Kragujevac Journal of Mathematics
Volume 50(8) (2026), Pages 1221–1242.

# ON GENERALIZED MADDOX SPACES VIA DIFFERENCE OPERATOR

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Abstract. In the present work, we begin our investigation with some dynamic properties of new generalized difference operator $\Delta_h^{\alpha,\beta,\gamma}$, defined in Baliarsingh [5]. Combining this operator with the well known Cesàro operator, we also introduce new classes of generalized Cesàro summable difference sequence spaces $w(\Delta_h^{\alpha,\beta,\gamma}, p), w_0(\Delta_h^{\alpha,\beta,\gamma}, p)$ and $w_\infty(\Delta_h^{\alpha,\beta,\gamma}, p)$, which are the natural extension of the spaces $w^p$, $w_0^p$ and $w_\infty^p$ defined in [12] and $w_0(p)$, $w(p)$, and $w_\infty(p)$ defined by Maddox [21]. We establish various topological properties on these spaces along with some inclusion relations with other basic sequence spaces. Further, our investigation is carried out to determine $\alpha^*$- and $\beta^*$- duals and characterize matrix transformations on these spaces.

## 1. Introduction, Definitions and Preliminaries

One of the potential fields of sequence spaces as well as the growing achievements of summability theory is the difference sequence spaces, which enriched with the existence of sequence spaces via various difference operators providing wide range of applications in both pure and applied area of mathematics. Moreover, ideal concepts of difference operators and the related spaces are stimulating diverse fields of research like approximation theory [11, 19], spectral theory and linear algebra [1, 7, 15], numerical analysis [8], compact operator theory [25, 31], fractional calculus [5, 6] and matrix theory [13, 26, 29], etc.

A sequence space is a subspace of linear space $w$ (the space of all real or complex valued sequences), for instance, $c$, $c_0$ and $\ell_\infty$ are the spaces of all convergent, null,

*Key words and phrases.* Sequence spaces, difference operators, topological properties, matrix transformations, dual spaces.
2020 *Mathematics Subject Classification.* Primary: 46A45. Secondary: 46A35, 47B45.
https://doi.org/10.46793/KgJMat2608.1221M
*Received*: May 12, 2024.
*Accepted*: November 13, 2024.

and bounded sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. As the initial development of this theory, in 1981, the idea of difference sequence spaces $X(\Delta) = \{x = (x_k) \in w : \Delta x \in X\}$, where $X \in \{c_0, c, \ell_\infty\}$ had been considered by Kızmaz [20], by using the forward difference operator

$$\Delta x_k = x_k - x_{k+1}, \quad k \in \mathbb{N}_0 \text{ (the set of all non negative integers)}. \tag{1.1}$$

Recently, the difference space $bv_p$ consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been studied in the case $0 < p < 1$ by Altay and Başar [2], and in the case $1 \leq p \leq +\infty$ by Başar and Altay [13]. The reader also refer to the recent monograph [29], and references therein, devoted to the matrix transformations and related topics.

In 1995, these spaces had been extended to the case of integer order '$m$' by Et and Çolak [17], using the difference operator $\Delta^m$, defined by

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}, \quad k \in \mathbb{N}_0. \tag{1.2}$$

Latter, Baliarsingh [4] (see, also [8]) extended these spaces by using fractional order difference operator, $\Delta^\alpha$ for a positive proper fraction $\alpha$, i.e.,

$$\Delta^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha+1-i)} x_{k+i}, \quad k \in \mathbb{N}_0, \tag{1.3}$$

where $\Gamma(\alpha)$ denotes the Euler gamma function. Quite recently, the idea has been extended to the case of arbitrary order $\alpha, \beta, \gamma$ by Baliarsingh [5] by defining

$$\Delta_h^{\alpha,\beta,\gamma} x_k = \sum_{i=0}^{\infty} \frac{(-\alpha)_i (-\beta)_i}{(-\gamma)_i \ i! \ h^{\alpha+\beta-\gamma}} x_{k \pm i}, \tag{1.4}$$

where $(\sigma)_k$ is the Pochhamer symbol or shifted factorial for $\sigma \in \mathbb{R}$, defined by

$$(\sigma)_k = \begin{cases} 1, & \sigma = 0 \text{ or } k = 0, \\ \sigma(\sigma+1)\cdots(\sigma-k+1), & k \in \mathbb{N}. \end{cases}$$

In particular, if $h = 1$ the above operators include the cases, such as the operator $\Delta^1$ ([20]) for $\alpha = 1$, $\beta = \gamma$, the operator $\Delta^{(m)}$ ([17]) for $\alpha = m \in \mathbb{N}, \gamma = \beta$ and the operator $\Delta^\alpha$ ([4]) for $\beta = \gamma$. Note that matrix representation of the operators $\Delta = (\delta_{nk})$, $\Delta^m = (\delta^m_{nk})$, $\Delta^\alpha = (\delta^\alpha_{nk})$ and $\Delta_h^{\alpha,\beta,\gamma} = (\delta^{a,b,c}_{h,nk})$ (defined in (1.1), (1.2), (1.3) and (1.4)), respectively, are as follows:

$$\delta_{nk} = \begin{cases} 1, & k = n, \\ -1, & k = n-1, \\ 0, & k > n, \end{cases} \quad \delta^m_{nk} = \begin{cases} 1, & k = n, \\ (-1)^{n-k}\binom{m}{n-k}, & 0 \leq k < n, \\ 0, & k > n, \end{cases}$$

$$\delta^\alpha_{nk} = \begin{cases} 1, & k = n, \\ \frac{(-\alpha)_{n-k}}{(n-k)!}, & 0 \leq k < n, \\ 0, & k > n, \end{cases} \quad \text{and} \quad \delta^{a,b,c}_{h,nk} = \begin{cases} 1, & k = n, \\ \frac{(-\alpha)_{n-k}(-\beta)_{n-k}}{(-\gamma)_{n-k} \ (n-k)!}, & 0 \leq k < n, \\ 0, & k > n. \end{cases}$$

Several authors including Mursaleen, Altay and Başar [27], Mursaleen and Başar [29], Dutta and Baliarsingh [6], Tripathy and Sarma [30], Aydin and Polat [26], Vardwaj and Gupta [14], Et [16], etc., have given their valuable attempts to construct new sequence spaces by combining the notion of difference operators with some functions (like Modulus function, Orlicz function, $\phi$ function, etc.), and certain means (Cesàro mean, Nörlund mean, Riesz mean, Euler mean, etc.). In this regard, various authors studied the topological properties on defined sequence spaces along with their matrix transformations, inclusion relations and also determined their dual spaces. It has been remarked that the above works will be more convenient if the used difference operators defined by (1.1), (1.2), (1.3) and (1.4) are well defined and the related difference sequences are convergent. In fact, the behavior of the operators mentioned earlier on any arbitrary sequences is completely dynamic in nature. At this stage, we want to emphasize this dynamic nature by following examples.

*Example* 1.1. Let $\alpha > \gamma > 0$, $\beta = 0$, $h = 1$ and define a sequence $x = (x_k) = (\frac{1}{5^k})$, $k \in \mathbb{N}_0$. Then, we have

$$\begin{aligned}\Delta_h^{\alpha,\beta,\gamma} x_k = \sum_{i=0}^{+\infty} \frac{(-\alpha)_i}{(-\gamma)_i\ i!} x_{k+i} &= \frac{1}{5^k} \sum_{i=0}^{+\infty} \frac{(-\alpha)_i}{(-\gamma)_i\ i!} \cdot \frac{1}{5^i}\\ &= \frac{1}{5^k} \sum_{i=0}^{+\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-i+1)}{\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-i+1)i!} \cdot \frac{1}{5^i}\\ &= \frac{1}{5^k} \sum_{i=0}^{+\infty} t_i.\end{aligned}$$

By applying D'Almbert's ratio test, we obtain the right hand side of above series

$$\lim_{i\to+\infty} \left|\frac{t_{i+1}}{t_i}\right| = \lim_{i\to+\infty} \left|\frac{(\alpha-i)}{(\gamma-i)(i+1)5}\right| = 0.$$

This ultimately makes the sequence $\Delta_h^{\alpha,\beta,\gamma} x$ converges. So, the above series is convergent.

*Example* 1.2. Let $x = e = (1, 1, 1, \dots)$ and $\alpha > \gamma > 0$ and $\beta = 0$, $h = 1$. Then, clearly, the sequence $\Delta_1^{\alpha,\beta,\gamma} x$ converges. But, for $\beta = \gamma$, $h = 1$, it can be observed that

$$\Delta^{-\alpha} x_k = \alpha + \frac{\alpha(\alpha+1)}{2} + \cdots \to +\infty, \quad \text{as } k \to +\infty,$$

which is divergent.

