text{if } \gamma\tau(x) < N, \\ +\infty, & \text{if } \gamma\tau(x) \geq N, \end{cases} \quad \text{for all } x \in \overline{\Omega},$$

then, the embedding

$$(2.12) \quad \mathcal{H}_{\tau(x),0}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

is compact and there is a constant $c_0 > 0$, such that $\|u\|_{L^{r(x)}} \leq c_0\|u\|$.

2.3. Genus Theory. We introduce fundamental concepts related to Krasnoselskii's genus (refer to [9]) which will be employed in the proof of our main results. Let X be a real Banach space and

$$\mathfrak{R} := \{A \subset X \setminus \{0\} : A \text{ is compact and symmetric}\}.$$

Definition 2.1. Let $A \in \mathfrak{R}$ and $X = \mathbb{R}^k$. We define the genus of A as follows:

$$\mathfrak{G}(A) := \inf \left\{ k \geq 1 : \text{exists } g \in C(A, \mathbb{R}^k \setminus \{0\}), g \text{ is odd} \right\}$$

and $\mathfrak{G}(A) = +\infty$, if does not exist such a map for any $k > 0$.

Theorem 2.1 ([9]). *Let $\Omega \subset \mathbb{R}^N$ be bounded symmetric with boundary $\partial\Omega$. Assume that $0 \in \Omega$, then $\mathfrak{G}(\partial\Omega) = N$.*

Corollary 2.1 ([9]). *The genus of unit sphere $\mathbb{S}^{N-1}$ of the space $\mathbb{R}^N$ is N , i.e., $\mathfrak{G}(\mathbb{S}^{N-1}) = N$.*

Definition 2.2. Let X be a real Banach space, and $\Upsilon \in C^1(X, \mathbb{R})$. We say that Υ satisfies the Palais-Smale condition ((PS) for short) if any sequence $\{u_n\}_{n \in \mathbb{N}} \subset X$ such that $\{\Upsilon(u_n)\}_{n \in \mathbb{N}}$ is bounded and $\Upsilon'(u_n) \rightarrow 0$ as $n \rightarrow +\infty$, admits a convergent subsequence.

Theorem 2.2 ([10]). *Let $\Upsilon \in C^1(X, \mathbb{R})$ and let it satisfies the (PS) condition. Additionally, we assume the following conditions.*

(i) Υ is bounded from below and even.

(ii) *There is a compact set $N \in \mathfrak{R}$ such that $\mathfrak{G}(N) = k$ and $\sup_{x \in N} \Upsilon(x) < \Upsilon(0)$.*

Then, Υ has at least k pairs of distinct critical points, and their corresponding critical values are less than $\Upsilon(0)$.

3. MAIN RESULT

In this section, we will proving the existence and multiplicity results for problems (1.1) relies on the utilization of the critical points theorem together with genus theory and variational approach.

Definition 3.1. We say that $u \in \mathcal{H}_{\tau(x),0}(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned} & \left(\alpha + \beta \mathcal{L}(u) \right) \int_{\Omega} \left(\left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u + \frac{\left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{2\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u}{\sqrt{1 + \left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{2\tau(x)}}} \right) D_{0+}^{\gamma, \kappa; \psi} \varphi \, dx \\ &= \xi \int_{\Omega} |u|^{r(x)-2} u \varphi \, dx - \int_{\Omega} h(x) |u|^{\tau(x)-2} \varphi \, dx, \end{aligned}$$

for all $\varphi \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Let us introduce the energy functional $\mathfrak{E} : \mathcal{H}_{\tau(x),0}(\Omega) \rightarrow \mathbb{R}$ associated to problem (1.1)

$$\mathfrak{E}(u) = \alpha \mathcal{L}(u) + \frac{\beta}{2} (\mathcal{L}(u))^2 - \xi \int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} \, dx + \int_{\Omega} \frac{h(x)}{\tau(x)} |u|^{\tau(x)} \, dx,$$

for all $u \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Observe that $\mathfrak{E} \in C^1(\mathcal{H}_{\tau(x),0}(\Omega), \mathbb{R})$ and it is noteworthy that the critical points of $\mathfrak{E}$ correspond to weak solutions of (1.1) and its Gateaux derivative is

