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EXISTENCE RESULTS FOR DOUBLE PHASE PROBLEM INVOLVING FRACTIONAL OPERATORS AND SINGULAR NONLINEARITY

RYM CHAMMEM¹, ABDELJABBAR GHANMI², AND MAHFOUDH MECHERGUI¹

ABSTRACT. The purpose of this paper is to extend some existing results related to the fractional Laplacian operator to a more general fractional operator. More precisely, we combine the variational method with the min-max problem, to study the existence of solutions for some singular problems with double-phase nonlocal operators. To validate our main results an illustrative example is presented.

1. INTRODUCTION

One of the key operators in the field of harmonic analysis and partial differential equations is the fractional Laplace operator which may be defined as

$$(1.1) \quad -(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy,$$

where $N > 2s$ and $s \in (0, 1)$. Indeed, the elliptic partial differential equations approach in a nonlocal framework has recently been used very often in Ambrosetti et al. [2], Ghanmi [19], Nezza et al. [28], Saoudi et al. [31], Servadei [32], Servadei and Valdinoci [33] and Servadei et al. [34]. Also, we note that the interest in this operator has increased considerably, given its importance and its rigorous applications in several fields including, in particular, the mechanics of continuous media, phase transition phenomena, population dynamics, for more applications, financial mathematics, see for example Autuori and Pucci [3], Barrios et al. [5], Cabre and Tan [7], Caffarelli and Silvestre [8], Capella [9], Dong and Kim [15], Servadei [32] and references therein.

Key words and phrases. Elliptic equation, variational method, min-max, nonlocal operator, singular problem.

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Many authors use several methods to study problems of the type:

$$(1.2) \quad \begin{cases} (-\Delta)^s u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $N > 2s$, $0 < s < 1$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, we refer the interested readers to the papers of Teng [35] (Critical point theory for non-differentiable functionals), [36] (Extending three-critical-points theorem), Wu et al. [38] (Stampacchia truncation method and the Leray-Schauder fixed point theorem).

Recently, several researchers have been concentrated on the generalization of the fractional Laplace operator Servadei and Valdinoci [33] use a more general non-local operator $\mathcal{L}_K$ to study the following elliptic problem

$$(1.3) \quad \begin{cases} \mathcal{L}_K u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $K : \mathbb{R}^N \setminus \Omega \rightarrow (0, +\infty)$ is a function satisfying some appropriate assumptions, $f \in C^1(\bar{\Omega} \times \mathbb{R})$, and $\mathcal{L}_K$ is a nonlocal operator which generalizes the fractional Laplacian one and defined as follows:

$$(1.4) \quad \mathcal{L}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x)) K(y) dy.$$

By using the Mountain Pass Theorem they showed the existence of a non-trivial solution.

We note that the operator $\mathcal{L}_K$ is a generalization of the operator $-(-\Delta)^s$ in the sense that if $K(x) = |x|^{-(N+2s)}$ then $\mathcal{L}_K = -(-\Delta)^s$.

Next, many authors developed some problems like (1.3), we mention the works of Caffarelli and Silvestre [8], Chammem et al. [10, 11], Saoudi et al. [31], Servadei [32].

Very recently, Ghanmi and Saoudi [21] studied the following singular problem

$$(1.5) \quad \begin{cases} \mathcal{L}_K u = \frac{a(x)}{u^\gamma} + \lambda f(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < \gamma < 1$, $\lambda > 0$, $a \in C(\Omega)$ and $h \in C^1(\bar{\Omega} \times \mathbb{R})$. They showed (using the method of Nehari manifold) the existence of λ_0 such that for all $\lambda \in (0, \lambda_0)$, the problem (1.5) admits two nontrivial weak solutions. For the interested readers, other singular problems like (1.5) can be found in [12, 16, 17, 20–23, 27, 37].

In recent years, problems involving mixed mixed-fractional Laplacian (double phase problems) operators attracted the attention of several researchers. In particular, Huyuan et al. [25] considered the following problem

$$\begin{cases} (-\Delta)^{s_1} u = \lambda ((-\Delta)^{s_2} u + \mu u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s_2 < s_1 < 1$ and $N > 2s_1$. They proved that the above problem admits a sequence of nontrivial solutions. More recently, Giacomoni et al. [24] developed the

following interesting problem

$$(1.6) \quad \begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $1 < q \leq p < +\infty$, $0 < s_2 \leq s_1 < 1$ and $f \in L_{loc}^\infty(\Omega)$. The fractional p -Laplace operator $(-\Delta)_p^s$ is defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+ps}} dy.$$

They study the Hölder continuity results and maximum principle for weak solutions to the problems (1.6).

We note that in general, double phase differential operators and corresponding energy functionals appear in several physical applications for example in the elasticity theory see Zhikov [39] and in other applications which can be found in the works of Bahroumi et al. [4]. Also, we note that double-phase problems appear in rigorous journals, we cite for example the papers [14, 18, 29, 30, 40], and the references therein.

Inspired by the above-mentioned works, in this paper we study the following problem

$$(Q_\lambda) \quad \begin{cases} \mathcal{L}_{K_1} u + \mathcal{L}_{K_2} u = \frac{b(x)}{u^\gamma} - \lambda h(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded smooth domain, $h \in C^1(\bar{\Omega} \times \mathbb{R})$, $\lambda \in \mathbb{R}$ and $0 < \gamma < 1$.

In the rest of this paper, the letter i will denote the integers one or two and we suppose that the function $K_i : \mathbb{R}^N \setminus \{0\} \rightarrow (0, +\infty)$ satisfies the following hypotheses:

(N1) $\min\{|x|, 1\} K_i \in L^1(\mathbb{R}^N)$;

(N2) there exists $\mu_i > 0$ such that for all $x \in \mathbb{R}^N \setminus \{0\}$ we have

$$K_i(x) > \mu_i |x|^{-(N+2s_i)}, \quad s_i \in (0, 1);$$

(N3) $K_i(x) = K_i(-x)$, for any $x \in \mathbb{R}^N \setminus \{0\}$.

We mentioned that the function $K_i(x) = |x|^{-(N+2s_i)}$ satisfies the above conditions. Also, we mentioned that the problem (Q_λ) is studied by M. Chhetri et al. [13] in the case when $\mathcal{L}_{K_1} = \mathcal{L}_{K_2} = |x|^{-(N+2s)}$, $b = 0$ and $\lambda < 0$.

Hereafter, we put $s = \min\{s_1, s_2\}$ and we assume that the function b satisfies:

$$(1.7) \quad 0 < b \in L^{\frac{\beta}{\beta+\gamma-1}}(\Omega), \quad 1 \leq \beta < 2^* := \frac{2N}{N-2s}.$$

The double-phase problem (Q_λ) is inspired by various models in mathematical physics. For example, consider the fourth-order relativistic operator

$$\varphi \mapsto \operatorname{div} \left(\frac{|\nabla \varphi|^2}{(1 - |\nabla \varphi|^4)^{\frac{3}{4}}} \nabla \varphi \right),$$

which characterizes a wide range of phenomena in relativistic quantum mechanics. If we apply Taylor's formula to the function $y^2(1-y^4)^{-\frac{3}{4}}$, then we can approximate the last operator by the following double-phase operator

$$\varphi \mapsto \Delta_4 \varphi + \frac{3}{4} \Delta_8 \varphi.$$

The main body of the paper is organized as follows. In Section 2, we recall some basic definitions and collect some variational settings. Next, we study the existence and the multiplicity of solutions of problem Q_λ , in Section 3 for the case $\lambda < 0$, and in Section 4 for the case $\lambda \geq 0$. To illustrate the validity of the main results, an example is given in Section 5.

2. PRELIMINARIES AND VARIATIONAL SETTING

In this section, we collect and introduce some results related to the variational setting for the problem (Q_λ) . First, we introduce the fractional Sobolev space

$$\mathcal{H}^s(\Omega) = \left\{ u \in L^2(\Omega) : [u]_{s,2} < +\infty \right\},$$

with the norm

$$\|u\|_{\mathcal{H}^s(\Omega)} = |u|_{L^2(\Omega)} + [u]_{s,2},$$

where

$$[u]_{s,2} = \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}$$

is the Gagliardo norm. For a detailed account on the properties of $\mathcal{H}^s$, we refer the reader to [1, 28].

Denote $\mathbf{T} = \mathbb{R}^{2N} \setminus \mathcal{D}$, with $\mathcal{D} = (\Omega^c) \times (\Omega^c) \subset \mathbb{R}^{2N}$, such as : $(\Omega^c) = \mathbb{R}^N \setminus \Omega$. We denote by $\mathbf{O} = \{u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ Lebesgue measurable}\}$. We define the space

$$X_i = \left\{ u \in \mathbf{O} : u|_{\Omega} \in L^2(\Omega) \text{ and } (u(x) - u(y))\sqrt{K_i(x-y)} \in L^2(\mathbf{T}) \right\},$$

with the norm

$$\|u\|_{X_i} = \|u\|_{L^2(\Omega)} + \left(\int_{\mathbf{T}} |u(x) - u(y)|^2 K_i(x-y) dx dy \right)^{\frac{1}{2}}.$$

We consider the following space:

$$F_i = \{u \in X_i : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \{\Omega\}\},$$

with the norm

$$\|u\|_{F_i} = \left(\int_{\mathbf{T}} |u(x) - u(y)|^2 K_i(x-y) dx dy \right)^{\frac{1}{2}}.$$

We stress that $(F_i, \|\cdot\|_{F_i})$ is a Hilbert space [33]. Also, $C_0^2(\Omega) \subset F_i$, $X_i \subset \mathcal{H}^s(\Omega)$ and $F_i \subset \mathcal{H}^s(\mathbb{R}^N)$.

Theorem 2.1. *For all $1 \leq \beta < 2^*$, the injection $F_i \hookrightarrow L^\beta(\Omega)$ is continuous and compact.*

Put $F = F_1 \cap F_2$. Then, we obviously have that $(F, \|\cdot\|_F)$ is a Hilbert space, with

$$\|u\|_F = \|u\|_{F_1} + \|u\|_{F_2}.$$

Moreover, for $1 \leq \beta < 2^*$, the embedding

$$(2.1) \quad F \hookrightarrow L^\beta(\Omega) \text{ is continuous and compact.}$$

Remark 2.1. Let $u \in F$. Then, we have

- (i) $\|u\|_{F_i} \leq \|u\|_F$;
- (ii) $\frac{1}{2}\|u\|_F^2 \leq \|u\|_{F_1}^2 + \|u\|_{F_2}^2 \leq \|u\|_F^2$;
- (iii) according to 2.1, if we note that S_β is the best Sobolev constant, then we have

$$\int_{\Omega} |u|^\beta dx \leq S_\beta \|u\|_F^\beta.$$

To use the variational method to prove the existence of solutions in F , we define the energy functional $\phi_\lambda : F \rightarrow \mathbb{R}$ associated to problem (Q_λ) by:

$$(2.2) \quad \phi_\lambda(u) = \frac{1}{2}\|u\|_{F_1}^2 + \frac{1}{2}\|u\|_{F_2}^2 + \lambda \int_{\Omega} H(x, u) dx - \frac{1}{1-\gamma} \int_{\Omega} b(x) |u|^{1-\gamma} dx,$$

where

$$(2.3) \quad H(x, u) = \int_0^u h(x, s) ds.$$

We say that $u \in F$ is a weak solution of problem (Q_λ) , for every $v \in F$, we have

$$\begin{aligned} & \int_{\mathbf{T}} (u(x) - u(y))(v(x) - v(y))(K_1(x - y) + K_2(x - y)) dx dy \\ &= \int_{\Omega} b(x) u^{-\gamma} v(x) dx - \lambda \int_{\Omega} h(x, u(x)) v(x) dx. \end{aligned}$$

Now, we aim to study the existence and the multiplicity of solutions to problem (Q_λ) by using variational methods [6, 26].

3. EXISTENCE RESULT FOR $\lambda < 0$

In this section, we will present and prove the first main result of this paper. For this aim, we shall consider the following hypotheses.

- (M1) There exist $f \in L^r(\Omega)$ continuous function and $1 < q < 2 < \frac{N}{s} < r$ such that

$$h(x, t) \leq C f(x) |t|^{q-2} t, \quad \text{a.e. } x \in \Omega, t \in \mathbb{R},$$

where C is a positive constant.

- (M2) There exists $\Omega' \subset \Omega$ with $|\Omega'| > 0$ such that

$$h(x, t), f(x) > 0, \quad \text{in } \Omega'.$$

The first main result of this paper is the following theorem.

Theorem 3.1. *Assume that (M1) and (M2) hold. If $\lambda < 0$, then problem (Q_λ) has at least one nontrivial weak solution with negative energy.*

To prove Theorem 3.1, we need to prove several lemmas.

Lemma 3.1. *Let $\lambda < 0$ and under assumption (M1), the functional ϕ_λ is coercive and bounded below on F .*

Proof. Let $u \in F$. Then, using (1.7) and (2.1) and Hölders inequality, we obtain

$$(3.1) \quad \int_{\Omega} b(x)u^{1-\gamma}dx \leq \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_{\beta}^{1-\gamma} \leq S_{\beta}^{1-\gamma} \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_F^{1-\gamma}.$$

Consequently, from (3.1), we have

$$(3.2) \quad \begin{aligned} \phi_{\lambda}(u) &\geq \frac{1}{2}\|u\|_{F_1}^2 + \frac{1}{2}\|u\|_{F_2}^2 + \lambda \int_{\Omega} H(x, u)dx - S_{\beta}^{1-\gamma} \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_F^{1-\gamma} \\ &\geq \frac{1}{4}\|u\|_F^2 + \lambda \int_{\Omega} H(x, u)dx - S_{\beta}^{1-\gamma} \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_F^{1-\gamma}. \end{aligned}$$

On the other hand, from (M1) and Hölders inequality, we get

$$(3.3) \quad \int_{\Omega} H(x, u)dx \leq \frac{C}{q} \int_{\Omega} f(x)|u|^q(x)dx \leq \frac{C}{q} \|f\|_r \|u\|_{qr'}^q,$$

where r' is such that $\frac{1}{r} + \frac{1}{r'} = 1$.

Note that, from (M1), we have

$$(3.4) \quad 2^* - qr' = \frac{Nr(2-q) + 2(qrs - N)}{(N-2s)(r-1)} > 0.$$

From (3.3), applying (2.1), we get

$$(3.5) \quad \int_{\Omega} H(x, u)dx \leq C_1 \|f\|_r \|u\|_F^q,$$

for some positive constant C_1 .

Finally, combining (3.2) and (3.5) and since $\lambda < 0$, we get

$$\phi_{\lambda}(u) \geq \frac{1}{4}\|u\|_F^2 + \lambda C_1 \|f\|_r \|u\|_F^q - S_{\beta}^{1-\gamma} \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_F^{1-\gamma}.$$

Since, $2 > q > 1 - \gamma$, we infer that $\phi_{\lambda} \rightarrow +\infty$ if $\|u\|_F \rightarrow +\infty$. So, the functional ϕ_{λ} is coercive and bounded from below on F . That gives the proof of Lemma 3.1. $\square$

Lemma 3.2. *Let $\lambda < 0$ and assume that assumption (M2) holds, then there exists $e \in E$ with $e > 0$ such that $\phi_{\lambda}(te) < 0$ for all $t > 0$ small enough.*

Proof. Let $e \in C_0^\infty$ such that $\text{supp}(e) \subset \Omega' \subset \Omega$, $e = 1$ in a subset $\Omega'' \subset \text{supp}(e)$ and $0 \leq e \leq 1$ in Ω' . Using assumption (M2), Remark 2.1 and the fact that $\lambda < 0$, we get

$$\begin{aligned} \phi_{\lambda}(te) &= \frac{1}{2}\|te\|_{F_1}^2 + \frac{1}{2}\|te\|_{F_2}^2 + \lambda \int_{\Omega} H(x, te)dx - \frac{1}{1-\gamma} \int_{\Omega} b(x)|te|^{1-\gamma}dx \\ &\leq \frac{1}{2}t^2\|e\|_F^2 + \lambda \int_{\Omega} H(x, te)dx - \frac{1}{1-\gamma}t^{1-\gamma} \int_{\Omega} b(x)|e|^{1-\gamma}dx \\ &\leq \frac{1}{2}t^2\|e\|_F^2 + \lambda \int_{\Omega'} H(x, te)dx - \frac{1}{1-\gamma}t^{1-\gamma} \int_{\Omega} b(x)|e|^{1-\gamma}dx. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \phi_\lambda(te) &\leq t^2 \frac{1}{2} \|e\|_F^2 - t^{1-\gamma} \frac{1}{1-\gamma} \int_\Omega b(x) |e|^{1-\gamma} dx \\ (3.6) \quad &\leq t^{1-\gamma} \left(t^{2+\gamma-1} \frac{1}{2} \|e\|_F^2 - \frac{1}{1-\gamma} \int_\Omega b(x) |e|^{1-\gamma} dx \right). \end{aligned}$$

Since $\|e\|_F^2 > 0$ (if $\|e\|_F^2 = 0$, then $e = 0$ in Ω , which is a contradiction). Finally, if t is such that $0 < t < \min\{1, t_0\}$, then from equation (3.6), we get $\phi_\lambda(te) < 0$, where t_0 is given by

$$t_0 = \left(\frac{\frac{1}{1-\gamma} \int_\Omega b(x) |e|^{1-\gamma} dx}{\frac{1}{2} \|e\|_F^2} \right)^{\frac{1}{2+\gamma-1}}.$$

□

In the sequel, we put

$$(3.7) \quad m_\lambda = \inf_{u \in F} \phi_\lambda(u).$$

Lemma 3.3. *Let $\lambda < 0$. Under assertions (1.7) and (M1)-(M2), the function ϕ_λ reaches its global minimizer in F , that is, there exists $v_\lambda \in F$ such that*

$$\phi_\lambda(v_\lambda) = m_\lambda < 0.$$

Proof. Let $\{v_n\}$ be a minimizing sequence, then $\phi_\lambda(v_n) \rightarrow m_\lambda$. We claim that $\{v_n\}$ is bounded in F .

Indeed, if $\{v_n\}$ is not bounded in F , so $\|v_n\|_F \rightarrow +\infty$ as $n \rightarrow +\infty$. Since ϕ_λ is coercive,

$$\phi_\lambda(v_n) \rightarrow +\infty, \quad \text{as } \|v_n\|_F \rightarrow +\infty.$$

This contradicts the fact that $\{v_n\}$ is a minimizing sequence. Therefore, $\{v_n\}$ is bounded in the reflexive space F_i . Then, from Theorem 2.1, up to a subsequence, still denoted by $\{v_n\}$ there exists $v_\lambda \in F_i$, such that

$$\begin{cases} v_n \rightharpoonup v_\lambda \text{ weakly in } F_i, \\ v_n \rightarrow v_\lambda \text{ strongly in } L^\beta(\Omega), \\ v_n \rightarrow v_\lambda \text{ a.e in } \Omega. \end{cases}$$

We begin by proving that ϕ_λ is weakly lower semi-continuous on F . So, we begin by remarking that from the fact that $v_n \rightharpoonup v_\lambda$ weakly in F_i , then we have

$$(3.8) \quad \|v_\lambda\|_{F_i} \leq \liminf_{n \rightarrow +\infty} \|v_n\|_{F_i}.$$

Next, by Vitali's Theorem [39], we can claim that

$$(3.9) \quad \lim_{n \rightarrow +\infty} \int_\Omega b(x) |v_n|^{1-\gamma} dx = \int_\Omega b(x) |v_\lambda|^{1-\gamma} dx.$$

Indeed, we only need to prove that

$$\left\{ \int_\Omega b(x) |v_n|^{1-\gamma} dx : n \in \mathbb{N} \right\},$$

is equi-absolutely-continuous.

Let $\varepsilon > 0$. Then, by the absolutely-continuity of $\int_{\Omega} |b(x)|^{\frac{\beta}{\beta+\gamma-1}} dx$, there exists $\alpha > 0$ such that

$$\int_{\Omega_3} |b(x)|^{\frac{\beta}{\beta+\gamma-1}} dx < \varepsilon^{\frac{\beta}{\beta+\gamma-1}}, \quad \text{for every } \Omega_3 \subset \Omega, \text{ with } |\Omega_3| < \alpha.$$

Note that $\{v_n\}$ is bounded, by the Sobolev embedding theorem, there exists a constant $A > 0$, such that $\|v_n\|_F \leq A$.

Consequently, by Hölder's inequality, we have

$$\int_{\Omega_3} b(x)|v_n|^{1-\gamma} dx \leq \|v_n\|_F^{1-\gamma} \left(\int_{\Omega_3} |b(x)|^{\frac{\beta}{\beta+\gamma-1}} dx \right)^{\frac{\beta+\gamma-1}{\beta}} \leq A^{1-\gamma} \varepsilon.$$

Finally, we will prove that

$$(3.10) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} H(x, v_n) dx = \int_{\Omega} H(x, v_{\lambda}) dx.$$

Let $\epsilon > 0$. Then, from (M1), there exists $C_{\epsilon} > 0$ such that

$$H(x, v_n(x)) \leq \frac{C_{\epsilon}}{q} |f(x)| \cdot |v_n|^q.$$

Since $v_n \rightharpoonup v_{\lambda}$ in F and from (3.4), we obtain the strong convergence for a subsequence of $\{v_n\}$ in $L^{r'q}(\Omega)$, still denoted by $\{v_n\}$. Therefore, we get $v_n \rightarrow v_{\lambda}$ a.e. in Ω , from [6, Theorem IV-9] there exists $l \in L^{r'q}(\Omega)$ such that

$$(3.11) \quad |v_n(x)| \leq l(x).$$

Then, assumption (M1), equations (2.3), (3.11) and Hölder's inequality imply that

$$\begin{aligned} \int_{\Omega} H(x, v_n) dx &\leq \frac{C_{\epsilon}}{q} \int_{\Omega} f(x) |v_n|^q dx \leq \frac{C_{\epsilon}}{q} \int_{\Omega} f(x) l^q(x) dx \\ &\leq \frac{C_{\epsilon}}{q} \|f\|_r \|l\|_{q^{r'}}^q \leq C_3 \|f\|_r \|l\|_F^q. \end{aligned}$$

Hence, by the dominated convergence theorem, (3.10) holds.

Finally, by combining (2.2), (3.9), (3.8) with (3.10), we get ϕ_{λ} is weakly lower semi-continuous on F . Therefore,

$$(3.12) \quad \phi_{\lambda}(v_{\lambda}) \leq \liminf_{n \rightarrow +\infty} \phi_{\lambda}(v_n) = m_{\lambda}.$$

On the other hand, from (3.7), we have

$$(3.13) \quad \phi_{\lambda}(v_{\lambda}) \geq m_{\lambda}.$$

Combining (3.12) and (3.13), we conclude that $\phi_{\lambda}(v_{\lambda}) = m_{\lambda}$. □

Proof of Theorem 3.1. From Lemma 3.3, v_λ is a global minimum for ϕ_λ . Let $\varphi \in F$, then, for sufficiently small $t > 0$, we have

$$\begin{aligned} 0 &\leq \phi_\lambda(v_\lambda + t\varphi) - \phi_\lambda(v_\lambda) \\ &= \frac{1}{2} \left(\|v_\lambda + t\varphi\|_{F_1}^2 - \|v_\lambda\|_{F_1}^2 + \|v_\lambda + t\varphi\|_{F_2}^2 - \|v_\lambda\|_{F_2}^2 \right) \\ &\quad + \lambda \int_\Omega H(x, v_\lambda + t\varphi) - H(x, v_\lambda) dx - \frac{1}{1-\gamma} \int_\Omega b(x) (|v_\lambda + t\varphi|^{1-\gamma} - |v_\lambda|^{1-\gamma}) dx, \end{aligned}$$

dividing this by $t > 0$ and letting $t \rightarrow 0^+$, we obtain

$$\begin{aligned} 0 &\leq \int_\Omega (v_\lambda(x) - v_\lambda(y))(\varphi(x) - \varphi(y))(K_1(x-y) + K_2(x-y)) dx dy \\ &\quad - \int_\Omega b(x)|v_\lambda|^{-\gamma}\varphi(x) dx + \lambda \int_\Omega h(x, v_\lambda(x))\varphi(x) dx. \end{aligned}$$

Since φ is arbitrary in F , we can replace φ by $-\varphi$. Thus, the equality holds for all $\varphi \in F$, which means that v_λ is a weak solution of problem (Q_λ) . Moreover, the fact that $\phi_\lambda(v_\lambda) < 0$ implies that v_λ is nontrivial. $\square$

4. EXISTENCE RESULT FOR $\lambda > 0$

In this section, we will present and prove the second main result of this paper. For this aim, we shall consider the following hypothesis.

(M3) There exist $\Omega_2 \subset \Omega$ with $|\Omega_2| > 0$ and a nonnegative function f_1 on Ω_2 such that $f_1 \in L^{r_1}(\Omega)$ with

$$\lim_{|t| \rightarrow 0} \frac{h(x, t)}{f_1(x)|t|^{p-1}} = 0, \quad \text{for } x \in \Omega \text{ uniformly.}$$

(M4) There exists a nonnegative function f_2 on Ω such that $f_2 \in L^{r_2}(\Omega)$ and

$$\lim_{|t| \rightarrow +\infty} \frac{h(x, t)}{f_2(x)|t|^{p'-1}} = 0, \quad \text{for } x \in \Omega \text{ uniformly,}$$

where

$$(4.1) \quad 1 < \max\{p, p'\} < 2 < \frac{N}{s} < \min\{r_1, r_2\}.$$

(M5) There exists $B > 0$ such that

$$\int_\Omega H(x, t) dx \geq 0, \quad \text{for any } t > B.$$

The second main result of this paper is the following theorem.

Theorem 4.1. *Assume that (M3), (M4) and (M5) hold. If $\lambda > 0$, then problem (Q_λ) has at least one nontrivial weak solution with negative energy.*

To prove Theorem 4.1, we need to prove several lemmas.

Lemma 4.1. *Under assumption (M5) and the fact that $\lambda > 0$, the functional ϕ_λ is coercive and bounded below on F .*

Proof. Let $u \in F$ and $\lambda > 0$. Then, using (M5), (2.1) and (3.1) for any $\|u\| > \max\{1, B\}$, we get

$$\begin{aligned}\phi_\lambda(u) &= \frac{1}{2}\|u\|_{F_1}^2 + \frac{1}{2}\|u\|_{F_2}^2 + \lambda \int_{\Omega} H(x, u)dx - \frac{1}{1-\gamma} \int_{\Omega} b(x)|u|^{1-\gamma}dx \\ &\geq \frac{1}{4}\|u\|_F^2 + \lambda \int_{\Omega} H(x, u)dx - S_\beta^{1-\gamma} \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_F^{1-\gamma} \\ &\geq \frac{1}{4}\|u\|_F^2 - S_\beta^{1-\gamma} \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_F^{1-\gamma}.\end{aligned}$$

Now, since $2 > 1 - \gamma$, we have

$$\lim_{\|u\|_F \rightarrow +\infty} \frac{1}{4}\|u\|_F^2 - S_\beta^{1-\gamma} \|b\|_{\frac{\beta}{\beta+\gamma-1}} \|u\|_F^{1-\gamma} = +\infty.$$

So, the functional ϕ_λ is coercive and bounded from below on F which gives the proof of Lemma 4.1. $\square$

Lemma 4.2. *Under assumption (M3) and for $\lambda > 0$, there exists $\zeta \in F$ with $\zeta > 0$ such that $\phi_\lambda(t\zeta) < 0$ for all sufficiently small $t > 0$.*

Proof. Let $\zeta \in C_0^\infty(\Omega)$ such that $\text{supp}(\zeta) \subset \Omega_2 \subset \Omega$, $\zeta = 1$ in subset $\Omega'_2 \subset \text{supp}(\zeta)$ and $0 < \zeta \leq 1$ in Ω_2 . Using (2.3) and assumption (M3), there exists $C_2 > 0$ such that

$$(4.2) \quad \int_{\Omega} H(x, t\zeta)dx \leq C_2 |t|^p \int_{\Omega_2} f_1(x) |\zeta(x)|^p dx.$$

Assume that $0 < t < 1$, then by (4.2) and Remark 2.1, we get

$$\begin{aligned}\phi_\lambda(t\zeta) &= \frac{1}{2}\|t\zeta\|_{F_1}^2 + \frac{1}{2}\|t\zeta\|_{F_2}^2 + \lambda \int_{\Omega} H(x, t\zeta)dx - \frac{1}{1-\gamma} \int_{\Omega} b(x)|t\zeta|^{1-\gamma}dx \\ &\leq \frac{1}{2}t^2\|\zeta\|_F^2 + \lambda C_2 |t|^p \int_{\Omega_2} f_1(x) |\zeta(x)|^p dx - \frac{1}{1-\gamma} t^{1-\gamma} \int_{\Omega} b(x) |\zeta|^{1-\gamma} dx \\ &\leq t^{\min\{2,p\}} \left[\frac{1}{2}\|\zeta\|_F^2 + \lambda C_2 \int_{\Omega_2} f_1(x) |\zeta|^p dx \right] - t^{1-\gamma} \frac{1}{1-\gamma} \int_{\Omega} b(x) |\zeta|^{1-\gamma} dx \\ (4.3) \quad &\leq t^{1-\gamma} \left(\frac{1}{2}\|\zeta\|_F^2 + \lambda C_2 \int_{\Omega_2} f_1(x) |\zeta|^p dx \right) \left(t^{\min\{2,p\}+\gamma-1} - t_0^{\min\{2,p\}+\gamma-1} \right),\end{aligned}$$

where

$$t_0 = \left(\frac{\frac{1}{1-\gamma} \int_{\Omega} b(x) |\zeta|^{1-\gamma} dx}{\frac{1}{2}\|\zeta\|_F^2 + \lambda C_2 \int_{\Omega_2} f_1(x) |\zeta|^p dx} \right)^{\frac{1}{\min\{2,p\}+\gamma-1}}.$$

Since $\lambda > 0$, $\zeta \neq 0$ and $f_1(x) \geq 0$, then

$$\frac{1}{2}\|\zeta\|_F^2 + \lambda C_2 \int_{\Omega_2} f_1(x) \zeta^p dx > 0.$$

We see from (4.3), that if $0 < t < \min\{1, t_0\}$, then $\phi_\lambda(t\zeta) < 0$, which gives the proof of Lemma 4.2. $\square$

In the sequel, we put

$$(4.4) \quad m'_\lambda = \inf_{u \in F} \phi_\lambda(u).$$

Lemma 4.3. *Let $\lambda > 0$. Under assertions (1.7) and (M3)-(M5), the function ϕ_λ reaches its global minimizer in F , that is, there exists $\nu_\lambda \in F$ such that*

$$\phi_\lambda(\nu_\lambda) = m'_\lambda < 0.$$

Proof. Let $\{\nu_n\}$ be a minimizing sequence, then, $\phi_\lambda(\nu_n) \rightarrow m'_\lambda$. $\{\nu_n\}$ is bounded in F .

Indeed, if $\{\nu_n\}$ is not bounded in F , so $\|\nu_n\|_F \rightarrow +\infty$ as $n \rightarrow +\infty$. Since ϕ_λ is coercive

$$\phi_\lambda(\nu_n) \rightarrow +\infty, \quad \text{as } \|\nu_n\|_F \rightarrow +\infty.$$

This contradicts the fact that $\{\nu_n\}$ is a minimizing sequence. Therefore, $\{\nu_n\}$ is bounded in the reflexive space F_i . Then from Theorem 2.1, up to a subsequence, still denoted by $\{\nu_n\}$ there exists $\nu_\lambda \in F_i$ such that

$$\begin{cases} \nu_n \rightharpoonup \nu_\lambda \text{ weakly in } F_i, \\ \nu_n \rightarrow \nu_\lambda \text{ strongly in } L^\beta(\Omega), \\ \nu_n \rightarrow \nu_\lambda \text{ a.e in } \Omega. \end{cases}$$

We will prove that

$$(4.5) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} H(x, \nu_n) dx = \int_{\Omega} H(x, \nu_\lambda) dx.$$

Assume that (M3), (M4) and (2.3), that for all $\delta > 0$ there exists A_δ such that

$$|H(x, \nu_n)| \leq \delta \frac{C_3}{p} \int_{\Omega} f_1(x) |\nu_n|^p dx + A_\delta \frac{C_3'}{p'} \int_{\Omega} f_2(x) |\nu_n|^{p'} dx.$$

Since $\nu_n \rightarrow \nu_\lambda$ strongly in $L^\beta(\Omega)$, $1 \leq \beta < 2^*$, from [6, Theorem IV-9] there exists $l_1 \in L^\beta(\Omega)$ such that

$$(4.6) \quad |\nu_n(x)| \leq l_1(x), \quad \text{a.e in } \Omega.$$

Then, by Hölder's inequality, we obtain

$$\begin{aligned}
 |H(x, \nu_n)| &\leq \delta \frac{C_3}{p} \int_{\Omega} f_1(x) |\nu_n|^p dx + A_{\delta} \frac{C'_3}{p'} \int_{\Omega} f_2(x) |\nu_n|^{p'} dx \\
 &\leq \delta \frac{C_3}{p} \left(\int_{\Omega} |f_1(x)|^{r_1} dx \right)^{\frac{1}{r_1}} \left(\int_{\Omega} |\nu_n|^{pr'_1} dx \right)^{\frac{1}{r'_1}} \\
 &\quad + A_{\delta} \frac{C'_3}{p'} \left(\int_{\Omega} |f_2(x)|^{r_2} dx \right)^{\frac{1}{r_2}} \left(\int_{\Omega} |\nu_n|^{p'r'_2} dx \right)^{\frac{1}{r'_2}} \\
 &\leq \delta \frac{C_3}{p} \left(\int_{\Omega} |f_1(x)|^{r_1} dx \right)^{\frac{1}{r_1}} \left(\int_{\Omega} |l_1|^{pr'_1} dx \right)^{\frac{1}{r'_1}} \\
 &\quad + A_{\delta} \frac{C'_3}{p'} \left(\int_{\Omega} |f_2(x)|^{r_2} dx \right)^{\frac{1}{r_2}} \left(\int_{\Omega} |l_1|^{p'r'_2} dx \right)^{\frac{1}{r'_2}} \\
 (4.7) \quad &\leq \delta \frac{C_3}{p} \|f_1\|_{r_1} \|l_1\|_{pr'_1}^p + A_{\delta} \frac{C'_3}{p'} \|f_2\|_{r_2} \|l_1\|_{p'r'_2}^{p'},
 \end{aligned}$$

where $\frac{1}{r_1} + \frac{1}{r'_1} = 1$ and $\frac{1}{r_2} + \frac{1}{r'_2} = 1$.

Note that, from (4.1), we get

$$(4.8) \quad 2^* - pr'_1 = \frac{Nr_1(2-p) + 2(pr_1s - N)}{(N-2s)(r_1-1)} > 0$$

and

$$2^* - p'r'_2 = \frac{Nr_2(2-p') + 2(p'r_2s - N)}{(N-2s)(r_2-1)} > 0.$$

From (4.7) applying (2.1), we get

$$|H(x, \nu_n)| \leq \delta \frac{C_3}{p} \|f_1\|_{r_1} \|l_1\|_F^p + A_{\delta} \frac{C'_3}{p'} \|f_2\|_{r_2} \|l_1\|_F^{p'}.$$

Hence, by the dominated convergence theorem, (4.5) holds.

By combining (2.2), (3.9), (3.8) and (4.5), we get that ϕ_{λ} is weakly lower semi-continuous. Therefore,

$$(4.9) \quad \phi_{\lambda}(\nu_{\lambda}) \leq \liminf_{n \rightarrow +\infty} \phi_{\lambda}(\nu_n) = m'_{\lambda}.$$

On the other hand, from the definition of m'_{λ} , we have

$$(4.10) \quad I_{\lambda}(\nu_{\lambda}) \geq m'_{\lambda}.$$

Combining (4.9) and (4.10), we conclude that $\phi_{\lambda}(\nu_{\lambda}) = m_{\lambda}$. □

Proof of Theorem 4.1. Let $\lambda > 0$ and from Lemma 4.3, ν_λ is a global minimum for ϕ_λ . Let $\varphi \in F$, then, for sufficiently small $t > 0$,

$$\begin{aligned} 0 &\leq \phi_\lambda(\nu_\lambda + t\varphi) - \phi_\lambda(\nu_\lambda) \\ &= \frac{1}{2} \left(\|\nu_\lambda + t\varphi\|_{F_1}^2 - \|\nu_\lambda\|_{F_1}^2 + \|\nu_\lambda + t\varphi\|_{F_2}^2 - \|\nu_\lambda\|_{F_2}^2 \right) \\ &\quad + \lambda \int_{\Omega} (H(x, \nu_\lambda + t\varphi) - H(x, \nu_\lambda)) dx - \frac{1}{1-\gamma} \int_{\Omega} b(x) (|\nu_\lambda + t\varphi|^{1-\gamma} - |\nu_\lambda|^{1-\gamma}) dx, \end{aligned}$$

dividing this by $t > 0$ and letting $t \rightarrow 0^+$, we obtain

$$\begin{aligned} 0 &\leq \int_{\mathbf{T}} (\nu_\lambda(x) - \nu_\lambda(y)) (\varphi(x) - \varphi(y)) (K_1(x-y) + K_2(x-y)) dx dy \\ &\quad - \int_{\Omega} b(x) |\nu_\lambda|^{-\gamma} \varphi(x) dx + \lambda \int_{\Omega} h(x, \nu_\lambda(x)) \varphi(x) dx. \end{aligned}$$

Since φ is arbitrary in F , we can replace φ by $-\varphi$. Thus, the equality holds for all $\varphi \in F$, which means that ν_λ is a weak solution of problem (Q_λ) . Moreover, the fact that $\phi_\lambda < 0$, implies that ν_λ is nontrivial. $\square$

Remark 4.1. Let $\lambda = 0$, Lemmas 3.1 and 3.2 are true, then by Lemma 3.3 the function ϕ_0 reaches its global minimizer in F , that is, there exists $v_0 \in F$ such that

$$\phi_0(v_\lambda) = m_0 < 0.$$

We follow the same proof of Theorem 3.1 for $\lambda = 0$, we conclude that v_0 is a weak solution of problem (Q_λ) . Moreover, the fact that $\phi_0(v_0) < 0$ implies that v_0 is nontrivial.