*Example* 1.3. Let the sequence $x = (x_k)$ be defined by $x_k = k^{\frac{5}{2}}$, $k \in \mathbb{N}_0$. Then, it can be easily observed that

$$\Delta x_k = k^{\frac{5}{2}} - (k+1)^{\frac{5}{2}} = -\frac{5}{2} - \mathcal{O}(k^{\frac{1}{2}}) \to -\infty, \quad \text{as } k \to +\infty,$$

which is divergent. But, for $\alpha = 3$, $\beta = \gamma$, $h = 1$ we obtain

$$\Delta^3 x_k = k^{\frac{5}{2}} - 3(k+1)^{\frac{5}{2}} + 3(k+2)^{\frac{5}{2}} - (k+3)^{\frac{5}{2}}$$

$$=k^{\frac{5}{2}} - 3\left\{k^{\frac{5}{2}} + \frac{5}{2}k^{\frac{3}{2}} + \frac{15}{8}k^{\frac{1}{2}} + \mathcal{O}(k^{-\frac{1}{2}})\right\}$$
$$+ 3\left\{k^{\frac{5}{2}} + 5k^{\frac{3}{2}} + \frac{15}{2}k^{\frac{1}{2}} + \mathcal{O}(k^{-\frac{1}{2}})\right\} - \left\{k^{\frac{5}{2}} + \frac{15}{2}k^{\frac{3}{2}} + \frac{135}{2}k^{\frac{1}{2}} + \mathcal{O}(k^{-\frac{1}{2}})\right\} \to 0,$$
as $k \to +\infty$.

As a result, we conclude that the sequence $\Delta^3 x$ is convergent.

*Example* 1.4. Let $x = (x_k)$ be the oscillating sequence defined by $x_k = \frac{1}{3} + \frac{(-1)^k}{6}$, $k \in \mathbb{N}_0$. Now, after applying the operator $\Delta^\alpha$ on $x$ we have,

$$\Delta^\alpha x_k = \begin{cases} \frac{2^{\alpha-1}}{3}, & k \text{ is even,} \\ -\frac{2^{\alpha-1}}{3}, & k \text{ is odd,} \end{cases}$$

which is a divergent sequence.

Furthermore, the study on convergence of difference sequences up to the case of fractional order, is found in the work of Baliarsingh [6]. But, convergence analysis in general case is quite challenging, which to be discussed in this sections. Moreover, our aim is to construct some new class of difference sequence spaces by combining the generalized difference operator with Cesàro operator of order one and also investigate their topological properties with some inclusion relations. Usually, the Cesàro operator $C_1$ of order one, and its inverse $(C_1)^{-1}$, respectively, in matrix form are given by $C_1 = (c_{nk})$, and $C_1^{-1} = (s_{nk})$, where

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \le k \le n, \\ 0, & \text{elsewhere,} \end{cases} \quad \text{and} \quad s_{nk} = \begin{cases} n+1, & k = n, \\ -(k+1), & k = n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

Now, combining the generalized difference operator $\Delta^{\alpha,\beta,\gamma}_{1,-}$ and Cesàro operator $C_1$, we have the generalized Cesàro difference operator $(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})$ as

$$(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})_{nk} = \begin{cases} \frac{1}{n+1}\sum_{j=0}^{n-k} \frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!}, & \text{if } 0 \le k < n, \\ \frac{1}{n+1}, & k = n, \\ 0, & k > n. \end{cases}$$

Using Theorem 2 and Remark 1 of [9], we determine the inverse of $C_1\Delta^{\alpha,\beta,\gamma}_{1,-}$ explicitly as

$$(C_1\Delta^{a,b,c}_{1,-})^{-1}_{nk} = \begin{cases} (-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}), & \text{if } 0 \le k < n, \\ n+1, & k = n, \\ 0, & k > n, \end{cases}$$

where
(1.5)

$$\bar{D}_n^k(C_1\Delta_{1,-}^{\alpha,\beta,\gamma}) = \begin{vmatrix} \frac{1}{k+2}\sum_{j=k}^{k+1}\frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{k+2} & 0 & \cdots \\ \frac{1}{k+3}\sum_{j=k}^{k+2}\frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{k+3}\sum_{j=k}^{k+1}\frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{k+3} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1}\sum_{j=k}^{n}\frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \frac{1}{n+1}\sum_{j=k}^{n-1}\frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} & \cdots & \frac{1}{n+1}\sum_{j=k}^{k+1}\frac{(-\alpha)_j(-\beta)_j}{(-\gamma)_j j!} \end{vmatrix}.$$

This follows from the fact that $(C_1\Delta_{1,-}^{\alpha,\beta,\gamma})^{-1} = (\Delta_{1,-}^{\alpha,\beta,\gamma})^{-1}C_1^{-1}$. Note that $C_1^{-1}$ is as stated above and

$$(\Delta_{1,-}^{\alpha,\beta,\gamma})^{-1}_{nk} = \begin{cases} 1, & k = n, \\ (-1)^{n-k}\tilde{D}_{n-k}^k(\Delta_{1,-}^{\alpha,\beta,\gamma}), & 0 \le k \le n-1, \\ 0, & k > n, \end{cases}$$

where

$$\tilde{D}_n^k(\Delta_{1,-}^{\alpha,\beta,\gamma}) = \begin{vmatrix} \tilde{d}_{10} & \tilde{d}_{11} & \cdots & \tilde{d}_{1,n-1} \\ \tilde{d}_{20} & \tilde{d}_{21} & \cdots & \tilde{d}_{2,n-1} \\ \tilde{d}_{30} & \tilde{d}_{31} & \cdots & \tilde{d}_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n0} & d_{n1} & \cdots & \tilde{d}_{n,n-1} \end{vmatrix},$$

where $\tilde{d}_{10} = -\frac{\alpha\beta}{\gamma},\ \tilde{d}_{11} = 1,\ \tilde{d}_{1,n-1} = 0,\ \tilde{d}_{20} = \frac{\alpha(\alpha-1)\beta(\beta-1)}{2!\gamma(\gamma-1)},\ \tilde{d}_{21} = -\frac{\alpha\beta}{\gamma},\ \tilde{d}_{2,n-1} = 0,$

$$\begin{aligned} \tilde{d}_{30} =& \frac{-\alpha(\alpha-1)(\alpha-2)\beta(\beta-1)(\beta-2)}{3!\gamma(\gamma-1)(\gamma-2)}, \\ \tilde{d}_{31} =& \frac{\alpha(\alpha-1)\beta(\beta-1)}{2!\gamma(\gamma-1)}, \quad \tilde{d}_{3,n-1} = 0, \\ \tilde{d}_{n0} =& (-1)^n\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\beta(\beta-1)(\beta-2)\cdots(\beta-n+1)}{\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-n+1)n!}, \\ \tilde{d}_{n,1} =& (-1)^{n-1}\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+2)\beta(\beta-1)(\beta-2)\cdots(\beta-n+2)}{\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-n+2)(n-1)!}, \\ \tilde{d}_{n,n-1} =& \frac{-\alpha\beta}{\gamma}. \end{aligned}$$

Let $p = (p_k)$ be bounded sequence of positive real numbers and $M = \sup_k\{1, p_k\}$ and $0 < \inf p_k$. Then, the spaces $w_0(p),\ w(p)$ and $w_\infty(p)$ of all sequences that are strongly summable to zero, summable and bounded sequences, respectively defined in [21, 22] as follows:

$$\begin{aligned} w_0(p) =& \Big\{x = (x_k) \in w : \lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n}|x_k|^{p_k} = 0\Big\}, \\ w(p) =& \Big\{x = (x_k) \in w : \lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n}|x_k - l|^{p_k} = 0,\ \text{for some } l \in \mathbb{C}\Big\}, \end{aligned}$$

$$w_\infty(p) = \left\{ x = (x_k) \in w : \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^{p_k} \right) < +\infty \right\},$$

which are complete paranormed space with respect to the paranorm [12]

$$g(x) = \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^{p_k} \right)^{\frac{1}{M}}.$$

Motivated by the idea above, we define following new spaces by combining the generalized difference operator $\Delta_h^{\alpha,\beta,\gamma}$ with Cesàro operator of order one $C_1$ as

$$\begin{aligned} w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) =& \left\{ x = (x_k) \in w : \lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = 0 \right\}, \\ w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) =& \left\{ x = (x_k) \in w : \lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k - l|^{p_k} = 0, \right. \\ & \left. \text{for some } l \in \mathbb{C} \right\}, \\ w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) =& \left\{ x = (x_k) \in w : \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} \right) < +\infty \right\}, \end{aligned}$$

where $\Delta_{h,-}^{\alpha,\beta,\gamma} x_k = \sum_{i=0}^{+\infty} \frac{(-\alpha)_i (-\beta)_i}{(-\gamma)_i \; i! \; h^{\alpha+\beta-\gamma}} x_{k-i}$. In particular, these spaces includes the following spaces.

(a) For $\alpha = 0, \ \beta = \ \gamma, \ h = 1$, the above spaces are reduced to the basic spaces defined by [21].
(b) For $(p_k) = (p)$, where $1 \leq p < +\infty$, $\alpha = 0, \ \beta = \gamma, \ h = 1$, the above spaces are reduce to the basic spaces $w_\infty^p, \ w_0^p$ and $w^p$ defined by [12].

Now, we provide certain definitions which are used in the sequel.