$$\begin{aligned} & \langle \mathfrak{E}'(u), v \rangle \\ &= \left(\alpha + \beta \mathcal{L}(u) \right) \int_{\Omega} \left(\left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u + \frac{\left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{2\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u}{\sqrt{1 + \left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{2\tau(x)}}} \right) D_{0+}^{\gamma, \kappa; \psi} v \, dx \\ & \quad - \xi \int_{\Omega} |u|^{r(x)-2} u v \, dx + \int_{\Omega} h(x) |u|^{\tau(x)-2} u v \, dx, \end{aligned}$$

for all $u, v \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Note that the functional $\mathcal{L} \in C^1(\mathcal{H}_{\tau(x),0}(\Omega), \mathbb{R})$ and its derivative operator in weak sense $\mathcal{L}' : \mathcal{H}_{\tau(x),0}(\Omega) \rightarrow (\mathcal{H}_{\tau(x),0}(\Omega))^*$ is such that

$$\langle \mathcal{L}'(u), v \rangle = \int_{\Omega} \left(\left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u + \frac{\left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{2\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u}{\sqrt{1 + \left| D_{0+}^{\gamma, \kappa; \psi} u \right|^{2\tau(x)}}} \right) D_{0+}^{\gamma, \kappa; \psi} v \, dx,$$

for all $u, v \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Theorem 3.1. Problem (1.1) admits at least k pairs of different critical points if (1.2) holds.

Proposition 3.1 ([1]). The functional $\mathcal{L}$ is a convex. The mapping $\mathcal{L}' : \mathcal{H}_{\tau(x),0}(\Omega) \rightarrow (\mathcal{H}_{\tau(x),0}(\Omega))^*$ is bounded homeomorphism and strictly monotone operator, and is a mapping of type (S_+) , i.e., if $u_n \rightharpoonup u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$ and $\lim_{n \rightarrow +\infty} \langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$.

Lemma 3.1. The functional $\mathfrak{E}$ satisfies the (PS) condition.

Proof. Let show that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{H}_{\tau(x),0}(\Omega)$. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_{\tau(x),0}(\Omega)$ be a (PS) sequence. Employing a proof by contradiction, we assume that, possibly after considering a sub-sequence, still denote by $\{u_n\}_{n \in \mathbb{N}}$, one has $\|u_n\| \rightarrow +\infty$ as $n \rightarrow +\infty$. Let us choose $0 < \omega < \left\{ \frac{1}{r^+}, \frac{1}{\tau^+}, \frac{\tau^-}{2(\tau^+)^2} \right\}$. According to Proposition 2.3, for sufficiently large n , one has

$$\begin{aligned} c + \|u_n\| &\geq \mathfrak{E}(u_n) - \omega \langle \mathfrak{E}'(u_n), u_n \rangle \\ &\geq \alpha \left(\frac{1}{\tau^+} - \omega \right) \int_{\Omega} |D_{0+}^{\gamma,\kappa;\psi} u_n|^{\tau(x)} dx + \beta \left(\frac{1}{2(\tau^+)^2} - \frac{\omega}{\tau^-} \right) \left(\int_{\Omega} |D_{0+}^{\gamma,\kappa;\psi} u_n|^{\tau(x)} dx \right)^2 \\ &\quad - \xi \left(\frac{1}{r^-} - \omega \right) \int_{\Omega} |u_n|^{r(x)} dx + \left(\frac{1}{\tau^+} - \omega \right) \int_{\Omega} h(x) |u_n|^{\tau(x)} dx \\ &\geq \alpha \left(\frac{1}{\tau^+} - \omega \right) \|u_n\|^{\tau^-} + \beta \left(\frac{1}{2(\tau^+)^2} - \frac{\omega}{\tau^-} \right) \|u_n\|^{2\tau^-} - \xi c_0 \left(\frac{1}{r^-} - \omega \right) \|u_n\|. \end{aligned}$$

Dividing the aforementioned inequality by $\|u_n\|$ and taking the limit as $n \rightarrow +\infty$, we arrive at a contradiction. It is implied by (1.2) that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{H}_{\tau(x),0}(\Omega)$.