5. AN EXAMPLE

In this section, we give an example to illustrate our main result.

Example 5.1. Let $\Omega \subset \mathbb{R}^N$, $N > 2$, be a bounded domain. We consider the following problem:

$$(Q_\lambda) \begin{cases} L_1 u + L_2 u = \frac{b(x)}{u^\gamma} - \lambda h(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where λ is a parameter, b is a function that is almost everywhere positive in Ω provided that

$$b \in L^{\frac{\beta}{\beta+\gamma-1}}(\Omega), \quad 1 \leq \beta < 2^*, \quad \text{with } 2^* = \frac{2N}{N-2s}.$$

Case 1. Let $\lambda < 0$ and let h be the function defined respectively by:

$$h(x, t) = C f(x) |t|^{q-2} t, \quad \text{for all } x \in \Omega, \quad t \in \mathbb{R},$$

where, $f \in L^r(\Omega)$ is a continuous function and $1 < q < 2 < \frac{N}{s} < r$.

Then, assumptions (M1) and (M2) hold. So, for any $\lambda < 0$ problem (Q_λ) has at least one nontrivial weak solution in F with negative energy.

Case 2. Let $\lambda > 0$ and let h be the function defined respectively by:

$$h(x, t) = \begin{cases} f_1(x)|t|^{p-1}, & |t| \leq 1, \\ f_2(x)|t|^{p'-1}, & |t| > 1, \end{cases}$$

where $f_1 \in L^{r_1}(\Omega)$ and $f_2 \in L^{r_2}(\Omega)$ are nonnegative functions with

$$1 < \max\{p, p'\} < 2 < \frac{N}{s} < \min\{r_1, r_2\}.$$

Then, assumptions (M3), (M4) and (M5) hold. So for all $\lambda > 0$, problem (Q_λ) has at least one nontrivial weak solution in F with negative energy.

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SOLVABILITY OF GENERALIZED FRACTIONAL FUNCTIONAL-INTEGRAL EQUATIONS IN BANACH SPACES BY PETRYSHYN'S FIXED POINT THEOREM

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ABSTRACT. We study the solvability of fractional functional-integral equations of product type in Banach spaces. We use Petryshyn's fixed point theorem associated with measure of non-compactness to obtain our results. Our results extend some existing results in the literature. We illustrate the applicability of our results with examples.

1. INTRODUCTION

Functional integral equations are a powerful tool for studying real-world problems. In recent years, there has been a growing interest in fractional integral equations. These equations are more general than traditional integral equations, and they can be used to model a wider range of phenomena. They have been used in a variety of fields, including mathematics, engineering, physics, mechanics, and modeling etc. [15, 33, 39].

Fixed point theorems are a powerful tool for studying fractional integral equations. They provide a solid theoretical foundation to prove the existence and uniqueness of solutions of these equations. This is supported by many books and articles that have been published on the subject, such as [20, 27, 40, 52]. In this paper, we discuss the

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solvability of the following fractional integral functional integral equation (FFIE):

$$(1.1) \quad u(\zeta) = \left(f(\zeta, u(\zeta), u(\alpha(\zeta))) + F\left(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right) \\ \times \left(g(\zeta, u(\zeta), u(\eta(\zeta))) + G\left(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))) ds \right) \right),$$

for $\zeta \in I_b = [0, b]$, $0 < h \leq 1$.

We found the following integral equations are particular types of the equation (1.1).

- Darwish [18] studied the existence of solutions for FFIE

$$(1.2) \quad u(\zeta) = F\left(\zeta, u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^\zeta \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds \right) \\ \times G\left(t, u(\varphi(t)), u(\zeta) \int_0^1 q(\zeta, s, u(s)) ds \right), \quad \zeta \in [0, 1].$$

- Banaś and Rzepka [8] studied the existence of solutions for FFIE

$$(1.3) \quad u(\zeta) = A(\zeta) + \frac{\tilde{f}(\zeta, u(\zeta))}{\Gamma(h)} \int_0^\zeta \frac{p(s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, 1].$$

- Banaś and Regan [10] studied the existence of solutions for FFIE

$$(1.4) \quad u(\zeta) = A(\zeta) + \frac{\tilde{f}(\zeta, u(\zeta))}{\Gamma(h)} \int_0^\zeta \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, +\infty).$$

- Darwish and Henderson [19] studied the existence of solutions for FFIE

$$(1.5) \quad u(\zeta) = f(\zeta, u(\zeta)) + \frac{\tilde{f}(\zeta, u(\zeta))}{\Gamma(h)} \int_0^\zeta \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, +\infty).$$

- Balachandran et al. [2] studied the FFIE

$$(1.6) \quad u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \frac{\tilde{f}(\zeta, u(\beta(\zeta)))}{\Gamma(h)} \int_0^\zeta \frac{p(\zeta, s, u(\mu(s)))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, +\infty).$$

- Darwish [17] studied the existence of solutions for FFIE

$$(1.7) \quad u(\zeta) = A(\zeta) + \frac{u(\zeta)}{\Gamma(h)} \int_0^\zeta \frac{p(s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in I_b.$$

- Deep et al. [21] studied the existence of solution for FIE

$$(1.8) \quad u(\zeta) = \left(A(\zeta) + f(\zeta, u(\zeta)) + F\left(\zeta, \int_0^\zeta p(\zeta, s, u(\mu(s))) ds, u(\beta(\zeta)) \right) \right) \\ \times \left(B(\zeta) + g(\zeta, u(\zeta)) + G\left(\zeta, \int_0^b q(\zeta, s, u(\nu(s))) ds, u(\varphi(\zeta)) \right) \right), \quad \zeta \in I_b.$$

- Deepmala [26] analyzed the following FIE

$$(1.9) \quad u(\zeta) = \left(f(\zeta, u(\zeta)) + F\left(\zeta, \int_0^\zeta p(\zeta, s, u(s))ds, u(\beta(\zeta))\right) \right) \\ \times \left(G\left(\zeta, \int_0^b q(\zeta, s, u(s))ds, u(\varphi(\zeta))\right) \right), \quad \zeta \in I_b.$$

- Banaś [5] as well as Maleknejad et al. [42] analyzed the following FIE

$$(1.10) \quad u(\zeta) = F\left(\zeta, \int_0^\zeta p(\zeta, s, u(\mu(s)))ds, u(\beta(\zeta))\right) \times G\left(\zeta, \int_0^b q(\zeta, s, u(\nu(s)))ds, u(\varphi(\zeta))\right), \\ \zeta \in I_b.$$

- Caballero et al. [12] analyzed the following FIE

$$(1.11) \quad u(\zeta) = F\left(\zeta, \int_0^\zeta p(\zeta, s, u(s))ds, u(\beta(\zeta))\right) \times G\left(\zeta, \int_0^b u(s)q(\zeta, s, u(s))ds, u(\varphi(\zeta))\right), \\ \zeta \in I_b.$$

- Hu and Yan [32] examined the existence of solutions for FIE

$$(1.12) \quad u(\zeta) = F\left(\zeta, u(\zeta), \int_0^\zeta p(\zeta, s, u(s))ds\right), \quad \zeta \in [0, +\infty).$$

- Maleknejad et al. [43, 44] examined the existence result of FIEs

$$(1.13) \quad u(\zeta) = f(\zeta, u(\zeta)) + F\left(\zeta, \int_0^\zeta p(\zeta, s, u(s))ds, u(\beta(\zeta))\right),$$

$$(1.14) \quad u(\zeta) = \tilde{f}(\zeta, u(\beta(\zeta))) \int_0^\zeta p(\zeta, s, u(s))ds, \quad \zeta \in I_b.$$

- Banaś [6, 11] analyzed the following FIEs

$$(1.15) \quad u(\zeta) = \tilde{f}(\zeta, u(\zeta)) \int_0^\zeta p(\zeta, s, u(s))ds,$$

$$(1.16) \quad u(\zeta) = c(\zeta) + \tilde{f}(\zeta, u(\zeta)) \int_0^\zeta p(\zeta, s, u(s))ds, \quad \zeta \in [0, +\infty).$$

- Dhage [28] analyzed the following FIE

$$(1.17) \quad u(\zeta) = c(\zeta) \int_0^b q(\zeta, s, u(s))ds + \int_0^\zeta p(\zeta, s, u(s))ds \times \int_0^b q(\zeta, s, u(s))ds, \quad \zeta \in I_b.$$

- Çakan [13], Özdemir et al. [46], Özdemir [47] analyzed the following FIE

$$(1.18) \quad u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \tilde{f}(\zeta, u(\beta(\zeta))) \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds,$$

$$(1.19) \quad u(\zeta) = f(\zeta, u(\alpha(\zeta))) + F\left(\zeta, \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, u(\beta(\zeta))\right), \quad \zeta \in I_b.$$

- Srivastava et al. [51], considered following FIE of two variables:

$$(1.20) \quad x(l, h) = \left(u(l, h, x(l, h)) + F\left(l, h, \int_0^l \int_0^h P(l, h, r, u, x(r, u))drdu, x(l, h)\right) \right) \\ \times G\left(l, h, \int_0^a \int_0^a Q(l, h, r, u, x(r, u))drdu, x(l, h)\right), \quad \text{for } l, h \in [0, a].$$

- Aghajani and Jalilian [1] have studied the equation

$$(1.21) \quad u(\zeta) = F\left(\zeta, u(\beta(\zeta)), \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds\right), \quad \zeta \in [0, +\infty).$$

- Cichon and Metwali [16] have studied the equation

$$(1.22) \quad u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \tilde{f}(\zeta, u(\zeta)) \int_0^1 p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0, 1].$$

- Vetro and Vetro [53] have studied the equation

$$(1.23) \quad u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0, +\infty).$$

- Hashem and Rwaily [29] have studied the equation

$$(1.24) \quad u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \tilde{f}(\zeta, u(\beta(\zeta))) \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0, +\infty).$$

- Banaś and Dhage [9] have studied the equation

$$(1.25) \quad u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0, +\infty).$$

Further, some well-known first-order equations, such as the Volterra equation, Urysohn equation, Abel equation, and Chandrasekhar type [15], have the form

$$(1.26) \quad u(\zeta) = F(\zeta, u(\beta(\zeta))), \\ u(\zeta) = A(\zeta) + \frac{1}{\Gamma(h)} \int_0^\zeta \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds,$$

$$(1.27) \quad u(\zeta) = A(\zeta) + \int_0^\zeta p(\zeta, s, u(s))ds,$$

$$(1.28) \quad u(\zeta) = A(\zeta) + \int_0^b p(\zeta, s, u(s))ds,$$

$$(1.29) \quad u(\zeta) = 1 + u(\zeta) \int_0^b \frac{\zeta}{\zeta + s} \phi(s) u(s) ds.$$

It is worth noting that equation (1.1) is more general than equations (1.2)–(1.29). For example, equation (1.2) can be derived from equation (1.1) if

$$f(\zeta, u_1, u_2) = g(\zeta, u_1, u_2) = 0, \\ F(\zeta, u_1, u_2, u_3) = F(\zeta, u_2, u_3), \quad G(\zeta, u_1, u_2, u_3) = G(\zeta, u_2, u_1, u_3)$$

and $\phi(\zeta) = \mu(\zeta) = \nu(\zeta) = \zeta$.

Similarly, if $f(\zeta, u_1, u_2) = A(\zeta)$, $g(\zeta, u_1, u_2) = 0$, $G(\zeta, u_1, u_2, u_3) = 1$, $\phi(\zeta) = \mu(\zeta) = \zeta$ and $F(\zeta, u_1, u_2, u_3) = \tilde{f}(\zeta, u_1)u_3$, then equation (1.1) will reduce to equation (1.4). Also, if

$$f(\zeta, u_1, u_2) = g(\zeta, u_1, u_2) = 0, \quad h = 1, \quad \phi(\zeta) = \zeta,$$

and

$$F(\zeta, u_1, u_2, u_3) = F(\zeta, u_3, u_2), \quad G(\zeta, u_1, u_2, u_3) = G(\zeta, u_3, u_2),$$

then equation (1.10) is obtained from equation (1.1).

For

$$f(\zeta, u_1, u_2) = A(\zeta), \quad g(\zeta, u_1, u_2) = 0, \quad \phi(\zeta) = \mu(\zeta) = \zeta,$$

and

$$F(\zeta, u_1, u_2, u_3) = u_3, \quad G(\zeta, u_1, u_2, u_3) = 1,$$

then equation (1.26) is reduced to equation (1.1). Also equation (1.1) reduces to equation (1.27) through $h = 1$, $f(\zeta, u_1, u_2) = A(\zeta)$, $g(\zeta, u_1, u_2) = 0$, $G(\zeta, u_1, u_2, u_3) = 1$, $\phi(\zeta) = \mu(\zeta) = \zeta$ and $F(\zeta, u_1, u_2, u_3) = u_3$ and so on.

To solve different types of functional integral equations, many authors have investigated these equations using the concepts of measure of noncompactness (MNC) with fixed point theorems in Banach function spaces. For more information, we refer to [4, 7, 14, 21–26, 30, 31, 34–38, 41–44, 49, 50] and references therein.

The paper is organized into 4 sections with the introduction. In Section 2, we recall preliminaries and specify the idea of MNC. Section 3 is involved to stating and proving a theorem for equation (1.1) including densifying operators using by Petryshyn's fixed point theorem. In the last section, we show some instances that demonstrate the validity of this class of FFIE.

2. PRELIMINARIES

First, we recall the idea of fractional integral of order h for $u(\zeta)$.

Definition 2.1 ([40]). Let $u \in C[a, b]$ and $a < \zeta < b$. Then,

$$I_{a+}^h u(\zeta) = \frac{1}{\Gamma(h)} \int_a^\zeta \frac{u(s)}{(\zeta - s)^{1-h}} ds, \quad h > 0,$$

is known as the Riemann-Liouville fractional integral of order h . Γ the gamma function is defined as

$$\Gamma(h) = \int_0^{+\infty} s^{h-1} e^{-s} ds.$$

Let E be a real Banach space and B_{r_θ} denote closed ball with center at θ and radius r_θ . $\partial B_{r_\theta} = \{u \in E : \|u\| = r_\theta\}$ for the sphere in E around θ with radius $r_\theta > 0$. The MNC is a helpful tool to apply fixed point theory in non-linear analysis in any Banach space E .

Definition 2.2 ([3]). Let H be a non-empty and bounded subset of E then the Kuratowski MNC is defined as:

$$\hat{\alpha}(H) = \inf \left\{ \epsilon > 0 : H = \bigcup_{i=1}^n H_i \text{ with } \text{diam} H_i \leq \epsilon, i = 1, 2, \dots, n \right\}.$$

Definition 2.3 ([3]). The Hausdroff MNC is defined as:

$$\psi(H) = \inf \{ \epsilon > 0 : \text{exists a finite } \epsilon\text{-net for } H \text{ in } E \},$$

where a finite ϵ -net for H in E is a set $\{u_1, u_2, \dots, u_n\} \subset E$ such that the balls $B_\epsilon(E, u_1), B_\epsilon(E, u_2), \dots, B_\epsilon(E, u_m)$ cover H .

These MNCs are respectively similar in the sense that

$$\psi(H) \leq \hat{\alpha}(H) \leq 2\psi(H),$$

for any bounded set $H \subset E$.

Theorem 2.1. Let $H, \hat{H} \subset E$ be bounded sets and $\lambda \in \mathbb{R}$. Then,

- (i) $\psi(H \cup \hat{H}) = \max\{\psi(H), \psi(\hat{H})\};$
- (ii) $H \subseteq \hat{H}$ implies $\psi(H) \leq \psi(\hat{H});$
- (iii) $\psi(\bar{co}H) = \psi(H);$
- (iv) $\psi(H) = 0$ if and only if H is relatively-compact;
- (v) $\psi(\lambda H) = |\lambda|\psi(H);$
- (vi) $\psi(H + \hat{H}) \leq \psi(H) + \psi(\hat{H}).$

Further, Banach space $C[0, b]$ is the set of all real valued continuous functions on $[0, b]$ with the sup norm

$$\|u\| = \max\{|u(\zeta)| : \zeta \in [0, b]\}.$$

Also, space $E = C[0, b]$ is the formation of Banach algebra.

Suppose $H \subset C[0, b]$ be a fix set. For $\epsilon > 0$ and $u \in H$, the modulus of continuity of u is defined by

$$\omega(u, \epsilon) = \sup\{|u(\zeta_2) - u(\zeta_1)| : \zeta_2, \zeta_1 \in [0, b], |\zeta_2 - \zeta_1| \leq \epsilon\}.$$

Further,

$$\omega(H, \epsilon) = \sup\{\omega(u, \epsilon) : u \in H\}, \quad \omega_0(H) = \lim_{\epsilon \rightarrow 0} \omega(H, \epsilon).$$

Theorem 2.2 ([34]). The Hausdorff MNC is equivalent to

$$\psi(H) = \lim_{\epsilon \rightarrow 0} \sup_{u \in H} \omega(u, \epsilon),$$

for all bounded sets $H \subset C[0, b]$.

Definition 2.4 ([45]). Let $T : E \rightarrow E$ be a continuous mapping of E . T is said to be a k -set contraction if for all $P \subset E$ with P bounded, $T(P)$ is bounded and $\hat{\alpha}(TP) \leq k\hat{\alpha}(P)$, $k \in (0, 1)$. If $\hat{\alpha}(TP) < \hat{\alpha}(P)$ for all $\hat{\alpha}(P) > 0$, then T is called the densifying or condensing mapping.

Theorem 2.3 ([48]). *Let $T : B_{r_0} \rightarrow E$ be a condensing mapping, which fulfills the boundary condition if $T(u) = ku$, for some $u \in \partial B_{r_0}$ then $k \leq 1$. Then, $\mathbf{F}(T)$ (the set of fixed points of T) in B_{r_0} is non-empty.*

Lemma 2.1 ([45]). *Let F be a Banach space. If the operators F and G each fulfil the Petryshyn's condition on a bounded set P of F with constants k_1 and k_2 , respectively, then the operator $T = F.G$ fulfil the Petryshyn's condition (condensing map) on P with the constant $\|F(P)\|_{k_2} + \|G(P)\|_{k_1}$. Particularly, if $\|F(P)\|_{k_2} + \|G(P)\|_{k_1} < 1$, then T is a contraction with respect to the measure ψ and it has at least one fixed point in the set P .*

3. MAIN RESULTS

We analyze the equation (1.1) under the following conditions.

- (1) $f, g \in C(I_b \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $F, G \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, C_1] \times \mathbb{R}, \mathbb{R})$, $q \in C(I_b \times [0, C_2] \times \mathbb{R}, \mathbb{R})$, and $\phi, \theta : I_b \rightarrow \mathbb{R}^+$, $\mu : [0, C_1] \rightarrow I_b$, $\nu : [0, C_2] \rightarrow I_b$, $\alpha, \beta, \eta, \varphi : I_b \rightarrow I_b$ are continuous, $\phi(\zeta) \leq C_1$, $\theta(\zeta) \leq C_2$ for every $\zeta \in I_b$.
- (2) There exist the constants h_i , $i = 1, 2, \dots, 10$, such that $h_1 + h_2 + h_5 + h_6 < 1$, $h_3 + h_4 + h_8 + h_9 < 1$ and

$$\begin{aligned} |f(\zeta, u_1, u_2) - f(\zeta, v_1, v_2)| &\leq h_1|u_1 - v_1| + h_2|u_2 - v_2|, \\ |g(\zeta, u_1, u_2) - g(\zeta, v_1, v_2)| &\leq h_3|u_1 - v_1| + h_4|u_2 - v_2|, \\ |F(\zeta, u_1, u_2, u_3) - F(\zeta, v_1, v_2, v_3)| &\leq h_5|u_1 - v_1| + h_6|u_2 - v_2| + h_7|u_3 - v_3|, \\ |G(\zeta, u_1, u_2, u_3) - G(\zeta, v_1, v_2, v_3)| &\leq h_8|u_1 - v_1| + h_9|u_2 - v_2| + h_{10}|u_3 - v_3|. \end{aligned}$$

- (3) There exists $r_0 > 0$ such that the following bounded conditions are fulfilled: $\sup\{(f' + M_1) \times (g' + M_2)\} \leq r_0$ and $\{(f' + M_1)(h_3 + h_4 + h_8 + h_9) + (g' + M_2)(h_1 + h_2 + h_5 + h_6)\} < 1$. Here,

$$\begin{aligned} f' &= \sup\{|f(\zeta, u_1, u_2)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0]\}, \\ g' &= \sup\{|g(\zeta, u_1, u_2)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0]\}, \\ M_1 &= \sup\left\{|F(\zeta, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \text{ and } \right. \\ &\quad \left. u_3 \in \left[-\frac{H_1 C_1^h}{\Gamma(h+1)}, \frac{H_1 C_1^h}{\Gamma(h+1)}\right]\right\}, \\ M_2 &= \sup\left\{|G(s, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \text{ and } \right. \\ &\quad \left. u_3 \in [-H_2 C_2, H_2 C_2]\right\}, \\ H_1 &= \sup\{|p(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_1] \text{ and } u \in [-r_0, r_0]\}, \\ H_2 &= \sup\{|q(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_2] \text{ and } u \in [-r_0, r_0]\}. \end{aligned}$$

Theorem 3.1. *Under the assumptions (1)–(3), the equation (1.1) has at least one solution in $E = C(I_b)$.*

Proof. Define the operators $P, S : B_{r_0} \rightarrow E$ in the following form

$$(Pu)(\zeta) = \left(f(\zeta, u(\zeta), u(\alpha(\zeta))) + F\left(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right),$$

$$(Su)(\zeta) = \left(g(\zeta, u(\zeta), u(\eta(\zeta))) + G\left(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))) ds \right) \right),$$

for $\zeta \in [0, b]$.

Further, we set the operator T such that $Tu = (Pu)(Su)$. Now, we prove that P is continuous on B_{r_0} . To do this, let $\epsilon > 0$ and any $u, z \in B_{r_0}$ such that $\|u - z\| < \epsilon$. Then,

$$\begin{aligned} & |(Pu)(\zeta) - (Pz)(\zeta)| \\ &= \left| f(\zeta, u(\zeta), u(\alpha(\zeta))) + F\left(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right. \\ &\quad \left. - f(\zeta, z(\zeta), z(\alpha(\zeta))) - F\left(\zeta, z(\zeta), z(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, z(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right| \\ &\leq h_1|u(\zeta) - z(\zeta)| + h_2|u(\alpha(\zeta)) - z(\alpha(\zeta))| + h_5|u(\zeta) - z(\zeta)| + h_6|u(\alpha(\zeta)) - z(\alpha(\zeta))| \\ &\quad + h_7 \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{|p(\zeta, s, u(\mu(s))) - p(\zeta, s, z(\mu(s)))|}{(\phi(\zeta) - s)^{1-h}} ds \\ &\leq (h_1 + h_2 + h_5 + h_6)|u(\zeta) - z(\zeta)| + \frac{h_7}{\Gamma(h+1)} C_1^h \omega(p, \epsilon), \end{aligned}$$

where

$$\omega(p, \epsilon) = \sup\{|p(\zeta, s, u) - p(\zeta, s, z)| : \zeta \in I_b, s \in [0, C_1], u, z \in [-r_0, r_0], |u - z| \leq \epsilon\}.$$

By the uniform continuity of $p = p(\zeta, s, u)$ on $I_b \times [0, C_1] \times [-r_0, r_0]$, we indicate that $\omega(p, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. From the above inequality, we prove that P is continuous on B_{r_0} . Further,

$$\begin{aligned} & |(Su)(\zeta) - (Sz)(\zeta)| \\ &= \left| g(\zeta, u(\zeta), u(\eta(\zeta))) + G\left(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))) ds \right) \right. \\ &\quad \left. - g(\zeta, z(\zeta), z(\eta(\zeta))) - G\left(\zeta, z(\zeta), z(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, z(\nu(s))) ds \right) \right| \\ &\leq h_3|u(\zeta) - z(\zeta)| + h_4|u(\eta(\zeta)) - z(\eta(\zeta))| + h_8|u(\zeta) - z(\zeta)| + h_9|u(\varphi(\zeta)) - z(\varphi(\zeta))| \\ &\quad + h_{10} \int_0^{\theta(\zeta)} |q(\zeta, s, u(\nu(s))) - q(\zeta, s, z(\nu(s)))| ds \\ &\leq (h_3 + h_4 + h_8 + h_9)|u(\zeta) - z(\zeta)| + h_{10} C_2 \omega(q, \epsilon), \end{aligned}$$

where

$$\omega(q, \epsilon) = \sup\{|q(\zeta, s, u) - q(\zeta, s, z)| : \zeta, s \in I_b, u, z \in [-r_0, r_0], |u - z| \leq \epsilon\}.$$

By the uniform continuity of $q = q(\zeta, s, u)$ on $I_b \times I_b \times [-r_0, r_0]$, we indicate that $\omega(q, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. By the inequality above, we can prove that S is continuous on B_{r_0} . Hence, T is a continuous operator on B_{r_0} .

Next, we show that the P fulfill the condensing condition on B_{r_0} with respect to ψ . To do this, take any subset H of B_{r_0} . Select $\epsilon > 0$ and $\zeta_1, \zeta_2 \in I_b$ such that $|\zeta_1 - \zeta_2| \leq \epsilon$.

We get

$$\begin{aligned} & |(Pu)(\zeta_2) - (Pu)(\zeta_1)| \\ &= \left| f(\zeta_2, u(\zeta_2), u(\alpha(\zeta_2))) + F\left(\zeta_2, u(\zeta_2), u(\beta(\zeta_2)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_2)} \frac{p(\zeta_2, s, u(\mu(s)))}{(\phi(\zeta_2) - s)^{1-h}} ds\right) \right. \\ &\quad \left. - f(\zeta_1, u(\zeta_1), u(\alpha(\zeta_1))) - F\left(\zeta_1, u(\zeta_1), u(\beta(\zeta_1)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right| \\ &\leq |f(\zeta_2, u(\zeta_2), u(\alpha(\zeta_2))) - f(\zeta_2, u(\zeta_1), u(\alpha(\zeta_1)))| + |f(\zeta_2, u(\zeta_1), u(\alpha(\zeta_1))) \\ &\quad - f(\zeta_1, u(\zeta_1), u(\alpha(\zeta_1)))| + \left| F\left(\zeta_2, u(\zeta_2), u(\beta(\zeta_2)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_2)} \frac{p(\zeta_2, s, u(\mu(s)))}{(\phi(\zeta_2) - s)^{1-h}} ds\right) \right. \\ &\quad \left. - F\left(\zeta_2, u(\zeta_2), u(\beta(\zeta_2)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right| \\ &\quad + \left| F\left(\zeta_2, u(\zeta_2), u(\beta(\zeta_2)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right. \\ &\quad \left. - F\left(\zeta_2, u(\zeta_2), u(\beta(\zeta_1)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right| \\ &\quad + \left| F\left(\zeta_2, u(\zeta_2), u(\beta(\zeta_1)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right. \\ &\quad \left. - F\left(\zeta_2, u(\zeta_1), u(\beta(\zeta_1)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right| \\ &\quad + \left| F\left(\zeta_2, u(\zeta_1), u(\beta(\zeta_1)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right. \\ &\quad \left. - F\left(\zeta_1, u(\zeta_1), u(\beta(\zeta_1)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds\right) \right| \\ &\leq h_1|u(\zeta_2) - u(\zeta_1)| + h_2|u(\alpha(\zeta_2)) - u(\alpha(\zeta_1))| + \omega_f(I_b, \epsilon) \\ &\quad + \frac{h_7}{\Gamma(h)} \left| \int_0^{\phi(\zeta_2)} \frac{p(\zeta_2, s, u(\mu(s)))}{(\phi(\zeta_2) - s)^{1-h}} ds - \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds \right| \\ &\quad + h_5|u(\zeta_2) - u(\zeta_1)| + h_6|u(\beta(\zeta_2)) - u(\beta(\zeta_1))| + \omega_F(I_b, \epsilon) \\ &\leq h_1\omega(u, \epsilon) + h_2\omega(u, \omega(\alpha, \epsilon)) + \omega_f(I_b, \epsilon) + h_5\omega(u, \epsilon) + h_6\omega(u, \omega(\beta, \epsilon)) + \omega_F(I_b, \epsilon) \end{aligned}$$

$$\begin{aligned}
& + \frac{h_7}{\Gamma(h)} \left[\left| \int_0^{\phi(\zeta_2)} \frac{p(\zeta_2, s, u(\mu(s)))}{(\phi(\zeta_2) - s)^{1-h}} ds - \int_0^{\phi(\zeta_2)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_2) - s)^{1-h}} ds \right| \right. \\
& + \left| \int_0^{\phi(\zeta_2)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_2) - s)^{1-h}} ds - \int_0^{\phi(\zeta_2)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds \right| \\
& + \left. \left| \int_0^{\phi(\zeta_2)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds - \int_0^{\phi(\zeta_1)} \frac{p(\zeta_1, s, u(\mu(s)))}{(\phi(\zeta_1) - s)^{1-h}} ds \right| \right] \\
& \leq h_1\omega(u, \epsilon) + h_2\omega(u, \omega(\alpha, \epsilon)) + \omega_f(I_b, \epsilon) + h_5\omega(u, \epsilon) + h_6\omega(u, \omega(\beta, \epsilon)) + \omega_F(I_b, \epsilon) \\
& + \frac{h_7}{\Gamma(h)} \left[\int_0^{\phi(\zeta_2)} \frac{|p(\zeta_2, s, u(\mu(s))) - p(\zeta_1, s, u(\mu(s)))|}{(\phi(\zeta_2) - s)^{1-h}} ds \right. \\
& + \int_0^{\phi(\zeta_2)} |p(\zeta_1, s, u(\mu(s)))| [(\phi(\zeta_2) - s)^{h-1} - (\phi(\zeta_1) - s)^{h-1}] ds \\
& + \left. \int_{\phi(\zeta_1)}^{\phi(\zeta_2)} \frac{|p(\zeta_1, s, u(\mu(s)))|}{(\phi(\zeta_1) - s)^{1-h}} ds \right] \\
& \leq h_1\omega(u, \epsilon) + h_2\omega(u, \omega(\alpha, \epsilon)) + \omega_f(I_b, \epsilon) + h_5\omega(u, \epsilon) + h_6\omega(u, \omega(\beta, \epsilon)) + \omega_F(I_b, \epsilon) \\
& + \frac{h_7}{\Gamma(h+1)} \omega_p(I_b, \epsilon) (\phi(\zeta_2))^h + \frac{h_7}{\Gamma(h+1)} H_1 [(\phi(\zeta_2))^h - (\phi(\zeta_1))^h] \\
& \leq h_1\omega(u, \epsilon) + h_2\omega(u, \omega(\alpha, \epsilon)) + \omega_f(I_b, \epsilon) + h_5\omega(u, \epsilon) + h_6\omega(u, \omega(\beta, \epsilon)) + \omega_F(I_b, \epsilon) \\
& + \frac{h_7}{\Gamma(h+1)} \omega_p(I_b, \epsilon) (\phi(\zeta_2))^h + \frac{h_7}{\Gamma(h+1)} H_1 [(\phi(\zeta_2)) - (\phi(\zeta_2))]^h,
\end{aligned}$$

where

$$\begin{aligned}
\omega_F(I_b, \epsilon) &= \sup \left\{ |F(\zeta_2, u_1, u_2, u_3) - F(\zeta_1, u_1, u_2, u_3)| : \zeta_2, \zeta_1 \in I_b, \right. \\
&\quad \left. u_3 \in \left[-\frac{H_1 C_1^h}{\Gamma(h+1)}, \frac{H_1 C_1^h}{\Gamma(h+1)} \right], u_1, u_2 \in [-r_0, r_0], |\zeta_2 - \zeta_1| \leq \epsilon \right\}, \\
\omega_p(I_b, \epsilon) &= \sup \{ |p(\zeta_2, s, u) - p(\zeta_1, s, u)| : |\zeta_2 - \zeta_1| \leq \epsilon, \zeta_2, \zeta_1 \in I_b, u \in [-r_0, r_0], \\
&\quad s \in [0, C_1] \}, \\
\omega_f(I_b, \epsilon) &= \sup \{ |f(\zeta_2, u_1, u_2) - f(\zeta_1, u_1, u_2)| : |\zeta_2 - \zeta_1| \leq \epsilon, \zeta_2, \zeta_1 \in I_b, \\
&\quad u_1, u_2 \in [-r_0, r_0] \}, \\
\omega(\alpha, \epsilon) &= \sup \{ |\alpha(\zeta_2) - \alpha(\zeta_1)| : \zeta_2, \zeta_1 \in I_b, |\zeta_2 - \zeta_1| \leq \epsilon \}, \\
\omega(\beta, \epsilon) &= \sup \{ |\beta(\zeta_2) - \beta(\zeta_1)| : \zeta_2, \zeta_1 \in I_b, |\zeta_2 - \zeta_1| \leq \epsilon \}, \\
\omega(\phi, \epsilon) &= \sup \{ |\phi(\zeta_2) - \phi(\zeta_1)| : \zeta_2, \zeta_1 \in I_b, |\zeta_2 - \zeta_1| \leq \epsilon \}.
\end{aligned}$$

From the above estimate,

$$\omega(FH, \epsilon) \leq h_1\omega(u, \epsilon) + h_2\omega(H, \omega(\alpha, \epsilon)) + \omega_f(I_b, \epsilon)$$

$$\begin{aligned}
& + h_5 \omega(u, \epsilon) + h_6 \omega(H, \omega(\beta, \epsilon)) + \omega_F(I_b, \epsilon) \\
& + \frac{h_7}{\Gamma(h+1)} \omega_p(I_b, \epsilon) (C_1)^h + \frac{h_7}{\Gamma(h+1)} H_1 [\omega(\phi, \epsilon)]^h.
\end{aligned}$$

Taking limit as $\epsilon \rightarrow 0$, we obtain $\omega_0(FH) \leq (h_1 + h_2 + h_5 + h_6) \omega_0(H)$. We get

$$(3.1) \quad \psi(FH) \leq (h_1 + h_2 + h_5 + h_6) \psi(H).$$

Hence, F is a condensing map.