- Let $g$ be a function from a linear space to $\mathbb{R}$, is said to be a paranorm if it satisfies the following axioms
  (a) $g(x) = 0$ if $x = 0$;
  (b) $g(-x) = g(x)$;
  (c) $g(x + y) \leq g(x) + g(y)$;
  (d) If $(\mu_n)$ be a sequence of scalars with $\mu_n \to \mu$ as $n \to +\infty$, and $(x_n)_{n\in\mathbb{N}}, \ x \in X$. Then, $g(\mu_n x_n - \mu x) \to 0$ as $n \to +\infty$.

  Moreover, $(X, g)$ is called the paranormed space (cf. [12])
- A sequence space $E$ is said to be solid, if $(\lambda_k x_k) \in E$, for all sequence of scalar $|\lambda_k| \leq 1$, for $(x_k) \in E$.
- A sequence space $E$ is said to be separable if there exists a countable dense subspace for this space.

## 2. Main Results

This section includes some results associated with convergence of generalized difference sequence $\Delta_{h,-}^{\alpha,\beta,\gamma}x$, and topological properties of newly defined spaces. Before proceeding further, we required the following definition (cf. [6]),

A sequence $x=(x_k)\in w$ is said to be of order $\nu>0$, i.e., $x=\mathcal{O}(k^\nu)$ if there exits a positive constant $\mathcal{C}$ such that

$$|x_k|\leq \mathcal{C}k^\nu,\quad k=0,1,2,\ldots$$

Motivated by the work in [6], we generalize the following theorem.

**Theorem 2.1.** *Let $\alpha,\beta,\gamma\in\mathbb{R}^+$ (the set of positive real numbers) and $x=(x_k)$, be a sequence of order $\nu$ such that $\alpha+\beta-\gamma>\nu$. Then, the difference sequence $(\Delta_{h,-}^{\alpha,\beta,\gamma}x_k)$ is absolutely convergent.*

*Proof.* Suppose $x=(x_k)$ is a sequence of order $\nu$. Then there exists a positive constant $\mathcal{M}$, such that,

$$|x_k|\leq \mathcal{M}k^\nu,\quad k\in\mathbb{N}_0.$$

Now, the series defined in equation (1.4) becomes

$$\begin{aligned}
|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k| =&\left|\sum_{i=0}^{+\infty}\frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}}x_{k-i}\right|\\
\leq&\sum_{i=0}^{+\infty}\left|\frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}}x_{k-i}\right|\\
\leq&\mathcal{M}\sum_{i=0}^{+\infty}\left|\frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}}\right|\cdot|k-i|^\nu\\
=&\mathcal{M}\sum_{i=0}^{+\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-(i-1))\beta(\beta-1)\cdots(\beta-(i-1))}{i!h^{\alpha+\beta-\gamma}\gamma(\gamma-1)\cdots(\gamma-(i-1))}\right|\cdot|k-i|^\nu\\
=&\frac{\mathcal{M}}{h^{a+b-c}}\sum_{i=0}^{+\infty}\left|\frac{(i-(\alpha+1))(i-(\alpha+2))\cdots(i-(\alpha+i))}{i!}\right|\\
&\times\left|\frac{(i-(\beta+1))(i-(\beta+2))\cdots(i-(\beta+i)))}{(i-(\gamma+1))(i-(\gamma+2))\cdots(i-(\gamma+i)))}\right|\cdot|k-i|^\nu\\
\leq&\frac{\mathcal{M}}{h^{\alpha+\beta-\gamma}}\sum_{i=0}^{+\infty}\left|\frac{\mathcal{O}(i^{i-(\alpha+1)})\mathcal{O}(i^{i-(\beta+1)})}{i!\mathcal{O}(i^{i-(\gamma+1)})}\right|\cdot|k-i|^\nu\\
=&\frac{\mathcal{M}}{h^{\alpha+\beta-\gamma}}\sum_{i=0}^{+\infty}\left|\frac{\mathcal{O}(i^{2i-(\alpha+\beta)-2})}{\mathcal{O}(i^{2i-(\gamma+1)})}\right|i^\nu\left|\frac{k}{i}-1\right|^\nu\\
=&\frac{M}{h^{\alpha+\beta-\gamma}}\sum_{i=0}^{+\infty}\left|\mathcal{O}(i^{\nu-(\alpha+\beta)+\gamma-1})\right|\cdot\left|\frac{k}{i}-1\right|^\nu.
\end{aligned}$$

Note that for $i\to+\infty$, $|\frac{k}{i}-1|^\alpha\to1$, and the right hand side converges if $\nu-(\alpha+\beta)+\gamma-1<-1$ which implies $\nu<\alpha+\beta-\gamma$.

For more clarity, we mention the plots of $\Delta_{h,-}^{\alpha,\beta,\gamma} x_k$, for the set of sequences $x_k = k^{1.05},\ k^{1.75},\ k^{1.25},\ k^{1.15},\ k^{1.95},\ k^2$, with $\alpha = 1.58,\ \beta = 3.11,\ \gamma = 3.12,\ h = 1$ as below. $\square$

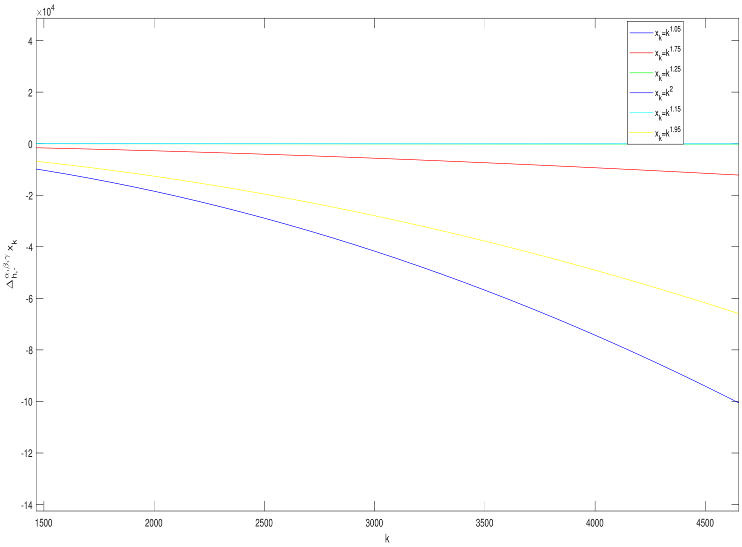


FIGURE 1. Convergence of $(\Delta_{h,-}^{\alpha,\beta,\gamma} x_k)$, using the condition as in Theorem 1

**Theorem 2.2.** *For every convergent sequence $x = (x_k)$, the difference sequence $(\Delta_{h,-}^{\alpha,\beta,\gamma} x_k)$ converges if $\alpha + \beta > \gamma$.*

*Proof.* Suppose $x = (x_k)$ is a convergent sequence. Then, for each $\epsilon > 0$, there exist $l \in \mathbb{C}$ and $K \in \mathbb{N}_0$ such that

$$|x_k - l| < \epsilon, \quad \text{for all } k \geq K.$$

So, the series,

$$\begin{aligned}
|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| =& \Big| \sum_{i=0}^{+\infty} \frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}} x_{k-i} \Big| \\
=& \Big| \sum_{i=0}^{+\infty} \frac{(-\alpha)_i(-\beta)_i}{i!(-c\gamma)_i h^{\alpha+\beta-\gamma}} x_{k-i} - l + l \Big| \\
\leq& \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i(-\beta)_i}{i!(-c)_i h^{\alpha+\beta-\gamma}} \right| \cdot |x_{k-i} - l| + |l| \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i(-\beta)_i}{i!(-\gamma)_i h^{\alpha+\beta-\gamma}} \right| \\
=& (|l| + \epsilon) \sum_{i=0}^{+\infty} \left| \frac{(-\alpha)_i(-\beta)_i}{i!(-c)_i h^{\alpha+\beta-\gamma}} \right|.
\end{aligned}$$

Note that the convergence of the difference sequence depends on the convergence of series in right hand side, above. For simplicity we take $h = 1$. Then, applying Raabe's test in the above series, we obtain

$$\begin{aligned}
\lim_{i\to+\infty}\Big|i\Big(\frac{a_i}{a_{i+1}}-1\Big)\Big| &= \lim_{i\to+\infty}\Big|\frac{(-\alpha)_i(-\beta)_i}{(-\gamma)_i\ i!}\cdot\frac{(-\gamma)_{i+1}\ (i+1)!}{(-\alpha)_{i+1}(-\beta)_{i+1}}\Big| \\
&= \lim_{i\to+\infty}\Big|\frac{-\alpha(-\alpha+1)\cdots(-\alpha+i-1)}{\alpha(-\alpha+1)(-\alpha+2)\cdots(-\alpha+i)}\Big| \\
&\quad\times\Big|\frac{-\beta(-\beta+1)\cdots(-\beta+i-1)(i+1)-\gamma(-\gamma+1)\cdots(-\gamma+i)}{-\beta(-\beta+1)\cdots(-\beta+i)-\gamma(-\gamma+1)\cdots(-\gamma+i-1)}\Big| \\
&= \lim_{i\to+\infty}\Big|i\Big(\frac{(i+1)(-\gamma+i)}{(-\alpha+i)(-\beta+i)}-1\Big)\Big| \\
&= \lim_{i\to+\infty}\Big|i\frac{-i\gamma+i^2-\gamma+i-\alpha\beta+i\alpha+i\beta-i^2}{i(-\frac{\alpha}{i}+1)(-\frac{\beta}{i}+1)}\Big| \\
&= \lim_{i\to+\infty}\left|\frac{(\alpha+\beta-\gamma+1)-\frac{\alpha\beta+\gamma}{i}}{(-\frac{\alpha}{i}+1)(-\frac{\beta}{i}+1)}\right| = \alpha+\beta-\gamma+1.
\end{aligned}$$

This implies that the difference sequence converges while $\alpha + \beta - \gamma + 1 > 1$, i.e., $\alpha + \beta > \gamma$, which concludes the proof. □

*Remark* 2.1. It is pertinent to note here that,

$$(\Delta_{1,-}^{\alpha_1,\beta_1,\gamma_1}(\Delta_{1,-}^{\alpha_2,\beta_2,\gamma_2}(x_k)) \neq (\Delta_{1,-}^{\alpha_1+\alpha_2+\beta_1+\beta_2,\gamma_1+\gamma_2}(x_k)),$$

which can be countered by following example.