Moreover, based on Proposition 2.3, we can assume that

$$(3.1) \quad \begin{cases} u_n \rightarrow u \text{ strongly in } L^{r(x)}(\Omega), \\ u_n(x) \rightarrow u(x) \text{ a.e in } \Omega, \\ u_n \rightharpoonup u \text{ weakly in } \mathcal{H}_{\tau(x),0}(\Omega). \end{cases}$$

Using Holder's inequality and (3.1), one has

$$\begin{aligned} \left| \int_{\Omega} |u_n|^{\tau(x)-2} u_n (u_n - u) dx \right| &\leq \int_{\Omega} |u_n|^{\tau(x)-1} |u_n - u| dx \\ &\leq \|u_n\|^{\frac{\tau^+-1}{\tau(x)-1}} \|u_n - u\|_{\tau(x)} \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Therefore,

$$(3.2) \quad \int_{\Omega} |u_n|^{\tau(x)-2} u_n (u_n - u) dx \rightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

and

$$(3.3) \quad \int_{\Omega} |u_n|^{r(x)-2} u_n (u_n - u) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Then,

$$\langle \mathfrak{E}'(u_n), u_n - u \rangle \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

This implies that

$$\begin{aligned} &\langle \mathfrak{E}'(u_n), u_n - u \rangle \\ &= (\alpha + \beta \mathcal{L}(u_n)) \int_{\Omega} \left(|D_{0+}^{\gamma,\kappa;\psi} u_n|^{\tau(x)-2} D_{0+}^{\gamma,\kappa;\psi} u_n + \frac{|D_{0+}^{\gamma,\kappa;\psi} u_n|^{2\tau(x)-2} D_{0+}^{\gamma,\kappa;\psi} u_n}{\sqrt{1 + |D_{0+}^{\gamma,\kappa;\psi} u_n|^{2\tau(x)}}} \right) D_{0+}^{\gamma,\kappa;\psi} (u_n - u) dx \\ &\quad - \xi \int_{\Omega} |u_n|^{r(x)-2} u_n (u_n - u) dx + \int_{\Omega} h(x) |u_n|^{\tau(x)-2} u_n (u_n - u) dx \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Thus, we can infer from equations (3.2) and (3.3) that

$$\begin{aligned} & (\alpha + \beta \mathcal{L}(u_n)) \int_{\Omega} \left(|D_{0+}^{\gamma, \kappa; \psi} u_n|^{\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u_n + \frac{|D_{0+}^{\gamma, \kappa; \psi} u_n|^{2\tau(x)-2} D_{0+}^{\gamma, \kappa; \psi} u_n}{\sqrt{1 + |D_{0+}^{\gamma, \kappa; \psi} u_n|^{2\tau(x)}}} \right) \\ & \times \left(D_{0+}^{\gamma, \kappa; \psi} u_n - D_{0+}^{\gamma, \kappa; \psi} u \right) dx \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Therefore, according to Proposition 3.1, $u_n \rightarrow u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$. Hence, we conclude the proof. $\square$

Lemma 3.2. *The functional $\mathfrak{E}$ is coercive and bounded from below.*

Proof. For any $u \in \mathcal{H}_{\tau(x),0}(\Omega)$, we have

$$\begin{aligned} \mathfrak{E}(u) & \geq \frac{\alpha}{\tau^+} \int_{\Omega} |D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)} + \frac{\beta}{2(\tau^+)^2} \left(\int_{\Omega} |D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)} dx \right)^2 - \frac{\xi}{r^-} \int_{\Omega} |u|^{r(x)} dx \\ & \quad + \frac{1}{\tau^+} \int_{\Omega} h(x) |u|^{\tau(x)} dx. \end{aligned}$$

Using Propositions 2.1 and 2.3, we have two cases.

Case 1. If $\|u\|_{L^{\tau(x)}} > 1$, then

$$\mathfrak{E}(u) \geq \frac{\alpha}{\tau^+} \|u\|^{\tau^-} + \frac{\beta}{2(\tau^+)^2} \|u\|^{2\tau^-} - \frac{\xi c_0}{r^-} \|u\|^{r^+}.$$

According to (1.2), $\mathfrak{E}$ is coercive and bounded from below.