Similarly,

$$\begin{aligned}
& |(Su)(\zeta_2) - (Su)(\zeta_1)| \\
& = \left| g(\zeta_2, u(\zeta_2), u(\eta(\zeta_2))) + G\left(\zeta_2, u(\zeta_2), u(\varphi(\zeta_2)), \int_0^{\theta(\zeta_2)} q(\zeta_2, s, u(\nu(s))) ds\right) \right. \\
& \quad \left. - g(\zeta_1, u(\zeta_1), u(\eta(\zeta_1))) - G\left(\zeta_1, u(\zeta_1), u(\varphi(\zeta_1)), \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) \right| \\
& \leq |g(\zeta_2, u(\zeta_2), u(\varphi(\zeta_2))) - g(\zeta_2, u(\zeta_1), u(\varphi(\zeta_1)))| + |g(\zeta_2, u(\zeta_1), u(\varphi(\zeta_1))) \\
& \quad - g(\zeta_1, u(\zeta_1), u(\varphi(\zeta_1)))| + \left| G\left(\zeta_2, u(\zeta_2), u(\varphi(\zeta_2)), \int_0^{\theta(\zeta_2)} q(\zeta_2, s, u(\nu(s))) ds\right) \right. \\
& \quad \left. - G\left(\zeta_2, u(\zeta_2), u(\varphi(\zeta_2)), \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) \right| + \left| G\left(\zeta_2, u(\zeta_2), u(\varphi(\zeta_2)), \right. \right. \\
& \quad \left. \times \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) - G\left(\zeta_2, u(\zeta_2), u(\varphi(\zeta_1)), \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) \Big| \\
& \quad + \left| G\left(\zeta_2, u(\zeta_2), u(\varphi(\zeta_1)), \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) - G\left(\zeta_2, u(\zeta_1), u(\varphi(\zeta_1)), \right. \right. \\
& \quad \left. \times \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) \Big| + \left| G\left(\zeta_2, u(\zeta_1), u(\varphi(\zeta_1)), \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) \right. \\
& \quad \left. - G\left(\zeta_1, u(\zeta_1), u(\varphi(\zeta_1)), \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds\right) \right| \\
& \leq h_3 |u(\zeta_2) - u(\zeta_1)| + h_4 |u(\eta(\zeta_2)) - u(\eta(\zeta_1))| + \omega_g(I_b, \epsilon) \\
& \quad + h_{10} \left| \int_0^{\theta(\zeta_2)} q(\zeta_2, s, u(\nu(s))) ds - \int_0^{\theta(\zeta_1)} q(\zeta_1, s, u(\nu(s))) ds \right| \\
& \quad + h_8 |u(\zeta_2) - u(\zeta_1)| + h_9 |u(\varphi(\zeta_2)) - u(\varphi(\zeta_1))| + \omega_G(I_b, \epsilon) \\
& \leq h_3 \omega(u, \epsilon) + h_4 (\omega(u, \eta, \epsilon)) + \omega_g(I_b, \epsilon) + h_8 \omega(u, \epsilon) + h_9 (\omega(u, \varphi, \epsilon)) + \omega_g(I_b, \epsilon) \\
& \quad + h_{10} \int_0^{\theta(\zeta_1)} |q(\zeta_2, s, u(\nu(s))) - q(\zeta_1, s, u(\nu(s)))| ds + h_{10} \int_{\theta(\zeta_1)}^{\theta(\zeta_2)} |q(\zeta_2, s, u(\nu(s)))| ds \\
& \leq h_3 \omega(u, \epsilon) + h_4 (\omega(u, \eta, \epsilon)) + \omega_g(I_b, \epsilon) + h_8 \omega(u, \epsilon) + h_9 (\omega(u, \varphi, \epsilon)) + \omega_g(I_b, \epsilon) \\
& \quad + h_{10} C_2 \omega_q(I_b, \epsilon) + h_{10} H_2 \omega(\theta, \epsilon),
\end{aligned}$$

where

$$\begin{aligned}\omega_G(I_b, \epsilon) &= \sup\{|G(\zeta_2, u_1, u_2, u_3) - G(\zeta_1, u_1, u_2, u_3)| : \zeta_2, \zeta_1 \in I_b, \\ &\quad u_3 \in [-C_2 H_2, C_2 H_2], u_1, u_2 \in [-r_0, r_0], |\zeta_2 - \zeta_1| \leq \epsilon\}, \\ \omega_q(I_b, \epsilon) &= \sup\{|q(\zeta_2, s, u) - q(\zeta_1, s, u)| : |\zeta_2 - \zeta_1| \leq \epsilon, \zeta_2, \zeta_1 \in I_b, u \in [-r_0, r_0], \\ &\quad s \in [0, C_2]\}, \\ \omega_g(I_b, \epsilon) &= \sup\{|g(\zeta_2, u_1, u_2) - g(\zeta_1, u_1, u_2)| : |\zeta_2 - \zeta_1| \leq \epsilon, \zeta_2, \zeta_1 \in I_b, \\ &\quad u_1, u_2 \in [-r_0, r_0]\}, \\ \omega(\eta, \epsilon) &= \sup\{|\eta(\zeta_2) - \eta(\zeta_1)| : \zeta_2, \zeta_1 \in I_b, |\zeta_2 - \zeta_1| \leq \epsilon\}, \\ \omega(\varphi, \epsilon) &= \sup\{|\varphi(\zeta_2) - \varphi(\zeta_1)| : \zeta_2, \zeta_1 \in I_b, |\zeta_2 - \zeta_1| \leq \epsilon\}, \\ \omega(\theta, \epsilon) &= \sup\{|\theta(\zeta_2) - \theta(\zeta_1)| : \zeta_2, \zeta_1 \in I_b, |\zeta_2 - \zeta_1| \leq \epsilon\}.\end{aligned}$$

From above estimate

$$\begin{aligned}\omega(GH, \epsilon) &\leq h_3 \omega(u, \epsilon) + h_4(H, \omega(\eta, \epsilon)) + \omega_g(I_b, \epsilon) \\ &\quad + h_8 \omega(u, \epsilon) + h_9(H, \omega(\varphi, \epsilon)) + \omega_g(I_b, \epsilon) + h_{10} C_2 \omega_q(I_b, \epsilon) + h_{10} H_2 \omega(\theta, \epsilon).\end{aligned}$$

Taking limit as $\epsilon \rightarrow 0$, we obtain

$$\omega_0(GH) \leq (h_3 + h_4 + h_8 + h_9) \omega_0(H).$$

This gives the following estimate

$$(3.2) \quad \psi(GH) \leq (h_3 + h_4 + h_8 + h_9) \psi(H).$$

Hence, G is a condensing map. So, from (3.1), (3.2) and Lemma (2.1), we get T is a condensing mapping with constant $(f' + M_1)(h_3 + h_4 + h_8 + h_9) + (g' + M_2)(h_1 + h_2 + h_5 + h_6) < 1$.

Now, let $u \in \partial B_{r_0}$ and if $Tu = ku$, then $\|Tu\| = k\|u\| = kr_0$ and by assumption (3),

$$\begin{aligned}\|Tu(\zeta)\| &= \left\| \left(f(\zeta, u(\zeta), u(\alpha(\zeta))) + F\left(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right) \right. \\ &\quad \times \left. \left(g(\zeta, u(\zeta), u(\eta(\zeta))) + G\left(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))) ds \right) \right) \right\| \\ &\leq r_0, \quad \text{for all } \zeta \in I_b.\end{aligned}$$

Hence, $\|Tu\| \leq r_0$, i.e., $k \leq 1$. □

The following corollary is a direct consequence of Theorem 3.1 under the following assumptions (1') – (3').

- (1') $f \in C(I_b \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $F, G \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, C_1] \times \mathbb{R}, \mathbb{R})$, $q \in C(I_b \times [0, C_2] \times \mathbb{R}, \mathbb{R})$, $\phi, \theta : I_b \rightarrow \mathbb{R}^+$, $\mu : [0, C_1] \rightarrow I_b$, $\nu : [0, C_2] \rightarrow I_b$, $\alpha, \beta, \varphi : I_b \rightarrow I_b$ are continuous such that $\phi(\zeta) \leq C_1, \theta(\zeta) \leq C_2$ for every $\zeta \in I_b$.

- (2') There exist constants h_i , $i = 1, 2, \dots, 8$, such that $h_1 + h_2 + h_3 + h_4 < 1$, $h_6 + h_7 < 1$ and

$$\begin{aligned} |f(\zeta, u_1, u_2) - f(\zeta, v_1, v_2)| &\leq h_1|u_1 - v_1| + h_2|u_2 - v_2|, \\ |F(\zeta, u_1, u_2, u_3) - F(\zeta, v_1, v_2, v_3)| &\leq h_3|u_1 - v_1| + h_4|u_2 - v_2| + h_5|u_3 - v_3|, \\ |G(\zeta, u_1, u_2, u_3) - G(\zeta, v_1, v_2, v_3)| &\leq h_6|u_1 - v_1| + h_7|u_2 - v_2| + h_8|u_3 - v_3|. \end{aligned}$$

- (3') There exists $r_0 > 0$ such that the following bounded condition fulfilled

$$\sup\{(f' + M_1) \times (M_2)\} \leq r_0,$$

and $(f' + M_1)(h_6 + h_7) + (M_2)(h_1 + h_2 + h_3 + h_4) < 1$. Here,

$$f' = \sup\{|f(\zeta, u_1, u_2)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0]\},$$

$$M_1 = \sup\left\{|F(\zeta, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0]\right.$$

$$\left. \text{and } u_3 \in \left[-\frac{H_1 C_1^h}{\Gamma(h+1)}, \frac{H_1 C_1^h}{\Gamma(h+1)}\right]\right\},$$

$$M_2 = \sup\{|G(s, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0], u_3 \in [-H_2 C_2, H_2 C_2]\},$$

$$H_1 = \sup\{|p(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_1] \text{ and } u \in [-r_0, r_0]\},$$

$$H_2 = \sup\{|q(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_2] \text{ and } u \in [-r_0, r_0]\}.$$

Corollary 3.1. Under the assumptions (1')–(3'), the equation

$$\begin{aligned} u(\zeta) = & \left(f(\zeta, u(\zeta), u(\alpha(\zeta))) + F\left(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right) \\ & \times \left(G\left(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))) ds \right) \right) \end{aligned}$$

has at least one solution in $C(I_b)$.

Proof. We can omit the detailed proof here, as it closely resembles to the proof of Theorem 3.1. $\square$

The next corollary is also derived from Theorem 3.1 under the following assumptions.

- (1'') $F, G \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, C_1] \times \mathbb{R}, \mathbb{R})$, $q \in C(I_b \times [0, C_2] \times \mathbb{R}, \mathbb{R})$, $\phi, \theta : I_b \rightarrow \mathbb{R}^+$, $\mu : [0, C_1] \rightarrow I_b$, $\nu : [0, C_2] \rightarrow I_b$, $\beta, \varphi : I_b \rightarrow I_b$ are continuous, $\phi(\zeta) \leq C_1$, $\theta(\zeta) \leq C_2$ for every $\zeta \in I_b$.
 (2'') There exist constants h_i , $i = 1, 2, \dots, 6$, such that $h_1 + h_2 < 1$, $h_4 + h_5 < 1$ and

$$\begin{aligned} |F(\zeta, u_1, u_2, u_3) - F(\zeta, v_1, v_2, v_3)| &\leq h_1|u_1 - v_1| + h_2|u_2 - v_2| + h_3|u_3 - v_3|, \\ |G(\zeta, u_1, u_2, u_3) - G(\zeta, v_1, v_2, v_3)| &\leq h_4|u_1 - v_1| + h_5|u_2 - v_2| + h_6|u_3 - v_3|. \end{aligned}$$

(3'') There exists $r_0 > 0$ such that the following bounded condition is fulfilled

$$\sup\{(M_1) \times (M_2)\} \leq r_0,$$

and $M_1(h_4 + h_5) + M_2(h_1 + h_2) < 1$, where

$$M_1 = \sup \left\{ |F(\zeta, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \right.$$

$$\left. \text{and } u_3 \in \left[-\frac{H_1 C_1^h}{\Gamma(h+1)}, \frac{H_1 C_1^h}{\Gamma(h+1)} \right] \right\},$$

$$M_2 = \sup\{|G(s, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0], u_3 \in [-H_2 C_2, H_2 C_2]\},$$

$$H_1 = \sup\{|p(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_1] \text{ and } u \in [-r_0, r_0]\},$$

$$H_2 = \sup\{|q(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_2] \text{ and } u \in [-r_0, r_0]\}.$$

Corollary 3.2. *Under the above assumptions (1'')–(3''), there exists at least one solution in $C(I_b)$ for the equation:*

$$u(\zeta) = \left(F\left(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right. \\ \left. \times \left(G\left(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))) ds \right) \right) \right).$$

Proof. To avoid redundancy, we can skip the proof here, as it closely resembles the proof of Theorem 3.1. $\square$

4. EXAMPLES

In this section, we present some examples of equations to illustrate the usefulness of our study.

Example 4.1. Let us consider following fractional integral equation

(4.1)

$$u(\zeta) = \left(\frac{1}{4} e^{-\zeta} + \frac{\zeta^2}{3 + 3\zeta^2} \ln(1 + |u(\zeta^3)|) + \frac{1}{(5 + \sin(|u(\sqrt{\zeta})|))\Gamma(\frac{1}{3})} \right. \\ \times \int_0^\zeta \frac{e^{-3\zeta^2} (e^\zeta + \zeta \cos(1+s) + \frac{1}{3}(u(\sqrt{s})))}{(\zeta - s)^{\frac{2}{3}}} ds \Big) \times \left(\frac{1}{3} \cos(u(1 - \zeta)) + \frac{1}{4(e^t + |\cos(u(\zeta))|)} \right. \\ \left. \times \int_0^{\zeta^3} \left[e^{-2\zeta^2} \left(e^\zeta + t \cos(s) + \sin\left(\frac{u(s)}{1 + u(s)}\right) \right) \right] ds \right), \quad \zeta \in [0, 1].$$

Here $f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $F, G : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\alpha, \beta, \mu, \eta, \varphi, \theta, \phi, v : [0, 1] \rightarrow [0, 1]$, $p, q : [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and comparing (4.1) with equation (1.1), we obtain

$$\alpha(\zeta) = \theta(\zeta) = \zeta^3, \quad \beta = \mu = \sqrt{\zeta}, \quad \varphi = \phi = v = \zeta, \quad \eta(\zeta) = 1 - \zeta,$$

$$h = \frac{1}{3}, \quad C_1 = C_2 = 1,$$

$$f(\zeta, u_1, u_2) = \frac{1}{4}e^{-\zeta} + \frac{\zeta^2}{3 + 3\zeta^2} \ln(1 + |u_2|), \quad g(\zeta, u_1, u_2) = \frac{1}{3} \cos(u_2),$$

$$F(\zeta, u_1, u_2, z) = \frac{z}{5 + \sin(|u_2|)}, \quad z = \frac{1}{\Gamma(\frac{1}{3})} \int_0^\zeta \frac{p(\zeta, s, u(\mu(s)))}{(\zeta - s)^{\frac{2}{3}}} ds,$$

$$G(\zeta, u_1, u_2, w) = \frac{w}{4(e^t + |\cos(u_2)|)}, \quad w = \int_0^{\zeta^3} q(\zeta, s, u(v(s))) ds,$$

$$p(\zeta, s, u(\mu(s))) = e^{-3\zeta^2} (e^\zeta + \zeta \cos(1 + s) + \frac{1}{3}(u(\sqrt{s}))), \quad |p(\zeta, s, u)| \leq e + 1 + \frac{1}{3}|u|,$$

$$q(\zeta, s, u(v(s))) = e^{-2\zeta^2} \left(e^\zeta + t \cos(s) + \sin \left(\frac{u(s)}{1 + u(s)} \right) \right), \quad |q(\zeta, s, u)| \leq e + 2,$$

for all $\zeta \in [0, 1]$.

Above functions fulfill the assumptions (1) and (2) of Theorem 3.1. We have

$$\begin{aligned} |u(\zeta)| &= \left| \left(\frac{1}{4}e^{-\zeta} + \frac{\zeta^2}{3 + 3\zeta^2} \ln(1 + |u(\zeta^3)|) + \frac{1}{(5 + \sin(|u(\sqrt{\zeta}|))\Gamma(\frac{1}{3}))} \right. \right. \\ &\quad \times \int_0^\zeta \frac{e^{-3\zeta^2} (e^\zeta + \zeta \cos(1 + s) + \frac{1}{3}(u(\sqrt{s})))}{(\zeta - s)^{\frac{2}{3}}} ds \Bigg) \times \left(\frac{1}{3} \cos(u(1 - \zeta)) \right) \\ &\quad \left. + \frac{1}{4(e^t + |\cos(u(\zeta))|)} \int_0^{\zeta^3} \left[e^{-2\zeta^2} (e^\zeta + t \cos(s) + \sin \left(\frac{u(s)}{1 + u(s)} \right)) \right] ds \right| \\ &\leq \left(\frac{1}{4} + \frac{1}{3}r_0 + \frac{1}{5\Gamma(\frac{4}{3})} \left(\frac{1}{3}r_0 + e + 1 \right) \right) \left(\frac{1}{3} + \frac{1}{4}(e + 2) \right), \quad \text{for all } \zeta \in [0, 1]. \end{aligned}$$

Assume $\|u\| \leq r_0$, $r_0 > 0$, then

$$\left(\frac{1}{4} + \frac{1}{3}r_0 + \frac{1}{5\Gamma(\frac{4}{3})} \left(\frac{1}{3}r_0 + e + 1 \right) \right) \left(\frac{1}{3} + \frac{1}{4}(e + 2) \right) \leq r_0.$$

Thus, assumption (3) of Theorem 3.1 holds if $r_0 \geq 4.2796$. It is easy to see that $(f' + M_1)(h_3 + h_4 + h_8 + h_9) + (g' + M_2)(h_1 + h_2 + h_5 + h_6) < 1$. Hence, from Theorem 3.1, the equation (4.1) has at least one solution in $C[0, 1]$.

Example 4.2. Let us consider the following fractional integral equation

(4.2)

$$\begin{aligned} u(\zeta) &= \left(\frac{1}{2}\zeta e^{-(\zeta+4)} + \frac{\zeta^2(u(\zeta) + 2u(\sqrt{\zeta}))}{9(1 + \zeta^2)} + \frac{1}{4(1 + \sin^2(u(\zeta^2)))\Gamma(\frac{1}{2})} \right. \\ &\quad \times \int_0^\zeta \frac{\sin(u(1 - s)) + \frac{1}{3}\zeta \arctan \left(\frac{|u(1-s)|}{1+|u(1-s)|} \right)}{(\zeta - s)^{\frac{1}{2}}} ds \Bigg) \times \left(\frac{\zeta}{4(1 + \zeta)} \sin(u(\sqrt{\zeta})) \right) \end{aligned}$$

$$+ \frac{\zeta^2}{2+2\zeta^2} \int_0^1 \left[\left(\frac{\zeta}{1+\zeta+\ln(1+s)} \right) \sin \left(\frac{u(\sqrt{s})}{1+x(\sqrt{s})} \right) + \frac{u(\sqrt{s})}{2} \right] ds, \zeta \in [0, 1].$$

Here,

$$\beta(\zeta) = \zeta^2, \quad \alpha(\zeta) = \eta(\zeta) = \sqrt{\zeta}, \quad \varphi = 0, \quad \phi(\zeta) = \zeta, \quad \theta(\zeta) = 1, \quad v(s) = \sqrt{s},$$

$$\mu(s) = 1-s, \quad h = \frac{1}{2}, \quad C_1 = C_2 = 1, \quad \text{for all } \zeta \in [0, 1],$$

$$f(\zeta, u_1, u_2) = \frac{1}{2}\zeta e^{-(\zeta+4)} + \frac{\zeta^2(u_1+2u_2)}{9(1+\zeta^2)}, \quad g(\zeta, u_1, u_2) = \frac{\zeta}{4(1+\zeta)} \sin(u_2),$$

$$F(\zeta, u_1, u_2, z) = \frac{z}{4(1+\sin^2(u_2))}, \quad z = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\zeta \frac{p(\zeta, s, u(\mu(s)))}{(\zeta-s)^{\frac{1}{2}}} ds,$$

$$G(\zeta, u_1, u_2, w) = \frac{w\zeta^2}{2+2\zeta^2}, \quad w = \int_0^1 q(\zeta, s, u(v(s))) ds,$$

$$p(\zeta, s, u(\mu(s))) = \sin(u(1-s)) + \frac{1}{3}\zeta \arctan \left(\frac{|u(1-s)|}{1+|u(1-s)|} \right),$$

$$|p(\zeta, s, u)| \leq 1 + \frac{1}{3}|u|,$$

$$q(\zeta, s, u(v(s))) = \left(\frac{\zeta}{1+\zeta+\ln(1+s)} \right) \sin \left(\frac{u(\sqrt{s})}{1+x(\sqrt{s})} \right) + \frac{u(\sqrt{s})}{2},$$

$$|q(\zeta, s, u)| \leq \frac{1}{2}(1+|u|), \quad \text{for all } \zeta \in [0, 1].$$

Above functions fulfill the assumptions (1) and (2) of Theorem 3.1. Now, we review that assumption (3) also holds. Assume $\|u\| \leq r_0$, $r_0 > 0$, then

$$\begin{aligned} |u(\zeta)| &= \left| \left(\frac{1}{2}\zeta e^{-(\zeta+4)} + \frac{\zeta^2(u(\zeta)+2u(\sqrt{\zeta}))}{9(1+\zeta^2)} \right) + \frac{1}{4(1+\sin^2(u(\zeta^2)))\Gamma(\frac{1}{2})} \right. \\ &\quad \times \int_0^\zeta \frac{\sin(u(1-s)) + \frac{1}{3}\zeta \arctan \left(\frac{|u(1-s)|}{1+|u(1-s)|} \right)}{(\zeta-s)^{\frac{1}{2}}} ds \Bigg) \times \left(\frac{\zeta}{4(1+\zeta)} \sin(u(\sqrt{\zeta})) \right) \\ &\quad \left. + \frac{\zeta^2}{2+2\zeta^2} \int_0^1 \left[\left(\frac{\zeta}{1+\zeta+\ln(1+s)} \right) \sin \left(\frac{u(\sqrt{s})}{1+x(\sqrt{s})} \right) + \frac{u(\sqrt{s})}{2} \right] ds \right| \\ &\leq r_0, \quad \text{for all } \zeta \in [0, 1]. \end{aligned}$$

Hence, (3) holds if,

$$\left(\frac{1}{2} + \frac{1}{3}r_0 + \frac{1}{4\Gamma(\frac{3}{2})} \left(1 + \frac{1}{3}r_0 \right) \right) \left(\frac{1}{4} + \frac{1}{4}(1+r_0) \right) \leq r_0.$$

Thus, (3) holds if $r_0 \in [0.76875, 4.7608]$. It is easy to verify that $(f' + M_1)(h_3 + h_4 + h_8 + h_9) + (g' + M_2)(h_1 + h_2 + h_5 + h_6) < 1$. Hence, by Theorem 3.1, there exists at least one solution of (4.2) in $C[0, 1]$.

5. CONCLUSIONS

We introduced a novel approach based on the MNC and Petryshyn's fixed point theorem for solving FFIEs. The method is shown to be effective in solving a variety of FFIEs, and the results are validated using numerical examples. The results have several implications, including the provision of a new tool for solving FFIEs, the demonstration of the effectiveness of the MNC in solving FFIEs, and the provision of a theoretical foundation for numerical methods for solving FFIEs. Interested researcher could extend it to other types of fractional equations, like those involving Caputo and Riemann-Liouville derivatives, which are commonly used in modeling physical processes. Another possibility is exploring higher-dimensional problems and developing numerical methods that could make our theoretical findings more applicable. Finally, applying our results to fields like control theory and optimization could lead to significant advancements, particularly in understanding the stability and control of systems described by FFIEs.

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ON THE BRYANT-SCHNEIDER GROUP OF RIGHT CHEBAN LOOP

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ABSTRACT. In this study, we investigate the Bryant-Schneider group of right Cheban loops. The study started with establishing some results on the algebraic properties of right Cheban loop. Consequently, it is established that every right pseudo-automorphism of a right Cheban loop is an element of the Bryant-Schneider group. It is shown that the middle inner mapping T_a is an automorphism of a right Cheban loop if and only if $a \in N_\mu$ of the right Cheban loop. R_a and L_a are subsequently shown to be elements of the Bryant-Schneider group. The Crypto-automorphism group of right Cheban loop is investigated. A two-middle pseudo-automorphism group, the crypto-automorphism group and the Bryant-Schneider group of a loop are found to coincide.

1. INTRODUCTION

Cheban loops are loops of the generalized Bol-Moufang type. Both left Cheban loops and Cheban loops were introduced by Cheban [19]. The author also showed that Cheban loops are generalized Moufang loops. The structural properties of left Cheban loops and Cheban loops were studied in [12, 19]. Construction of a right Cheban loop of small orders was presented in [10] and the holomorph of the right Cheban loops was studied in [11]. Some recent results in the theory of loops, especially Bol-Moufang and non-Bol-Moufang types, can be found in [4]. Robinson [21] introduced the idea of Bryant-Schneider group of a loop and the study was motivated by the work of Bryant and Schneider [5]. Since the advent of the Bryant-Schneider group, some studies by

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Adeniran [2, 3], and Chiboka [9] have been done on it relative to CC-loops, C-loops, and extra loops after Robinson [21] studied the Bryant-Schneider group of a Bol loop. The study of subgroup of Bryant-Schneider group of a Basarab loop was recently announced by Osoba et al. [17]. As mentioned in [21], the Bryant-Schneider group of a loop is extremely useful in investigating isotopy condition(s) in loops. Adeniran, in [1, 2], established a condition under which an element of the Bryant-Schneider group of a C-loop will form an automorphism. Chiboka in [8] got a similar result for extra loops but the condition was necessary and sufficient which is not the case for C-loops in [2]. Adeniran [2] went further to show that elements of the Bryant-Schneider group of a C-loop can be expressed as a product of pseudo-automorphism and right translations of elements of the nucleus of the loop but Chiboka [8] corresponding result on extra loops was a general form of this and was also possible for left translations. The two authors confirmed that the Bryant-Schneider group of the two loops is a kind of generalized holomorph of the loops. In fact, this result is true for Bol loops [21]. The crypto-automorphism groups of some quasigroups were studied in [16] and the set of all crypto-automorphism groups of a quasigroup with left and right identity elements was shown to form a group. Consequently, different characterizations of subgroups of the crypto-automorphism groups of a middle Bol loop were presented and the crypto-automorphism groups and the Bryant-Schneider groups of a loop were found to coincide. In this study, we investigate the Bryant-Schneider groups of right Cheban loops. It is established that every right pseudo-automorphism of a right Cheban loop is an element of the Bryant-Schneider group. It is shown that the middle inner mapping T_a is an automorphism of right Cheban loop if and only if $a \in N_\mu$ of the right Cheban loop. R_a and L_a are subsequently shown to be elements of the Bryant-Schneider group. Crypto-automorphism group of right Cheban loop is investigated. A two-middle pseudo-automorphism groups, the crypto-automorphism groups and the Bryant-Schneider groups of a loop are found to coincide.

2. PRELIMINARIES

In this section, we give definitions of terminologies and present some previous results used in this work. Let Q be a non-empty set. Define a binary operation $\cdot$ on Q . If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid. If the system of equations:

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b.$$

have unique solutions in Q for x and y respectively, then $(Q, \cdot)$ is called a quasigroup. If there exists a unique element $e \in Q$, called the identity element, such that for all $x \in Q$

$$x \cdot e = e \cdot x = x.$$

$(Q, \cdot)$ is called a loop. Let x be a fixed element in a groupoid $(Q, \cdot)$. The left and right translation maps of Q , L_x and R_x are, respectively, defined by

$$yL_x = x \cdot y \quad \text{and} \quad yR_x = y \cdot x.$$

Also, a quasigroup is a groupoid in which the left and right translations defined on the groupoids are bijections. Since left and right translation mappings of a loop are bijectives, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x \backslash y = yL_x^{-1} \quad \text{and} \quad y / x = yR_x^{-1},$$

and note that, $x \backslash y = z$ if and only if $x \cdot z = y$ and $x / y = z$ if and only if $z \cdot y = x$. A loop identity is of Bol-Moufang type if two variables of its three variables occur once on each side, the third variable occurs twice on each side and the order in which the variables appear on both sides is the same, for instance, $(xy \cdot z)y = x(yz \cdot y)$. Phillips and Vojtechovsky in [20] showed that there are fourteen such varieties, they also extended and completed Fenyves work in [13]. The identity of Bol-Moufang type can be generalized by assigning different variable orders on either side of the identity, for instance, $(y \cdot xx)z = (y \cdot zx)x$. The varieties of loops axiomatized by a single identity of this type are said to be of generalized Bol-Moufang type [12]. Right, left and Cheban loops are loops of the generalized Bol-Moufang type. Both Left Cheban and Cheban loops were introduced by Cheban in [6]. He showed that Cheban loops are generalized Moufang loops. Phillips and Shcherbacov in [19] also studied on the structural properties of left Cheban and Cheban loops. They established that left Cheban loops are left conjugacy closed (LCC). Furthermore, they proved that Cheban loops have weak inverse property, are power associative and are conjugacy closed loops.

Definition 2.1. A loop satisfying the identity

$$(2.1) \quad (z \cdot yx)x = zx \cdot xy$$

is called a *Right Cheban loop* (RChL). *Left Cheban loops* on the other hand are loops satisfying the mirror identity

$$(2.2) \quad x(xy \cdot z) = yx \cdot xz.$$

Loops that are both right and left Cheban are called *Cheban loops*. Cheban loops can also be characterized as those loops that satisfy the identity

$$(2.3) \quad x(xy \cdot z) = (y \cdot zx)x.$$

Definition 2.2. A loop $(Q, \cdot)$ is called a left inverse property loop if it satisfies the left inverse property (LIP) given by: $x^\lambda(xy) = y$.

Definition 2.3. A loop $(Q, \cdot)$ is called a right inverse property loop if it satisfies the right inverse property (RIP) given by: $(yx)x^\rho = y$.

A loop is called an IP loop if it is both LIP-loop and RIP-loop.

Definition 2.4. Let $(Q, \cdot)$ be a loop and $BS(Q, \cdot)$ be a set of all permutations θ of Q such that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta)$ is an autotopism of $(Q, \cdot)$ for some $f, g \in Q$. Then, $BS(Q, \cdot)$ is called the *Bryant-Schneider group* of the loop $(Q, \cdot)$.

Definition 2.5. Let $BS_\lambda(Q, \cdot) = \{\theta \in SYM(Q, \cdot) : \exists g \in Q \ni (\theta R_g^{-1}, \theta, \theta) \in AUT(Q, \cdot)\}$, that is the set of all special maps in a loop, then $BS_\lambda \leq BS(Q, \cdot)$ is called the left Bryant-Schneider group of the $(Q, \cdot)$ and

$$BS_\rho(Q, \cdot) = \{\theta \in SYM(Q, \cdot) : \exists g \in Q \ni (\theta, \theta L_g^{-1}, \theta) \in AUT(Q, \cdot)\},$$

that is the set of all special maps in a loop, then $BS_\rho \leq BS(Q, \cdot)$ is called the right Bryant-Schneider group of the $(Q, \cdot)$.

Definition 2.6. A loop $(Q, \cdot)$ is called an automorphic inverse property loop if it satisfies the automorphic inverse identity given by $(xy)^{-1} = x^{-1}y^{-1}$.

Definition 2.7. A triple (U, V, W) of bijections from a set G onto a set H is called an *isotopism* of a groupoid $(G, \cdot)$ onto groupoid $(H, \circ)$ provided $xU \circ yV = (xy)W$ for all $x, y \in G$. $(H, \circ)$ is then called an isotope of $(G, \cdot)$, and groupoids $(G, \cdot)$ and $(H, \circ)$ are said to be *isotopic* to each other. An isotopism of $(G, \cdot)$ onto $(G, \cdot)$ is called an *autotopism* of $(G, \cdot)$.

Definition 2.8. A bijection U on Q is called a *right pseudo-automorphism* of a quasigroup $(Q, \cdot)$ if there exists at least one element $c \in Q$ such that

$$xU \cdot (yU \cdot c) = (xy)U \cdot c, \quad \text{for all } x, y \in Q.$$

The element c is then called a *companion* of U .

Definition 2.9. The inner mapping group I is generated by the permutations $R_{(x,y)} = R_x R_y R_{(xy)^{-1}}$, $L_{(x,y)} = L_x L_y L_{(yx)^{-1}}$ and $T_x = R_x L_{x^{-1}}$. These permutations are called inner mappings.

Definition 2.10. The left nucleus of $(Q, \cdot)$ denoted by

$$N_\lambda(Q, \cdot) = \{a \in Q : a \cdot xy = ax \cdot y \text{ for all } x, y \in Q\}.$$

The right nucleus of $(Q, \cdot)$ denoted by

$$N_\rho(Q, \cdot) = \{a \in Q : xy \cdot a = x \cdot ya \text{ for all } x, y \in Q\}.$$

The middle nucleus of $(Q, \cdot)$ denoted by

$$N_\mu(Q, \cdot) = \{a \in Q : xa \cdot y = x \cdot ay \text{ for all } x, y \in Q\}.$$

The nucleus of $(Q, \cdot)$ denoted by

$$N(Q, \cdot) = N_\lambda(Q, \cdot) \cap N_\rho(Q, \cdot) \cap N_\mu(Q, \cdot).$$

The centrum of $(Q, \cdot)$ denoted by

$$C(Q, \cdot) = \{a \in Q : ax = xa \text{ for all } x \in Q\}.$$

The center of $(Q, \cdot)$ denoted by

$$Z(Q, \cdot) = N(Q, \cdot) \cap C(Q, \cdot).$$

Definition 2.11. Let $(Q, \cdot)$ be a quasigroup. A mapping $\phi \in SYM(Q)$ will be called a two-middle pseudo-automorphism if there exist $a, b \in Q$ such that $(\phi R_a^{-1}, \phi L_{b\lambda}^{-1}, \phi) \in AUT(Q, \cdot)$. The set of two-middle pseudo-automorphisms will be represented by $PS_{\mu_2}(Q, \cdot)$. Note that $PS_\mu(Q, \cdot) \subseteq PS_{\mu_2}(Q, \cdot)$.

Lemma 2.1. *Let Q be a loop and a its element. Then,*

- (i) $a \in N_\lambda$ if and only if $(L_a, I, L_a) \in \text{AUT}(Q)$;
- (ii) $a \in N_\rho$ if and only if $(I, R_a, R_a) \in \text{AUT}(Q)$;
- (iii) $a \in N_\mu$ if and only if $(R^{-1}, L_a, I) \in \text{AUT}(Q)$.

Theorem 2.1 ([18]). *Let (U, V, W) be an autotopism of a loop $(Q, \cdot)$.*

- (i) *If $(Q, \cdot)$ is an RIPL, then (W, JVJ, U) is also an autotopism of $(Q, \cdot)$.*
- (ii) *If $(Q, \cdot)$ is an LIPL, then (JUJ, W, V) is also an autotopism of $(Q, \cdot)$.*

Theorem 2.2. *A loop Q is a Right Conjugacy Closed if and only if (R_x, T_x, R_x) is an autotopism for all $x \in Q$.*

Theorem 2.3 ([7]). *A loop Q is left Cheban if and only if it is Left Conjugacy Closed (LCC) and $R_x^2 = L_x^2$, for all $x \in Q$.*

Theorem 2.4 ([16]). *The set of crypto-automorphisms $\text{CAUM}(Q, \cdot)$ of a quasigroup $(Q, \cdot)$ with right and left identity elements forms a group.*

Theorem 2.5 ([16]). *Let $(Q, \cdot)$ be a loop. Then, $\text{BS}(Q, \cdot) = \text{CAUM}(Q, \cdot)$.*

3. ALGEBRAIC PROPERTIES OF RIGHT CHEBAN LOOP

Theorem 3.1. *Let $(Q, \cdot)$ be a RChL. Then, the left and right inverses coincide and thus, $x^\lambda = x^\rho = x^{-1}$.*

Proof. If $(Q, \cdot)$ is a RChL, then we have $(z \cdot yx)x = zx \cdot xy$. Setting $y = x^\rho$ in the RChL identity, we obtain

$$(z \cdot x^\rho x)x = zx \cdot xx^\rho, \quad (z \cdot x^\rho x)x = zx, \quad (z \cdot x^\rho x) = z, \quad x^\rho x = e, \quad x^\rho x = e.$$

This implies $x^\rho = x^\lambda$. Therefore, $x^\lambda = x^\rho = x^{-1}$. □

Remark 3.1. This result implies that every element in a right Cheban loop has a unique inverse.

Theorem 3.2. *Commutative RChL are Abelian groups.*

Proof. Let $(Q, \cdot)$ be a commutative RChL. Then,

$$(3.1) \quad (z \cdot yx)x = zx \cdot xy = zx \cdot yx \quad (\text{by commutativity}).$$

Setting $yx = a$ in (3.1), we have $az \cdot x = za \cdot x = zx \cdot a$. This implies $az \cdot x = a \cdot zx$ for all $a, x, z \in Q$. This shows that associativity holds in $(Q, \cdot)$. Hence, $(Q, \cdot)$ is an Abelian group. □

Theorem 3.3. *A loop $(Q, \cdot)$ is a RChL if and only if $(R_x, R_x^{-1}L_x, R_x) \in \text{AUT}(Q, \cdot)$.*

Proof. Suppose $(Q, \cdot)$ is a RChL. Then,

$$(z \cdot yx)x = zx \cdot xy, \quad (z \cdot yR_x)R_x = zR_x \cdot yL_x.$$

Setting $y = yR_x^{-1}$, we have

$$(zy)R_x = zR_x \cdot yR_x^{-1}L_x, \quad (R_x, R_x^{-1}L_x, R_x) \in \text{AUT}(Q, \cdot).$$

Conversely, suppose $(R_x, R_x^{-1}L_x, R_x) \in \text{AUT}(Q, \cdot)$. Then, letting this autotopism take on the product yz as argument, we have $yR_x \cdot zR_x^{-1}L_x = (yz)R_x$. Setting $z = zR_x$ to give

$$yR_x \cdot zL_x = (y \cdot zR_x)R_x, \quad yx \cdot xz = (y \cdot zx)x.$$

□

Theorem 3.4. *Let $(Q, \cdot)$ is a RChL. Then, $[R_{x^2}, R_x] = I$ for all $x \in Q$.*

Proof. Let $(Q, \cdot)$ be a RChL. Then, $(z \cdot yx)x = zx \cdot xy$. Setting $y = x$ in the above equation, we have $(z \cdot x^2)x = zx \cdot x^2$. This implies $zR_{x^2}R_x = zR_xR_{x^2}$. Thus, $[R_{x^2}, R_x] = I$ (i.e., R_{x^2} and R_x commute). □

Corollary 3.1. *Right Cheban loops are 3-power associative.*

Proof. By Theorem 3.4, $R_{x^2}R_x = R_xR_{x^2}$. Then, $eR_{x^2}R_x = eR_xR_{x^2}$. Thus, $x^2 \cdot x = x \cdot x^2$. It then follows that $x^3 = xxx$. □

Remark 3.2. Theorem 3.4 and Corollary 3.1 demonstrate a weaker form of Power associativity.