*Example* 2.1. Suppose the sequence $x = (x_k)$ is defined by $x_k = 1,\ k \in \mathbb{N}_0$. Then it can be easily observed that

$$\Delta_{1,-}^{1,2,2}x_k = \Delta_{1,-}^{0+1,-1+3,5-3}x_k = \Delta(x_k) = 0, \quad \text{as } k \to +\infty.$$

But, for $\alpha_1 = 0,\ \alpha_2 = 1,\ \beta_1 = -1,\ \beta_2 = 3,\ \gamma_1 = -3,\ \gamma_2 = 5$ and $h = 1$ and we obtain,

$$\Delta_{1,-}^{1,3,5}x_k = \sum_{i=0}^{+\infty}\frac{(-1)_i(-3)_i}{(-5)_i\ i!},$$

which is a divergent sequence as $k \to +\infty$, from Rabee's test of a series. Consequently, the sequence $(\Delta_{1,-}^{0,-1,-3}(\Delta_{1,-}^{1,3,5}x_k))$ is divergent.

*Remark* 2.2. The spaces $X(\Delta_{h,-}^{\alpha,\beta,\ \gamma}, p),\ X \in \{w, w_0, w_\infty\}$, respectively, form linear spaces, provided $\Delta_{h,-}^{\alpha,\beta,\gamma}x_k$ exists for all $k \in \mathbb{N}_0$.

*Proof.* The proof is trivial, hence omitted. □

**Theorem 2.3.** *The spaces $X(\Delta_{h,-}^{\alpha,\beta,\ \gamma}, p),\ X \in \{w, w_0\}$ form complete paranormed spaces with respect to the paranorm*

$$g(x) = \sup_n\Big(\frac{1}{n+1}\sum_{k=0}^{n}\Big|\Delta_{h,-}^{\alpha,\ \beta,\ \gamma}x_k\Big|^{p_k}\Big)^{\frac{1}{M}},$$

*where* $M = \max\{1, H\}$ *and* $H = \sup_k p_k$*. However, the space* $w_\infty(\Delta^{\alpha,\beta,\gamma}_{h,-}, p)$ *is a paranormed space if and only if* $\inf p_k > 0$*.*

*Proof.* Clearly, if $x = \theta = (0, 0, \dots)$, then $g(x) = 0$. Now, we prove the theorem for $X = w$.

Again,

$$\begin{aligned} g(x) =& \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| (-1)\Delta^{\alpha,\ \beta,\ \gamma}_{h,-}(-x_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ =& \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta^{\alpha,\ \beta,\ \gamma}_{h,-} x_k \right|^{p_k} \right)^{\frac{1}{M}} = g(-x). \end{aligned}$$

Also,

$$\begin{aligned} g(x+y) =& \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta^{\alpha,\beta,\gamma}_{h,-}(x_k + y_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ \leq& \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta^{\alpha,\ \beta,\ \gamma}_{h,-} x_k \right| \right)^{\frac{1}{M}} + \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \Delta^{\alpha,\ \beta,\ \gamma}_{h,-} y_k \right| \right)^{\frac{1}{M}} \\ =& g(x) + g(y). \end{aligned}$$

Now, let $(q^n)$ be a sequence of scalar with $q^n \to q$ as $n \to +\infty$ and $x^n \in w(\Delta^{\alpha,\ \beta,\gamma}_{h,-}, p)$. Then,

$$\begin{aligned} g(q^n x^n - qx) =& \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta^{\alpha,\ \beta,\ \gamma}_{h,-}(q^n x^n_k - q x_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ \leq& \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta^{\alpha,\beta,\gamma}_{h,-}(q^n x^n_k - q^n x_k + q^n x_k - q x_k) \right|^{p_k} \right)^{\frac{1}{M}} \\ =& \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta^{\alpha,\ \beta,\ \gamma}_{h,-}(q^n(x^n_k - x_k) + x_k(q^n - q) \right|^{p_k} \right)^{M} \to 0, \end{aligned}$$

as $n \to +\infty$. As $(q^n)$ and $(x^n)$ converge to $q$ and $x$ respectively, the right hand side tends to zero.

For the case of $w_\infty(\Delta^{\alpha,\beta,\gamma}_{h,-},\ p)$ the proofs of condition 1, 2 and 3 of the paranorm are similar, hence we omit these. For last condition, we require the followings.

Let $\lambda = (\lambda_n)$ be the sequence of scalar with $\lambda_n \to 0$ and $x$ be fixed (implies $\lambda x \to \theta$ (null sequence)). We need to prove $\inf p_k > 0$ to form a paranormed space.

On contrary, suppose $\inf p_k < 0$ (Theorem 2 in [23]). Then, there exist $k_1, k_2, \dots, k_i$, $\dots$ such that

$$p_k < \frac{1}{i^2},$$

where $2^1 < k_1 < 2^2,\ 2^2 < k_2 < 2^3, \dots, 2^r < k < 2^{r+1}$. Again consider,

$$h(x) = \sup_r \left\{ \frac{1}{2^r} \sum_r \left| \Delta^{\alpha,\beta,\gamma}_{h,-} x_k \right|^{p_k} \right\}^{\frac{1}{M}}.$$

Let $(x_k)$ be the sequence such that, $\Delta_{h,-}^{\alpha,\beta,\gamma} x_k \to 1$ as $k \to +\infty$. Now, let us define the sequence $\bar{x} = (\bar{x}_k)$ by

$$\bar{x}_k = \begin{cases} \Delta_{h,-}^{\alpha,\beta,\gamma} x_k, & k = k_i, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly, $\bar{x} = (\bar{x}_k) \in w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$. Again,

$$h(\lambda \bar{x}) = \frac{1}{2^{r_i}} \sum_{r_i} |\Delta_{h,-}^{\alpha,\beta,\gamma} \lambda x_{k_i}|^{p_{k_i}} = |\lambda|^{p_{k_i}} \leq |\lambda|^{\frac{1}{i^2}} \to 1, \quad \text{as } i \to +\infty$$

Moreover,

$$\frac{g(x)}{2} < h(x) < 2g(x), \tag{2.1}$$

provided, the series for the operator $\Delta_{h,-}^{\alpha,\beta,\gamma}$ exists. This implies $g(\lambda \bar{x}) \geq \frac{1}{2}$, which is a contradiction. So, this proves the condition.

Secondly, our claim is now reduced to show the completeness property. So, let $(x^i)$ be any arbitrary Cauchy sequence in $w(\Delta_h^{\alpha,\beta,\gamma}, p)$.

By definition, for each $\epsilon > 0$, there exists $I \in \mathbb{N}_0$ such that

$$g(x^i - x^j) < \epsilon, \quad \text{for all } i, j \geq I$$

implies $\sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta_{h,-}^{\alpha,\beta,\gamma} (x_k^i - x_k^j) \right|^{p_k} \right)^{\frac{1}{M}} < \epsilon$, for all $i, j > I$. This implies that for each sufficiently large $k \in \mathbb{N}_0$ the sequence $(x_k^i)$, forming a Cauchy sequence in $\mathbb{C}$, which is complete and so convergent to $l^i$ (say), i.e., $|x_k^i - l_k^i|^{p_k} < \frac{\epsilon}{F}$, where let $\left| \Delta_{h,-}^{\alpha,\beta,\gamma} x_k \right| = F$.

Now,

$$g(x^i - l^i) = \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \Delta_{h,-}^{\alpha,\beta,\gamma} (x_k^i - l_k^i) \right|^{p_k} \right)^{\frac{1}{M}} \leq \frac{\epsilon}{F} \sum_{k=0}^{n} \left| \Delta_{h,-}^{\alpha,\beta,\gamma} \right| < \left( \frac{\epsilon \ F}{F} \right)^{\frac{1}{M}} < \epsilon,$$

for all $i > I$, for all $k$. Now $(x^i)$ is the Cauchy sequence with strong limit $l^i$ ultimately gives $l^i \in w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ ([24], page 320). □

**Theorem 2.4.** *The space $w(\Delta_{h,-}^{\alpha,\beta,\gamma},\ p)$ is not separable in general.*

*Proof.* The proof follows from [14, Theorem 3.7]. Let us take the sequence $x = (x_k)$, such that for each $\varphi > 0 \in \mathbb{R}$

$$x_k = k^\varphi + r, \quad \text{where } r \in \mathbb{R}.$$

Obviously, for each $p \in \mathbb{R}$ , $(x_k) \in w(\Delta_{1,-}^{\alpha,\beta,\beta},\ p)$, (for $\alpha > \varphi,\ \beta = \gamma$).