Case 2. If $\|u\|_{L^{\tau(x)}} < 1$, then

$$\mathfrak{E}(u) \geq \frac{\alpha}{\tau^+} \|u\|^{\tau^+} + \frac{\beta}{2(\tau^+)^2} \|u\|^{2\tau^+} - \frac{\xi c_0}{r^-} \|u\|^{r^-}.$$

Since $2\tau^+ > \tau^+$ and $2\tau^+ > \tau^-$, this implies that $\mathfrak{E}$ is coercive and bounded from below. $\square$

Proof of Theorem 3.1. Let $(s_n)_{n=1}^{\infty}$ be a Schauder basis for the space $\mathcal{H}_{\tau^+,0}(\Omega)$ and $Y_k = \text{span}\{s_1, s_2, \dots, s_k\}$, the subspace of $\mathcal{H}_{\tau^+,0}(\Omega)$ generated by $s_1, s_2, \dots, s_k$. Clearly, Y_k is subspace of $\mathcal{H}_{\tau^+,0}(\Omega)$. Then, since $\mathcal{H}_{\tau^+,0}(\Omega) \subset \mathcal{H}_{\tau(x),0}(\Omega) \subset L^{r(x)}(\Omega)$ we have $Y_k \subset L^{r(x)}(\Omega)$. Also, since Y_k is a finite-dimensional space, the norms $\|\cdot\|$ and $\|\cdot\|_{L^{r(x)}}$ are equivalent on Y_k . Therefore, there exists a positive constant c_k such that

$$\|u\|_{L^{r(x)}} \geq c_k \|u\|, \quad \text{for all } u \in Y_k.$$

Note that since $\sqrt{1 + |D_{0+}^{\gamma, \kappa; \psi} u|^{2\tau(x)}} < 2|D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)}$, we have

$$(3.4) \quad \mathcal{L}(u) \leq \frac{3}{\tau^-} \int_{\Omega} |D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)} dx.$$

Let $u \in Y_k$ such that $\|u\| < 1$, then using (3.4) and $(\mathbf{H}_0)$ one has

$$\mathfrak{E}(u) = \alpha \mathcal{L}(u) + \frac{\beta}{2} (\mathcal{L}(u))^2 - \xi \int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx + \int_{\Omega} \frac{h(x)}{\tau(x)} |u|^{\tau(x)} dx$$

$$\begin{aligned}
&\leq \frac{3\alpha}{\tau^-} \int_{\Omega} |D_{0+}^{\gamma, \kappa; \psi} u|^{\tau(x)} dx + \frac{9\beta}{2(\tau^-)^2} \int_{\Omega} |D_{0+}^{\gamma, \kappa; \psi} u|^{2\tau(x)} dx \\
&\quad - \frac{\xi}{r^+} \int_{\Omega} |u|^{r(x)} dx + \frac{1}{\tau^-} \int_{\Omega} h(x) |u|^{\tau(x)} dx \\
&\leq c_1 \left(\|u\|^{\tau^-} + \|u\|^{2\tau^-} \right) - c_2 \|u\|_{L^{r(x)}^+}^{r^+} + \frac{1}{\tau^-} \|h\|_{L^\infty} \|u\|^{\tau^-} \\
&\leq c_3 \left(\|u\|^{\tau^-} + \|u\|^{2\tau^-} \right) - c_4 \|u\|^{r^+} \\
&= \|u\|^{r^+} \left[c_3 \left(\|u\|^{\tau^- - r^+} + \|u\|^{2\tau^- - r^+} \right) - c_4 \right].
\end{aligned}$$

There exists $\lambda \in (0, 1)$ sufficiently small such that $\lambda^{r^+} < 1$ and $c_3 \lambda^{\tau^- - r^+} + c_3 \lambda^{2\tau^- - r^+} \leq \frac{c_4}{2}$. Let consider $\mathbf{S}_\lambda^k := \{u \in Y_k : \|u\| = \lambda\}$. We have

$$\mathfrak{E}(u) \leq \lambda^{r^+} \left(c_3 \lambda^{\tau^- - r^+} + c_3 \lambda^{2\tau^- - r^+} - c_4 \right),$$

for all $u \in \mathbf{S}_\lambda^k$. Thus,

$$\sup_{u \in \mathbf{S}_\lambda^k} \mathfrak{E}(u) \leq \left(\frac{c_4}{2} - c_4 \right) = -\frac{c_4}{2} < 0 = \mathfrak{E}(0).$$

Since Y_k and $\mathbb{R}^k$ are isomorphic, $\mathbf{S}_\lambda^k$ and $\mathbb{S}^{k-1}$ are homeomorphic, thus $\mathfrak{G}(\mathbf{S}_\lambda^k) = k$. According to Theorem 2.2, $\mathfrak{E}$ has at least k pairs of different critical points. $\square$

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