4. BRYANT-SCHNEIDER GROUP OF RIGHT CHEBAN LOOP

In the foregoing, we give different characterisations of right Cheban loops in terms of its automorphisms and pseudo-automorphisms with associated companion(s). These characterisations are later used to study Bryant-Schneider group of right Cheban loop.

Lemma 4.1. *Let $(Q, \cdot)$ be a RChL. Then, the following are equivalent:*

- (1) *flexibility;*
- (2) *left inverse property.*

Proof. By setting $y = y/x$ in RChL identity (2.1), one obtain

$$(4.1) \quad (z \cdot y)x = zx \cdot x(y/x).$$

(1) if and only if (2). Set $z = e$ in a RChL (4.1),

$$(4.2) \quad yR_x = yR_x^{-1}L_xL_x, \quad R_xL_x^{-1} = R_x^{-1}L_x, \quad L_x^{-1}R_x = L_xR_x^{-1}.$$

Now, setting $z = x^\lambda$ in (4.1), to get

$$(4.3) \quad (x^\lambda y)x = x(y/x), \quad R_x^{-1}L_x = L_{x^\lambda}R_x.$$

Then, in a RChL, (4.2) = (4.3) if and only if $L_xR_x^{-1} = R_x^{-1}L_x$ if and only if $yL_xR_x^{-1} = yR_x^{-1}L_x$ if and only if $(xy)/x = x \cdot (y/x)$ if and only if $xy \cdot x = x \cdot yx$. So, a RChL is flexible if and only if $L_x^{-1}R_x = L_{x^\lambda}R_x$, if and only if $L_x^{-1} = L_{x^\lambda}$ if and only if $yL_x^{-1} = yL_{x^\lambda}$ if and only if $x^\lambda y = x \setminus y$ if and only if $x^\lambda \cdot xy = y$. □

Theorem 4.1. *Let $(Q, \cdot)$ be a RChL with flexibility property and let $x \in N_\mu$ for all $x \in Q$. if $\alpha \in BS(Q, \cdot)$. Then, $\alpha \in \text{AUM}(Q, \cdot)$.*

Proof. Let $(Q, \cdot)$ be a RChL and $x \in N_\mu$ implies that $H = (R_x, L_x^{-1}, I) \in \text{AUT}(Q, \cdot)$ for all $x \in Q$. $\alpha \in \text{BS}(Q, \cdot)$ if and only if $J = (\alpha R_t^{-1}, \alpha L_k^{-1}, \alpha) \in \text{AUT}(Q, \cdot)$ for some $k, t \in Q$. Then, the product $HJ = (\alpha R_t^{-1} R_x, \alpha L_k^{-1} L_x^{-1}, \alpha) \in \text{AUT}(Q, \cdot)$ for some $k, t \in Q$ and for all $x \in Q$.

Setting $t = x$, we have $(\alpha, \alpha L_k^{-1} L_x^{-1}, \alpha) \in \text{AUT}(Q, \cdot)$ for some $k, t \in Q$ and for all $x \in Q$. Since $(Q, \cdot)$ is flexible if and only if it satisfies LIP from Lemma 4.1, then $(\alpha, \alpha L_k^{-1} L_x^{-1}, \alpha) \in \text{AUT}(Q, \cdot)$ for all $x \in Q$ and for some $k \in Q$. Setting $k = x^{-1}$ to get $(\alpha, \alpha L_{x^{-1}}^{-1} L_x^{-1}, \alpha) \in \text{AUT}(Q, \cdot)$. Hence, $\alpha \in \text{AUM}(Q, \cdot)$. $\square$

Theorem 4.2. *Let $(Q, \cdot)$ be a RChL with flexibility property. If $\alpha \in \text{BS}(Q, \cdot)$, then α is a pseudo-automorphism with companion x^{-1} only if $t = k = x$.*

Proof. Suppose $(Q, \cdot)$ is a RChL. Then, $(Q, \cdot)$ is an RCC and this implies that, $H = (R_x, R_x L_x^{-1}, R_x)$ is an autotopism for all $x \in Q$. Then, $H^{-1} = (R_x^{-1}, L_x R_x^{-1}, R_x^{-1})$ is also an autotopism for all $x \in Q$. But for a right Cheban loop $(Q, \cdot)$, $N = (R_x, R_x^{-1} L_x, R_x) \in \text{AUT}(Q, \cdot)$. Hence, $NH^{-1} = (I, R_x^{-1} L_x L_x R_x^{-1}, I)$ is an autotopism for all $x \in Q$.

Let $\alpha \in \text{BS}(Q, \cdot)$. Then, $F = (\alpha R_t^{-1}, \alpha L_k^{-1}, \alpha)$ is an autotopism of $(Q, \cdot)$ for some $t, k \in Q$. Thus,

$$FNH^{-1} = (\alpha R_t^{-1}, \alpha L_k^{-1} R_x^{-1} L_x L_x R_x^{-1}, \alpha).$$

Now using $t = k = x$ and by using the dual of Theorem 2.3, we have,

$$FNH^{-1} = (\alpha R_x^{-1}, \alpha L_x^{-1} R_x^{-1} L_x^2 R_x^{-1}, \alpha) = (\alpha R_x^{-1}, \alpha L_x^{-1} R_x^{-1} R_x^2 R_x^{-1}, \alpha).$$

Hence, we have

$$(\alpha R_x^{-1}, \alpha L_x^{-1} R_x^{-1} R_x R_x R_x^{-1}, \alpha) = (\alpha R_x^{-1}, \alpha L_{x^{-1}}, \alpha),$$

and by Theorem 2.1, we have

$$(4.4) \quad (J\alpha R_x^{-1} J, \alpha, \alpha L_{x^{-1}}) \in \text{AUT}(Q, \cdot), \quad \text{for all } x \in Q.$$

Therefore, for all $y, z \in Q$ we get, $yJ\alpha R_x^{-1} J \cdot z\alpha = (yz)\alpha L_{x^{-1}}$, and this becomes $y\alpha R_x^{-1} J = \alpha L_{x^{-1}}$ when $z = e$. This implies that $(\alpha L_{x^{-1}}, \alpha, \alpha L_{x^{-1}}) \in \text{AUT}(Q, \cdot)$ for all $x \in Q$. Thus, α is an automorphism of $(Q, \cdot)$. $\square$

Remark 4.1. Robinson [21] considered the Bryant-Schneider group of a Bol loop and found out that they can be expressed as a product of pseudo-automorphisms and right translations. Theorems 4.1 and 4.2 show that the Bryant-Schneider group of right Cheban loop can also be expressed in the same way.

Theorem 4.3. *Let $(Q, \cdot)$ be a RChL and let θ be a bijection on Q such that θR_x^{-1} is an automorphism of Q , then θ is an element of the Bryant-Schneider group of $(Q, \cdot)$, $\text{BS}(Q, \cdot)$.*

Proof. Suppose $(Q, \cdot)$ is a RChL and θ is a bijection on Q such that θR_x^{-1} is an automorphism of Q , then $(\theta R_x^{-1}, \theta R_x^{-1}, \theta R_x^{-1}) \in \text{AUT}(Q, \cdot)$. By Theorem 2.2, it follows that the product

$$(\theta R_x^{-1}, \theta R_x^{-1}, \theta R_x^{-1})(R_x, R_x L_x^{-1}, R_x) = (\theta, \theta L_x^{-1}, \theta) \in \text{AUT}(Q, \cdot).$$

Hence, $\theta \in BS_\rho(Q, \cdot)$, but $BS_\rho(Q, \cdot) \leq BS(Q, \cdot)$ and thus, $\theta \in BS(Q, \cdot)$. $\square$

Theorem 4.4. *Let $(Q, \cdot)$ be a flexible RChL. Then, $a \in N_\mu(Q, \cdot)$ if and only if the middle inner mapping of Q , $T_a = R_a L_a^{-1}$, is an automorphism of $(Q, \cdot)$.*

Proof. Suppose $(Q, \cdot)$ is a flexible RChL and let $a \in N_\mu(Q, \cdot)$. Then it is a left inverse property loop and thus, $(J R_a J, R_a, R_a^{-1} L_a) = (L_a^{-1}, R_a, R_a^{-1} L_a) = (L_a^{-1}, R_a, R_a^{-1} L_a) \in AUT(Q, \cdot)$ and it follows that $(R_a, L_a^{-1}, I) \in AUT(Q, \cdot)$ for every $a \in Q$. By Theorem 2.2, it follows also

$$(L_a^{-1}, R_a, R_a^{-1} L_a)(R_a, L_a^{-1}, I) = (L_a^{-1} R_a, R_a L_a^{-1}, R_a^{-1} L_a) \in AUT(Q, \cdot).$$

Thus,

$$(R_a L_a^{-1}, R_a L_a^{-1}, R_a^{-1} L_a) = (T_a, T_a, T_a) \in AUT(Q, \cdot)$$

Hence, it follows that $T_a \in AUM(Q, \cdot)$. The converse is achieved by reversing the previous steps. $\square$

Corollary 4.1. *Let $(Q, \cdot)$ be a RChL. Then, for every $a \in N_\mu(Q, \cdot)$, L_a is an element of $BS(Q, \cdot)$.*

Proof. By Theorem 4.4, $R_a L_a^{-1} \in AUM(Q, \cdot)$ implies that $(R_a L_a^{-1})^{-1} = L_a R_a^{-1} \in AUM(Q, \cdot)$. Since $AUM(Q, \cdot)$ is a group then it follows, from Theorem 4.3, that $L_a \in BS(Q, \cdot)$. $\square$

Corollary 4.2. *Let $(Q, \cdot)$ be a RChL. Then, for every $a \in N_\mu(Q, \cdot)$, $R_a L_a^{-1} \in BS(Q, \cdot)$.*

Proof. By Theorem 4.4, $R_a L_a^{-1} \in AUM(Q, \cdot)$. Then, the result follows from the fact that every automorphism of $(Q, \cdot)$ is an element of $BS(Q, \cdot)$. $\square$

Corollary 4.3. *Let $(Q, \cdot)$ be a RChL. Then, for every $a \in N_\mu(Q, \cdot)$, R_a is an element of $BS(Q, \cdot)$.*

Proof. Since $BS(Q, \cdot)$ is a group, the result follows from Corollaries 4.1 and 4.2 that $R_a L_a^{-1} \cdot L_a = R_a \in BS(Q, \cdot)$. $\square$

5. CRYPTO-AUTOMORPHISM GROUP OF RIGHT CHEBAN LOOPS

Lemma 5.1. *Let $(Q, \cdot)$ be a quasigroup. A map $\phi \in SYM(Q)$ is a crypto-automorphism of $(Q, \cdot)$ with companion (a, b^λ) if and only if $(R_a \phi, L_{b^\lambda} \phi, \phi) \in AUT(Q, \cdot)$.*

Proof. It follows from Definition 2.11. $\square$

Theorem 5.1. *Let $(Q, \cdot)$ be a quasigroup. Then, $PS_{\mu_2}(Q, \cdot) = CAUM(Q, \cdot)$.*

Proof. If ϕ is a two-middle pseudo-automorphism of $(Q, \cdot)$, then $(\phi R_a^{-1}, \phi L_{b^\lambda}^{-1}, \phi) \in AUT(Q, \cdot)$ with companion (a, b) . For all $u, v \in Q$, we have

$$(5.1) \quad u \phi R_a^{-1} \cdot v \phi L_{b^\lambda}^{-1} = (uv) \phi.$$

Let $u\phi R_a^{-1} = f$ if and only if $fa = u\phi$ if and only if $u = (fa)\phi^{-1}$. Analogously, set $v\phi L_b^{-1} = g$ if and only if $v\phi = b^\lambda g$ if and only if $v = (b^\lambda g)\phi^{-1}$. Substitute u and v in (5.1), to get $(fa)R_a^{-1} \cdot (b^\lambda g)L_b^{-1} = ((fa)\phi^{-1} \cdot (b^\lambda g)\phi^{-1})\phi$ if and only if $(f \cdot g)\phi^{-1} = (f \cdot a)\phi^{-1} \cdot (b^\lambda \cdot g)\phi^{-1}$ if and only if $(R_a\phi^{-1}, L_b\phi^{-1}, \phi^{-1}) \in AUT(Q, \cdot)$. Hence, $\phi^{-1} \in CAUM(Q, \cdot)$. So, by Theorem 2.4, $\phi \in CAUM(Q, \cdot)$. Conversely, we reverse the steps to get $\phi \in PS_{\mu_2}(Q, \cdot)$. $\square$

Theorem 5.2. *Let U be a crypto-automorphism of $RChL(Q, \cdot)$ with companion (a, b) . Then, $R_x U = \varphi \in BS(Q, \cdot)$.*

Proof. $(Q, \cdot)$ is a $RChL$ if and only if $P = (R_x, R_x^{-1}L_x, R_x)$ is an autotopism of $(Q, \cdot)$. By Theorem 3.3, $P = (R_x, R_x L_x^{-1}, R_x) \in AUT(Q, \cdot)$ for all $x \in Q$. Let U be a crypto-automorphism of $(Q, \cdot)$. Then, $T = (R_a U, L_b U, U)$ is an autotopism of $(Q, \cdot)$. The product

$$(5.2) \quad PT = (R_x, R_x L_x^{-1}, R_x)(R_a U, L_b U, U) = (R_x R_a U, R_x L_x^{-1} L_b U, R_x U) \in AUT(Q, \cdot).$$

$R_x U = \varphi$ in the last autotopism implies that

$$(5.3) \quad (R_x R_a U, R_x L_x^{-1} L_b U, \varphi) \in AUT(Q, \cdot).$$

Writing this in identical relation, for all $y, z \in Q$, we have

$$(5.4) \quad y R_x R_a U \cdot z R_x L_x^{-1} L_b U = (y \cdot z) \varphi.$$

Put $y = e$ in (5.4) to obtain $e R_x R_a U \cdot z R_x L_x^{-1} L_b U = e \cdot z \varphi$, this implies

$$(ex) R_a U \cdot z R_x L_x^{-1} L_b U = z \varphi$$

and

$$(5.5a) \quad z R_x L_x^{-1} L_b U L_{xa} U = z \varphi, \quad z R_x L_x^{-1} L_b U = z \varphi L_{xa}^{-1}, \quad R_x L_x^{-1} L_b U = \varphi L_{xa}^{-1}.$$

Also, put $z = e$ in (5.4) to have $y R_x R_a U \cdot e R_x L_x^{-1} L_b U = y \varphi$. Then,

$$y R_x R_a U \cdot (x \setminus (ex)) L_b U = y \varphi$$

and

$$(5.5) \quad y R_x R_a U R_{bU} = y \varphi, \quad y R_x R_a U = y \varphi R_{bU}^{-1}, \quad R_x R_a U = \varphi R_{bU}^{-1}.$$

Using (5.5a) and (5.5) in (5.3), we have $(\varphi R_{bU}^{-1}, \varphi L_{xa}^{-1}, \varphi) \in AUT(Q, \cdot)$. Thus, $\varphi \in BS(Q, \cdot)$ with $(a, b) = (bU, xaU)$. $\square$

Corollary 5.1. *Let $(Q, \cdot)$ be a $RChL$, and let $R_x U = \varphi$ be a middle pseudo-automorphism with companion (a, b) . Then, φ^{-1} is a crypto-automorphism with companion (bU, xaU) .*

Proof. Consequence of Theorem 5.1 and Theorem 5.2. $\square$

Theorem 5.3. *Let U be a crypto-automorphism of $RChL(Q, \cdot)$ with companion (a, b) . Then, $R_x R_a U = \varphi$ is a right pseudo-automorphism with companion xU , where $R_x^{-1} \varphi = R_a U$.*

Proof. Let $R_x R_a U = \varphi$ in (5.2) of Theorem 5.2 which implies that

$$(5.6) \quad (\varphi, R_x L_x^{-1} L_b U, R_x U)$$

is an autotopism of $(Q, \cdot)$. For all $y, z \in Q$, we have

$$(5.7) \quad y\varphi \cdot z R_x L_x^{-1} L_b U = (y \cdot z) R_x U.$$

Put $y = e$ in (5.6) the identity element of Q to get

$$e\varphi \odot z R_x L_x^{-1} L_b U = z R_x U, \quad z R_x L_x^{-1} L_b U = z R_x U, \quad R_x L_x^{-1} L_b U = R_x U,$$

for all $z \in Q$. Substituting the last equality in (5.6) to get

$$(5.8) \quad (\varphi, R_x U, R_x U) \in AUT(Q, \cdot).$$

Note that $e = e R_x R_a U = e R_x R_x^{-1} \varphi = e\varphi$. Thus, $e\varphi = e$. For all $y, z \in Q$, we have $y\varphi \cdot z R_x U = (y \cdot z) R_x U$. Put $z = e$ to get

$$y\varphi \cdot e R_x U = y R_x U, \quad y\varphi \cdot (e x U) = y R_x U, \quad y\varphi R_{(xU)} = y R_x U, \quad \varphi R_{(xU)} = R_x U.$$

Substituting the last equality in (5.8), we get that $(\varphi, \varphi R_{xU}, \varphi R_{xU})$ is an autotopism of $(Q, \cdot)$. Thus, φ is a right pseudo-automorphism with companion xU . $\square$

Theorem 5.4. *Let $(Q, \cdot)$ be a loop. Then, $PS_{\mu_2}(Q, \cdot) = CAUM(Q, \cdot) = BS(Q, \cdot)$.*

Proof. This follows from Theorem 5.1 and Theorem 2.5. $\square$

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ABOUT COMPACTNESS TYPE OF UNIFORM SPACES

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ABSTRACT. This paper introduces uniform analogues of compact, finally compact and countable compact spaces and investigates their connection with other uniform properties of compactness type.

1. INTRODUCTION AND PRELIMINARIES

It is well known that compact, finally compact and countable compact spaces play an important role in set-theoretical topology. Finding and studying uniform analogues of these classes of compactness types is an important and interesting problem in uniform topology.

Throughout this paper, all uniform spaces are assumed to be Hausdorff, topological space are Tychonoff and mappings are uniformly continuous.

A topological space X is called finally compact if every open cover has a countable open refinement [1]; a topological space X is called countable compact, if every countable open cover has a finite open refinement [1]; a uniform space (X, U) is called precompact if it has a base consisting of finite covers [2, 4, 17]; a uniform space (X, U) is called strongly uniformly paracompact if every finitely additive open cover has a star finite uniform refinement [5]; the filter F is called Cauchy filter if $\alpha \cap F \neq \emptyset$ for any $\alpha \in U$ [1, 17]; a uniformly continuous mapping $f : (X, U) \rightarrow (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called a precompact, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and finite uniform cover $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [1]; a continuous mapping $f : X \rightarrow Y$ of topological space X to a topological space Y

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is called ω -mapping, if for each point $y \in Y$ there exist neighborhood O_y and $W \in \omega$, such that $f^{-1}O_y \subset W$ [5].

For a cover α of a set X and $x \in X$, $M \subset X$, we have: $St(\alpha, x) = \{A \in \alpha : x \in A\}$, $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, M) = \{A \in \alpha : A \cap M \neq \emptyset\}$, $\alpha(M) = \bigcup St(\alpha, M)$. For covers α and β of the set X , the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β , i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, for covers α and β of a set X , we have: $\alpha \wedge \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. For a cover α of a set X let $\alpha^\wedge = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha - \text{is finite}\}$. The cover α is finitely additive if $\alpha^\wedge = \alpha$. For the uniformity U by τ_U we denote the topology generated by the uniformity and symbol U_X means the universal uniformity.

2. COMPACTNESS TYPE OF UNIFORM SPACES

This paper introduces and studies u-uniformly compact, u-uniformly finally compact and u-uniformly countable compact spaces.

The following lemma will serve as the basis for our definitions.

Lemma 2.1. *A topological space (X, τ) is compact (countable compact, finally compact) if and only if every finitely additive arbitrary (countable, arbitrary) open cover has an open finite (finite, countable) refinement.*

Proof. Necessity. Let α be an arbitrary (countable) open cover of (X, τ) . Then, for the finitely additive (countable) open cover $\alpha^\wedge$ there exists a finite (finite, countable) open cover β such that $\beta \succ \alpha^\wedge$. For each $B \in \beta$ choose $A_B \in \alpha^\wedge$ such that $B \subset A_B$, $A_B = \bigcup_{i=1}^n A_i$, $A_i \in \alpha$, $i = 1, 2, \dots, n$. Let $\alpha_0 = \bigcup \{A_B : B \in \beta\}$, $\alpha_0 = \{B \cap A_i : i = 1, 2, \dots, n\}$. Then α_0 is finite (finite, countable) open cover of the topological space (X, τ) such that $\alpha_0 \succ \alpha$. Consequently, the topological space (X, τ) is compact (countable compact, finally compact).

The sufficiency is obvious. □

Definition 2.1. A uniform space (X, U) is said to be u-uniformly compact if every finitely additive open cover has a finite uniform refinement.

Definition 2.2. A uniform space (X, U) is said to be u-uniformly countable compact if every finitely additive countable open cover has a finite uniform refinement.

Definition 2.3. A uniform space (X, U) is said to be u-uniformly finally compact if every finitely additive open cover has a countable uniform refinement.

Proposition 2.1. *If a uniform space (X, U) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact), then topological space (X, τ_U) is compact (countable compact, finally compact). Conversely, if a topological space (X, τ) is compact (countable compact, finally compact), then the space (X, U_X) is compact (countable compact, finally compact), where U_X is the universal uniformity on topological space (X, τ) such that $\tau_{U_X} = \tau$.*

Proof. Let α be an arbitrary (countable) open cover of the space (X, τ_U) . Then for a finitely additive (countable) open cover $\alpha^\angle$ of (X, U) there exist finite (finite, countable) uniform cover $\beta \in U$ such that $\beta \succ \alpha^\angle$. It is known that the interior $\langle \beta \rangle$ of uniform cover β is a uniform cover. Denote $\gamma = \langle \beta \rangle$. Then γ is a finite (finite, countable) open uniform cover of the uniform space (X, U) . For each $\Gamma \in \gamma$ we choose $A_\Gamma \in \alpha^\angle$ such that $\Gamma \subset A_\Gamma$, where $A_\Gamma = \bigcup_{i=1}^n A_i$, $A_i \in \alpha$, $i = 1, 2, \dots, n$. Let's put $\alpha_0 = \bigcup \{\alpha_\Gamma : \Gamma \in \gamma\}$, $\alpha_\Gamma = \{\Gamma \cap A_i : i = 1, 2, \dots, n\}$. Then α_0 is finite (finite, countable) open cover of the space (X, τ_U) and $\alpha_0 \succ \alpha$. So (X, τ_U) is compact (countable compact, finally compact).

Conversely, let (X, τ) be compact (countable compact, finally compact). Then the set of all open covers form the base of the universal uniformity U_X of the topological space (X, τ) . It is evident that the uniform space (X, U_X) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact). $\square$

The following theorem gives a characterization of u-uniformly compactness (u-uniformly finally compactness) in the spirit of Tamano.

Theorem 2.1. *Let (X, U) be a uniform space and bX be a certain compact Hausdorff extensions of the space (X, τ_U) . A uniform space (X, U) is u-uniformly compact (u-uniformly finally compact) if and only if for compact $K \subset bX \setminus X$ there exist a finite (countable) uniformly cover $\alpha \in U$, such that $[A]_{bX} \cap K = \emptyset$ for any $A \in \alpha$.*

Proof. Necessity. Let (X, U) be a u-uniformly compact (u-uniformly finally compact) space and $K \subset bX \setminus X$ be an arbitrary compact. For each point $x \in X$ there is an open neighborhood O_x of the compact Hausdorff extensions bX of the space (X, τ_U) , such that $[O_x]_{bX} \cap K = \emptyset$. Denote $\gamma = \{O_x \cap X : x \in X\}$. Obviously γ is open cover of the space (X, U) . Let $\beta \in U$ be a finite (countable) uniform cover of the space (X, U) such that $\beta \succ \gamma^\angle$, then $[B]_{bX} \subset [\bigcup_{i=1}^n (O_{x_i} \cap X)]_{bX} \subset \bigcup_{i=1}^n [O_{x_i}]_{bX}$. Since $[O_{x_i}]_{bX} \cap K = \emptyset$, $i = 1, 2, \dots, n$, then $[B]_{bX} \cap K = \emptyset$, $B \in \beta$.

Sufficiency. Let α be an arbitrary finitely additive open cover of the space (X, U) . Then there exists an open system β of the compact Hausdorff extensions bX , such that $\beta \wedge \{X\} = \alpha$. Denote $K = bX \setminus \bigcup \beta$. It is easy to see that K is compact. Accordingly to the condition of the theorem, there exists a finite (countable) cover $\gamma \in U$, such that $[\Gamma]_{bX} \cap K = \emptyset$, $\Gamma \in \gamma$. Since $[\Gamma]_{bX}$ is compact in bX , there exist $B_1, B_2, \dots, B_n \in \beta$, such that $[\Gamma]_{bX} \subset \bigcup_{i=1}^n B_i$. Then $\Gamma \subset \bigcup_{i=1}^n A_i$, $\bigcup_{i=1}^n A_i \in \alpha$. Thus, the space (X, U) is u-uniformly compact (u-uniformly finally compact). $\square$

Proposition 2.2. *Any compact uniform space (X, U) is u-uniformly compact.*

Proof. Let α be an arbitrary (countable) open cover of (X, U) . As is known, every open cover of a compact is a uniform cover. It follows that $\alpha \in U$. Since the space (X, U) is precompact, there exists a finite uniform cover $\beta \in U$ such that $\beta \succ \alpha$. $\square$

Proposition 2.3. *Any u-uniformly compact uniform space (X, U) is u-uniformly finally compact (u-uniformly countable compact).*

Proof. Let α be an arbitrary (countable) open cover of (X, U) . Since the space (X, U) is u -uniformly compact, there exists a finite uniform cover $\beta \in U$ such that $\beta \succ \alpha$. Thus, (X, U) is u -uniformly finally compact (u -uniformly countable compact). $\square$

Proposition 2.4. *Any u -uniformly finally compact uniform space (X, U) is complete.*

Proof. Let (X, U) be a u -uniformly compact space and F be an arbitrary Cauchy filter. Suppose that a Cauchy filter F does not converge at any point of the space (X, U) . Then, each point $x \in X$ has a open neighborhood O_x and $B \in F$ such that $O_x \cap B = \emptyset$. Denote $\alpha = \{O_x : x \in X\}$. Since the space (X, U) is u -uniformly finally compact, then $\alpha^\angle \in U$, therefore $\alpha^\angle \cap F \neq \emptyset$, i.e., there are $A^\angle \in \alpha^\angle$ such that $A^\angle \in F$, where $A^\angle = \bigcup_{i=1}^n O_{x_i}$, $O_{x_i} \in \alpha$. Hence, $(\bigcap_{i=1}^n B_{x_i}) \cap (\bigcup_{i=1}^n O_{x_i}) \neq \emptyset$. Then, there exists $i_0 \leq n$, such that $B_{x_{i_0}} \cap O_{x_{i_0}} \neq \emptyset$. A contradiction. Thus, (X, U) is complete. $\square$

Corollary 2.1. *Any u -uniformly compact uniform space (X, U) is complete.*

Proposition 2.5. *For a uniform space (X, U_X) the following conditions are equivalent:*

1. (X, U_X) is u -uniformly compact;
2. (X, U_X) is compact.

Proof. 1. $\Rightarrow$ 2. The completeness of (X, U_X) implies in Proposition 2.4. Let α be an arbitrary uniform cover of the space (X, U_X) . Then, there exists a finite uniform cover $\beta \in U$, such that $\beta \succ \alpha^\angle$. For each $B_i \in \beta$ we choose $A_{B_i} \in \alpha^\angle$ such that $B_i \subset A_{B_i}$, where $A_{B_i} = \bigcup_{j=1}^k A_j$, $A_j \in \alpha$, $j = 1, 2, \dots, k$. Denote $\alpha_0 = \bigcup \{\alpha_{B_i} : i = 1, 2, \dots, n\}$, $\alpha_{B_i} = \{B_i \cap A_j : j = 1, 2, \dots, k\}$. Then, α_0 is a finite open cover of the space (X, U_X) and $\alpha_0 \succ \alpha$. Since U_X is universal uniformity, then $\alpha_0 \in U_X$. Thus, (X, U_X) is compact.

2. $\Rightarrow$ 1. Is implied in Proposition 2.2. $\square$

In the class of precompact uniform spaces compactness, u -uniform compactness and u -uniform finally compactness are equivalent.

Proposition 2.6. *Any u -uniformly compact space (X, U) is strongly uniformly paracompact.*

Proof. Let α be an arbitrary finitely additive open cover of the space (X, U) . Then, there exists a finite uniform cover β of the space (X, U) such that the cover β is refined in a cover α . It is easily to see that the cover β is star finite uniform cover of the space (X, U) . Thus, the uniform space (X, U) is strongly uniformly paracompact. $\square$

If (X, τ) is a strongly paracompact but not compact space, then a uniform space (X, U_X) according to Proposition 2.1. is strongly uniformly paracompact but not u -uniformly compact.

Example 2.1. Real line $\mathbb{R}$ with the natural uniformity is u -uniformly finally compact.

Proof. Let α be an arbitrary finitely additive cover of $\mathbb{R}$ and $\beta = \{(n-1, n+1) : n = 0, \pm 1, \pm 2, \dots\}$. Then, $\beta \in U$ is countable uniform cover of $\mathbb{R}$. Since $[n-1, n+1]$ is compactum, then there exists a finite family $\{A_1, A_2, \dots, A_n\}$, $A_i \in \alpha$, $i = 1, 2, \dots, n$, such that $(n-1, n+1) \subset [n-1, n+1] \subset \bigcup_{i=1}^n A_i$, i.e., $\beta \succ \alpha^\angle = \alpha$. Hence, $\mathbb{R}$ is u-uniformly finally compact, but not u-uniformly compact. $\square$

Let (X, τ) be a countable compact, but not compact space. Then a uniform space (X, U_X) according to Proposition 2.1. is a u-uniformly countable compact but not u-uniformly compact.

u-uniformly compactness (u-uniformly countable compactness, u-uniformly finally compactness) inherented by taking closed subspaces.

Theorem 2.2. *Any closed subspace M of u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact) space (X, U) is u-uniformly compact (u-uniformly finally compact) space.*

Proof. Let γ_M be an arbitrary finitely additive (countable) open cover of the subspace (M, U_M) . Then there exists a finitely additive (countable) open family γ of the space (X, U) , such that $\gamma \wedge \{M\} = \gamma_M$. Put $\hat{\gamma} = \{\gamma, X \setminus M\}$. Obviously the cover $\hat{\gamma}$ is an finitely additive (countable) open cover of the space (X, U) . Then, there exists a finite (finite, countable) uniform cover $\beta \in U$, such that $\beta \succ \hat{\gamma}$. Denote $\beta_M = \beta \wedge \{M\}$. Then, it is easy to see that β_M is finite (finite, countable) uniform cover of the subspace (M, U_M) and $\beta_M \succ \gamma_M$. Thus, (M, U_M) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact). $\square$

Theorem 2.3. *Let $f : (X, U) \rightarrow (Y, V)$ be a ω -mapping between uniform spaces (X, U) and (Y, V) . If (Y, V) is u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space, then the uniform space (X, U) is also u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space.*

Proof. Let ω be an arbitrary finitely additive (countable) open cover of the space (X, U) and $f : (X, U) \rightarrow (Y, V)$ be a ω -mapping of the uniform space (X, U) onto u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact) space (Y, V) . Then, for each point $y \in Y$ there exists a neighborhood O_y and $W \in \omega$, such that $f^{-1}O_y \subset W$. Denote $\beta = \{O_y : y \in Y\}$, $\beta^\angle = \{\bigcup \beta_0 : \beta_0 \subset \beta \text{ is finite}\}$. Since the space (Y, V) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact), there exists a finite (finite, countable) uniform cover $\gamma \in V$, such that $\gamma \succ \beta^\angle$. Then finite (finite, countable) uniform cover $f^{-1}\gamma$ is refined of cover ω . Thus, (X, U) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact). $\square$

Theorem 2.4. *Let $f : (X, U) \rightarrow (Y, V)$ be a perfect mapping between uniform spaces (X, U) and (Y, V) . If (Y, V) is u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space then the uniform space (X, U) is also u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space.*

Proof. Let α be an arbitrary finite additive open cover of uniform space (X, U) . It is clear that the cover $\{f^{-1}y : y \in Y\}$ is refined of the cover α . Then, $\beta = f^\# \alpha = \{f^\# A : A \in \alpha\}$, where $f^\# A = Y \setminus f(X \setminus A)$ is an open cover of the space (Y, V) . Since the space (Y, V) is u -uniformly compact (u -uniformly finally compact, u -uniformly countable compact) space, there exists a finite (countable) cover $\gamma \in V$ such that $\gamma \succ \beta^\angle$. It is easy to see that $f^{-1}\gamma \succ \alpha$ and $f^{-1}\gamma$ is finite (countable) cover of the space (X, U) . Consequently, the space (X, U) is u -uniformly compact (u -uniformly finally compact, u -uniformly countable compact). $\square$

Corollary 2.2. *Let $f : (X, U) \rightarrow (Y, V)$ be a uniformly perfect mapping between uniform spaces. If (Y, V) is u -uniformly compact (u -uniformly finally compact, u -uniformly countable compact) space, then the uniform space (X, U) is also u -uniformly compact (u -uniformly finally compact, u -uniformly countable compact) space.*

Proposition 2.7. *Product $(X, U) \times (Y, V)$ of a u -uniformly compact (u -uniformly countable compact, u -uniformly finally compact) space (X, U) by a compact space (Y, V) is u -uniformly compact (u -uniformly countable compact, u -uniformly finally compact).*

Proof. Let (X, U) be a u -uniformly compact (u -uniformly countable compact, u -uniformly finally compact) and let (Y, V) be compact. It is known that projection $\pi_X : (X, U) \times (Y, V) \rightarrow (X, U)$ is uniformly perfect mapping of the product $(X, U) \times (Y, V)$ onto u -uniformly compact (u -uniformly countable compact, u -uniformly finally compact) space (Y, V) . Then, projection π_X is ω -mapping for any finitely additive (countable) an open cover ω . Then, according to Theorem 2.3., the product $(X, U) \times (Y, V)$ is u -uniformly compact (u -uniformly countable compact, u -uniformly finally compact). $\square$

Theorem 2.5. *A uniform space (X, U) is u -uniformly compact if and only if (X, U) is u -uniformly finally compact and u -uniformly countable compact.*

Proof. Necessity follows from Proposition 2.3.

Sufficiency. Let α be an arbitrary finally additive open cover of the space (X, U) . Since the space (X, U) is u -uniformly finally compact space, there exists a countable cover $\beta \in U$ such that $\beta \succ \alpha$. Since the space (X, U) is u -uniformly countable compact there exists a finite uniform cover $\gamma \in U$, such that $\gamma \succ \beta^\angle$. Then, $\gamma \succ \beta^\angle \succ \alpha^\angle = \alpha$, this means that $\gamma \succ \alpha$. Thus, (X, U) is u -uniformly compact. $\square$

Corollary 2.3. *A Tychonoff space (X, τ) is compact if and only if (X, τ) is finally compact and countable compact.*

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ON $F(p, q, s)$ SPACES AND WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS ON THEM

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ABSTRACT. In this work, we first consider weighted differentiation composition operator from a general family of analytic functions into Bloch-type space and find an approximation for the essential norm. Then, we characterize the boundedness of the operator into Zygmund-type space and finally, essential norm of this operator is estimated.

1. INTRODUCTION

Let f be an analytic function on the open unit disk in the complex plane and φ be an analytic self-map of the unit disk. The well-known composition operator denoted by C_φ is defined by

$$C_\varphi f = f \circ \varphi.$$

This operator can also be defined on the space of measurable functions. Composition operators are used in the study of isometries, backward shifts of all multiplicities, commutates of multiplication operators and more general operators and De Branges' proof of the Bieberbach conjecture. It is an interesting problem to describe the operator properties of C_φ in terms of the function properties of the symbol φ when the operator C_φ acts on several spaces of analytic functions on the unit disk. It is clear that C_φ is linear and bounded by Littlewood's subordination principle on the classical Hardy and Bergman spaces, see [5, 8].

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The composition operators can be generalized through multiplication to the weighted composition operators and more generally weighted differentiation composition operators. This type of operators contain many other operators that are well known in this field. We are going to investigate the properties of weighted differentiation composition operators on the general space of analytic functions $F(p, q, s)$ to weighted Bloch and Zygmund spaces. The space $F(p, q, s)$ includes several classical spaces of analytic functions such as Bergman, Bloch, Hardy, BMOA and $\mathcal{Q}_p$. So the results of the paper can be stated for several operators and spaces.