Now consider $p = (p_k) = (e)$ and the set, $A = \{k^\varphi + r, k^\varphi + s, k^\varphi + t, \dots\} = \{x^r, x^s, x^t, \dots\}$, where $|r - s| > \frac{1}{3}$.

Now, $g(x^r - x^s) = \sup_n \frac{1}{n+1} \sum_{k=0}^{n} |\Delta_{1,-}^{\alpha,\beta,\beta} (x_k^r - x_k^s)| = |r - s| > \frac{1}{3}$.

Clearly, $A$ is uncountable. Again, consider $D$ be any arbitrary dense subset of $w(\Delta_{1,-}^{\alpha,\beta,\beta},\ p)$.

Moreover, for each $x^r = (x_k^r)_{k\in\mathbb{N}_0} \in w(\Delta_{1,-}^{\alpha,\beta,\beta},\ p)$, we can find $y^r = (y_k^r)_{k\in\mathbb{N}_0} \in D$ such that $g(x^r - y^r) < \frac{1}{6}$.

Now, let us define the transformation $f$ as $f : A \to D$, $x^r = (x_k^r) \mapsto f(x^r) = (y_k^r)$. Again $g(x^r - y^s) = g(x^r - x^s + x^s - y^s) \geq g(x^r - y^s) - g(y^s - x^s) > \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$.

This inequality represents for $x^r \neq x^s$ implies that $y^r \neq y^s$ with respect to the paranorm. This shows that $f$ is one-to-one. Furthermore, $f(A) \subset D$ and $f(A)$ is hence uncountable. Since, $D$ is any arbitrary dense set so, the space $w(\Delta_{1,-}^{\alpha,\beta,\beta},\ p)$ is not dense in general. □

**Theorem 2.5.** *The space $w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma},\ p)$ forms a solid space. But, $w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ does not form solid in general.*

*Proof.* Suppose $(\lambda_k)$ is any sequence of scalar such that $|\lambda_k| \leq 1$, for all $k \in \mathbb{N}_0$.

Now, the sequence $(\lambda_k x_k)$, for $(x_k) \in w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$. So,

$$\begin{aligned}\sup_n \frac{1}{n+1}\sum_{k=0}^{n} |\Delta_{h,-}^{\alpha,\beta,\gamma}\lambda_k x_k|^{p_k} &= \sup_n \frac{1}{n+1}\sum_{k=0}^{n} |\lambda_k|^{p_k}|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k|^{p_k}\\ &\leq \sup_n \frac{1}{n+1}\sum_{k=0}^{n} |\Delta_{h,-}^{\alpha,\beta,\gamma}x_k|^{p_k} < +\infty.\end{aligned}$$

This implies that $(\lambda_k x_k) \in w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ and makes the space $w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ as a solid space. But the space $w(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ does not form solid in general, which can be proved by following counter example.

Let $(p_k) = (1, 1, 1, \dots)$, $\beta = \gamma$, $h = 1$ and the sequence of scalar $\lambda = (\lambda_k)$ be

$$\lambda_k = \begin{cases} \frac{1}{2}, & k \text{ is even},\\ 0, & \text{otherwise}.\end{cases}$$

Clearly, $|\lambda_k| < 1$, for all $k \in \mathbb{N}_0$. Suppose that, $x_k = c$ is any constant sequence, which gives us $x \in w(\Delta_{1,-}^{\alpha,\beta,\beta},\ p)$. But,

$$y_k = \lambda_k x_k = \begin{cases} \frac{c}{2}, & k \text{ is even},\\ 0, & k \text{ is odd},\end{cases} \quad \text{and} \quad \Delta_{1,-}^{\alpha,\beta,\beta} y_k = \begin{cases} c\ 2^{\alpha-2}, & k \text{ is even},\\ -c\ 2^{\alpha-2}, & k \text{ is odd},\end{cases}$$

i.e., we can not get an unique $l \in \mathbb{C}$ such that $(y_k) \notin w(\Delta_{1,-}^{\alpha,\beta,\beta},\ p)$. □

**Theorem 2.6.** *The space $w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$ does not form a sequence algebra, in general.*

*Proof.* Let $x = (x_k)$ and $y = (y_k)$ be two sequences such that, $x_k = k$ and $y_k = k^2$, for all $k \in \mathbb{N}_0$ with $(p_k) = e$. Clearly, $x = (x_k), y = (y_k) \in w_\infty(\Delta_{1,-}^{2,\beta,\beta}, p)$. But, for $z_k = x_k y_k = k^3$, for all $k \in \mathbb{N}_0$ and $\Delta_{1,-}^{2,\beta,\beta} z_k = 6k - 2$ .

Again, $\sup_n \left(\frac{1}{n+1}\sum_{k=0}^{n}(6k-2)\right) = +\infty$, which consequently, gives us $z = (z_k) \notin w_\infty(\Delta_{1,-}^{2,\beta,\beta}, p)$ and completes the proof. □

**Theorem 2.7.** (a) *If* $\alpha+\beta>\gamma$*, then* $c\subset w(\Delta_{h,-}^{\alpha,\beta,\gamma},p)$*.*

(b) *If* $\Delta_{h,-}^{\alpha,\beta,\gamma}(e)$ *exists, then* $\ell_\infty\subset w_\infty(\Delta_h^{\alpha,\beta,\gamma},p)$*.*

(c) $w_0(\Delta_{h,-}^{\alpha,\beta,\gamma},p)\subset w(\Delta_{h,-}^{\alpha,\beta,\gamma},p)\subset w_\infty(\Delta_{h,-}^{\alpha,\beta,\gamma},p)$*.*

(d) $c(\Delta_{h,-}^{\alpha,\beta,\gamma})\subset w(\Delta_{h,-}^{\alpha,\beta,\gamma},p)$*, for* $p_k<M$*, for all* $k\in\mathbb{N}_0$*, where* $c(\Delta_{h,-}^{\alpha,\beta,\gamma})=\{x\in w:(\Delta_{1,-}^{\alpha,\beta,\gamma}x_k)\in c\}$*.*

*Proof.* (a) Let $x\in c$ be any convergent sequence, with $\alpha+\beta>\gamma$. This implies, $(\Delta_{h,-}^{\alpha,\beta,\gamma}x_k)$ is convergent. Ultimately, we obtain

$$\lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k-l|^{p_k}=0,\quad\text{for some } l\in\mathbb{C},$$

i.e.,

$$x\in w(\Delta_{h,-}^{\alpha,\beta,\gamma},p)\implies c\subset w(\Delta_{h,-}^{\alpha,\beta,\gamma},p).$$

Furthermore, this inclusion is strict by the following example.

Let $x_k=k^a$, for all $k\in\mathbb{N}_0$, $a>1$, being $a$ fixed real number. If $\alpha+\beta-\gamma>a$, then $x=(x_k)\in w(\Delta_{h,-}^{\alpha,\beta,\gamma},p)$. But, $(x_k)$ is a divergent sequence. The proofs of (b) and (c) are similar as above.

(d) Let $x\in c(\Delta_{h,-}^{\alpha,\beta,\gamma})$, which implies $|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k-l|<\epsilon^{\frac{1}{M}}<\epsilon$, for some $l\in\mathbb{C}$. Clearly,

$$\lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k-l|^{p_k}<(\epsilon^{\frac{1}{M}})^M=\epsilon,$$

i.e., $x\in w(\Delta_h^{\alpha,\beta,\gamma},p)$.

This inclusion being strict by taking $(x_k)=(1,3,1,3,1,\dots)$ (see [14, Theorem 3.3]), such that $w(\Delta_{1,-}^{1,\beta,\beta},e)\not\subset c(\Delta_{1,-}^{1,\beta,\beta})$.

This completes the proof. □

We obtain our next results by following to Maddox [22]. Hence consider the set $w_0(\Delta_h^{\alpha,\beta,\gamma},p)$ denoting the set such that,

$$2^{-r}\Big|\sum_r\Delta_{h,-}^{\alpha,\beta,\gamma}x_k\Big|^{p_k}\to 0,\tag{2.2}$$

where $\sum_r$ is the sum over the $k$ for $k\in[2^r,2^{r+1})$ and $r$ being any integer.