Information about $F(p, q, s)$ can be found in [12, 13]. Also for some equivalent norms in the case of the unit ball, one can refer to [11]. Weighted composition operators from $F(p, q, s)$ to H_μ^∞ have been studied by Zhu in [15]. Yang investigated generalized weighted composition operators from $F(p, q, s)$ space to the Bloch-type space in [10]. Volterra-type operators from $F(p, q, s)$ space to Bloch–Orlicz and Zygmund–Orlicz spaces were characterized in [4]. Recently, the essential norm of t -generalized composition operators from $F(p, q, s)$ to iterated weighted-type Banach space has been approximated in [1]. For further research on this subject, see [2, 3, 6, 7, 9].

2. BACKGROUND MATERIAL

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$ and $\mathcal{S}(\mathbb{D})$ be the set of all analytic self-maps of $\mathbb{D}$. Let $u \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. The generalized weighted composition operator, weighted differentiation composition operator, is given by

$$D_{u,\varphi}^n f(z) = u(z) f^{(n)}(\varphi(z)), \quad f \in \mathcal{H}(\mathbb{D}), z \in \mathbb{D},$$

where $n \in \mathbb{N}_0$. This type of operators include composition operator C_φ , multiplication operator M_u , DC_φ , $C_\varphi D$ and some other classical operators. While the operator $D_{u,\varphi}^n$ acts on some subspaces of $\mathcal{H}(\mathbb{D})$, it is a natural question to find conditions under which the operator has operator properties such as boundedness, compactness, and others. Instead of using classical subspaces individually, we deal with a general function space $F(p, q, s)$. Let $0 < p < +\infty$, $0 \leq s < +\infty$ and $-2 < q < +\infty$. The space $F(p, q, s)$ is the space of all analytic functions in $\mathbb{D}$ for which

$$\|f\|_{p,q,s}^p = |f(0)|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < +\infty.$$

Here dA is normalized area measure on $\mathbb{D}$, $g(z, a) = \log \left| \frac{1}{\varphi_a(z)} \right|$, $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. The above space was introduced in [12] and becomes a Banach space with $\|\cdot\|_{p,q,s}$. For the properties and the other spaces included in it see [13]. We bring here some spaces.

- When $s > 1$, $F(p, q, s) = \mathcal{B}^{(q+2)/p}$, the Bloch space.
- $F(2, 0, s) = \mathcal{Q}_s$ and $F(2, 0, 1) = \text{BMOA}$.
- When $0 < p < +\infty$ and $-1 < q < +\infty$, $F(p, q, 0) = D_q^p$, Dirichlet space and if $1 < p < +\infty$, then $F(p, p-2, 0) = B_p$, the analytic Besov space.
- $F(2, 1, 0) = H^2$, Hardy space.
- When $0 < p < +\infty$, $F(p, p+q, 0) = A_q^p$, the Bergman space.

If $q + s \leq -1$, then $F(p, q, s)$ is the space of constant functions, see [12]. For $0 < p, s < +\infty$, $-2 < q < +\infty$ and $q + s > -1$, if $f \in F(p, q, s)$, then

$$(2.1) \quad |f^{(m)}(z)| \leq C \frac{\|f\|_{p,q,s}}{(1 - |z|^2)^{\frac{2+q-p}{p}+m}}, \quad m \in \mathbb{N},$$

see [1]. Here we use weighted Bloch and Zygmund spaces. A positive continuous function μ on $\mathbb{D}$ is called a weight. For a weight μ which is radial, $\mu(z) = \mu(|z|)$, the weighted Zygmund space $\mathcal{Z}_\mu$, Zygmund-type space, is the space of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}} \mu(|z|) |f''(z)| < +\infty.$$

The above statement is a semi-norm and it will be a norm if we add $|f(0)| + |f'(0)|$. So,

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} \mu(|z|) |f''(z)|.$$

With this norm, $\mathcal{Z}_\mu$ is a Banach space. The weighted Bloch space $\mathcal{B}_\mu$ is defined by

$$\mathcal{B}_\mu = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z \in \mathbb{D}} \mu(|z|) |f'(z)| < +\infty\}.$$

Note that if $\mu(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, then we obtain the Zygmund space $\mathcal{Z}^\alpha$ and the Bloch space $\mathcal{B}^\alpha$.

Throughout the paper we set $\gamma = \frac{q+2}{p}$ and

$$A(u, t) = \frac{\mu(|z|) |u'(z)|}{(1 - |\varphi(z)|^2)^t}.$$

All positive constants will be denoted by C which may vary from one occurrence to another. By $A \succeq B$ we mean there exists a constant C such that $A \geq CB$ and $A \approx B$ means that $A \succeq B \succeq A$.

3. MAIN RESULTS

Lemma 3.1. *Let $0 < p, s < +\infty$, $-2 < q < +\infty$, $q + s > -1$, $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ be a radial weight. If $\rho = \|\varphi\|_\infty < 1$ and $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{B}_\mu$ is bounded, then $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{B}_\mu$ is compact.*

Proof. We need to show that for any bounded sequence $\{f_i\}$ in $F(p, q, s)$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\|D_{u,\varphi}^n f_i\|_{\mathcal{B}_\mu}$ converges to 0 as $i \rightarrow +\infty$.

The boundedness of the operator implies that for any $f \in F(p, q, s)$, $\|D_{u,\varphi}^n f\|_{\mathcal{B}_\mu} \leq C \|f\|_{p,q,s}$. Apply this to the functions z^n and z^{n+1} and using triangle inequality we obtain

$$(3.1) \quad \begin{aligned} C_1 &= \sup_{z \in \mathbb{D}} \mu(|z|) |u'(z)| < +\infty, \\ C_2 &= \sup_{z \in \mathbb{D}} \mu(|z|) |u(z) \varphi'(z)| < +\infty. \end{aligned}$$

Therefore,

$$\begin{aligned}
\|D_{u,\varphi}^n f_i\|_{\mathcal{B}_\mu} &= |(D_{u,\varphi}^n(f_i))(0)| + \sup_{z \in \mathbb{D}} \mu(|z|) |(D_{u,\varphi}^n(f_i))'(z)| \\
&\leq |u(0)| |f_i^{(n)}(\varphi(0))| + \sup_{z \in \mathbb{D}} \mu(|z|) |u'(z)| |f_i^{(n)}(\varphi(z))| \\
&\quad + \sup_{z \in \mathbb{D}} \mu(|z|) |u(z)\varphi'(z)| |f_i^{(n+1)}(\varphi(z))| \\
&\leq |u(0)| |f_i^{(n)}(\varphi(0))| + \sup_{|\varphi(z)| \leq \rho} \mu(|z|) |u'(z)| |f_i^{(n)}(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)| \leq \rho} \mu(|z|) |u(z)\varphi'(z)| |f_i^{(n+1)}(\varphi(z))| \\
&\leq |u(0)| |f_i^{(n)}(\varphi(0))| + C_1 \sup_{|\varphi(z)| \leq \rho} |f_i^{(n)}(\varphi(z))| + C_2 \sup_{|\varphi(z)| \leq \rho} |f_i^{(n+1)}(\varphi(z))|.
\end{aligned}$$

All the three terms above tend to 0 and $\|D_{u,\varphi}^n f_i\|_{\mathcal{B}_\mu}$ tends to 0 and so the proof is completed. $\square$

Theorem 3.1. Suppose that $\varphi \in \mathcal{S}(\mathbb{D})$ with $\|\varphi\|_\infty = 1$, $u \in \mathcal{H}(\mathbb{D})$ and μ is a normal weight on $[0, 1)$ as in [10]. Also suppose that $0 < p, s < +\infty$, $-2 < q < +\infty$ and $q + s > -1$. If $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{B}_\mu$ is bounded, then

$$(3.2) \quad \|D_{u,\varphi}^n\|_e \approx \limsup_{|\varphi(z)| \rightarrow 1} A(u', \gamma + n - 1) + \limsup_{|\varphi(z)| \rightarrow 1} A(u\varphi', \gamma + n).$$

Proof. Define the sequences of functions as follows

$$g_i(z) = \frac{1 - |\varphi(z_i)|}{(1 - \overline{\varphi(z_i)}z)^\gamma}, \quad f_i(z) = \frac{1 - |\varphi(z_i)|}{(1 - \overline{\varphi(z_i)}z)^\gamma} - \frac{2 + q}{2 + q + pn} \cdot \frac{(1 - |\varphi(z_i)|)^2}{(1 - \overline{\varphi(z_i)}z)^{\gamma+1}}.$$

Here $\{z_i\} \subset \mathbb{D}$ is a sequence that $|\varphi(z_i)| \rightarrow 1$. By performing a simple calculation on the derivatives of the g_i and f_i , one can see that

$$\begin{aligned}
g_i^{(n)}(\varphi(z_i)) &= \beta \frac{\overline{\varphi(z_i)}^n}{(1 - |\varphi(z_i)|^2)^{\gamma+n-1}}, \quad g_i^{(n+1)}(\varphi(z_i)) = \beta(\gamma + n) \frac{\overline{\varphi(z_i)}^{n+1}}{(1 - |\varphi(z_i)|^2)^{\gamma+n}}, \\
f_i^{(n)}(\varphi(z_i)) &= 0, \quad f_i^{(n+1)}(\varphi(z_i)) = \beta \frac{\overline{\varphi(z_i)}^{n+1}}{(1 - |\varphi(z_i)|^2)^{\gamma+n}},
\end{aligned}$$

where $\beta = \gamma(\gamma + 1) \cdots (\gamma + n - 1)$. From [10], we see that both sequences are bounded in $F(p, q, s)$ and converge to zero uniformly on compact subsets of $\mathbb{D}$. So for any compact operator K from $F(p, q, s)$ to $\mathcal{B}_\mu$, $\|Kg_i\|_{\mathcal{B}_\mu} \rightarrow 0$ and $\|Kf_i\|_{\mathcal{B}_\mu} \rightarrow 0$, as $i \rightarrow +\infty$. Noting to the definition of the essential norm we get

$$(3.3) \quad \|D_{u,\varphi}^n\|_{e, F(p,q,s) \rightarrow \mathcal{B}_\mu} = \inf_K \|D_{u,\varphi}^n - K\|_{F(p,q,s) \rightarrow \mathcal{B}_\mu}.$$

So, it will be sufficient to compute the operator norm of $D_{u,\varphi}^n - K : F(p, q, s) \rightarrow \mathcal{B}_\mu$. Therefore,

$$(3.4) \quad \|D_{u,\varphi}^n - K\|_{F(p,q,s) \rightarrow \mathcal{B}_\mu} = \sup_{\|f\|_{p,q,s} \leq 1} \|(D_{u,\varphi}^n - K)f\|_{\mathcal{B}_\mu} \geq \|(D_{u,\varphi}^n - K)f_i\|_{\mathcal{B}_\mu}.$$

Then,

$$\begin{aligned}
 \|D_{u,\varphi}^n - K\|_{F(p,q,s) \rightarrow \mathcal{B}_\mu} &\succeq \limsup_{i \rightarrow +\infty} \|(D_{u,\varphi}^n - K)f_i\|_{\mathcal{B}_\mu} = \limsup_{i \rightarrow \infty} \|D_{u,\varphi}^n f_i\|_{\mathcal{B}_\mu} \\
 &\geq \limsup_{i \rightarrow +\infty} \sup_{z \in \mathbb{D}} \mu(|z|) |(D_{u,\varphi}^n f_i)'(z)| \\
 &\geq \limsup_{i \rightarrow +\infty} \mu(|z_i|) |(D_{u,\varphi}^n f_i)'(z_i)| \\
 &= \limsup_{i \rightarrow +\infty} \beta \frac{\mu(|z_i|) |u(z_i) \varphi'(z_i)| |\varphi(z_i)|^{n+1}}{(1 - |\varphi(z_i)|^2)^{\gamma+n}} \\
 &= \limsup_{i \rightarrow +\infty} \beta \frac{\mu(|z_i|) |u(z_i) \varphi'(z_i)|}{(1 - |\varphi(z_i)|^2)^{\gamma+n}}.
 \end{aligned}$$

From (3.3) and (3.4) we have

$$(3.5) \quad \|D_{u,\varphi}^n\|_{e, F(p,q,s) \rightarrow \mathcal{B}_\mu} \succeq \beta \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|) |u(z) \varphi'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n}}.$$

Now we consider the sequence $\{g_i\}$ and get to the following

$$\begin{aligned}
 \|D_{u,\varphi}^n\|_{e, F(p,q,s) \rightarrow \mathcal{B}_\mu} &\succeq \limsup_{i \rightarrow +\infty} \|D_{u,\varphi}^n g_i\|_{\mathcal{B}_\mu} \\
 &\geq \limsup_{i \rightarrow +\infty} \sup_{z \in \mathbb{D}} \mu(|z|) |(D_{u,\varphi}^n g_i)'(z)| \\
 &\geq \limsup_{i \rightarrow +\infty} \mu(|z_i|) |(D_{u,\varphi}^n g_i)'(z_i)| \\
 &\geq \limsup_{i \rightarrow +\infty} \mu(|z_i|) |u'(z_i)| |g_i^{(n)}(\varphi(z_i))| \\
 &\quad - \limsup_{i \rightarrow +\infty} \mu(|z_i|) |u(z_i) \varphi'(z_i)| |g_i^{(n+1)}(\varphi(z_i))| \\
 &= \beta \limsup_{i \rightarrow +\infty} \frac{\mu(|z_i|) |u'(z_i)| |\varphi(z_i)|^n}{(1 - |\varphi(z_i)|^2)^{\gamma+n-1}} \\
 &\quad - \beta(\gamma + n) \limsup_{i \rightarrow +\infty} \frac{\mu(|z_i|) |u(z_i) \varphi'(z_i)| |\varphi(z_i)|^{n+1}}{(1 - |\varphi(z_i)|^2)^{\gamma+n}} \\
 &= \beta \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|) |u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} \\
 &\quad - \beta(\gamma + n) \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|) |u(z) \varphi'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n}} \\
 &\succeq \beta \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|) |u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} - (\gamma + n) \|D_{u,\varphi}^n\|_{e, F(p,q,s) \rightarrow \mathcal{B}_\mu},
 \end{aligned}$$

where we use (3.5) in the last line. Then,

$$(3.6) \quad (\gamma + n + 1) \|D_{u,\varphi}^n\|_{e, F(p,q,s) \rightarrow \mathcal{B}_\mu} \succeq \beta \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|) |u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}}.$$

Now (3.5) and (3.6) imply that

$$\|D_{u,\varphi}^n\|_{e,F(p,q,s)\rightarrow\mathcal{B}_\mu} \succeq \limsup_{|\varphi(z)|\rightarrow 1} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} + \limsup_{|\varphi(z)|\rightarrow 1} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}},$$

and the lower estimate for the essential norm is obtained. Now, suppose that $\{r_j\}$ is a sequence in $(0, 1)$ and $r_j \rightarrow 1$ such that $j \rightarrow +\infty$. Since $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{B}_\mu$ is bounded, then from Theorem 1 of [10] we obtain

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} < +\infty, \quad \sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}} < +\infty.$$

So,

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u'(z)|}{(1-|r_j\varphi(z)|^2)^{\gamma+n-1}} < +\infty, \quad \sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|r_j\varphi(z)|^2)^{\gamma+n}} < +\infty.$$

Again Theorem 1 of [10] implies that $D_{u,r_j\varphi}^n : F(p, q, s) \rightarrow \mathcal{B}_\mu$ is bounded. Also, since $\|r_j\varphi\|_\infty < 1$, then using Lemma 3.1, $D_{u,r_j\varphi}^n : F(p, q, s) \rightarrow \mathcal{B}_\mu$ is a compact operator for any j . So,

$$\|D_{u,\varphi}^n\|_{e:F(p,q,s)\rightarrow\mathcal{B}_\mu} \leq \|D_{u,\varphi}^n - D_{u,r_j\varphi}^n\| = \sup_{\|f\|_{p,q,s} \leq 1} \|(D_{u,\varphi}^n - D_{u,r_j\varphi}^n)f\|_{\mathcal{B}_\mu}, \quad j \in \mathbb{N}.$$

Let $f \in F(p, q, s)$ and $\|f\|_{p,q,s} \leq 1$. Then,

$$\begin{aligned} \|(D_{u,\varphi}^n - D_{u,r_j\varphi}^n)f\|_{\mathcal{B}_\mu} &= |((D_{u,\varphi}^n - D_{u,r_j\varphi}^n)f)(0)| + \sup_{z \in \mathbb{D}} \mu(|z|)|((D_{u,\varphi}^n - D_{u,r_j\varphi}^n)f)'(z)| \\ &\leq |u(0)f^{(n)}(\varphi(0)) - u(0)f^{(n)}(r_j\varphi(0))| \\ &\quad + \sup_{z \in \mathbb{D}} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D}} \mu(|z|)|u(z)\varphi'(z)||f^{(n+1)}(\varphi(z)) - r_jf^{(n+1)}(r_j\varphi(z))| \\ (3.7) \quad &= E + F + G. \end{aligned}$$

Then, $E \rightarrow 0$ as $j \rightarrow +\infty$. For F and G we have

$$\begin{aligned} F &\leq \sup_{|\varphi(z)| \leq \delta} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))| \\ &\quad + \sup_{|\varphi(z)| > \delta} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))| = F_1 + F_2, \\ G &\leq \sup_{|\varphi(z)| \leq \delta} \mu(|z|)|u(z)\varphi'(z)||f^{(n+1)}(\varphi(z)) - r_jf^{(n+1)}(r_j\varphi(z))| \\ &\quad + \sup_{|\varphi(z)| > \delta} \mu(|z|)|u(z)\varphi'(z)||f^{(n+1)}(\varphi(z)) - r_jf^{(n+1)}(r_j\varphi(z))| = G_1 + G_2, \end{aligned}$$

where $\delta \in (0, 1)$ is fixed. Hence $F_1, G_1 \rightarrow 0$ as $j \rightarrow +\infty$, due to the compactness of the set $\{|\varphi(z)| \leq \delta\}$. Therefore, we get from (2.1) that

$$F_2 = \sup_{|\varphi(z)| > \delta} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))|$$

$$\begin{aligned} &\preceq \sup_{|\varphi(z)| > \delta} \frac{\mu(|z|)|u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} + \sup_{|\varphi(z)| > \delta} \frac{\mu(|z|)|u'(z)|}{(1 - |r_j \varphi(z)|^2)^{\gamma+n-1}} \\ &\leq 2 \sup_{|\varphi(z)| > \delta} \frac{\mu(|z|)|u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}}. \end{aligned}$$

Also, for G_2 we get

$$G_2 \preceq 2 \sup_{|\varphi(z)| > \delta} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n}}.$$

From (3.7) and above discussion we have

$$\|(D_{u,\varphi}^n - D_{u,r_j\varphi}^n)f\|_{\mathcal{B}_\mu} \preceq \sup_{|\varphi(z)| > \delta} \frac{\mu(|z|)|u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} + \sup_{|\varphi(z)| > \delta} \frac{\mu(|z|)|u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}},$$

and letting $\delta \rightarrow 1$ and using the definition of essential norm, we obtain

$$\|D_{u,\varphi}^n\|_{e,F(p,q,s) \rightarrow \mathcal{B}_\mu} \preceq \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|)|u'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} + \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n}}.$$

□

In the following theorem we find a necessary and sufficient condition for the operator $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ to be bounded.

Theorem 3.2. *Suppose that $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ is a radial weight. Also suppose that $0 < p, s < +\infty$, $-2 < q < +\infty$ and $q + s > -1$. Then, $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ is bounded if and only if the functions $A(u'', \gamma + n - 1)$, $A(2u'\varphi' + u\varphi'', \gamma + n)$ and $A(u\varphi'^2, \gamma + n + 1)$ are bounded on $\mathbb{D}$.*

Proof. Suppose that the above functions are bounded on $\mathbb{D}$. To prove that the operator is bounded, we compute its norm. To do that, we have

$$\begin{aligned} \|D_{u,\varphi}^n\|_{F(p,q,s) \rightarrow \mathcal{Z}_\mu} &= \sup_{\|f\|_{p,q,s} \leq 1} \|D_{u,\varphi}^n f\|_{\mathcal{Z}_\mu} \\ &= |(D_{u,\varphi}^n f)(0)| + |(D_{u,\varphi}^n f)'(0)| + \sup_{z \in \mathbb{D}} \mu(|z|)|D_{u,\varphi}^n f''(z)| \\ &= |u(0)f^{(n)}(\varphi(0))| + |u'(0)f^{(n)}(\varphi(0)) + u(0)\varphi'(0)f^{(n+1)}(\varphi(0))| \\ &\quad + \sup_{z \in \mathbb{D}} \mu(|z|)|u''(z)f^{(n)}(\varphi(z)) + (2u'(z)\varphi'(z) \\ &\quad + u(z)\varphi''(z))f^{(n+1)}(\varphi(z)) + u(z)\varphi'^2(z)f^{(n+2)}(\varphi(z))| \\ &\preceq \frac{|u(0)|}{(1 - |\varphi(0)|^2)^{\gamma+n-1}} + \frac{|u'(0)|}{(1 - |\varphi(0)|^2)^{\gamma+n-1}} + \frac{|u(0)\varphi'(0)|}{(1 - |\varphi(0)|^2)^{\gamma+n}} \\ &\quad + \sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} + \sup_{z \in \mathbb{D}} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n}} \\ &\quad + \sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'^2(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n+1}} < +\infty. \end{aligned}$$

So, $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ is bounded. Now suppose that $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ is bounded. Then there exists a positive constant C such that for every $f \in F(p, q, s)$, $\|D_{u,\varphi}^n f\|_{\mathcal{Z}_\mu} \leq C\|f\|_{p,q,s}$. Applying this to the polynomials z^n, z^{n+1}, z^{n+2} and using some basic techniques, the followings are obtained

$$(3.8) \quad \sup_{z \in \mathbb{D}} A(u'', 0) < +\infty, \quad \sup_{z \in \mathbb{D}} A(2u'\varphi' + u\varphi'', 0) < +\infty, \quad \sup_{z \in \mathbb{D}} A(u\varphi'^2, 0) < +\infty.$$

By [3] and Lemma 3 in [9], there exist functions $g_{i,w} \in F(p, q, s)$, $i \in \{n, n+1, n+2\}$ and $w \in \mathbb{D}$, such that

$$g_{i,w}^{(j)}(w) = \frac{\delta_{i,j} \bar{w}^j}{(1 - |w|^2)^{\gamma+j-1}}, \quad j \in \{n, n+1, n+2\}.$$

Also, $\sup_{w \in \mathbb{D}} \|g_{i,w}\|_{p,q,s} < +\infty$. Here $\delta_{i,j}$ is the Kronecker delta. We prove that $A(u'', \gamma + n - 1)$ is bounded, the others are similar. Noting that by the boundeness of the operator we have

$$\begin{aligned} C &\geq \|D_{u,\varphi}^n g_{n,\varphi(w)}\|_{\mathcal{Z}_\mu} \geq \sup_{z \in \mathbb{D}} \mu(|z|) |(D_{u,\varphi}^n g_{n,\varphi(w)})''(z)| \geq \mu(|w|) |(D_{u,\varphi}^n g_{n,\varphi(w)})''(w)| \\ &= \frac{\mu(|w|) |u''(w)| |\varphi(w)|^n}{(1 - |\varphi(w)|^2)^{\gamma+n-1}}. \end{aligned}$$

Then,

$$(3.9) \quad \sup_{|\varphi(w)| > \delta} \frac{\mu(|w|) |u''(w)|}{(1 - |\varphi(w)|^2)^{\gamma+n-1}} < +\infty,$$

where $\delta \in (0, 1)$ is fixed. Also

$$(3.10) \quad \sup_{|\varphi(w)| \leq \delta} \frac{\mu(|w|) |u''(w)|}{(1 - |\varphi(w)|^2)^{\gamma+n-1}} < \frac{1}{(1 - \delta^2)^{\gamma+n-1}} \sup_{w \in \mathbb{D}} A(u'', 0) < +\infty.$$

From the above discussion and using the fact that w is arbitrary, we get

$$A(u'', \gamma + n - 1) = \sup_{w \in \mathbb{D}} \frac{\mu(w) |u''(w)|}{(1 - |\varphi(w)|^2)^{\gamma+n-1}} < +\infty.$$

Applying the functions $g_{n+1,\varphi(w)}$ and $g_{n+2,\varphi(w)}$, the boundedness of the functions $A(2u'\varphi' + u\varphi'', \gamma + n)$ and $A(u\varphi'^2, \gamma + n + 1)$ is obtained. $\square$

The proof of the following lemma is analogues to the proof of Lemma 3.1.

Lemma 3.2. *Let $0 < p, s < +\infty$, $-2 < q < +\infty$, $q + s > -1$, $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ is a radial weight. If $\|\varphi\|_\infty < 1$ and $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ be bounded, then $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ is compact.*

Theorem 3.3. *Suppose that $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ is a radial weight. Also suppose that $0 < p, s < +\infty$, $-2 < q < +\infty$ and $q + s > -1$. If $D_{u,\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ is bounded, then*

$$\|D_{u,\varphi}^n\|_e \approx \limsup_{|\varphi(z)| \rightarrow 1} \left\{ A(u'', \gamma + n - 1) + A(2u'\varphi' + u\varphi'', \gamma + n) + A(u\varphi'^2, \gamma + n + 1) \right\}.$$

Proof. Let $\{w_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(w_k)| \rightarrow 1$, as $k \rightarrow +\infty$. By [3] and Lemma 3 in [9], there exist functions $g_{i,w_k} \in F(p, q, s)$, $i \in \{n, n+1, n+2\}$ such that

$$g_{i,w_k}^{(j)}(w_k) = \frac{\delta_{i,j} \overline{w_k^j}}{(1 - |w_k|^2)^{\gamma+j-1}}, \quad j \in \{n, n+1, n+2\}.$$

Also $\sup_{w_k \in \mathbb{D}} \|g_{i,w_k}\|_{p,q,s} < +\infty$. Here $\delta_{i,j}$ is the Kronecker delta. Moreover $g_{i,w_k} \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, as $k \rightarrow +\infty$. So, for any compact operator $T : F(p, q, s) \rightarrow \mathcal{Z}_\mu$, $\|Tg_{i,w_k}\|_{\mathcal{Z}_\mu} \rightarrow 0$. If we use the sequence $\{g_{n,w_k}\}$, then we have

$$\begin{aligned} \|D_{u,\varphi}^n - T\|_{F(p,q,s) \rightarrow \mathcal{Z}_\mu} &\succeq \limsup_{k \rightarrow +\infty} \|(D_{u,\varphi}^n - T)g_{n,w_k}\|_{\mathcal{Z}_\mu} = \limsup_{k \rightarrow +\infty} \|D_{u,\varphi}^n g_{n,w_k}\|_{\mathcal{Z}_\mu} \\ &\geq \limsup_{k \rightarrow +\infty} \sup_{z \in \mathbb{D}} \mu(|z|) |(D_{u,\varphi}^n g_{n,w_k})''(z)| \\ &\geq \limsup_{k \rightarrow +\infty} \mu(|w_k|) |(D_{u,\varphi}^n g_{n,w_k})''(w_k)| \\ &= \limsup_{k \rightarrow +\infty} \frac{\mu(|w_k|) |u''(w_k)| |\varphi(w_k)|^n}{(1 - |\varphi(w_k)|^2)^{\gamma+n-1}} \\ &= \limsup_{k \rightarrow +\infty} \frac{\mu(|w_k|) |u''(w_k)|}{(1 - |\varphi(w_k)|^2)^{\gamma+n-1}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|D_{u,\varphi}^n\|_{e,F(p,q,s) \rightarrow \mathcal{Z}_\mu} &= \inf_{T \text{ compact}} \|D_{u,\varphi}^n - T\|_{F(p,q,s) \rightarrow \mathcal{Z}_\mu} \\ &\succeq \|D_{u,\varphi}^n - T\|_{F(p,q,s) \rightarrow \mathcal{Z}_\mu} \succeq \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(|z|) |u''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} \\ &= \limsup_{|\varphi(z)| \rightarrow 1} A(u'', \gamma + n - 1). \end{aligned}$$

In the same way, using the sequences $\{g_{n+1,w_k}\}$ and $\{g_{n+2,w_k}\}$, the following is obtained

$$\begin{aligned} \|D_{u,\varphi}^n\|_{e,F(p,q,s) \rightarrow \mathcal{Z}_\mu} &\succeq \limsup_{|\varphi(z)| \rightarrow 1} A(2u'\varphi' + u\varphi'', \gamma + n), \\ \|D_{u,\varphi}^n\|_{e,F(p,q,s) \rightarrow \mathcal{Z}_\mu} &\succeq \limsup_{|\varphi(z)| \rightarrow 1} A(u\varphi'^2, \gamma + n + 1), \end{aligned}$$

and the lower estimate for the essential norm is obtained.

Let $\{r_k\}$ be a sequence in $(0, 1)$ such that $\lim_{k \rightarrow +\infty} r_k = 1$. Using the boundedness of $D_{u,\varphi}^n$ and Theorem 3.2, the operator $D_{u,r_k\varphi}^n : F(p, q, s) \rightarrow \mathcal{Z}_\mu$ is bounded and then it is compact by Lemma 3.2, since $\|r_k\varphi\|_\infty < 1$. Therefore,

$$(3.11) \quad \|D_{u,\varphi}^n\|_e \leq \|D_{u,\varphi}^n - D_{u,r_k\varphi}^n\| = \sup_{\|f\|_{p,q,s} \leq 1} \|(D_{u,\varphi}^n - D_{u,r_k\varphi}^n)f\|_{\mathcal{Z}_\mu}.$$

Noting that $f - f_{r_k} \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, $f_r(z) = f(rz)$, and employing (3.8) we have

$$\limsup_{k \rightarrow +\infty} |u(0)f^{(n)}(\varphi(0)) - u(0)f^{(n)}(r_k\varphi(0))| = 0,$$

$$\begin{aligned}
\limsup_{k \rightarrow +\infty} |u'(0)(f^{(n)}(\varphi(0)) - f^{(n)}(r_k \varphi(0)))| &= 0, \\
\limsup_{k \rightarrow +\infty} |u(0)\varphi'(0)(f^{(n+1)}(\varphi(0)) - r_k f^{(n+1)}(r_k \varphi(0)))| &= 0, \\
\limsup_{k \rightarrow +\infty} \sup_{|\varphi(z)| \leq \delta} \mu(|z|)|u''(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_k \varphi(z))| &= 0, \\
\limsup_{k \rightarrow +\infty} \sup_{|\varphi(z)| \leq \delta} \mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)||f^{(n+1)}(\varphi(z)) - r_k f^{(n+1)}(r_k \varphi(z))| &= 0, \\
\limsup_{k \rightarrow +\infty} \sup_{|\varphi(z)| \leq \delta} \mu(|z|)|u(z)\varphi'^2(z)||f^{(n+2)}(\varphi(z)) - r_k^2 f^{(n+2)}(r_k \varphi(z))| &= 0,
\end{aligned}$$

where $\delta \in (0, 1)$ is fixed. Also, (2.1) implies that for any f with $\|f\|_{p,q,s} \leq 1$,

$$\begin{aligned}
& \sup_{\delta < |\varphi(z)| < 1} \mu(|z|)|u''(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_k \varphi(z))| \\
& \leq \sup_{\delta < |\varphi(z)| < 1} \mu(|z|)|u''(z)||f^{(n)}(\varphi(z))| + \sup_{\delta < |\varphi(z)| < 1} \mu(|z|)|u''(z)||f^{(n)}(r_k \varphi(z))| \\
& \preceq \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|u''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} + \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|u''(z)|}{(1 - |r_k \varphi(z)|^2)^{\gamma+n-1}} \\
& \leq 2 \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|u''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \sup_{\delta < |\varphi(z)| < 1} \mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)||f^{(n+1)}(\varphi(z)) - r_k f^{(n+1)}(r_k \varphi(z))| \\
& \preceq \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n}}
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{\delta < |\varphi(z)| < 1} \mu(|z|)|u(z)\varphi'^2(z)||f^{(n+2)}(\varphi(z)) - r_k^2 f^{(n+2)}(r_k \varphi(z))| \\
& \preceq \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|u(z)\varphi'^2(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n+1}}.
\end{aligned}$$

We get from the above discussion that

$$\begin{aligned}
\sup_{\|f\|_{p,q,s} \leq 1} \|(D_{u,\varphi}^n - D_{u,r_k \varphi}^n)f\|_{z_\mu} & \preceq \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|u''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n-1}} \\
& + \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n}} \\
& + \sup_{\delta < |\varphi(z)| < 1} \frac{\mu(|z|)|u(z)\varphi'^2(z)|}{(1 - |\varphi(z)|^2)^{\gamma+n+1}}.
\end{aligned}$$

Now it follows from (3.11) along with $\delta \rightarrow 1$ that

$$\|D_{u,\varphi}^n\|_e \preceq \limsup_{|\varphi(z)| \rightarrow 1} \left\{ A(u'', \gamma + n - 1) + A(2u'\varphi' + u\varphi'', \gamma + n) + A(u\varphi'^2, \gamma + n + 1) \right\}.$$

□

Remark 3.1. As we noted in the introduction, the operators we use here contain some other well-known operators and also $F(p, q, s)$ spaces include several classical spaces of analytic functions. So some corollaries can be stated here which we do not bring here.

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THE SOLVABILITY OF p -KIRCHHOFF TYPE PROBLEMS WITH CRITICAL EXPONENT

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ABSTRACT. The article aims to study new and current problems in the theory of nonclassical partial differential equations and their applications, proving the existence and nonexistence of solutions to p -Kirchhoff type problems with critical exponent of Sobolev in $\mathbb{R}^n$, which are of great interest in the study of mathematical physics equations. We show the existence of a local minimizer with negative/positive energy by using variational methods. More precisely, we considered a minimization of E_λ constrained in a neighborhood of zero using the Ekeland variational principle, then we found the first critical point of E_λ which achieves the local minimum of E_λ whose level is negative; next around the zero point, using the mountain pass theorem, we also obtained a critical point whose level is positive. In addition, we studied the case of $\lambda = 0$, where there is no non-trivial solution using the contradiction principle. We also established infinite solutions and discuss the different cases.

1. INTRODUCTION AND POSITION OF PROBLEM

In $\mathbb{R}^n$, we are concerned with the following problem

$$(1.1) \quad - \left(a \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^{\theta-1} + b \right) \operatorname{div} (|\nabla u|^{p-2} \nabla u) = |u|^{p^*-2} u + \lambda f(x),$$

where $1 < p < n$, $0 \leq a, b$, $0 < a + b$, $\theta > 1$, λ is a parameter, $p^* = pn/(n-p)$ is the critical Sobolev exponent and $f \in W^* \setminus \{0\}$. Here, W^* is the dual space of $W^{1,p}(\mathbb{R}^n)$.

Key words and phrases. Variational methods, Kirchhoff equations, critical Sobolev exponent, Lyapunov functions, nonlinear equations.

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Since equation (1.1) contains an integral over $\mathbb{R}^n$, it is no longer a point-wise identity. Therefore, it is often called a nonlocal problem. It is also called non-degenerate if $0 < b$ and $a \geq 0$, while it is degenerate if $b = 0$ and $0 < a$.

The non-local elliptic problem (1.1) is related to the original Kirchhoff equation in [1] which was first introduced by Kirchhoff as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the strings produced by transverse vibrations.

Problems involving non-local operators have been widely studied due to their numerous and relevant applications in various fields of science. In particular, Kirchhoff-type problems proved to be valuable tools for modeling several physical and biological phenomena, and many works have been made to ensure the existence of solutions for such problems. We quote in particular the article of Lions [2]. Since this famous paper, very fruitful developments are given rise to many works in this promising direction, and in most of them, with most relying on topological methods. However, only a few improvements have been made concerning the multiplicity of solutions. In this regard, the variational approach was sought instead of topological methods to solve these kinds of problems and also to prove the existence of multiple solutions (see [3, 4]).

In the last few years, great attention has been paid to the study of Kirchhoff problems involving critical nonlinearities. This problem creates many difficulties in applying variational methods. It is worth mentioning that the semilinear Laplace equation of elliptic type involving the critical Sobolev exponent was investigated in the crucial paper of Brézis and Nirenberg [5]. After that, many researchers have dedicated themselves to the study of several kinds of elliptic equations with a critical exponent in a bounded domain or in the whole space. For $p = 2$ and $a = 0$, Tarantello [6] treated the problem (1.1) in a bounded domain of $\mathbb{R}^n$ and proved the existence of at least two solutions using Nehari manifold methods. The first work on the Kirchhoff-type problem with the critical Sobolev exponent was by Alves, Corrêa and Figueiredo in [7]. Naimen in [8] showed a Brézis-Nirenberg type result for the Kirchhoff problem in a bounded domain. In [9], He et al. considered the following problem

$$\begin{aligned} & - \left(a \int_{\mathbb{R}^n} |\nabla u|^p dx + b \right) \operatorname{div} (|\nabla u|^{p-2} \nabla u) = f(u) + h \text{ in } \Omega, \\ & u = 0 \text{ on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $0 \leq h \in L^2(\Omega)$ and $f \in C(\mathbb{R}, \mathbb{R})$. They obtained the existence of at least one or two positive solutions using the monotonicity technique and the non-existence criterion by using the corresponding Pohožaev identity. Also, they showed nonexistence properties for the 3-sublinear case and the critical case. Under general assumptions on the nonlinearity, they also established the existence result for the whole space $\mathbb{R}^3$ by using the properties of the Pohožaev identity and some delicate analysis.

In [10], Miyagaki et al. obtained the existence of infinitely many solutions for the following problem

$$-(a \int_{\mathbb{R}^n} |\nabla u|^p dx + b) \operatorname{div} (|\nabla u|^{p-2} \nabla u) = \lambda g(x, u) + \mu |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^n,$$

where $\lambda > 0$, $0 < \mu < \mu_*$, with μ_* a positive constant, $n \geq 2p$ and $g(x, u)$ satisfies certain subcritical growth conditions.

Ke et al. in [11] considered the problem (1.1) with $\theta = 2$ and $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n)$. They obtained the existence of infinitely many solutions for problem (1.1) and the multiplicity of solutions for the non-degenerate case ($0 < b$) with $p^* = 2p$ and $0 < a < S^{-2}$ and the degenerate case ($b = 0$) with $p^* > 2p$ and $0 < a$.

Recently, Benaissa et al. in [12] discussed the problem $(\mathcal{P}_\lambda)$ with $\theta = 2$. When $p^* > 2p$, the authors showed the existence of $\lambda_* > 0$ such that for $0 < \lambda < \lambda_*$ (1.1) has at least two solutions in a particular dimension $n = 3p/2$.

The main results in the present paper can be considered as an extension of the work of [11] and [12] for a more general non-local problem (degenerate or non-degenerate problem) with a large range of n .

To our knowledge, many of the results are new for $p > 1$ and even in the case $\theta = 2$. Our results and setting are more general and delicate; it is not difficult to obtain the second solution in the degenerate case where $\theta < \frac{p^*}{p}$.

In the case where $a = \lambda = 0$, $b = 1$ and $1 < p < n$, our main problem can be reduced to the following problem

$$(1.2) \quad -\operatorname{div} (|\nabla u|^{p-2} \nabla u) = |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^n.$$

Sciunzi in [13] provided that if u is a positive solution of (1.2) then $u(x) = v_{\varepsilon, x_0}(x)$ where

$$(1.3) \quad v_{\varepsilon, x_0}(x) := \left[\frac{\varepsilon^{\frac{1}{p-1}} n^{\frac{1}{p}} \left(\frac{n-p}{p-1} \right)^{\frac{p-1}{p}}}{\varepsilon^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}} \right]^{\frac{n-p}{p}}, \quad \varepsilon > 0, x_0 \in \mathbb{R}^n.$$

Consequently,

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p/p^*}}$$

satisfies

$$(1.4) \quad \|v_{\varepsilon, x_0}\|^p = \int_{\mathbb{R}^n} |v_{\varepsilon, x_0}|^{p^*} dx = S^{\frac{p^*}{p^*-p}}.$$

For more details, see [14].

This paper is composed of four sections in addition to the introduction and the conclusion with discussion. In Section 2, we give some abstract conditions when the functional E_λ satisfies the Palais Smale condition, so to overcome the lack of compactness, we need to determine a good level of the Palais Smale condition. We state and prove our first main contribution results regarding the existence of local

minimizer with negative/positive energy (Theorem 3.1 and Theorem 3.2) in Section 3, by using variational methods. More precisely, we consider a minimization of E_λ constrained in a neighborhood of zero using the Ekeland variational principle (see [15]). Then we can find the first critical point of E_λ that reaches the local minimum of E_λ whose level is negative, next around the zero point, using the mountain pass theorem (see [16]). We also obtain a critical point whose level is positive. The proof of the second results (Theorem 4.1 and Theorem 4.2) is given and proved in Section 4, where we study the case of $\lambda = 0$.

2. PRELIMINARIES AND TOOLS

In this section, we state several preliminary results needed.

First, we make use of the following assumptions:

- ($\mathcal{H}_0$) $1 < \theta < \frac{p^*}{p}$, $0 < a$ and $0 < b$;
- ($\mathcal{H}_1$) $1 < \theta < \frac{p^*}{p}$, $a \geq 0$, $0 \leq b$ and $a + 0 < b$;
- ($\mathcal{H}_2$) $\theta = \frac{p^*}{p}$, $a \geq 0$ and $0 < b$;
- ($\mathcal{H}_3$) $\theta = \frac{p^*}{p}$, $0 \leq a < S^{-\theta}$ and $0 < b$;
- ($\mathcal{H}_4$) $\theta = \frac{p^*}{p}$, $a > S^{-\theta}$ and $b = 0$;
- ($\mathcal{H}_5$) $\theta = \frac{p^*}{p}$, $a \geq S^{-\theta}$ and $0 < b$;
- ($\mathcal{H}_6$) $\theta = \frac{p^*}{p}$, $0 < a$ and $b = 0$;
- ($\mathcal{H}_7$) $\theta > \frac{p^*}{p}$, $0 < a$ and $b = b^*$, where

$$b^* = \frac{\theta p - p^*}{(\theta - 1)p} \left(\frac{(\theta - 1)p}{p^* - p} a \right)^{-\frac{p^* - p}{\theta p - p^*}} S^{-\frac{(\theta - 1)p^*}{\theta p - p^*}};$$

- ($\mathcal{H}_8$) $\frac{p^*}{p} < \theta$, $0 < a$ and $b > b^*$;
- ($\mathcal{H}_9$) $\theta > \frac{p^*}{p}$, $0 < a$ and $b < b^*$.

The Sobolev space $W^{1,p}(\mathbb{R}^n)$ is the space of measurable functions $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that u and the distributional gradient $\nabla u = (\partial_1 u, \dots, \partial_n u)$ are in $L^p(\mathbb{R}^n)$.

Definition 2.1. Let $c \in \mathbb{R}$. A sequence $(u_n) \subset W^{1,p}(\mathbb{R}^n)$ is a Palais Smale sequence at level c , so called $(PS)_c$ sequence, if

$$E_\lambda(u_n) \rightarrow c \quad \text{and} \quad E'_\lambda(u_n) \rightarrow 0.$$

Here, E_λ verifies the $(PS)_c$ condition at level c if any $(PS)_c$ sequence has a convergent subsequence in $W^{1,p}(\mathbb{R}^n)$.

Lemma 2.1. Suppose that $f \in W^* \setminus \{0\}$ and assume that $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds. Let $c \in \mathbb{R}$ and $(u_n) \subset W^{1,p}(\mathbb{R}^n)$ be a $(PS)_c$ sequence for E_λ . Then,

$$u_n \rightharpoonup u, \quad \text{in } W^{1,p}(\mathbb{R}^n),$$

for some $u \in W^{1,p}(\mathbb{R}^n)$ with $E'_\lambda(u) = 0$.

Proof. We have

$$E_\lambda(u_n) \rightarrow c \quad \text{and} \quad E'_\lambda(u_n) \rightarrow 0.$$

Then,

$$c + o_n(1) = E_\lambda(u_n) \quad \text{and} \quad o_n(1) \|v\| = \langle E'_\lambda(u_n), v \rangle,$$

for all $v \in W^{1,p}(\mathbb{R}^n)$, where $o_n(1)$ denotes any quantity that tends to zero as $n \rightarrow +\infty$.

Since $n \rightarrow +\infty$, we have

$$\begin{aligned} c + o_n(1) - \frac{1}{p^*} o_n(1) \|u_n\| &= E_\lambda(u_n) - \frac{1}{p^*} \langle E'_\lambda(u_n), u_n \rangle \\ &= a \frac{p^* - \theta p}{\theta p p^*} \|u_n\|^{\theta p} + b \frac{p^* - p}{p p^*} \|u_n\|^p - \lambda \frac{p^* - 1}{p^*} \int_{\mathbb{R}^n} f(x) u_n dx \\ &\geq a \frac{p^* - \theta p}{\theta p p^*} \|u_n\|^{\theta p} + b \frac{p^* - p}{p p^*} \|u_n\|^p - \lambda \frac{p^* - 1}{p^*} \|f\|_{W^*} \|u_n\|, \end{aligned}$$

that is, (u_n) is bounded in $W^{1,p}(\mathbb{R}^n)$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds. Up to a subsequence if necessary, there exists a function $u \in W^{1,p}(\mathbb{R}^n)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$ and in $L^{p^*}(\mathbb{R}^n)$, $u_n \rightarrow u$ a.e. in $\mathbb{R}^n$, and $\int_{\mathbb{R}^n} f(x) u_n dx \rightarrow \int_{\mathbb{R}^n} f(x) u dx$. Then,

$$\langle E'_\lambda(u_n), v \rangle = 0, \quad \text{for all } v \in C_0^\infty(\mathbb{R}^n).$$

Thus, $E'_\lambda(u) = 0$. This completes the proof. $\square$

In order to introduce the local Palais Smale condition, we must state the following lemma, which can be considered a key step in obtaining a solution with positive energy (a mountain pass-type solution).

Lemma 2.2. *Let $\theta > 1$, $0 \leq a, b$, $0 < a + b$, $\sigma \geq 0$ and $\tilde{x} = \left(\frac{\sigma}{a} S^{-\theta}\right)^{\frac{1}{1-\sigma}}$ for $\sigma \neq 1$. For $x \geq 0$, let us consider the function $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^*$, where*

$$\Psi(x) = S^{-1} x^\sigma - a S^{\theta-1} x - b.$$

Then, the following hold.

(1) *If $\sigma = 1$, $S^{-\theta} > a \geq 0$ and $0 < b$, then the equation $\Psi(x) = 0$ has a unique positive solution such that*

$$x_1 = \frac{b}{(S^{-\theta} - a) S^{\theta-1}},$$

and $\Psi(x) \geq 0$ for all $x \geq x_1$.

(2) *If $\sigma > 1$, the equation $\Psi(x) = 0$ has a unique positive solution $x_2 > \tilde{x}$ and $\Psi(x) \geq 0$ for all $x \geq x_2$.*

(3) *If $\sigma < 1$, then we have the following cases.*

i) *If $\Psi(\tilde{x}) < 0$, then we have $\Psi(x) \neq 0$ for all $x \geq 0$.*

ii) *If $\Psi(\tilde{x}) = 0$, then we have*

$$b = S^{-1} (1 - \sigma) \left(\frac{\sigma}{a} S^{-\theta}\right)^{\frac{\sigma}{1-\sigma}},$$

and $\Psi(x) \neq 0$ for all $x \neq \tilde{x}$.

iii) If $\Psi(\tilde{x}) > 0$, then we have

$$b < S^{-1}(1 - \sigma) \left(\frac{\sigma}{a} S^{-\theta} \right)^{\frac{\sigma}{1-\sigma}},$$

and $\Psi(x) = 0$ for two different points.

Proof. (1) For $\sigma = 1$, $S^{-\theta} > a \geq 0$ and $0 < b$, we have

$$\Psi(x) = S^{\theta-1}(S^{-\theta} - a)x - b.$$

That is, equation $\Psi(x) = 0$ has a unique positive solution

$$x_1 = \frac{b}{(S^{-\theta} - a)S^{\theta-1}},$$

and $\Psi(x) \geq 0$ for all $x \geq x_1$.

(2) For $\sigma > 1$, we have $\Psi'(x) = \sigma S^{-1}x^{\sigma-1} - aS^{\theta-1}$ and

$$\Psi''(x) = \sigma(\sigma - 1)S^{-1}x^{\sigma-2} > 0, \quad \text{for all } x > 0.$$

Then, $\Psi'(\tilde{x}) = 0$, $\Psi'(x) < 0$ for $y < \tilde{x}$ and $\Psi'(y) > 0$ for $x > \tilde{x}$. Hence, Ψ is a concave function and

$$(2.1) \quad \min_{x \geq 0} \Psi(x) = \Psi(\tilde{x}) = -(\sigma - 1)S^{-1} \left(\frac{a}{\sigma} S^{\theta} \right)^{\frac{\sigma}{\sigma-1}} < 0.$$

Moreover, we have $\Psi(\tilde{x}) < 0$ and $\lim_{x \rightarrow +\infty} \Psi(x) = +\infty$. Thus, from (2.1) and the concavity of Ψ , we can conclude that the equation $\Psi(x) = 0$ has a unique positive solution $x_2 > \tilde{x}$ and $\Psi(x) \geq 0$ for all $x \geq x_2$.

(3) For $\sigma < 1$, we have $\Psi'(\tilde{x}) = 0$, Ψ is increasing for $0 < x < \tilde{x}$ and Ψ is decreasing when $x > \tilde{x}$. Moreover, from $\Psi(0) = -b < 0$, we obtain i), ii) and iii). $\square$

3. EXISTENCE OF LOCAL MINIMIZER WITH NEGATIVE/POSITIVE ENERGY

3.1. With negative energy.

Definition 3.1. We say that $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$ is a weak solution of equation (1.1) if

$$\left(a \|u\|^{(\theta-1)p} + b \right) \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla v dx - \int_{\mathbb{R}^n} (|u|^{p^*-2} u - \lambda f(x)) v dx = 0,$$

for any $v \in W^{1,p}(\mathbb{R}^n)$.

Next, we define the energy functional.

$$E_{\lambda}(u) = \frac{a}{\theta p} \|u\|^{\theta p} + \frac{b}{p} \|u\|^p - \frac{1}{p^*} \int_{\mathbb{R}^n} |u|^{p^*} dx - \lambda \int_{\mathbb{R}^n} f(x) u dx,$$

associated to problem (1.1), for all $u \in W^{1,p}(\mathbb{R}^n)$.

Notice that the functional E_{λ} is well defined in $W^{1,p}(\mathbb{R}^n)$, and belongs to

$$C^1(W^{1,p}(\mathbb{R}^n), \mathbb{R}),$$

and a critical point of E_λ is a weak solution of the problem (1.1). When $\lambda > 0$, we have the following result.

Theorem 3.1. *Suppose that $f \in W^* \setminus \{0\}$ and assume $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds. Then, there exist constants $\lambda_- > 0$ such that for any $\lambda \in (0, \lambda_-)$ problem (1.1) has a solution u_- with negative energy.*

Remark 3.1. If $\theta > \frac{p^*}{p}$, $a \geq 0$, $0 \leq b$ and $0 < a + b$ or $\theta = \frac{p^*}{p}$, $a = S^{-\theta}$ and $0 < b$ or $\theta = \frac{p^*}{p}$, $a > S^{-\theta}$ and $0 \leq b$, then for any $\lambda > 0$, we can easily show the existence of one solution which is a ground state solution.

Using the Ekeland variational principle, we now provide the proof of our first main result stated in Theorem 3.1.

Proof of Theorem 3.1. Let $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$, $0 < b$, $a \geq 0$ and $\theta \leq \frac{p^*}{p}$. By applying Hölder's and Sobolev inequalities, we have

$$\begin{aligned} E_\lambda(u) &= \frac{a}{\theta p} \|u\|^{\theta p} + \frac{b}{p} \|u\|^p - \frac{1}{p^*} \int_{\mathbb{R}^n} |u|^{p^*} dx - \lambda \int_{\mathbb{R}^n} f(x) u dx \\ &\geq \frac{b}{p} \|u\|^p + \frac{a}{\theta p} \|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*} \|u\|^{p^*} - \lambda \|f\|_{W^*} \|u\|. \end{aligned}$$

Now, we divide the proof into two cases.

Firstly, assume $0 < b$ and $a \geq 0$. If $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds, we get

$$E_\lambda(u) \geq \frac{b}{p} \|u\|^p - \frac{S^{-p^*/p}}{p^*} \|u\|^{p^*} - \left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*} \left(\frac{b}{2}\right)^{\frac{1}{p}} \|u\|.$$

It follows from the inequality $Xx \leq \frac{X^q}{q} + \frac{x^{q'}}{q'}$ for any $X, x \geq 0$ and $q, q' > 0$, with $\frac{1}{q} + \frac{1}{q'} = 1$, that

$$\begin{aligned} E_\lambda(u) &\geq \frac{b}{p} \|u\|^p - \frac{S^{-p^*/p}}{p^*} \|u\|^{p^*} - \frac{p-1}{p} \left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*} \right)^{\frac{p}{p-1}} - \frac{1}{p} \left(\left(\frac{b}{2}\right)^{\frac{1}{p}} \|u\| \right)^p \\ &\geq \frac{b}{2p} \|u\|^p - \frac{S^{-p^*/p}}{p^*} \|u\|^{p^*} - \frac{p-1}{p} \left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*} \right)^{\frac{p}{p-1}}. \end{aligned}$$

For $\rho \geq 0$, let us consider the function $h_1 \in C(\mathbb{R}^+, \mathbb{R}^*)$, given by

$$h_1(\rho) = \frac{b}{2p} \rho^p - \frac{S^{-p^*/p}}{p^*} \rho^{p^*}.$$

The direct calculation shows that

$$\max_{\rho \geq 0} h_1(\rho) = h_1(\rho_1) = \frac{p^* - p}{pp^*} S^{\frac{p^*}{p^*-p}} \left(\frac{b}{2}\right)^{\frac{p^*}{p^*-p}}, \quad \text{with } \rho_1 = \left[\frac{b}{2} S^{p^*/p}\right]^{\frac{1}{p^*-p}},$$

and $h_1(\rho) \geq 0$ for all $\rho \in B_{\rho_1}(0)$. Consequently,

$$(3.1) \quad E_\lambda(u)|_{B_{\rho_1}(0)} \geq -\frac{p-1}{p} \left(\left(\frac{b}{2} \right)^{\frac{-1}{p}} \lambda \|f\|_{W^*} \right)^{\frac{p}{p-1}}.$$

Moreover, for $\|u\| = \rho_1$ we have

$$\begin{aligned} E_\lambda(u) &\geq h_1(\rho_1) - \frac{p-1}{p} \left(\left(\frac{b}{2} \right)^{\frac{-1}{p}} \lambda \|f\|_{W^*} \right)^{\frac{p}{p-1}} \\ &\geq \frac{1}{p} h_1(\rho_1) + \frac{p-1}{p} h_1(\rho_1) - \frac{p-1}{p} \left(\left(\frac{b}{2} \right)^{\frac{-1}{p}} \lambda \|f\|_{W^*} \right)^{\frac{p}{p-1}} \geq \frac{1}{p} h_1(\rho_1) \\ &= \delta_1, \end{aligned}$$

for all $\lambda \in (0, \lambda_1)$, with

$$\lambda_1 = \left(\frac{p^* - p}{pp^*} S^{\frac{p^*}{p^* - p}} \right)^{\frac{p-1}{p}} \|f\|_{W^*}^{-1} \left(\frac{b}{2} \right)^{\frac{p^* - 1}{p^* - p}}.$$

We turn to the case where $0 < a$ and $0 \leq b$. If $(\mathcal{H}_1)$ holds, we obtain

$$\begin{aligned} E_\lambda(u) &\geq \frac{a}{\theta p} \|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*} \|u\|^{p^*} - \left(\left(\frac{a}{2} \right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*} \right) \left(\left(\frac{a}{2} \right)^{\frac{1}{\theta p}} \|u\| \right) \\ &\geq \frac{a}{\theta p} \|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*} \|u\|^{p^*} - \frac{\theta p - 1}{\theta p} \left(\left(\frac{a}{2} \right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*} \right)^{\frac{\theta p}{\theta p - 1}} - \frac{1}{\theta p} \left(\left(\frac{a}{2} \right)^{\frac{1}{\theta p}} \|u\| \right)^{\theta p} \\ &\geq \frac{a}{2\theta p} \|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*} \rho^{p^*} - \frac{\theta p - 1}{\theta p} \left(\left(\frac{a}{2} \right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*} \right)^{\frac{\theta p}{\theta p - 1}}. \end{aligned}$$

Now, we define the function $h_2 \in (\mathbb{R}^+, \mathbb{R}^*)$ as follows

$$h_2(\rho) = \frac{a}{2\theta p} \rho^{\theta p} - \frac{S^{-p^*/p}}{p^*} \rho^{p^*}.$$

Then,

$$\max_{\rho \geq 0} h_2(\rho) = h_2(\rho_2) = \left(\frac{1}{\theta p} - \frac{1}{p^*} \right) S^{-p^*/p} \left[\frac{a}{2} S^{p^*/p} \right]^{\frac{p^*}{p^* - \theta p}}, \quad \text{with } \rho_2 = \left[\frac{a}{2} S^{p^*/p} \right]^{\frac{1}{p^* - \theta p}},$$

and $h_2(\rho) \geq 0$ for all $\rho \in B_{\rho_2}(0)$. Consequently,

$$E_\lambda(u)|_{B_{\rho_2}(0)} \geq -\frac{\theta p - 1}{\theta p} \left(\left(\frac{a}{2} \right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*} \right)^{\frac{\theta p}{\theta p - 1}}.$$

Moreover, for $\|u\| = \rho_2$ we have

$$\begin{aligned} E_\lambda(u) &\geq h_2(\rho_2) - \frac{\theta p - 1}{\theta p} \left(\left(\frac{a}{2} \right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*} \right)^{\frac{\theta p}{\theta p - 1}} \\ &\geq \frac{\theta p - 1}{\theta p} h_2(\rho_2) + \frac{1}{\theta p} h_2(\rho_2) - \frac{\theta p - 1}{\theta p} \left(\left(\frac{a}{2} \right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*} \right)^{\frac{\theta p}{\theta p - 1}} \geq \frac{1}{\theta p} h_2(\rho_2) \\ &= \delta_2, \end{aligned}$$

for all $\lambda \in (0, \lambda_2)$, with

$$\lambda_2 = \left(\frac{p^* - \theta p}{\theta p p^*} S^{\frac{\theta p^*}{p^* - \theta p}} \right)^{\frac{\theta p - 1}{\theta p}} \left(\frac{a}{2} \right)^{\frac{p^* - 1}{p^* - \theta p}} \|f\|_{W^*}^{-1}.$$

We choose δ_* , ρ_* and λ_- such that

$$(3.2) \quad (\delta_*, \rho_*, \lambda_-) = \begin{cases} (\delta_1, \rho_1, \lambda_1), & \text{if } (\mathcal{H}_2) \text{ or } (\mathcal{H}_1) \text{ satisfies with } 0 < b, \\ (\delta_2, \rho_2, \lambda_2), & \text{if } (\mathcal{H}_1) \text{ satisfies with } 0 < a. \end{cases}$$

Then, for all $\lambda \in (0, \lambda_-)$, we have

$$(3.3) \quad E_\lambda(u)|_{\partial B_{\rho_*}(0)} \geq \delta_*$$

and

$$(3.4) \quad E_\lambda(u)|_{B_{\rho_*}(0)} \geq -C_\lambda,$$

with

$$(3.5) \quad C_\lambda := \begin{cases} \frac{p-1}{p} \left(\left(\frac{b}{2} \right)^{\frac{-1}{p}} \lambda \|f\|_{W^*} \right)^{\frac{p}{p-1}}, & \text{if } (\mathcal{H}_2) \text{ or } (\mathcal{H}_1) \text{ is satisfied with } 0 < b, \\ \frac{\theta p - 1}{\theta p} \left(\left(\frac{a}{2} \right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*} \right)^{\frac{\theta p}{\theta p - 1}}, & \text{if } (\mathcal{H}_1) \text{ is satisfied with } 0 < a. \end{cases}$$

Now, we define

$$(3.6) \quad c_- = \inf \{ E_\lambda(u), u \in \overline{B}_{\rho_*}(0) \}.$$

As $f \in W^* \setminus \{0\}$ we can choose $\varphi \in W^{1,p}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} f(x) \varphi dx > 0$. Then, for a fixed $\lambda \in (0, \lambda_-)$, there exists $t_0 > 0$ such that $\|t_0 \varphi\| < \rho_*$ and

$$c_- \leq E_\lambda(t_0 \varphi) < 0, \quad \text{for } t \in (0, t_0).$$

Hence, $c_- < E_\lambda(0) = 0$. Using the Ekeland variational principle, for the complete metric space $\overline{B}_{\rho_*}(0)$ with respect to the norm of $W^{1,p}(\mathbb{R}^n)$, we obtain the result that there exists a Palais Smale sequence $u_n \in \overline{B}_{\rho_*}(0)$ at level c_- . By Lemma 2.1, there exists $u_- \in \overline{B}_{\rho_*}(0)$ such that $u_n \rightharpoonup u_-$ in $W^{1,p}(\mathbb{R}^n)$ and $E'_\lambda(u_-) = 0$.

Now, we shall show that $u_n \rightarrow u_-$ in $W^{1,p}$. Suppose otherwise. Then, $\|u_-\| < \liminf_{n \rightarrow +\infty} \|u_n\|$, which implies that

$$\begin{aligned} c_- &\leq E_\lambda(u_-) \\ &= E_\lambda(u_-) - \frac{1}{p^*} \langle E'_\lambda(u_-), u_- \rangle \\ &= a \frac{p^* - \theta p}{\theta p p^*} \|u_-\|^{\theta p} + b \frac{p^* - p}{p p^*} \|u_-\|^p - \lambda \frac{p^* - 1}{p^*} \int_{\mathbb{R}^n} f(x) u_- dx \\ &< \liminf_{n \rightarrow +\infty} \left[a \frac{p^* - \theta p}{\theta p p^*} \|u_n\|^{\theta p} + b \frac{p^* - p}{p p^*} \|u_n\|^p - \lambda \frac{p^* - 1}{p^*} \int_{\mathbb{R}^n} f(x) u_n dx \right] \\ &= \liminf_{n \rightarrow +\infty} \left[E_\lambda(u_n) - \frac{1}{p^*} \langle E'_\lambda(u_n), u_n \rangle \right] \\ &= c_-. \end{aligned}$$

This is a contradiction. We conclude that $u_n \rightarrow u_-$ strongly in $W^{1,p}(\mathbb{R}^n)$. Therefore, $E'_\lambda(u_-) = 0$ and $E_\lambda(u_-) = c_- < 0 = E_\lambda(0)$. Hence, u_- is a nonzero solution of (1.1) with negative energy. $\square$

3.2. Existence of a mountain pass type solution. The next theorem guarantees a second solution of (1.1) of mountain pass type.

Theorem 3.2. *Suppose that $f \in W^* \setminus \{0\}$ such that $\int_{\mathbb{R}^n} f(x) v_{\varepsilon, x_0} dx \neq 0$. Assume that $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds. Then, there exists a constant $\lambda_+ \in (0, \lambda_-]$ such that for any $\lambda \in (0, \lambda_+)$ the problem (1.1) has a second solution u_+ with positive energy.*

Notice that the assumption $\int_{\mathbb{R}^n} f(x) v_{\varepsilon, x_0} dx \neq 0$ certainly holds if $f \in W^* \setminus \{0\}$ does not change its sign. Also, we have $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n)$ since $f \in W^* \setminus \{0\}$ and $u_-, u_+ \geq 0$ if $f \geq 0$. Furthermore, in Remark 1.2 [11], the authors mentioned that it is not easy to obtain the second solution for the case $p < n$, $0 < a$ and $0 < b$. For the special dimension $n = 3p/2$, this case is studied in [12]. In our work, we have overcome these difficulties and we give the results for general $\theta \in (1, \frac{p^*}{p}]$. Moreover, for $\theta = 2$, the condition $\theta \leq \frac{p^*}{p}$ is equivalent to $n \in]p, 2p]$ and $n \in \{3, 4\}$ for $p = 2$. Our results imply that suitable real θ can release the restriction on the spatial dimension n , for example. If $\theta = 1 + \varepsilon$ with $\varepsilon > 0$ is small enough, we have $\theta \leq \frac{p^*}{p}$ for a large range of n .

First, to ensure the local compactness of the Palais Smale sequence for E_λ , we have to prove an important result.

For $i \in \{1, 2\}$, let x_i be defined as in Lemma 2.2 and let

$$(3.7) \quad \begin{aligned} C_i &= a \left(\frac{1}{\theta p} - \frac{1}{p^*} \right) \left(S x_i^{\frac{1}{\theta-1}} \right)^\theta + b \left(\frac{1}{p} - \frac{1}{p^*} \right) S x_i^{\frac{1}{\theta-1}}, \\ C^* &= \begin{cases} C_1, & \text{if } \theta = \frac{p^*}{p}, \quad S^{-\theta} > a \geq 0, \quad 0 < b, \\ C_2, & \text{if } \theta < \frac{p^*}{p}, \quad a \geq 0, \quad 0 \leq b, \quad a + 0 < b. \end{cases} \end{aligned}$$

Lemma 3.1. *Suppose that $f \in W^* \setminus \{0\}$ and $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ hold. Let $(u_n) \subset W^{1,p}(\mathbb{R}^n)$ be a Palais Smale sequence for E_λ for some $c \in \mathbb{R}$. Then, either $u_n \rightarrow u$ or $c \geq E_\lambda(u) + C^*$.*

Proof. From the proof of Lemma 2.1, we have that (u_n) is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ and $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$ for some $u \in W^{1,p}(\mathbb{R}^n)$ with $E'_\lambda(u) = 0$. Furthermore, if we write $v_n = u_n - u$, we derive

$$(3.8) \quad \begin{cases} v_n \rightharpoonup 0 \text{ in } W^{1,p}(\mathbb{R}^n) \text{ and in } L^{p^*}(\mathbb{R}^n), \\ v_n \rightarrow 0 \text{ a.e. in } \mathbb{R}^n, \\ \int_{\mathbb{R}^n} f(x) v_n dx \rightarrow 0. \end{cases}$$

On the one hand, by using Brézis-Lieb's lemma [17], one has

$$(3.9) \quad \begin{cases} \|u\|^p = \|v_n\|^p + \|u\|^p + o_n(1), \\ \int_{\mathbb{R}^n} |u_n|^{p^*} dx = \int_{\mathbb{R}^n} |v_n|^{p^*} dx + \int_{\mathbb{R}^n} |u|^{p^*} dx + o_n(1). \end{cases}$$

Since $\langle E'_\lambda(u), u \rangle = 0$, we obtain by (3.9) that

$$(3.10) \quad o_n(1) = \langle E'_\lambda(u_n), u_n \rangle = \|v_n\|^p - \int_{\mathbb{R}^n} |v_n|^{p^*} dx,$$

and

$$\begin{aligned} c + o_n(1) &= E_\lambda(u_n) - \frac{1}{p^*} \langle E'_\lambda(u_n), u_n \rangle \\ &= a \left(\frac{1}{\theta p} - \frac{1}{p^*} \right) (\|v_n\|^p + \|u\|^p)^\theta + b \left(\frac{1}{p} - \frac{1}{p^*} \right) (\|v_n\|^p + \|u\|^p) \\ &\quad + \lambda \left(\frac{1}{p^*} - 1 \right) \int_{\mathbb{R}^n} f(x) v_n dx - \lambda \left(\frac{1}{p^*} - 1 \right) \int_{\mathbb{R}^n} f(x) u dx \\ &\geq a \left(\frac{1}{\theta p} - \frac{1}{p^*} \right) \|v_n\|^{\theta p} + b \left(\frac{1}{p} - \frac{1}{p^*} \right) \|v_n\|^p + E_\lambda(u) - \frac{1}{p^*} \langle E'_\lambda(u), u \rangle. \end{aligned}$$

Consequently,

$$(3.11) \quad c + o_n(1) \geq E_\lambda(u) + \left(\frac{a}{\theta p} - \frac{a}{p^*} \right) \|v_n\|^{\theta p} + \left(\frac{b}{p} - \frac{b}{p^*} \right) \|v_n\|^p.$$

Assume that $\lim_{n \rightarrow +\infty} \|v_n\| = l > 0$. Then, by (3.10) and the Sobolev inequality, we obtain

$$l^p \geq S \left(b l^p + a l^{\theta p} \right)^{\frac{p}{p^*}}.$$

This implies that

$$(3.12) \quad S^{-\frac{p^*}{p}} l^{p^*-p} - a l^{p(\theta-1)} - b \geq 0.$$

Let $x = S^{-(\theta-1)} l^{p(\theta-1)}$ and $\sigma = \frac{p^*-p}{(\theta-1)p}$. Then, by (3.12), we get

$$S^{-1} x^\sigma - a S^{\theta-1} x - 0 \leq b.$$

It is clear that $\sigma \geq 1$ thanks to $\theta \leq \frac{p^*}{p}$. Then, by definition of Ψ , we get $\Psi(x) \geq 0$. We are now in a position to discuss two cases.

Case 1. $\theta = \frac{p^*}{p}$, $S^{-\theta} > a \geq 0$ and $0 < b$. According to Lemma 2.2, we have $\Psi(x) \geq 0$ if $x \geq x_1$, with

$$x_1 = \frac{b}{(S^{-\theta} - a) S^{\theta-1}},$$

which implies that $l^p \geq S x_1^{\frac{1}{\theta-1}}$.

Case 2. $\theta < \frac{p^*}{p}$, $a \geq 0$, $0 \leq b$ and $a + 0 < b$. In this case, it follows from Lemma 2.2 that $\Psi(x) \geq 0$ if $x \geq x_2$, with

$$x_2 > \left(\frac{a(\theta-1)p}{p^*-p} S^\theta \right)^{\frac{(\theta-1)p}{p^*-\theta p}},$$

which implies that $l^p \geq S x_2^{\frac{1}{\theta-1}}$. Then, by (3.11), one has

$$\begin{aligned} c &\geq E_\lambda(u) + \left(\frac{a}{\theta p} - \frac{a}{p^*} \right) l^{\theta p} + \left(\frac{b}{p} - \frac{b}{p^*} \right) l^p \\ &\geq E_\lambda(u) + \begin{cases} b \frac{p^*-p}{pp^*} S x_1^{\frac{1}{\theta-1}}, & \text{if } \theta = \frac{p^*}{p}, S^{-\theta} > a \geq 0 \text{ and } 0 < b, \\ a \frac{p^*-\theta p}{\theta pp^*} \left(S x_2^{\frac{1}{\theta-1}} \right)^\theta + b \frac{p^*-p}{pp^*} S x_2^{\frac{1}{\theta-1}}, & \text{if } \theta < \frac{p^*}{p}, 0 \leq a, b \text{ and } a < b, \end{cases} \\ &= E_\lambda(u) + C^*. \end{aligned}$$

The proof of Lemma 3.1 is complete. $\square$

Next, we estimate the level energy.

Lemma 3.2. *Assume that all the conditions in Theorem 3.2 are fulfilled. Then, there exists $z_\varepsilon \in W^{1,p}(\mathbb{R}^n)$ and $\lambda^* > 0$ such that*

$$\sup_{t \geq 0} E_\lambda(t z_\varepsilon) < c_- + C^*, \quad \text{for all } \lambda \in (0, \lambda^*),$$

where c_- and C^* are given in (3.6) and (3.7), respectively.

Proof. Since $\int_{\mathbb{R}^n} f(x) v_{\varepsilon, x_0}(x) dx \neq 0$, there exists $z_\varepsilon = \pm v_{\varepsilon, x_0}$ that satisfies

$$\int_{\mathbb{R}^n} f(x) z_\varepsilon(x) dx > 0.$$

Given any $\lambda > 0$ and fixed $t > 0$, using (1.4), we have

$$\begin{aligned} E_\lambda(tz_\varepsilon) &= \frac{a}{\theta p} t^{\theta p} \|z_\varepsilon\|^{\theta p} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^n} |z_\varepsilon|^{p^*} dx - \lambda t \int_{\mathbb{R}^n} f(x) z_\varepsilon dx \\ &= \frac{a}{\theta p} t^{\theta p} S_{\frac{p^*}{p^*-p}}^{\frac{\theta p^*}{p^*}} + \frac{b}{p} t^p S_{\frac{p^*}{p^*-p}}^{\frac{p^*}{p^*}} - \frac{t^{p^*}}{p^*} S_{\frac{p^*}{p^*-p}}^{\frac{p^*}{p^*}} - \lambda t \int_{\mathbb{R}^n} f(x) z_\varepsilon dx. \end{aligned}$$

Define $g, h :]0, +\infty[\rightarrow \mathbb{R}$ by $g(t) = E_\lambda(tz_\varepsilon)$ and

$$h(t) = \frac{a}{\theta p} t^{\theta p} S_{\frac{p^*}{p^*-p}}^{\frac{\theta p^*}{p^*}} + \frac{b}{p} t^p S_{\frac{p^*}{p^*-p}}^{\frac{p^*}{p^*}} - \frac{t^{p^*}}{p^*} S_{\frac{p^*}{p^*-p}}^{\frac{p^*}{p^*}}.$$

Then,

$$h'(t) = -t^{p-1} S_{\frac{p^*}{p^*-p}}^{\frac{p^*}{p^*}} \left(t^{p^*-p} - a S_{\frac{p^*}{p^*-p}}^{\frac{(\theta-1)p^*}{p^*}} t^{(\theta-1)p} - b \right).$$

It follows from $h'(t) = 0$ that

$$(3.13) \quad a S_{\frac{p^*}{p^*-p}}^{\frac{(\theta-1)p^*}{p^*}} t^{(\theta-1)p} + b - t^{p^*-p} = 0.$$

So,

$$(3.14) \quad t^{p^*} = a S_{\frac{p^*}{p^*-p}}^{\frac{(\theta-1)p^*}{p^*}} t^{\theta p} + b t^p.$$

Let $x = S_{\frac{p^*}{p^*-p}}^{\frac{p(\theta-1)}{p^*-p}} t^{p(\theta-1)}$, $\sigma = \frac{p^*-p}{(\theta-1)p}$ and

$$x_* := \begin{cases} x_1, & \text{if } (\mathcal{H}_3) \text{ holds,} \\ x_2, & \text{if } (\mathcal{H}_1) \text{ holds.} \end{cases}$$

Then, by (3.13) and the definition of Ψ , we get

$$(3.15) \quad \Psi(x) = S^{-1} x^\sigma - a S^{\theta-1} x - b = 0.$$

By Lemma 2.2, we can conclude that $\Psi(x_*) = 0$, $\Psi(x) < 0$ for all $x \in]0, x_*[$ and $\Psi(x) > 0$ for all $x \in]x_*, +\infty[$. Therefore, $h'(t_*) = 0$, $h'(t) > 0$ for all $t \in]0, t_*[$ and $h'(t) < 0$ for all $t \in]t_*, +\infty[$, where

$$t_* := \begin{cases} S_{\frac{p^*}{p^*-p}}^{\frac{-1}{p^*-p}} x_1^{\frac{1}{(\theta-1)p}}, & \text{if } (\mathcal{H}_3) \text{ holds,} \\ S_{\frac{p^*}{p^*-p}}^{\frac{-1}{p^*-p}} x_2^{\frac{1}{(\theta-1)p}}, & \text{if } (\mathcal{H}_1) \text{ holds.} \end{cases}$$

Moreover, since $h(0) = 0$ and $\lim_{t \rightarrow +\infty} h(t) = -\infty$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds, then h attains its maximum at t_* .