**Theorem 2.8.** ([22, Theorem 7]) $w_0(\Delta_{h,-}^{\alpha,\beta,\gamma},p)\subset w_0(\Delta_{h,-}^{\alpha,\beta,\gamma},e)$ *if and only if*

(a) *there exists an integer* $N>1$ *such that*

$$B_r=\max_r M^{-\frac{1}{p_k}}2^{-r+\frac{r}{p_k}}=\mathcal{O}(1),$$

(b) $\inf_{s>1}\lim_{r\to\infty}\sup 2^{-r}M_r(s)=0$*, where* $M_r(s)$ *is the number of* $k$ *in* $[2^r,2^{r+1})$*, such that* $p_k\geq s$*.*

*Proof.* Let $x \in w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p)$, i.e., there exists $N \in \mathbb{N}_0$ for sufficiently large $r$, such that,

$$\frac{1}{2^r}\sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} < \frac{1}{M}.$$

Now, we may subdivide the sum as

$$2^{-r}\sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = \; 2^{-r}\sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} + 2^{-r}\sum_2 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k},$$

representing the sum $\sum_1$ and $\sum_2$ over $p_k < 1$ and $p_k \geq 1$. Let us consider $p_k < 1$ and $q_k = \frac{1}{p_k}$. We get the following inequalities

$$M^q|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k}\; 2^{-rq} \leq \; M|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} 2^{-r} \leq M\; 2^{-r}\sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k}.$$

Again,

$$2^{-r}\sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \leq \; 2^{-r}M^{-q_k}\sum_1 2^{rq_k}\; M^{q_k}|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|.$$

This implies,

$$\begin{aligned}
|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \leq& M^{1-q_k}|x_k|^{p_k}\; 2^{-r+rq},\\
2^{-r}\sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \leq& 2^{-2r}\; M\; \sum_1 |\Delta_{h,-}^{\alpha,\beta,\gamma}\; x_k|^{p_k}\; 2^{rq_k}\; M^{-q_k}\\
\leq& 2^{-r}M\; B_r\sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = \mathcal{O}(1)\cdot o(1) = o(1),
\end{aligned}$$

where the notation $\mathcal{O}$ is defined earlier and $o(f_2(y)) = f_1(y)$, for every positive constant $M$ and there exists a constant $y_0$, such that $0 \leq f_1(y) < Mf_2(y)$, for all $y \geq y_0$.

Our claim is now reduced to prove that, $2^{-r}\sum_2 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| = o(1)$.

Let $\epsilon > 0$ be arbitrary. Then, there exists $s > 1$ such that $2^{-r}N_r(s) < \epsilon$, for all sufficiently large $r$. We may write the sum as

$$\sum_2 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| = \; \sum_3 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} + \sum_4 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|,$$

over the sum $|\Delta_{h,-}^{\alpha,\beta,\gamma}| \leq 1$ and $|\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| > 1$, respectively. Again, $p_k \geq 1$, resulting

$$2^{-r}\sum_4 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| = 2^{-r}\sum_4 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} = o(1).$$

Let us subdivide the sum in two parts, i.e., $1 \leq p_k \leq s$ and $p_k > 1$, respectively, as $\sum_5$, $\sum_6$.

Moreover, we found

$$\sum_6 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| \leq \sum_6 1 = M_r(s).$$

This implies $2^{-r}\sum_6 |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k| < \epsilon$, for sufficiently large $r$. Now, only we want to show it for $\sum_5$.

Letting $r$ so large that

$$2^{-r}\sum_r |\Delta_{h,-}^{\alpha,\beta,\gamma} x_k|^{p_k} < \min\{\epsilon, \epsilon^s\}.$$

From [22] we get that for sufficiently large $r$ that,

$$2^{-r}\sum_{r}|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k| \leq 2^{-r}\sum_{r}|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k|^{p_k} + \left(2^{-r}\sum_{6}|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k|^{p_k}\right)^{\frac{1}{s}} < 2\epsilon.$$

So, all the inequalities obtained above ultimately resulting our claim that

$$2^{-r}\sum_{r}|\Delta_{h,-}^{\alpha,\beta,\gamma}x_k| \to 0, \quad \text{as } r \to +\infty,$$

i.e.,

$$x \in w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, e) \implies w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, p) \subset w_0(\Delta_{h,-}^{\alpha,\beta,\gamma}, e).$$

This completes the proof. □

## 3. Dual Spaces

Suppose $X$ and $Y$ are two non-empty sequence spaces, then the set

$$S(X,Y) = \Big\{z \in w \ : \ xz = (x_k z_k) \in Y, \text{ for every } x \in X\Big\},$$

is called the multiplier space of $X$ and $Y$. The special multiplier space that are $S(X,\ l_1)$, $S(X, cs)$ and $S(X, bs)$, respectively, called as $\alpha$-, $\beta$- and $\gamma$-duals of X (cf. [12]). In order to remove ambiguity, let us take $X^{\alpha^*} = S(X, l_1)$, $X^{\beta^*} = S(X, cs)$, $X^{\gamma^*} = S(X, bs)$.

**Theorem 3.1.** (a) $\{w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha^*} = D_1$, *where*

$$D_1 = \bigcap_{M>1}\Big\{a = (a_n) \in w : \sup_k \sum_n \Big|\sum_k (-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}_{n-k}^{k}(C_1\Delta_{1,-}^{\alpha,\beta,\gamma})a_n M^{\frac{1}{p_k}}\Big| < +\infty\Big\}.$$

(b) $\{w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha^*} = D_2$, *where*

$$D_2 = \bigcup_{M>1}\Big\{a = (a_n) \in w : \sup_k \sum_n \Big|\sum_k (-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}_{n-k}^{k}(C_1\Delta_{1,-}^{\alpha,\beta,\gamma})a_n M^{-\frac{1}{p_k}}\Big| < +\infty\Big\}.$$

(c) $\{w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p)\}^{\alpha^*} = D_3$, *where*

$$D_3 = D_2 \cap \bigcup_{M>1}\Big\{a = (a_n) \in w : \sup_k \sum_n \Big|\sum_k (-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}_{n-k}^{k}(C_1\Delta_{1,-}^{\alpha,\beta,\gamma})a_n\Big| < +\infty\Big\}.$$

*Proof.* The proof follows from Theorem 8 of [8]. Suppose $a \in D_1$ and $x \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\beta}, p)$. Now,

$$\begin{aligned}\sum_n |a_n x_n| &= \sum_n |a_n| \cdot |x_n| \\ &= \sum_n |a_n| M^{\frac{1}{p_k}} M^{-\frac{1}{p_k}} \sum_k \Big|(-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}_{n-k}^{k}(C_1\Delta_{1,-}^{\alpha,\beta,\gamma})\Big| \\ &\quad \times \Big|(-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}_{n-k}^{k}(C_1\Delta_{1,-}^{\alpha,\beta,\gamma})\Big|^{-1}|x_n|\end{aligned}$$

$$\leq \sup_n \left|\frac{1}{k+1}\sum_{j=0}^{k-n}\frac{(-\alpha)_j}{j!}\right| \cdot \left|M^{-\frac{1}{p_k}}|x_n| \sum_n \left|\sum_k \right|(-1)^{n-k}\right.$$
$$\times \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})a_n \left|M^{\frac{1}{p_k}}\right| < +\infty, \quad \text{for all } k \in \mathbb{N}_0.$$

So,

$$a \in \{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-}, p)\}^{\alpha^*} \implies D_1 \subset \{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-}, p)\}^{\alpha^*}.$$

Conversely, suppose $a \notin D_1$, Then, exists $M > 1$ such that,

$$\sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})a_n M^{\frac{1}{p_k}} \right| = +\infty.$$

Thus, we can find strictly increasing sequence of integer $n(s)$ such that,

$$\begin{aligned} n(1) =& 1 < n(2) < n(3) < \cdots < n(s) < n(s+1) \\ &< \cdots < \sum_{n=n(s)}^{n(s+1)} \sum_n \left| \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})M^{-\frac{1}{p_k}} \right| > 1. \end{aligned}$$

Now, let us define a sequence $x = (x_n)$ by

$$x_n = \begin{cases} 0, & \text{if } 1 < n < n(s), \\ M^{-\frac{1}{p_k}} \left| \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) \operatorname{sgn}(a_n) \right|, & n(s) < n < n(s+1). \end{cases}$$

Clearly, $x = (x_n) \in w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)$.

Moreover,

$$\begin{aligned} \sum_{n=0}^{+\infty} |a_n x_n| &= \sum_{n=n(1)}^{n(2)} |a_n y_n| + \sum_{n=n(2)}^{n(3)} |a_n y_n| + \cdots \\ &= \sum_{n=n(1)}^{n(2)} |a_n| \cdot \left| M^{-\frac{1}{p_k}} \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) \right| \\ &\quad + \sum_{n=n(2)}^{n(3)} |a_n| \cdot \left| M^{-\frac{1}{p_k}} \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) \right| + \cdots \\ &\geq \sum_{n=1}^{+\infty} 1 > +\infty. \end{aligned}$$

i.e., $a = (a_n) \notin \{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\alpha^*}$. By contra positively, we get $w_\infty\{(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\alpha^*} \subset D_1$, which consequently results $\{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\alpha^*} = D_1$.