So, from (3.14), we have

$$\begin{aligned}
 \max_{t \geq 0} h(t) &= h(t_*) \\
 &= \frac{a}{\theta p} t_*^{\theta p} S_{p^*-p}^{\frac{\theta p^*}{p}} + \frac{b}{p} t_*^p S_{p^*-p}^{\frac{p^*}{p}} - \frac{t_*^{p^*}}{p^*} S_{p^*-p}^{\frac{p^*}{p}} \\
 &= \frac{a}{\theta p} t_*^{\theta p} S_{p^*-p}^{\frac{\theta p^*}{p}} + \frac{b}{p} t_*^p S_{p^*-p}^{\frac{p^*}{p}} - \left(\frac{a}{p^*} t_*^{\theta p} S_{p^*-p}^{\frac{\theta p^*}{p}} + \frac{b}{p^*} t_*^p S_{p^*-p}^{\frac{p^*}{p}} \right) \\
 &= a \left(\frac{1}{\theta p} - \frac{1}{p^*} \right) t_*^{\theta p} S_{p^*-p}^{\frac{\theta p^*}{p}} + b \left(\frac{1}{p} - \frac{1}{p^*} \right) t_*^p S_{p^*-p}^{\frac{p^*}{p}} \\
 &= a \left(\frac{1}{\theta p} - \frac{1}{p^*} \right) S_{x_*}^{\theta} x_*^{\frac{\theta}{\theta-1}} + b \left(\frac{1}{p} - \frac{1}{p^*} \right) S_{x_*}^{\frac{1}{\theta-1}} \\
 &= C^*.
 \end{aligned}$$

By (3.4) and (3.6), we have $c_- \geq -C_\lambda$ for all $\lambda \in (0, \lambda_*)$. So, we can choose $\lambda_3 \leq \lambda_-$ such that for any $\lambda \in (0, \lambda_3)$ we have $C^* - c_- \geq C^* - C_\lambda > 0$. Hence, $C^* - c_- > 0$ for all $\lambda \in (0, \lambda_3)$.

Now, we consider the function $g(t) := E_\lambda(tz_\varepsilon)$, $t \geq 0$. Then, $g(t) = h(t) - \lambda t \int_{\mathbb{R}^n} f(x) z_\varepsilon dx$. So, for all $\lambda \in (0, \lambda_3)$, we have $g(0) = 0 < C^* - C_\lambda$. Hence, by the continuity of $g(t)$, there exists $t_1 > 0$ small enough such that $g(t) < C^* - C_\lambda$ for all $t \in (0, t_1)$.

We know also that $\lim_{t \rightarrow +\infty} g(t) = -\infty$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds. Then, for $t_2 > 0$ sufficiently large one has $g(t) < C^* - C_\lambda$, for all $t \in (t_2, +\infty)$. On the other hand, as $\int_{\mathbb{R}^n} f(x) z_\varepsilon dx > 0$, we can deduce from the above estimate on $h(t)$ that for all $t \in [t_1, t_2]$

$$g(t) < C^* - \lambda t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx.$$

Set

$$\lambda_4 = \begin{cases} \left(\frac{p}{p-1} t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx \right)^{p-1} \frac{b}{2} \|f\|_{W^*}^{-p}, & \text{if } (\mathcal{H}_3) \text{ or } (\mathcal{H}_1) \text{ with } 0 < b \text{ holds,} \\ \left(\frac{\theta p}{\theta p-1} t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx \right)^{\theta p-1} \frac{a}{2} \|f\|_{W^*}^{-\theta p}, & \text{if } (\mathcal{H}_1) \text{ with } 0 < a \text{ holds.} \end{cases}$$

Then, for any $\lambda \in (0, \lambda_4)$, one has

$$-\lambda t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx < -C_\lambda.$$

Taking $\lambda_+ = \min \{\lambda_-, \lambda_3, \lambda_4\}$, then we deduce that

$$\sup_{t \geq 0} E_\lambda(tz_\varepsilon) < C^* + c_-, \quad \text{for all } \lambda \in (0, \lambda_+).$$

This concludes the proof of Lemma 3.2. $\square$

Now, we are ready to prove the existence of the mountain pass-type solution and give the proof of Theorems 3.2 with the help of Theorem 3.1.

Proof of Theorem 3.2. Note that $E_\lambda(0) = 0$ and from (3.3), we have $E_\lambda(u)|_{\partial B_{\rho_*}(0)} \geq \delta_* > 0$ for all $\lambda \in (0, \lambda_-)$, where ρ_*, δ_* are defined in (3.2). We know also that $\lim_{t \rightarrow \infty} E_\lambda(tz_\varepsilon) = -\infty$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds, then $E_\lambda(Tz_\varepsilon) < 0$ for T large enough. Hence, E_λ satisfies the geometry conditions of the mountain pass theorem [16]. Then, there exists a Palais Smale sequence (u_n) at level c_+ , such that

$$E_\lambda(u_n) \rightarrow c_+ \quad \text{and} \quad E'_\lambda(u_n) \rightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

with

$$0 < c_+ = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_\lambda(\gamma(t)) \leq \sup_{t \geq 0} E_\lambda(tTz_\varepsilon) < C^* + c_-, \quad \text{for all } \lambda \in (0, \lambda_+),$$

where

$$\Gamma = \left\{ \gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \gamma(0) = 0, \gamma(1) = Tz_\varepsilon \right\}.$$

Using Lemma 2.1, we find that (u_n) has a subsequence, still denoted by (u_n) , such that $u_n \rightharpoonup u_+$ in $W^{1,p}(\mathbb{R}^n)$ as $n \rightarrow +\infty$. Hence, from Lemma 3.1 if $u_n \not\rightharpoonup u_+$ in $W^{1,p}(\mathbb{R}^n)$ as $n \rightarrow +\infty$, it holds

$$c_+ \geq E_\lambda(u_+) + C^* \geq c_- + C^*,$$

which is in contradiction with Lemma 3.2. Hence, $E'_\lambda(u_+) = 0$ and

$$E_\lambda(u_+) = c_+ > 0.$$

So, since $c_+ > 0 = E_\lambda(0)$, we conclude that u_+ is a non-zero solution of (1.1) with positive energy. This completes the proof of Theorem 3.2. $\square$

4. RESULTS IN THE CASE WHEN $\lambda = 0$

Now, in the case when $\lambda = 0$, we have the following results.

4.1. Infinitely solutions.

Theorem 4.1. *Let $\lambda = 0$, $0 < a$, $0 \leq b$, $1 < p < n$ and $\theta > 1$. For v_{ε, x_0} , given by (1.3), the following conclusions hold.*

(1) *If $\theta = \frac{p^*}{p}$, then under the hypothesis $(\mathcal{H}_3)$, problem (1.1) for $\lambda = 0$ has infinitely many nonnegative solutions and these solutions are*

$$\left(\frac{b}{1 - S^\theta a} \right)^{\frac{1}{p^* - p}} v_{\varepsilon, x_0}, \quad \text{for all } \varepsilon > 0,$$

and under the hypothesis $(\mathcal{H}_6)$, the problem (1.1) for $\lambda = 0$ has infinitely many positive solutions $\delta^{\frac{1}{(\theta-1)p}} v_{\varepsilon, x_0}$ (for any $\delta > 0$) if and only if $a = S^{-\theta}$.

(2) *If $\theta \neq \frac{p^*}{p}$, $b = 0$ and $0 < a$, then problem (1.1) has infinitely many nonnegative solutions and these solutions are*

$$\left(a S^{\frac{p^*(\theta-1)}{p^* - p}} \right)^{-\frac{1}{\theta p - p^*}} v_{\varepsilon, x_0}, \quad \text{for all } \varepsilon > 0.$$

(3) If $(\mathcal{H}_0)$ is satisfied, then there exists $\delta_2 > S^{-1} \left(\frac{a(\theta-1)p}{p^*-p} S^\theta \right)^{\frac{p^*-p}{p^*-\theta p}}$ such that $\delta_2^{\frac{1}{p^*-p}} v_{\varepsilon, x_0}$ are solutions of problem (1.1), for all $\varepsilon > 0$.

(4) If $(\mathcal{H}_7)$ is satisfied, then problem (1.1) has infinitely many nonnegative solutions and these solutions are

$$S^{-\frac{1}{p^*-p}} \left(\frac{p^*-p}{(\theta-1)pa} S^{-\theta} \right)^{\frac{1}{\theta p - p^*}} v_{\varepsilon, x_0}, \quad \text{for all } \varepsilon > 0.$$

(5) If $(\mathcal{H}_9)$ is satisfied, then there exist

$$\delta_3 \in \left(0, S^{-1} \left(\frac{p^*-p}{(\theta-1)pa} S^{-\theta} \right)^{\frac{p^*-p}{\theta p - p^*}} \right)$$

and

$$\delta_4 \in \left(S^{-1} \left(\frac{p^*-p}{(\theta-1)pa} S^{-\theta} \right)^{\frac{p^*-p}{\theta p - p^*}}, +\infty \right),$$

such that $\delta_3^{\frac{1}{p^*-p}} v_{\varepsilon, x_0}$ and $\delta_4^{\frac{1}{p^*-p}} v_{\varepsilon, x_0}$ are solutions of problem (1.1) for all $\varepsilon > 0$.

Remark 4.1. Ke et al. in [11] have obtained Theorem 4.1 for the case $\theta = 2$.

We give the proof of Theorem 4.1.

Proof of Theorem 4.1. For any $\delta > 0$, we define $V_{\varepsilon, \delta} = \delta^{\frac{1}{p^*-p}} v_{\varepsilon, x_0}$, where v_{ε, x_0} is given in (1.4). Since v_{ε, x_0} is a solution of problem (1.2), we have that $V_{\varepsilon, \delta}$ solves the following equation

$$-\delta \operatorname{div} \left(|\nabla V_{\varepsilon, \delta}|^{p-2} \nabla V_{\varepsilon, \delta} \right) = |V_{\varepsilon, \delta}|^{p^*-2} V_{\varepsilon, \delta}.$$

Moreover, according to (1.4), one has

$$\delta = a \|V_{\varepsilon, \delta}\|^{(\theta-1)p} + b = a \delta^{\frac{(\theta-1)p}{p^*-p}} \|v_{\varepsilon, x_0}\|^{(\theta-1)p} + b = a S^{\frac{p^*(\theta-1)}{p^*-p}} \delta^{\frac{(\theta-1)p}{p^*-p}} + b.$$

Therefore, the positive solution of the problem (1.1) corresponds to the solution of the following equation with respect to $\delta > 0$

$$(4.1) \quad \delta - a S^{\frac{p^*(\theta-1)}{p^*-p}} \delta^{\frac{(\theta-1)p}{p^*-p}} - b = 0.$$

(1) For $\theta = \frac{p^*}{p}$, equation (4.1) is equivalent to

$$\delta (1 - a S^\theta) - b = 0.$$

If $0 < b$ and $S^{-\theta} > a \geq 0$, we have that

$$\delta_0 = \frac{b}{1 - S^\theta a}$$

is a solution of equation (4.1). Hence, $V_{\varepsilon, \delta_0} = \delta_0^{\frac{1}{p^*-p}} v_{\varepsilon, x_0}$ satisfies the following equation in the weak sense

$$- \left(a \|u\|^{(\theta-1)p} + b \right) \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) = |u|^{p^*-2} u.$$

If $b = 0$ and $0 < a$, then equation (4.1) is equivalent to

$$\delta \left(1 - a S^\theta \right) = 0,$$

then, for $\delta > 0$, we get $1 - a S^\theta = 0$. Thus, when $\theta = \frac{p^*}{p}$, problem (1.1) has infinitely many positive solutions $V_{\varepsilon, \delta} = \delta^{\frac{1}{(\theta-1)p}} v_{\varepsilon, x_0}$ if and only if $a = S^{-\theta}$.

(2) For $\theta \neq \frac{p^*}{p}$, $b = 0$ and $0 < a$ it is easy to see that

$$\delta_1 = \left(a S^{\frac{p^*(\theta-1)}{p^*-p}} \right)^{-\frac{p^*-p}{\theta p - p^*}}$$

is a solution of equation (4.1). Then, problem (1.1) for $\lambda = 0$ has infinity many positive solutions $V_{\varepsilon, \delta_1} = \delta_1^{\frac{1}{p^*-p}} v_{\varepsilon, x_0}$.

(3) Let $x = (S\delta)^{\frac{(\theta-1)p}{p^*-p}}$, by (4.1) and Lemma 2.2, we get

$$(4.2) \quad S^{-1} x^{\frac{p^*-p}{(\theta-1)p}} - a S^{\theta-1} x - b = 0.$$

So, $\Psi(x) = 0$ with $\sigma = \frac{p^*-p}{(\theta-1)p}$. Hence, for $\theta < \frac{p^*}{p}$, according to Lemma 2.2, we have that $\Psi(x) = 0$ has a unique positive solution

$$x_2 > \left(\frac{a(\theta-1)p}{p^*-p} S^\theta \right)^{\frac{(\theta-1)p}{p^*-\theta p}}.$$

Thus, problem (1.1) has infinity many positive solutions

$$V_{\varepsilon, \delta_2} = \delta_2^{\frac{1}{p^*-p}} v_{\varepsilon, x_0},$$

with

$$\delta_2 = S^{-1} x_2^{\frac{p^*-p}{(\theta-1)p}} > S^{-1} \left(\frac{a(\theta-1)p}{p^*-p} S^\theta \right)^{\frac{p^*-p}{p^*-\theta p}}.$$

For $\theta > \frac{p^*}{p}$, by using (4.2) and according to Lemma 2.2, we have the following.

If $b = b^*$, then the equation $\Psi(x) = 0$ has a unique positive solution

$$\tilde{x} = \left(\frac{p^*-p}{(\theta-1)pa} S^{-\theta} \right)^{\frac{(\theta-1)p}{\theta p - p^*}}.$$

Thus, problem (1.1) has infinity many positive solutions

$$V_{\varepsilon, \tilde{\delta}} = \tilde{\delta}^{\frac{1}{p^*-p}} v_{\varepsilon, x_0},$$

with

$$\tilde{\delta} = S^{-1} \tilde{x}^{\frac{p^*-p}{(\theta-1)p}}.$$

This proves (4).

If $b < b^*$, then Ψ has two different zero points x_3 and x_4 with $0 < x_3 < \tilde{x} < x_4$. Consequently, problem (1.1) has infinitely many positive solutions

$$V_{\varepsilon, \delta_3} = \delta_3^{\frac{1}{p^* - p}} v_{\varepsilon, x_0}$$

and

$$V_{\varepsilon, \delta_4} = \delta_4^{\frac{1}{p^* - p}} v_{\varepsilon, x_0},$$

with

$$\delta_3 = S^{-1} x_3^{\frac{p^* - p}{(\theta - 1)p}} \in \left(0, S^{-1} \left(\frac{p^* - p}{(\theta - 1)pa} S^{-\theta} \right)^{\frac{p^* - p}{\theta p - p^*}} \right)$$

and

$$\delta_4 = S^{-1} x_4^{\frac{p^* - p}{(\theta - 1)p}} \in \left(S^{-1} \left(\frac{p^* - p}{(\theta - 1)pa} S^{-\theta} \right)^{\frac{p^* - p}{\theta p - p^*}}, +\infty \right).$$

This proves (5). □

4.2. Non-existence result.

Theorem 4.2. *Assume that one of the hypotheses $(\mathcal{H}_i)$ holds for $i \in \{4, 5, 8\}$. Then, problem (1.1) has no non-trivial solution for $\lambda = 0$.*

Remark 4.2. For $\theta = 2$, the authors in [11] proved the non-existence of solutions only in the case $p^* < 2p$, while the case $p^* = 2p$ is considered in our case. From this point of view, Theorem 4.2 could be viewed as some extension and completeness of the related results in [11].

We give the proof of Theorem 4.2.

Proof of Theorem 4.2. Suppose that $(\mathcal{H}_4)$ is satisfied and let $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$ be a solution of the problem (1.1). Then,

$$(4.3) \quad a \|u\|^{\theta p} = \int_{\mathbb{R}^n} |u|^{p^*} dx.$$

If $\theta = \frac{p^*}{p}$, $a > S^{-\theta}$ and $\int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\frac{p^*}{p}} \|u\|^{p^*}$, we have, by (4.3),

$$S^{-\theta} \|u\|^{\theta p} < a \|u\|^{\theta p} = \int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\theta} \|u\|^{\theta p},$$

which leads to a contradiction.

Suppose now that $(\mathcal{H}_5)$ is satisfied and let $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$ be a solution of (1.1). Then,

$$a \|u\|^{\theta p} + b \|u\|^p = \int_{\mathbb{R}^n} |u|^{p^*} dx.$$

If $\theta = \frac{p^*}{p}$, $a \geq S^{-\theta}$, $0 < b$ and

$$\int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\frac{p^*}{p}} \|u\|^{p^*},$$

we get

$$S^{-\theta} \|u\|^{\theta p} < a \|u\|^{\theta p} + b \|u\|^p = \int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\theta} \|u\|^{\theta p},$$

which is a contradiction.

In the same way as above, under the condition $(\mathcal{H}_8)$, suppose that $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$ is a solution of $(\mathcal{P}_0)$, that is,

$$a \|u\|^{\theta p} + b \|u\|^p = \int_{\mathbb{R}^n} |u|^{p^*} dx.$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^{p^*} dx &\leq S^{-\frac{p^*}{p}} \|u\|^{p^*} = \|u\|^{p^* - \frac{\theta p - p^*}{\theta - 1}} S^{-\frac{p^*}{p}} \|u\|^{\frac{\theta p - p^*}{\theta - 1}} \\ &= \left(\frac{(\theta - 1)p}{p^* - p} a \right)^{\frac{p^* - p}{(\theta - 1)p}} \|u\|^{\frac{\theta(p^* - p)}{\theta - 1}} \left(\frac{(\theta - 1)p}{p^* - p} a \right)^{-\frac{p^* - p}{(\theta - 1)p}} S^{-\frac{p^*}{p}} \|u\|^{\frac{\theta p - p^*}{\theta - 1}} \\ &\leq \frac{p^* - p}{(\theta - 1)p} \left(\left(\frac{(\theta - 1)p}{p^* - p} a \right)^{\frac{p^* - p}{(\theta - 1)p}} \|u\|^{\frac{\theta(p^* - p)}{\theta - 1}} \right)^{\frac{(\theta - 1)p}{p^* - p}} \\ &\quad + \frac{\theta p - p^*}{(\theta - 1)p} \left(\left(\frac{(\theta - 1)p}{p^* - p} a \right)^{-\frac{p^* - p}{(\theta - 1)p}} S^{-\frac{p^*}{p}} \|u\|^{\frac{\theta p - p^*}{\theta - 1}} \right)^{\frac{(\theta - 1)p}{\theta p - p^*}} \\ &\leq a \|u\|^{\theta p} + \frac{\theta p - p^*}{(\theta - 1)p} \left(\left(\frac{(\theta - 1)p}{p^* - p} a \right)^{-\frac{p^* - p}{(\theta - 1)p}} S^{-\frac{p^*}{p}} \right)^{\frac{(\theta - 1)p}{\theta p - p^*}} \|u\|^p \\ &= a \|u\|^{\theta p} + \frac{\theta p - p^*}{(\theta - 1)p} \left(\frac{(\theta - 1)p}{p^* - p} a \right)^{-\frac{p^* - p}{\theta p - p^*}} S^{-\frac{(\theta - 1)p^*}{\theta p - p^*}} \|u\|^p \\ &< a \|u\|^{\theta p} + b \|u\|^p \\ &= \int_{\mathbb{R}^n} |u|^{p^*} dx, \end{aligned}$$

which leads to a contradiction (see [18–22]). This completes the proof. $\square$

5. CONCLUSION

The objective of this article is to investigate contemporary challenges within the field of non-classical partial differential equations and to explore their practical applications. Specifically, it seeks to establish the presence or absence of solutions for p -Kirchhoff-type problems with critical Sobolev exponents in the Euclidean space $\mathbb{R}^n$. These investigations are pivotal for gaining a deeper understanding of mathematical physics equations and addressing pertinent issues in this domain. The proposed article is an expanded discussion of the question of existence and non-existence of solutions. We

found several new results related to the discussion of the existence of a solution with negative and positive energy using the latest methods in this field to support the rapidly developing literature. After that, we discussed different cases depending on the change in parameters to show that solutions do not exist.

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KEMENY’S CONSTANT OF A CYLINDER OCTAGONAL CHAIN

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ABSTRACT. If $A(G)$ is the adjacency matrix of a graph G with n vertices and $D^{-1/2}(G)$ is the diagonal matrix of reciprocals of square roots of vertex degrees, then the Kemeny’s constant of G is $K(G) = \sum_{i=2}^n \frac{1}{1-\lambda_i}$, where $\lambda_2, \lambda_3, \dots, \lambda_n$ are all but the largest eigenvalue of $D^{-1/2}(G)A(G)D^{-1/2}(G)$. We use an approach based on determinants of particular tridiagonal matrices admitting certain periodicity to provide a closed formula for the Kemeny’s constant of a cylinder octagonal chain graph, where a graph in question is obtained from a linear octagonal chain graph by identifying the lateral edges. In this way we present the correct result of [S. Zaman, A. Ullah, Kemeny’s constant and global mean first passage time of random walks on octagonal cell network, Math. Meth. Appl. Sci., 46 (2023), 9177–9186] that for the graphs in question calculated the multiple of Kirchhoff index instead.

1. INTRODUCTION

Let $G = (V, E)$ be an unoriented graph without loops or multiple edges. We write n for its order (i.e., the number of vertices), $A(G)$ for its adjacency matrix, and

$$D(G) = \text{diag}(1/\sqrt{d_1}, 1/\sqrt{d_2}, \dots, 1/\sqrt{d_n}),$$

for the diagonal matrix of reciprocals of square roots of vertex degrees.

Kemeny’s constant is a graph invariant that provides an interplay between Markov chains, random walks, and spectral invariants. It measures the expected number of time steps required for a Markov chain to transition from a starting state to a random destination state sampled from the Markov chain’s distribution. Equivalently, it measures an average of the mean first passage times in a random walk on the vertices of a graph. Consequently, it provides an information on graph shape and

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connectivity [4]. Actually, there are two definitions of this constant. One of them says that if π_i is the stationary probability for the vertex i and m_{ji} is the expected number of steps before the vertex i is visited in a random walk starting from the vertex j , then $\widetilde{K}(G) = \sum_{i=1}^n \pi_i m_{ji}$ is a constant not depending on the starting vertex j . The other definition says that $K(G) = \widetilde{K}(G) - 1$. Obviously, there is no essential difference between these definitions, and to avoid confusion in this study we take that the *Kemeny's constant* is $K(G)$. This approach agrees with the classical monograph of Kemeny and Snell [11]. It occurs that

$$(1.1) \quad K(G) = \sum_{i=2}^n \frac{1}{1 - \lambda_i},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are non-increasingly indexed eigenvalues of the symmetric matrix $D(G)^{-1/2}A(G)D(G)^{-1/2}$ [10, 12]. We know from [10] that the eigenvalues of the previous matrix lie in the segment $[-1, 1]$, along with $1 = \lambda_1 > \lambda_2$, which means that (1.1) is defined correctly.

We denote by L_n the linear octagonal chain of n octagons. The *cylinder octagonal chain* M'_n is obtained from L_n by identifying the lateral edges (see Figure 1). The main result of this paper reads as follows.

Theorem 1.1. *Let M'_n be a cylinder octagonal chain. The Kemeny's constant of M'_n is given by*

$$K(M'_n) = \frac{147n^2 - 19}{84} + \frac{37nU_{n-1}(8)}{8U_{n-1}(8) - 2U_{n-2}(8) - 2},$$

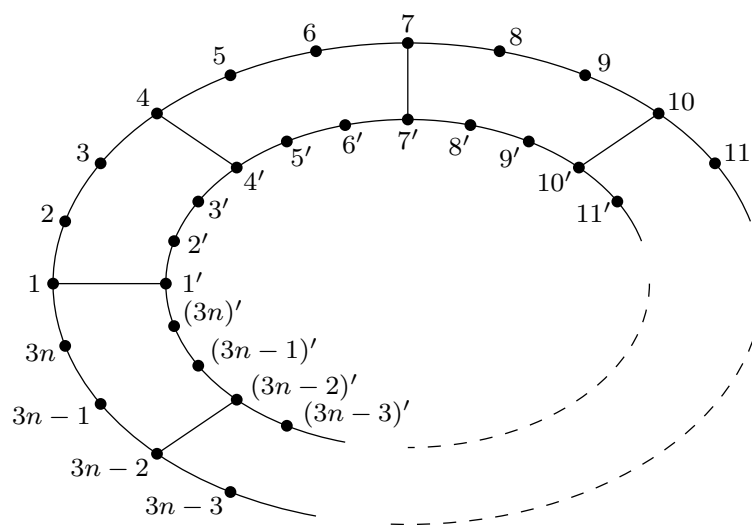
where $U_n(8) = \frac{(4+\sqrt{15})^{n+1} - (4-\sqrt{15})^{n+1}}{2\sqrt{15}}$.

We will see in the next sections that $U_n(\cdot)$, $n \geq 0$, is the Chebyshev polynomial of the second kind; in our result computed in the point 8.

In [9], the Kirchhoff index $n \sum_{i=2}^n \frac{1}{\mu_i}$, where $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix of a cylinder octagonal chain, was computed. Afterwards, in [16], the authors computed the Kemeny's constant of the same graph using a wrong expression stating that the Kemeny's constant of a connected graph is the Kirchhoff index multiplied by n . In addition, their computations go through identical lines as those in [9]. The same constant is dealt correctly in [17].

The aim of this paper is twofold. First, we present a correct closed formula for the Kemeny's constant of a cylinder octagonal chain. Secondly (better say, simultaneously), we show that some computations of [9, 16, 17] can be simplified by employing known but rarely used results of [15] concerning determinants of particular tridiagonal matrices admitting certain periodicity.

The remaining content is organized in the following way. In Section 2, we revisit two relevant results from [15]. In Section 3, we prove some particular results needed for the proof of Theorem 1.1. The proof of this theorem is finalized in Section 4.

FIGURE 1. The cylinder octagonal chain M'_n .

2. EMPHASIZING SOME RESULTS FROM MATRIX THEORY

In this section we recall some powerful, often neglected, results of [15] that give an explicit form for the computation of determinants of special kind of tridiagonal matrices, called periodic continuants. These results are also obtained in [8] using a different approach.

Accordingly,

$$D_n(a_i b_i) = \begin{pmatrix} a_1 & -b_1 & & & & \\ -b_1 & a_2 & -b_2 & & & \\ & -b_2 & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & -b_{n-1} \\ & & & & -b_{n-1} & a_n \end{pmatrix}$$

is called a *symmetric continuant matrix*. Its determinant is called a *continuant*. By $D_{i_1, i_2, \dots, i_p}$ we denote the minor of $D_n(a_i b_i)$ obtained from its $i_1, i_2, \dots, i_p$ rows and columns.

A symmetric continuant matrix of order $nm + r$, $0 \leq r \leq m - 1$, in which the elements a_k, b_k are periodical modulo m , i.e., $a_{k+m} = a_k$ and $b_{k+m} = b_k$ hold, is called a *periodic continuant matrix*. These matrices are also known as m -Toeplitz tridiagonal matrices (see [7]) with applications in different contexts (see [1–3, 5, 14]). The determinant of the corresponding matrix is called a *periodic continuant* of class r and period m and it is denoted by $P_n^{(m, r)}(a_k b_k)$, $k = 1, 2, \dots, m$. According to [15], $P_n^{(m, r)}(a_k b_k)$ can be expressed using Chebyshev polynomials of the second kind $U_n(x)$.

We recall that

$$\begin{aligned} U_n(x) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} \\ &= \frac{(x/2 + \sqrt{(x/2)^2 - 1})^{n+1} - (x/2 - \sqrt{(x/2)^2 - 1})^{n+1}}{2\sqrt{(x/2)^2 - 1}}, \quad x \neq 2, \end{aligned}$$

as well as $U_1(x) = x$ and $U_0 = 1$. We also have $U_n(2) = n + 1$.

The formula for $P_n^{(m,r)}(a_k b_k)$ reads as follows:

$$(2.1) \quad \begin{aligned} P_n^{(m,r)}(a_k b_k) &= (b_1 b_2 \cdots b_m)^n \\ &\times \left(D_{1,2,\dots,r} U_n(x) + \frac{b_m b_1 b_2 \cdots b_r}{b_{r+1} b_{r+2} \cdots b_{m-1}} D_{r+2,r+3,\dots,m-1} U_{n-1}(x) \right), \end{aligned}$$

where

$$x = \frac{D_{1,2,\dots,m} - b_m^2 D_{2,\dots,m-1}}{b_1 b_2 \cdots b_m}.$$

We set $D_{1,2,\dots,r} = 1$, for $r = 0$, $D_{r+2,r+3,\dots,m-1} = 1$, for $r = m-2$ and $D_{r+2,r+3,\dots,m-1} = 0$, for $r = m-1$.

Remark 2.1. We point out that present definition of $U_n(x)$ gives monic polynomials and does not coincide with the one used in Wolfram Mathematica. There $U_n(x)$ is computed using `ChebyshevU [n, \frac{x}{2}]`.

3. INITIAL RESULTS

In this and the next section, we write $A[i]$ and $A[i, j]$ to denote the matrix obtained from a matrix A by deleting the i th row and column, and the i th and the j th rows and columns, respectively. The following proposition is needed.

Proposition 3.1. ([6]) *Given an $n \times n$ symmetric matrix $A = (a_{ij})$ whose graph G is a cycle, say $(1, \dots, n, 1)$, the characteristic polynomial of A is*

$$\begin{aligned} \phi_A(x) &= (x - a_{ii})\phi_{A[i]}(x) - |a_{i-1,i}|^2 \phi_{A[i-1,i]}(x) \\ &\quad - |a_{i,i+1}|^2 \phi_{A[i,i+1]}(x) - 2(a_{12} \cdots a_{n-1,n} a_{n,1}). \end{aligned}$$

For the purpose of the current paper we will consider the periodic continuant matrices with period $m = 3$. Our starting point is the matrix of the following form:

$$(3.1) \quad A = \begin{pmatrix} a & -b & & & & -b \\ -b & c & -d & & & \\ & -d & c & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -d & a & -b \\ & & & & -b & c & -d \\ -b & & & & & -d & c \end{pmatrix}_{3n}.$$

In the sequel we compute $\det(A[i])$, and $\det(A[i, j])$; particular principal minors of A .

Lemma 3.1. *Let A be the matrix given in (3.1). Then,*

$$\det(A[i]) = (b^2d)^{n-1} \begin{cases} (ac - b^2)U_{n-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right), & \text{if } i \equiv 0 \pmod{3} \text{ or } i \equiv 2 \pmod{3}; \\ (c^2 - d^2)U_{n-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right), & \text{if } i \equiv 1 \pmod{3}. \end{cases}$$

Proof. The underlying graph of A is the cycle C_{3n} . The removal of the i th row and i th column in A results in a matrix whose graph is the path P_{3n-1} . We distinguish three cases depending on i .

(i) If $i \equiv 0 \pmod{3}$, then $A[i]$ is permutation similar to

$$(3.2) \quad \begin{pmatrix} a & -b & & & & \\ -b & c & -d & & & \\ & -d & c & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & -b & \\ & & & -b & a & -b \\ & & & & -b & c \end{pmatrix}_{3n-1}.$$

In this case we also have $3n - 1 = 3(n - 1) + 2$, and therefore $m = 3$ and $r = 2$. Therefore, by employing (2.1) (for these m and r), we arrive at

$$\det(A[i]) = (b^2d)^{n-1}(ac - b^2)U_{n-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right).$$

(ii) If $i \equiv 1 \pmod{3}$, then $A[i]$ is permutation similar to

$$(3.3) \quad \begin{pmatrix} c & -d & & & & \\ -d & c & -b & & & \\ & -b & a & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b & a & -b \\ & & & & -b & c & -d \\ & & & & & -d & c \end{pmatrix}_{3n-1}.$$

By (2.1),

$$\det(A[i]) = (b^2d)^{n-1}(c^2 - d^2)U_{n-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right).$$

(iii) For $i \equiv 2 \pmod{3}$, $A[i]$ is similar to

$$(3.4) \quad \begin{pmatrix} c & -b & & & & \\ -b & a & -b & & & \\ & -b & c & -d & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & -d & \\ & & & -d & c & -b \\ & & & & -b & a \end{pmatrix}_{3n-1},$$

which leads to

$$\det(A[i]) = (b^2d)^{n-1}(ac - b^2)U_{n-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right).$$

The proof is completed. $\square$

For the sake of conciseness, we introduce the following functions:

$$\begin{aligned} F_1^\ell(x) &= (b^2d)^{n-2}(ac - b^2)^2U_{\ell-1}(x)U_{n-\ell-1}(x), \\ F_2^\ell(x) &= (b^2d)^{n-2}(c^2 - d^2)^2U_{\ell-1}(x)U_{n-\ell-1}(x), \\ F_3^\ell(x) &= (b^2d)^{n-1}\left(cU_\ell(x) + dU_{\ell-1}(x)\right)\left(U_{n-\ell-1}(x) + \frac{c}{d}U_{n-\ell-2}(x)\right), \\ F_4^\ell(x) &= (b^2d)^{n-1}\left(aU_\ell(x) + \frac{b^2}{d}U_{\ell-1}(x)\right)\left(U_{n-\ell-1}(x) + \frac{ad}{b^2}U_{n-\ell-2}(x)\right). \end{aligned}$$

Lemma 3.2. *Let A be the matrix given in (3.1), $x = \frac{ac^2 - ad^2 - 2b^2c}{b^2d}$ and $\ell = \lfloor \frac{j-i}{3} \rfloor$, for $i < j$. Then $\det(A[i, j])$ is equal to*

$$\begin{cases} F_1^\ell(x), & \text{if } i, j \equiv 0 \pmod{3} \text{ or } i, j \equiv 2 \pmod{3}; \\ F_2^\ell(x), & \text{if } i, j \equiv 1 \pmod{3}; \\ F_3^\ell(x), & \text{if } i \equiv 1 \pmod{3}, j \equiv 0 \pmod{3} \text{ or } i \equiv 2 \pmod{3}, j \equiv 1 \pmod{3}; \\ F_3^{n-\ell-1}(x), & \text{if } i \equiv 0 \pmod{3}, j \equiv 1 \pmod{3} \text{ or } i \equiv 1 \pmod{3}, j \equiv 2 \pmod{3}; \\ F_4^\ell(x), & \text{if } i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3}; \\ F_4^{n-\ell-1}(x), & \text{if } i \equiv 2 \pmod{3}, j \equiv 0 \pmod{3}. \end{cases}$$

Proof. The underlying graph of $A[i, j]$ is either a path or two disjoint paths. We distinguish the following cases.

(i) $i, j \equiv 0 \pmod{3}$, or $i, j \equiv 2 \pmod{3}$.

Let $i = k$ and $j = k + 3\ell$, for some $\ell \in \{1, 2, \dots, n-1\}$. In this case, the graph of $A[i, j]$ is the disjoint union of two paths, whose matrices are of the form (3.2) with size $3(\ell-1) + 2$ and (3.3) with size $3(n-\ell-1) + 2$, respectively. Then,

$$\det(A[i, j]) = (b^2d)^{n-2}(ac - b^2)^2U_{\ell-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right)U_{n-\ell-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right).$$

(ii) $i, j \equiv 1 \pmod{3}$.

The graph of $A[i, j]$ is the union of two paths with matrices of the form given in (3.4). Their sizes are $3\ell + 2$ and $3(n - \ell - 2) + 2$, respectively. By fixing i and j to be as in the previous item, we obtain

$$\det(A[i, j]) = (b^2d)^{n-2}(c^2 - d^2)^2 U_{\ell-1}\left(\frac{ac^2 - 2b^2c - ad^2}{b^2d}\right) U_{n-\ell-1}\left(\frac{ac^2 - 2b^2c - ad^2}{b^2d}\right).$$

(iii) $i \equiv 1 \pmod{3}$, $j \equiv 0 \pmod{3}$.