So, it completes the proof. Similarly, we may prove the other two results. ☐

**Theorem 3.2.** (a) $\{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*} = D_4$, *where*

$$D_4 = \bigcap_{M>1} \left\{ a = (a_n) \in w : \lim_{n\to+\infty} \sum_k \left| (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})a_n M^{\frac{1}{p_k}} \right| \textit{ converges} \right\}.$$

(b) $\{w_0(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*} = D_5$, *where*

$$D_5 = \bigcup_{M>1} \Big\{ a=(a_n)\in w : \sup_n \sum_k \Big| (-1)^{n-k} \prod_{l=k}^{n} (l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) a_n M^{-\frac{1}{p_k}} \Big| < \infty \Big\}.$$

(c) $\{w(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*} = D_6 \bigcap D_7$, *where*

$$D_6 = \bigcup_{M>1} \Big\{ a=(a_n)\in w : \sup_n \sum_k \Big| (-1)^{n-k} \prod_{l=k}^{n} (l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) a_n M^{-\frac{1}{p_k}} - b_k \Big| = 0, \text{ for all } b_k \in \mathbb{C} \Big\}$$

*and*

$$D_7 = \Big\{ a=(a_n)\in w : \lim_{n\to+\infty} \sum_k \Big| (-1)^{n-k} \prod_{l=k}^{n} (l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) a_n - b_k \Big| = 0, \text{ for all } b_k \in \mathbb{C} \Big\}.$$

*Proof.* On contrary, let $a=(a_n)$ be a sequence such that $a \notin \{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*}$. Then, there exists $x=(x_n)\in w_\infty(\Delta^{\alpha,\beta,\beta}_{1,-},\ p)$ such that $\sum_n a_n x_n$ does not converge. Consequently, this leads that $\sum_n |a_n x_n|$ diverges to $\infty$. Furthermore,

$$\begin{aligned}\sum_n |a_n x_n| =& \sum_n \Big| a_n M^{\frac{1}{p_k}} M^{-\frac{1}{p_k}} \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) \frac{1}{k+1}\sum_{j=0}^{k-n}\frac{(-\alpha)_j}{j!} x_n \Big| \\ \le& \sup_n \sum_n \Big| \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) a_n M^{\frac{1}{p_k}} \\ &\times \Big| \frac{1}{k+1}\sum_{j=0}^{k-n}\frac{(-\alpha)_j}{j!} M^{-\frac{1}{p_k}} x_n \Big| \to \infty, \quad \text{for larger n.}\end{aligned}$$

Since, $x\in w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)$, implies

$$\sum_k \Big| (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) a_n M^{\frac{1}{p_k}} \Big| \to +\infty,$$

which is a contradiction to the fact that $a\in D_4$. So, we get $a \notin \{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*}$ and $D_4 \subset \{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*}$.

Conversely, suppose that $a\in \{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*}$. By definition $\sum_n a_n x_n$ converges, which necessarily gives $\lim_{n\to+\infty} a_n x_n = 0$. So, for each $\epsilon>0$, there exists $N\in\mathbb{N}_0$ such that $|a_n x_n| < \epsilon$, for all $n>N$. Again,

$$|a_n x_n| = \Big| a_n \sum_k (-1)^{n-k} \prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-}) M^{\frac{1}{p_k}} M^{-\frac{1}{p_k}} \frac{1}{k+1}\sum_{j=0}^{k-n}\frac{(-\alpha)_{k-n}}{(k-n)!} x_n \Big|$$

$$\leq \sup_{n,k>N}\left|\frac{1}{k+1}\sum_{j=0}^{k-n}\frac{(-\alpha)_j}{j!}M^{-\frac{1}{p_k}}x_n\right|\cdot\left|\sum_k(-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})\right.$$
$$\left.\times M^{\frac{1}{p_k}}a_n\right|<\epsilon,$$

for some $M>1$. For $x\in w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)$, it brings

$$\sum_k\left|(-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})M^{\frac{1}{p_k}}a_n\right|<\epsilon,$$

i.e., $a\in D_4$, which implies $\{w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},\ p)\}^{\beta^*}=D_4$.

Otherwise, for any larger $k\in\mathbb{N}$ by Theorem 5.1 in [18], there exists $b=(b_k)$ be any sequence of scalar in $\mathbb{C}$, such that

$$\lim_{n\to+\infty}\sum_k\left|(-1)^{n-k}\prod_{l=k}^{n}(l+1)\bar{D}^k_{n-k}(C_1\Delta^{\alpha,\beta,\gamma}_{1,-})M^{\frac{1}{p_k}}a_n-b_k\right|=0,$$

i.e., converges. So, it completes the proof. Similarly, we may obtain the other two results. □

## 4. Matrix Transformations

In this section, we characterize some matrix transformations among newly constructed sequence spaces as defined earlier. We characterize matrix transformations among spaces $w_\infty(\Delta^{\alpha,\beta,\gamma}_{1,-},p)$, $w_0(\Delta^{\alpha,\beta,\gamma}_{1,-},p)$ and $w(\Delta^{\alpha,\beta,\gamma}_{1,-},p)$ with classical sequence spaces $c(q)$, $c_0(q)$ and $\ell_\infty(q)$.

In brief, for $A$, be an infinite matrix from $X$ to $Y$, i.e., $A:X\to Y$, where for all $x=(x_n)\in X$, implies that

$$\Big\{(Ax)_n\Big\}=\Big\{\sum_{k\in\mathbb{N}_0}a_{nk}x_k\Big\}_{n\in\mathbb{N}_0}\in Y.$$

Suppose that $(X,Y)$ denote the set of all matrix transformations from $X$ to $Y$. Before proceeding further. We need the following propositions from [18], for $(q_k)_{k\in\mathbb{N}_0}$, non decreasing bounded sequence of positive real numbers. For more convenience, we can replace the $\sum_{k=0}^{\infty}$ by $\sum_k$.

(a) $A\in(\ell_\infty(p),\ \ell_\infty(q))$ if and only if for all $M$, such that

$$\sup_n\Big(\sum_k\Big|a_{nk}\Big|M^{\frac{1}{p_k}}\Big)^{q_n}<+\infty. \tag{4.1}$$

(b) $A\in(\ell_\infty(p),\ c(q))$ if and only if exists $\eta_k$, such that

$$\sup_n\sum_k|a_{nk}|M^{\frac{1}{p_k}}<+\infty,\quad \text{for all integers } M>1, \tag{4.2}$$

and

$$\lim_n\Big(\sum_k\Big|a_{nk}-\eta_k\Big|M^{-\frac{1}{p_k}}\Big)^{q_n}=0,\quad \text{for all } M>1. \tag{4.3}$$

(c) $A \in (c_0(p), \ell_\infty(q))$ if and only if exists $M$, such that

$$\sup_n \left( \sum_k \left| a_{nk} \right| M^{-\frac{1}{p_k}} \right)^{q_n} < +\infty. \tag{4.4}$$

(d) $A \in (c_0(p), c(q))$ if and only if (4.4) holds for all $L$ exist $M$, and $\eta_k \in \mathbb{R}$ such that

$$\lim_M \sup_n \sum_k |a_{nk}| M^{-\frac{1}{p_k}} < +\infty, \tag{4.5}$$

$$\sup_n L^{\frac{1}{q_n}} \sum_k \left| a_{nk} - \eta_k \right| M^{-\frac{1}{p_k}} < +\infty, \tag{4.6}$$

$$\lim_{n \to +\infty} |a_{nk} - \eta_k|^{q_n} = 0. \tag{4.7}$$

(e) $A \in (c_0(p), c_0(q))$ if and only if for all $L$, exists $M$, such that

$$\sup_n L^{\frac{1}{q_n}} \sum_k \left| a_{nk} \right| M^{-\frac{1}{p_k}} < +\infty. \tag{4.8}$$

(f) $A \in (c(p), \ell_\infty(q))$ if and only if (4.2) holds

$$\sup_n \left| \sum_k a_{nk} \right|^{q_n} < +\infty. \tag{4.9}$$

(g) $A \in (\ell_\infty(p),\ c_0(q))$ if and only if for all $M$,

$$\lim_n \left( \sum_k \left| a_{nk} - \eta_k \right| M^{-\frac{1}{p_k}} \right)^{q_n} = 0 < +\infty. \tag{4.10}$$

(h) $A \in (c(p), c(q))$ if and only if (4.2), (4.5), (4.6) hold, exists $\alpha \in \mathbb{R}$, such that

$$\lim_n \left| \sum_k a_{nk} - \eta \right|^{q_n} = 0. \tag{4.11}$$

(i) $A \in (c_0(p), \ell(q))$ if and only if exists $M \in \mathbb{N}$

$$\sup_k \sum_n \left| \sum_{k \in K} a_{nk} M^{-\frac{1}{p_k}} \right|^{q_n} < +\infty. \tag{4.12}$$

(j) $A \in (c(p), \ell(q))$ if and only if (4.11) holds and

$$\sum_n \left| \sum_k a_{nk} \right|^{q_n} < +\infty. \tag{4.13}$$

Consider the infinite matrix $\hat{A} = (\hat{a}_{nk})$ via the matrix $A = (a_{nk})$ as

$$\hat{a}_{nk} = \sum_{j=k}^{+\infty} (-1)^{k-i} \prod_{l=i}^{k} (l+1) \bar{D}^i_{k-i}(C_1 \Delta^{\alpha,\beta,\gamma}_{1,-}) a_{nj},$$

where $\bar{D}^i_{k-i}(C_1 \Delta^{\alpha,\beta,\gamma}_{1,-})$ is defined as in equation (1.5) and $\hat{A}$ is called the associated matrix of A.

**Theorem 4.1.** (a) $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$ *if and only if* (4.1) *holds, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$.

(b) $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c(q))$ *if and only if* (4.2) *and* (4.3) *hold, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$.

(c) $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c_0(q))$ *if and only if* (4.9) *holds and* (4.2), (4.3) *hold, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$, *with* $\eta_k = 0$.

**Theorem 4.2.** (a) $A \in (w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$ *if and only if* (4.4), (4.8) *hold, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$.

(b) $A \in (w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c(q))$ *if and only if* (4.5), (4.6) *and* (4.11) *hold, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$.

(c) $A \in (w(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c_0(q))$ *if and only if* (4.6), (4.7) *hold, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$ *with* $\eta_k = 0$.

**Theorem 4.3.** (a) $A \in (w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$ *if and only if* (4.4) *holds, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$.

(b) $A \in (w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c(q))$ *if and only if* (4.5), (4.6), (4.7) *hold, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$.

(c) $A \in (w_0(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), c_0(q))$ *if and only if* (4.1), (4.3) *holds and* (4.6), (4.7) *hold, with* $\hat{a}_{nk}$ *instead of* $a_{nk}$ *with* $\eta_k = 0$.

*Proof.* Suppose that $A \in (w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma}, p), \ell_\infty(q))$. Then, by the definition $\sum_{k=0}^{+\infty} a_{nk} y_k \in \ell_\infty(q)$, for every $y = (y_k) \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma},\ p)$. Again, $y = (y_k) \in w_\infty(\Delta_{1,-}^{\alpha,\beta,\gamma},\ p)$, if and only if $\bar{y} = C_1 \Delta_{1,-}^{\alpha,\beta,\gamma}(y) \in \ell_\infty(p)$. From Lemma 4.1 of [28], we get $\sum_{k=0}^{+\infty} \hat{a}_{nk} \bar{y}_k \in \ell_\infty(q)$, and the matrix $\hat{A} \in (\ell_\infty(p), \ell_\infty(q))$. Using the equation (4.1), $\hat{A} \in (\ell_\infty(p), \ell_\infty(q))$ if and only if $\sup_n \left( \sum_k \left| \hat{a}_{nk} \right| M^{\frac{1}{p_k}} \right)^{q_n} < +\infty$. This completes the proof of the first bit. We can prove the remaining parts of the theorem and the next theorems using similar argument. ☐

**Acknowledgements.** Authors express their sincere thanks to the anonymous referees for their valuable suggestions.

## References


[1] A. M. Akhmedov and F. Başar, *On the fine spectra of the difference operator* $\Delta$ *over the sequence space* $\ell_p$ $(1 \leq p < \infty)$, Demonstr. Math. **39**(3) (2006), 585–596. `https://doi.org/10.1515/dema-2006-0313`

[2] B. Altay and F. Başar, *The matrix domain and the fine spectrum of the difference operator* $\Delta$ *on the sequence space* $\ell_p$ $(0 < p < 1)$, Commun. Math. Anal. **2**(2) (2007), 1–11.

[3] S. Aydin and H. Polat, *Difference sequence spaces derived by using pascal transformation*, Fund. J. Math. Appl. **2**(1) (2019), 56–62. `https://doi.org/10.33401/fujma.541721`

[4] P. Baliarsingh, *Some new difference sequence spaces of fractional order and their dual spaces*, Appl. Math. Comput. **219**(18) (2013), 9737–9742. `https://doi.org/10.1016/j.amc.2013.03.073`

[5] P. Baliarsingh, *On a fractional difference operator*, Alex. Eng. J. **55**(2) (2016), 1811–1816. `https://doi.org/10.1016/j.aej.2016.03.037`

[6] P. Baliarsingh, *On certain dynamic properties of difference sequences and the fractioanl derivatives,* Math. Methods Appl. Sci. **44**(4) (2021), 3023–3035.`https://doi.org/10.1002/mma.6417`

[7] P. Baliarsingh, *On the spectrum of fractional difference operator,* Linear Multilinear Algebra **70**(21) (2022), 6568–6580. `https://doi.org/10.1080/03081087.2021.1960261`

[8] P. Baliarsingh and S. Dutta, *On the classes of fractional order difference sequence spaces and their matrix transformations,* Appl. Math. Comput. **250** (2015), 250–674. `https://doi.org/10.1016/j.amc.2014.10.121`

[9] P. Baliarsingh and S. Dutta, *On an explicit formula for inverse of triangular matrices,* J. Egypt. Math. Soc. **23** (2015), 297–302. `https://doi.org/10.1016/j.joems.2014.06.001`

[10] P. Baliarsingh and S. Dutta, *A unifying approach to the difference operators and their applications,* Bol. Soc. Paran. Math. **33**(1)(2015), 49–57. `https://doi.org/10.5269/bspm.v33i1.19884`

[11] P. Baliarsingh, U. Kadak and M. Mursaleen, *On statistical convergence of difference sequences of fractional order and related Korovokin type approximation theorems*, Quaest. Math. **41**(8) (2018), 1117–1133. `https://doi.org/10.2989/16073606.2017.1420705`

[12] F. Başar, *Summability Theory and Applications*, Chapman and Hall/CRC, 2022.

[13] F. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings,* (English, Ukrainian summary) Ukrain. Mat. Zh. **55**(1) (2003), 108-118. `https://doi.org/10.1023/A:1025080820961`

[14] V. K. Bhardwaj and S. Gupta, *Cesáro summable difference sequence space*, J. Inequal. Appl. **2013**(1) (2013), 1–9. `https://doi.org/10.1186/1029-242X-2013-315`

[15] S. Dutta and P. Baliarsingh, *On the fine spectra of generalized $r^{th}$ order difference operator $\Delta^r$ on the sequence space $\ell_1$,* Appl. Math. Comput. **219**(4) (2012), 1776–1784. `https://doi.org/10.1016/j.amc.2012.08.016`

[16] M. Et, *On new difference sequence spaces via Ceşaro mean,* ITM Web. Conf. **13** (2017).

[17] M. Et and R. Çolak, *On some generalized difference sequence spaces,* Soochow J. Math. **21**(4) (1995), 377–386.

[18] K. G. Grosseerdmann, *Matrix transformations between the sequence spaces of Maddox*, J. Math. Anal. Appl. **180**(1) (1993), 223–238. `https://doi.org/10.1006/jmaa.1993.1398`

[19] U. Kadak, *Generalized statistical convergence based on fractional order difference operator and its applications to approximation theorems,* Iran. J. Sci. Technol. Trans. A Sci. **43** (2019), 225–237. `https://doi.org/10.1007/s40995-017-0400-0`

[20] H. Kızmaz, *On certain sequence spaces,* Canad. Math. Bull. **24**(2) (1981), 169–176. `https://doi.org/10.4153/CMB-1981-027-5`

[21] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. **18**(1) (1967), 345–355. `https://doi.org/10.1093/qmath/18.1.345`

[22] I. J. Maddox,*On Kuttner's theorem,* J. Lond. Math. Soc. **1**(1) (1968), 285–290. `https://doi.org/10.1112/jlms/s1-43.1.285`

[23] I. J. Maddox, *Paranormed sequence spaces generated by infinite matrices,* Proc. Camb. Phil. Soc. **64** (1968), 335–340. `https://doi.org/10.1017/S0305004100042894`

[24] I. J. Maddox, *Some properties of paranormed sequence spaces,* J. Lond. Math. Soc. **1**(2) (1969), 316–322.

[25] A. Maji, A. Manna and P. Srivastava, *Some $m^{th}$ order difference sequence spaces of generalized means and compact operators*, Ann. Funct. Anal. **6**(1) (2015), 170–190. `https://doi.org/10.15352/afa/06-1-13`.

[26] E. Malkowsky and E. Savas, *Matrix transformations between sequence sppaces of generalized weighted means,* Appl. Math. Comput. **147**(2) (2004), 333–345. `https://doi.org/10.1016/S0096-3003(02)00670-7`

[27] M. Mursaleen, F. Başar and B. Altay, *On the Euler sequence spaces which include the spaces $\ell_p$ and $\ell_\infty$,* Inform. Sci. **176**(10) (2006), 1450–1462. https://doi.org/10.1016/j.ins.2005.05.008

[28] M. Mursaleen and A. K. Noman, *On generalized means and some related sequence spaces.* Comput. Math. Appl. **61**(4) (2011), 988–999. https://doi.org/10.1016/j.camwa.2010.12.047

[29] M. Mursaleen and F. Başar, *Sequence Spaces: Topics in Modern Summability Theory, Series: Mathematics and Its Applications, CRC Press/Taylor,* Francis Group, Boca Raton, London, New York, 2020.

[30] B. C. Tripathy and B. Sharma, *Statistical convergent difference double sequence spaces,* Acta Math. Sin. (Engl. Ser.) **24**(5) (2008), 737–742. https://doi.org/10.1007/s10114-007-6648-0

[31] T. Yaying, A. Das, B. Hazarika and P. Baliarsingh, *Compactness of binomial difference operator of fractional order and sequence spaces*, Rend. Circ. Mat. Palermo **68** (2019), 459–476. https://doi.org/10.1007/s12215-018-0372-8

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