Let $i = k$ and $j = k + 3\ell + 2$, for some $\ell \in \{0, 1, \dots, n - 1\}$. The underlying graph of $A[i, j]$ is either the union of an isolated vertex and a path or the union of two paths whose matrices are of the form

$$\begin{pmatrix} c & -d & & & & \\ -d & c & -b & & & \\ & -b & a & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -d & c & -b \\ & & & -b & a & -b \\ & & & -b & c & \end{pmatrix}_{3\ell+1}, \begin{pmatrix} a & -b & & & & \\ -b & c & -d & & & \\ & -d & c & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & -b & \\ & & & -b & c & -d \\ & & & -d & c & \end{pmatrix}_{3n-3\ell-3}.$$

In both cases, the determinant is given by

$$\begin{aligned} \det(A[i, j]) &= (b^2d)^{n-1} \left(cU_{\ell}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) + dU_{\ell-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) \right) \\ &\quad \times \left(U_{n-\ell-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) + \frac{c}{d}U_{n-\ell-2}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) \right). \end{aligned}$$

(iv) $i \equiv 2 \pmod{3}$, $j \equiv 1 \pmod{3}$.

Supposing that $i = k$ and $j = k + 3\ell + 2$, for some $\ell \in \{0, 1, \dots, n - 2\}$, similarly to the previous case we obtain

$$\begin{aligned} \det(A[i, j]) &= (b^2d)^{n-1} \left(cU_{\ell}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) + dU_{\ell-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) \right) \\ &\quad \times \left(U_{n-\ell-1}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) + \frac{c}{d}U_{n-\ell-2}\left(\frac{ac^2 - ad^2 - 2b^2c}{b^2d}\right) \right). \end{aligned}$$

(v) $i \equiv 0 \pmod{3}$, $j \equiv 2 \pmod{3}$ or $i \equiv 2 \pmod{3}$, $j \equiv 0 \pmod{3}$.

Assume that $i = k$ and $j = k + 3\ell + 2$, for some $\ell \in \{0, 1, \dots, n - 2\}$ or $i = k$ and $j = k + 3\ell + 1$, for some $\ell \in \{0, 1, \dots, n - 1\}$. The underlying graph of $A[i, j]$ is either a path or the union of two paths whose matrices are

$$\begin{pmatrix} a & -b & & & & \\ -b & c & -d & & & \\ & -d & c & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & -d & -b \\ & & & -d & c & -b \\ & & & -b & a & \end{pmatrix}_{3\ell+1}, \begin{pmatrix} c & -b & & & & \\ -b & a & -b & & & \\ & -b & c & -d & & \\ & & \ddots & \ddots & \ddots & \\ & & & -d & c & -b \\ & & & -b & a & -b \\ & & & -b & c & \end{pmatrix}_{3n-3\ell-3}.$$

In the former case, we have

$$\det(A[i, j]) = (b^2 d)^{n-1} \left(a U_\ell \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + \frac{b^2}{d} U_{\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right) \\ \times \left(U_{n-\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + \frac{ad}{b^2} U_{n-\ell-2} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right),$$

whereas in the latter one, it holds

$$\det(A[i, j]) = (b^2 d)^{n-1} \left(a U_{n-\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + \frac{b^2}{d} U_{n-\ell-2} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right) \\ \times \left(U_\ell \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + \frac{ad}{b^2} U_{\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right).$$

(vi) $i \equiv 1 \pmod{3}$, $j \equiv 2 \pmod{3}$.

Let $i = k$ and $j = k + 3\ell + 1$, for some $\ell \in \{0, 1, \dots, n-1\}$. The matrices of the corresponding paths are

$$\begin{pmatrix} c & -d & & & & \\ -d & c & -b & & & \\ & -b & a & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b & c & -d \\ & & & & -d & c & -b \\ & & & & & -b & a \end{pmatrix}_{3\ell}, \begin{pmatrix} c & -b & & & & \\ -b & a & -b & & & \\ & -b & c & -d & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & -b & \\ & & & -b & c & -d \\ & & & & -d & ce \end{pmatrix}_{3n-3\ell-2},$$

and, similarly to the previous cases, we obtain

$$\det(A[i, j]) = (b^2 d)^{n-1} \left(U_\ell \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + \frac{c}{d} U_{\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right) \\ \times \left(c U_{n-\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + d U_{n-\ell-2} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right).$$

(vii) $i \equiv 0 \pmod{3}$, $j \equiv 1 \pmod{3}$.

Let $i = k$ and $j = k + 3\ell + 1$, for some $\ell \in \{0, 1, \dots, n-2\}$. Then

$$\det(A[i, j]) = (b^2 d)^{n-1} \left(c U_{n-\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + d U_{n-\ell-2} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right) \\ \times \left(U_\ell \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) + \frac{c}{d} U_{\ell-1} \left(\frac{ac^2 - ad^2 - 2b^2 c}{b^2 d} \right) \right).$$

The proof is completed. $\square$

4. PROOF OF THEOREM 1.1

In this section, $\sigma(M)$ denotes the multiset of eigenvalues of a symmetric matrix M . We recall from the introductory section that M'_n denotes the cylinder octagonal chain with $6n$ vertices. If its vertices are labelled as in Figure 1, then its adjacency matrix

is

$$A(M'_n) = \left(\begin{array}{cccccccc|cccccccc} 0 & 1 & & & & & & & 1 & 1 & & & & & & \\ 1 & 0 & & 1 & & & & & & 0 & & & & & & & \\ & 1 & 0 & 1 & & & & & & & 0 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \ddots & & & & & \\ & & & & 1 & 0 & 1 & & & & & & 1 & & & & \\ & & & & & 1 & 0 & 1 & & & & & & 0 & & & \\ 1 & & & & & & 1 & 0 & 0 & & & & & & 0 & & \\ \hline 1 & & & & & & & & & 0 & 1 & & & & & 1 & \\ & 0 & & & & & & & & 1 & 0 & 1 & & & & & \\ & & 0 & & & & & & & 1 & 0 & 1 & & & & & \\ & & & \ddots & & & & & & \ddots & \ddots & \ddots & & & & & \\ & & & & 1 & & & & & & 1 & 0 & 1 & & & & \\ & & & & & 0 & & & & & & 1 & 0 & 1 & & & \\ & & & & & & 0 & 1 & & & & & 1 & 0 & & & \end{array} \right),$$

while the diagonal matrix of vertex degrees is

$$D(M'_n) = \text{diag}(\underbrace{3, 2, 2}_{2n}, \underbrace{3, 2, 2}_{2n}, \dots, \underbrace{3, 2, 2}_{2n}).$$

It can be easily verified that $I - D(M'_n)^{-1/2}A(M'_n)D(M'_n)^{-1/2}$ is equal to $S(M'_n)$ that is of the following form:

$$\left(\begin{array}{cccccccc|cccccccc} 1 & -\frac{1}{\sqrt{6}} & & & & & & -\frac{1}{\sqrt{6}} & -\frac{1}{3} & & & & & & & \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & & & & 0 & & & & & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & & & & & 0 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \ddots & & & & & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & & & & & -\frac{1}{3} & & & & \\ & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & & & & & 0 & & & \\ -\frac{1}{\sqrt{6}} & & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & -\frac{1}{2} & & & & & & & 0 & & \\ -\frac{1}{3} & & & & & & & 1 & 1 & -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{\sqrt{6}} & & \\ & 0 & & & & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & & & \\ & & 0 & & & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ & & & \ddots & & & & & & & \ddots & \ddots & \ddots & & & & \\ & & & & -\frac{1}{3} & & & & & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & \\ & & & & & 0 & & & & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & \\ & & & & & & 0 & & & & & & & -\frac{1}{2} & 1 & & \\ & & & & & & & 0 & -\frac{1}{\sqrt{6}} & & & & & & & 1 & \end{array} \right).$$

In a condensed form,

$$S(M'_n) = \begin{pmatrix} P'(M'_n) & R'(M'_n) \\ R'(M'_n) & P'(M'_n) \end{pmatrix},$$

according to the suggested block partition. Taking into account that both $P'(M'_n)$ and $R'(M'_n)$ are symmetric, it follows from [13, Lemma 4.1] that

$$\sigma(S(G)) = \sigma(P'(M'_n) + R'(M'_n)) \sqcup \sigma(P'(M'_n) - R'(M'_n)),$$

where $\sqcup$ denotes the sum of multisets, i.e., the multiset in which the multiplicity of an element is the sum of its multiplicities in the summands. We compute

$$P(M'_n) = P'(M'_n) + R'(M'_n) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{6}} & & & & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{\sqrt{6}} \\ & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\ -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{2} & 1 \end{pmatrix}$$

and

$$R(M'_n) = P'(M'_n) - R'(M'_n) = \begin{pmatrix} \frac{4}{3} & -\frac{1}{\sqrt{6}} & & & & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{2} & \frac{4}{3} & -\frac{1}{\sqrt{6}} \\ & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\ -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{2} & 1 \end{pmatrix}.$$

For both $P(M'_n)$ and $R(M'_n)$, the underlying graph is the cycle C_{3n} . In addition, $P(M'_n)$ is singular (confirmed later in this section). Let $\sigma(P(M'_n)) = \{\mu_1, \mu_2, \dots, \mu_{3n-1}, 0\}$ and $\sigma(R(M'_n)) = \{\nu_1, \nu_2, \dots, \nu_{3n}\}$. Taking into account (1.1), we obtain

$$K(M'_n) = \sum_{i=2}^{3n} \frac{1}{\mu_i} + \sum_{i=1}^{3n} \frac{1}{\nu_i}.$$

Let

$$\phi_{P(M'_n)}(t) = \det(tI - P(M'_n)) = t^{3n} + a_{3n-1}t^{3n-1} + \dots + a_1t$$

and

$$\phi_{R(M'_n)}(t) = \det(tI - R(M'_n)) = t^{3n} + b_{3n-1}t^{3n-1} + \dots + b_1t + b_0.$$

Then

$$\sum_{i=2}^{3n} \frac{1}{\mu_i} = -\frac{a_2}{a_1} \quad \text{and} \quad \sum_{i=1}^{3n} \frac{1}{\nu_i} = -\frac{b_1}{b_0}.$$

In addition,

$$a_1 = (-1)^{3n-1} \sum_{i=1}^{3n} \det(P(M'_n)[i]) \quad \text{and} \quad a_2 = (-1)^{3n-2} \sum_{1 \leq i < j \leq 3n} \det(P(M'_n)[i, j]),$$

along with

$$b_1 = (-1)^{3n-1} \sum_{i=1}^{3n} \det(R(M'_n)[i]) \quad \text{and} \quad b_0 = (-1)^{3n} \det(R(M'_n)).$$

Computations of these coefficients are separated in the following itemization. We apply Lemmas 3.1 and 3.2 for $a = \frac{2}{3}$ in $P(M'_n)$, and $a = \frac{4}{3}$ in $R(M'_n)$, and $b = \frac{1}{\sqrt{6}}$, $c = 1$, $d = \frac{1}{2}$ in both.

- By Lemma 3.1, for $\frac{ac^2-ad^2-2b^2c}{b^2d} = 2$, we obtain

$$\begin{aligned} a_1 &= (-1)^{3n-1} \sum_{i=1}^{3n} \det(P(M'_n)[i]) \\ &= (-1)^{3n-1} \left(\frac{n}{12^{n-1}} U_{n-1}(2) + \frac{3n}{4 \cdot 12^{n-1}} U_{n-1}(2) \right) \\ &= (-1)^{3n-1} \frac{7n}{4 \cdot 12^{n-1}} U_{n-1}(2) = (-1)^{3n-1} \frac{7n^2}{4 \cdot 12^{n-1}} = (-1)^{3n-1} \frac{21n^2}{12^n}. \end{aligned}$$

- Similarly,

$$\begin{aligned} b_1 &= (-1)^{3n-1} \sum_{i=1}^{3n} \det(R(M'_n)[i]) \\ &= (-1)^{3n-1} \frac{1}{12^{n-1}} \left(2n \cdot \frac{7}{6} U_{n-1}(8) + \frac{3}{4} n U_{n-1}(8) \right) \\ &= (-1)^{3n-1} \frac{37n}{12^n} U_{n-1}(8). \end{aligned}$$

- By Lemma 3.2,

$$\begin{aligned} (-1)^{3n-2} a_2 &= \sum_{\ell=1}^{n-1} (n-\ell) \left(2F_1^\ell(2) + F_2^\ell(2) \right) \\ &\quad + \sum_{\ell=0}^{n-1} (n-\ell) \left(F_3^\ell(2) + F_3^{n-\ell-1}(2) + F_4^{n-\ell-1}(2) \right) \\ &\quad + \sum_{\ell=0}^{n-2} (n-\ell-1) \left(F_3^\ell(2) + F_3^{n-\ell-1}(2) + F_4^\ell(2) \right) \\ &= \sum_{\ell=1}^{n-1} \frac{17\ell(n-\ell)^2}{16 \cdot 12^{n-2}} \\ &\quad + \sum_{\ell=0}^{n-1} \frac{(n-\ell)}{12^{n-1}} \left(\frac{(3\ell+2)(3(n-\ell)-2)}{2} + \frac{5(3\ell+1)(3(n-\ell)-1)}{6} \right) \\ &\quad + \sum_{\ell=0}^{n-2} \frac{(n-\ell-1)}{12^{n-1}} \left(\frac{5(3\ell+2)(3(n-\ell)-2)}{6} + \frac{(3\ell+1)(3(n-\ell)-1)}{2} \right) \\ &= \frac{17}{16 \cdot 12^{n-2}} \sum_{\ell=1}^{n-1} \ell(n-\ell)^2 \\ &\quad + \frac{1}{12^{n-1}} \sum_{\ell=0}^{n-2} \left(\frac{(3\ell+2)(3(n-\ell)-2)(8(n-\ell)-5)}{6} \right. \\ &\quad \left. + \frac{(3\ell+1)(3(n-\ell)-1)(8(n-\ell)-3)}{6} \right) + \frac{39n-23}{6 \cdot 12^{n-1}} \\ &= \frac{n^2(147n^2-19)}{48 \cdot 12^{n-1}}. \end{aligned}$$

• By Proposition 3.1 applied for $i = 1$, we have

$$\begin{aligned}
 b_0 &= (-1)^{3n} \left(-\frac{4}{3}(-1)^{3n-1} \det(R(M'_n)[1]) - \frac{1}{6}(-1)^{3n-2} \det(R(M'_n)[1, 3n]) \right. \\
 &\quad \left. - \frac{1}{6}(-1)^{3n-2} \det(R(M'_n)[1, 2]) - \frac{2(-1)^n}{12^n} \right) \\
 &= (-1)^{3n} \left(-\frac{4}{3}(-1)^{3n-1} \frac{3}{4 \cdot 12^{n-1}} U_{n-1}(8) - \frac{(-1)^{3n-2}}{6 \cdot 12^{n-1}} \left(U_{n-1}(8) + \frac{1}{2} U_{n-2}(8) \right) \right. \\
 &\quad \left. - \frac{(-1)^{3n-2}}{6 \cdot 12^{n-1}} \left(U_{n-1}(8) + \frac{1}{2} U_{n-2}(8) \right) - \frac{2(-1)^n}{12^n} \right) \\
 &= \frac{1}{12^{n-1}} U_{n-1}(8) - \frac{1}{6 \cdot 12^{n-1}} (2U_{n-1}(8) + U_{n-2}(8)) - \frac{2}{12^n} \\
 &= \frac{1}{12^n} (8U_{n-1}(8) - 2U_{n-2}(8) - 2).
 \end{aligned}$$

We also verify that $a_0 = 0$, i.e., that $P(M'_n)$ is singular:

$$\begin{aligned}
 a_0 &= (-1)^{3n} \left(-\frac{2}{3}(-1)^{3n-1} \det(R(M'_n)[1]) - \frac{1}{6}(-1)^{3n-2} \det(R(M'_n)[1, 3n]) \right. \\
 &\quad \left. - \frac{1}{6}(-1)^{3n-2} \det(R(M'_n)[1, 2]) - \frac{2(-1)^n}{12^n} \right) \\
 &= (-1)^{3n} \left(-\frac{2}{3}(-1)^{3n-1} \frac{3}{4 \cdot 12^{n-1}} U_{n-1}(2) - \frac{(-1)^{3n-2}}{6 \cdot 12^{n-1}} \left(U_{n-1}(2) + \frac{1}{2} U_{n-2}(2) \right) \right. \\
 &\quad \left. - \frac{(-1)^{3n-2}}{6 \cdot 12^{n-1}} \left(U_{n-1}(2) + \frac{1}{2} U_{n-2}(2) \right) - \frac{2}{12^n} \right) \\
 &= (-1)^{3n} \left(\frac{n(-1)^{3n-1}}{2 \cdot 12^{n-1}} - \frac{(-1)^{3n-2}}{6 \cdot 12^{n-1}} (2n + n - 1) - \frac{2}{12^n} \right) \\
 &= \frac{1}{6 \cdot 12^{n-1}} - \frac{2}{12^n} = 0.
 \end{aligned}$$

Gathering the previous results, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Taking into account the previous computation, we deduce

$$K(M'_n) = -\frac{a_2}{a_1} - \frac{b_1}{b_0} = \frac{147n^2 - 19}{84} + \frac{37(-1)^{3n} n U_{n-1}(8)}{8U_{n-1}(8) - 2U_{n-2}(8) - 2},$$

which completes the proof. $\square$

We conclude this section with an example.

Example 4.1. For $n = 4$, we obtain $K(M'_4) = \frac{9847}{210} \approx 46.8905$, whereas the value obtained in [16] from

$$\frac{9n^2 - 1}{12} + \frac{17\sqrt{15}n}{30} \cdot \frac{(4 + \sqrt{15})^n - (4 - \sqrt{15})^n}{(4 + \sqrt{15})^n + (4 - \sqrt{15})^n + 2}$$

gives approximately 20.6909. These values are different, due to the incorrect expression that was employed in [16]. It is worth mentioning that in [17], the correct expression was used, and there one can find the periodic continuants, as well. We believe that our approach can also simplify calculations of that paper.

We emphasise that a similar approach could be applied to compute the Kemeny's constant of cylinder k -gonal chains, for k even, constructed analogously. These graphs would be of order $(k - 2)n$ for some n . Clearly, all computations will be more complex due to increased number of cases modulo $k - 1$ (see Lemma 3.2) that should be considered.

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GENERALIZED VECTORIAL ALMOST PERIODICITY

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ABSTRACT. In this paper, we introduce and analyze several new classes of generalized vectorially almost periodic functions. We also analyze Σ -almost periodic type functions and the invariance of generalized vectorial almost periodicity under the actions of convolution products.

1. INTRODUCTION AND PRELIMINARIES

The notion of almost periodicity was introduced by the Danish mathematician H. Bohr around 1924–1926 and later generalized by many others (see the research monographs [6, 9, 11, 12, 14, 15, 20, 23] and the excellent survey article [2] for further information regarding almost periodic functions and their applications). Let $(X, \|\cdot\|)$ be a complex Banach space and let $F : \mathbb{R}^n \rightarrow X$ be a continuous function, where $n \in \mathbb{N}$. Then, $F(\cdot)$ is said to be almost periodic if for each $\epsilon > 0$ there exists $W > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W) \equiv \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{w}_0| \leq W\}$ such that

$$\|F(\mathbf{t} + \tau) - F(\mathbf{t})\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n,$$

here, $|\cdot - \cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$.

If the function $F : \mathbb{R}^n \rightarrow X$ is Lebesgue measurable and for every non-empty compact set $K \subseteq \mathbb{R}^n$, one has $\int_K \|F(\mathbf{t})\|^p d\mathbf{t} < +\infty$, where $p > 0$, then $F(\cdot)$ is said to be Stepanov- p -almost periodic if for every $\epsilon > 0$ there exists $W > 0$ such that for

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each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W)$ with

$$\|F(\mathbf{s} + \tau + \mathbf{w}) - F(\mathbf{s} + \mathbf{w})\|_{L^p([0,1]^n; X)} \leq \epsilon, \quad \mathbf{s} \in \mathbb{R}^n,$$

i.e.,

$$(1.1) \quad \int_{[0,1]^n} \|F(\mathbf{s} + \tau + \mathbf{w}) - F(\mathbf{s} + \mathbf{w})\|^p d\mathbf{w} \leq \epsilon^p, \quad \mathbf{s} \in \mathbb{R}^n.$$

Equivalently, the Bochner transform $\hat{F} : \mathbb{R}^n \rightarrow L^p([0,1]^n : X)$, defined by $[\hat{F}(\mathbf{t})](\mathbf{u}) := F(\mathbf{t} + \mathbf{u})$, $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{u} \in [0,1]^n$, is almost periodic; $F(\cdot)$ is said to be Stepanov almost periodic if $F(\cdot)$ is Stepanov-1-almost periodic. Any Bohr almost periodic function $F(\cdot)$ has to be Stepanov- p -almost periodic for any $p > 0$.

The class of almost automorphic functions was discovered by S. Bochner in 1955 ([7]). In a joint work with S. Abbas [1], we have recently investigated vectorially Weyl almost automorphic functions. This research article aims to examine several new classes of vectorially Stepanov- p -almost periodic type functions and vectorially (equi)-Weyl- p -almost periodic type functions, where $p > 0$. The novelty of our approach is that we use the vector-valued integration in the analysis of these classes of functions; unfortunately, we cannot consider the general value of exponent $p \neq 1$ in the pure vector-valued setting. These classes of functions extend the usually considered classes of Stepanov- p -almost periodic type functions and (equi)-Weyl- p -almost periodic type functions in the scalar-valued setting as well as the usually considered classes of Stepanov-1-almost periodic type functions and (equi)-Weyl-1-almost periodic type functions in the vector-valued setting. We also analyze Σ -almost periodic type functions and the invariance of generalized vectorial almost periodicity under some convolution transforms, which is incredibly important for applications to the abstract Volterra integro-differential equations. If A is the integral generator of an exponentially stable strongly continuous semigroup $(T(t))_{t \geq 0}$ and the function $f : \mathbb{R} \rightarrow X$ is Stepanov- p -almost periodic ($1 \leq p < +\infty$), for instance, then the function $u : \mathbb{R} \rightarrow X$, given by

$$u(t) = \int_{-\infty}^t T(t-s)f(s) ds, \quad t \in \mathbb{R},$$

is an almost periodic solution of the abstract Cauchy problem $u'(t) = Au(t) + f(t)$, $t \in \mathbb{R}$; moreover, if the function $h : \mathbb{R} \rightarrow X$ is Stepanov- p -almost periodic ($1 \leq p < +\infty$), the function $q : [0, +\infty) \rightarrow X$ vanishes at plus infinity and $f(t) = h(t) + q(t)$, $t \geq 0$, then the unique asymptotically almost periodic solution of the abstract Cauchy problem $u'(t) = Au(t) + f(t)$, $t \geq 0$; $u(0) = x$ is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s) ds, \quad t \geq 0,$$

if $f(\cdot)$ enjoys very mild regularity assumptions. Many similar applications of this type have been given to the abstract Volterra integro-differential equations and the abstract fractional integro-differential equations (cf. [13, 14] for more details).

The structure of this research article can be shortly depicted as follows. After fixing the notation, we analyze the notion of vectorial Stepanov almost periodicity in Section 2. The class of Σ -almost periodic functions is analyzed in Subsection 2.1, and the class of vectorially Stepanov almost periodic functions in general metric is analyzed in Subsection 2.2. Vectorially Weyl almost periodic type functions are analyzed in Section 3, while the invariance of vectorial Stepanov almost periodicity and vectorial Weyl almost periodicity under the actions of convolution products is analyzed in Section 4. The last section of paper is reserved for the final conclusions; for simplicity, we will not investigate here the extensions of vectorially generalized almost periodic functions and the corresponding composition principles as well as the vectorially Weyl almost periodic type sequences.

Notation and preliminaries. We assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X and $\mathbb{R}$ is a non-empty collection of sequences in $\mathbb{R}^n$. The standard assumption will be that, for every $y \in X$, there is a set $B \in \mathcal{B}$ such that $y \in B$. $L(X, Y)$ denotes the Banach space of all bounded linear operators from X into Y ; $L(X, X) \equiv L(X)$ and I denotes the identity operator on Y . Define $\mathbb{N}_0 := \{0, 1, \dots, m, \dots\}$. If C and D are non-empty sets, then we set $D^C := \{f \mid f : C \rightarrow D\}$; set $0^\zeta := 0$ for $\zeta > 0$.

Let $\emptyset \neq I \subseteq \mathbb{R}^n$, $\nu : I \rightarrow (0, +\infty)$, and let $1/\nu(\cdot)$ be locally bounded. In this paper, we deal with the vector space $C_{b,\nu}(I : X)$ consisting of those continuous functions $u : I \rightarrow X$ such that $\|u\| := \sup_{t \in I} \|\nu(t)u(t)\| < +\infty$. Then, $(C_{b,\nu}(I : X), \|\cdot\|)$ is a Banach space, as easily approved. Assuming that $\nu \equiv 1$, we also write $C_b(I : X)$ in place of $C_{b,\nu}(I : X)$.

2. VECTORIAL STEPANOV ALMOST PERIODICITY

The class of Stepanov- p -almost periodic functions, where $p \geq 1$, was introduced by W. Stepanoff in 1926 ([22]). For example, the function

$$f(s) = \sin\left(\frac{2}{2 + \cos s + \cos(\sqrt{2}s)}\right), \quad s \in \mathbb{R},$$

is bounded, continuous and Stepanov- p -almost periodic for any finite exponent $p \geq 1$ but it is not almost periodic. If there exists $c > 0$ such that the sequence $(\lambda_k)_{k \in \mathbb{Z}}$ of real numbers satisfies $\lambda_{k+1} - \lambda_k > c$ for all $k \in \mathbb{Z}$ and $(a_k)_{k \in \mathbb{Z}}$ is a real sequence such that $\sum_{k=-\infty}^{+\infty} |a_k|^2 < +\infty$, then

$$f(s) = \sum_{k=-\infty}^{+\infty} a_k e^{i\lambda_k s}, \quad s \in \mathbb{R},$$

is a Stepanov-2-almost periodic function; cf. [20, Theorem 5.3.2, pp. 214–216]. For further information, we refer the reader to [14–16] and references quoted therein.

Now we would like to introduce the following notion.

Definition 2.1. (i) A locally Lebesgue integrable function $F : \mathbb{R}^n \rightarrow X$ is vectorially Stepanov almost periodic if for every $\epsilon > 0$ there exists $W > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W)$ with

$$(2.1) \quad \left\| \int_{[0,1]^n} [F(\mathbf{s} + \tau + \mathbf{w}) - F(\mathbf{s} + \mathbf{w})] d\mathbf{w} \right\| \leq \epsilon, \quad \mathbf{s} \in \mathbb{R}^n.$$

(ii) A p -locally Lebesgue integrable function $F : \mathbb{R}^n \rightarrow \mathbb{C}$ is vectorially Stepanov- p -almost periodic if for every $\epsilon > 0$ there exists $W > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W)$ with

$$(2.2) \quad \left| \int_{[0,1]^n} [F(\mathbf{s} + \tau + \mathbf{w}) - F(\mathbf{s} + \mathbf{w})]^p d\mathbf{w} \right| \leq \epsilon^p, \quad \mathbf{s} \in \mathbb{R}^n.$$

It is clear that the notion of vectorial Stepanov- p -almost periodicity cannot be easily introduced if $p \neq 1$ and $X \neq \mathbb{C}$.

Immediately from the equations (1.1), (2.1) and (2.2), it follows that any Stepanov almost periodic function is vectorially Stepanov almost periodic as well as that any scalar-valued Stepanov- p -almost periodic function is vectorially Stepanov- p -almost periodic. The relations between Stepanov- p -almost periodic functions in norm (cf. [18, Definition 11]) and vectorial Stepanov- p -almost periodic functions are quite non-trivial ($p = 1$, general X ; $p > 0$ and $p \neq 1$, $X = \mathbb{C}$). Furthermore, the class of scalar-valued vectorially Stepanov- p -almost periodic functions behaves very badly if $p \neq 1$ and we will not examine this class in more detail henceforth. The use of vector-valued integration in Definition 2.1(i) may cause some unpleasant difficulties sometimes, as well; for example, it is not simple to consider the products of vectorially Stepanov almost periodic functions and the existence of Bohr-Fourier coefficients of vectorially Stepanov almost periodic functions (see, e.g., [14, Theorem 2.1.1]).

Example 2.1. On some function spaces, the Gaussian semigroup

$$(G(s)F)(y) := (4\pi s)^{-(n/2)} \int_{\mathbb{R}^n} F(y - z) e^{-\frac{|z|^2}{4s}} dz, \quad s > 0, \quad f \in Y, \quad y \in \mathbb{R}^n,$$

is a strongly continuous semigroup generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y . Let $F(\cdot)$ be bounded and vectorially Stepanov almost periodic, and let $s_0 > 0$ be fixed. Then, the function $\mathbb{R}^n \ni x \mapsto u(x, s_0) \equiv (G(s_0)F)(x) \in \mathbb{C}$ is bounded and vectorially Stepanov almost periodic, which is an essential consequence of the next calculus ($x \in \mathbb{R}^n$; $\tau \in \mathbb{R}^n$ satisfies the necessary

requirements):

$$\begin{aligned}
& \left| \int_{[0,1]^n} [(G(s_0)F)(v + \tau + z) - (G(s_0)F)(v + z)] dz \right| \\
&= \left| \int_{[0,1]^n} \left[\int_{\mathbb{R}^n} F(v + \tau + u - w) e^{-\frac{|w|^2}{4s_0}} dw - \int_{\mathbb{R}^n} F(v + u - w) e^{-\frac{|w|^2}{4s_0}} dw \right] du \right| \\
&= \left| \int_{[0,1]^n} \left[\int_{\mathbb{R}^n} [F(v + \tau + u - w) - F(v + u - w)] e^{-\frac{|w|^2}{4s_0}} dw \right] du \right| \\
&= \left| \int_{\mathbb{R}^n} e^{-\frac{|w|^2}{4s_0}} \left[\int_{[0,1]^n} [F(v + \tau + u - w) - F(v + u - w)] du \right] dw \right| \\
&\leq \int_{\mathbb{R}^n} e^{-\frac{|w|^2}{4s_0}} \left| \int_{[0,1]^n} [F(v + \tau + u - w) - F(v + u - w)] du \right| dy \leq \epsilon \int_{\mathbb{R}^n} e^{-\frac{|w|^2}{4t_0}} dw.
\end{aligned}$$

We can also consider here vectorially Stepanov c -almost periodic functions; see Definition 2.4 below with $\rho = cI$ and $c \in \mathbb{C} \setminus \{0\}$.

The proof of following result is relatively plain.

Proposition 2.1. *A locally integrable function $F : \mathbb{R}^n \rightarrow X$ is vectorially Stepanov almost periodic if and only if the function $G = \Sigma(F) : \mathbb{R}^n \rightarrow X$, given by $G(\mathbf{t}) := \int_{[0,1]^n} F(\mathbf{t} + \mathbf{u}) d\mathbf{u}$, $\mathbf{t} \in \mathbb{R}^n$, is almost periodic.*

In the one-dimensional setting, we have that $G(t) = F^{[1]}(t+1) - F^{[1]}(t)$, $t \in \mathbb{R}$, where $F^{[1]}(t) := \int_0^t F(s) ds$, $t \in \mathbb{R}$ is the first integral of function $F(\cdot)$. In particular, if $F^{[1]}(\cdot)$ is almost periodic, then $F(\cdot)$ is vectorially Stepanov almost periodic (see also Kadets's theorem [4, Theorem 4.6.11] and the research articles [3] by J. Andres, D. Pennequin, [8] by C. Budde, J. Kreulich, [10] by H.-S. Ding et al., and [21] by A. M. Samoilenko, S. I. Trofimchuk for further information concerning the integration of almost periodic functions). The following example justifies the introduction of notion in Definition 2.1.

Example 2.2. Let us recall that there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is bounded continuous and has the feature that the function $f^{[1]}(\cdot)$ is almost periodic and the function $f(\cdot)$ is not Stepanov almost periodic; cf. [5, Example 3.2]. Since $f^{[1]}(\cdot)$ is almost periodic, we have that $f(\cdot)$ is vectorially Stepanov almost periodic; see also [15, p. 62] for more details about this function.

Proposition 2.1 implies that vectorially Stepanov almost periodic functions form a vector space as well as that any vectorially Stepanov almost periodic function $F : \mathbb{R}^n \rightarrow X$ is vectorially Stepanov bounded in the sense that

$$\sup_{\mathbf{t} \in \mathbb{R}^n} \left\| \int_{t+[0,1]^n} F(\mathbf{w}) d\mathbf{w} \right\| < +\infty.$$

The notion of vectorial Stepanov boundedness and the usually considered Stepanov boundedness are different.

Example 2.3. Let $X := \mathbb{C}$ and $F(t) := t \sin(t^2)$, $t \in \mathbb{R}$. Then we have

$$\int_x^{x+1} \left| s \cdot \sin(s^2) \right| ds = \frac{1}{2} \int_{x^2}^{(x+1)^2} |\sin v| dv \geq c|x|, \quad x \in \mathbb{R},$$

for some $c > 0$. Hence,

$$\sup_{x \in \mathbb{R}} \left[(1 + |x|)^{-\sigma} \int_x^{x+1} |f(s)| ds \right] = +\infty, \quad \sigma \in (0, 1).$$

A similar argumentation shows that

$$\left| \int_x^{x+1} f(s) du \right| = \frac{1}{2} \left| \cos(x^2) - \cos((x+1)^2) \right| \leq 1, \quad x \in \mathbb{R},$$

which yields the required conclusion. Further on,

$$\begin{aligned} & \int_x^{x+1} [f(s+\tau) - f(s)] ds \\ &= \frac{1}{2} \left[\left(\cos((x+\tau)^2) - \cos((x+\tau+1)^2) \right) - \left(\cos(x^2) - \cos((x+1)^2) \right) \right], \quad x, \tau \in \mathbb{R}, \end{aligned}$$

which shows that $f(\cdot)$ cannot be vectorially Stepanov almost periodic.

The vector-valued integration can be useful in the studies of vectorial weighted ergodic components and vectorial weighted ergodic components in general metric, as well (see [15, Section 6.4] and [16, Section 5.2]).

2.1. Σ -almost periodicity. In the rest of this section, let us assume that $\emptyset \neq I \subseteq \mathbb{R}^n$, $P \subseteq Y^I$, $0 \in P$ and $\mathcal{P} = (P, d)$ is a pseudometric space; if $f \in P$, set $\|f\|_P := d(f, 0)$.

We recall that the notion of a (strongly) $(\mathbf{R}, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic function is introduced in [17, Definition 2.1]. Keeping in mind Proposition 2.1, we would like to introduce the following general notion.

Definition 2.2. Let $\emptyset \neq I \subseteq \mathbb{R}^n$, $\Sigma : Y^{I \times X} \rightarrow Y^{I \times X}$, and let $\mathbf{R}$ be an arbitrary non-empty collection of sequences in $\mathbb{R}^n$, $F : I \times X \rightarrow Y$, for each $B \in \mathcal{B}$ and $\mathbf{b} \in \mathbf{R}$ the set $L(B; \mathbf{b})$ is a collection of subsets of B , and we have the following:

$$(2.3) \quad \text{If } \mathbf{s} \in I, \mathbf{a} \in \mathbf{R} \text{ and } l \in \mathbb{N}, \text{ then we have } \mathbf{s} + \mathbf{a}(l) \in I.$$

Then $F(\cdot; \cdot)$ is said to be $(\Sigma, \mathbf{R}, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic, resp. strongly $(\Sigma, \mathbf{R}, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic in the case that $I = \mathbb{R}^n$, if $\Sigma(F)$ is $(\mathbf{R}, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic, resp. strongly $(\mathbf{R}, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic.

If for each $B' \in \mathcal{B}$ and $\mathbf{b}' \in \mathbf{R}$, we have $L(B'; \mathbf{b}') = \{B'\}$, then we omit the term “ L ” from the notation.

We can similarly define the notion of a (strong) $(\Sigma, \mathbf{R}_X, \mathcal{B}, \mathcal{P})$ -multi-almost periodic function (of type 1), where $\mathbf{R}_X$ is an arbitrary collection of sequences in $\mathbb{R}^n \times X$; cf. [17, Definition 2.2]. Once it is done, we can extend [17, Proposition 2.6] in the following way.

Proposition 2.2. *Let P be a vector structure with the usual operations, let P be complete and let the metric d be translation invariant. Let us assume that, for every $j \in \mathbb{N}$, the function $F_j : I \times X \rightarrow Y$ is $(\Sigma, R_X, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic of type 1 as well as that, for every sequence which belongs to R_X , any its subsequence also belongs to R_X . If $F : I \times X \rightarrow Y$ and for each $B' \in \mathcal{B}$, $(\mathbf{b}; \mathbf{x}) = ((b_k; x_k)) = ((b_k^1, b_k^2, \dots, b_k^n); x_k) \in R_X$, $B' \in L(B; (\mathbf{b}; \mathbf{x}))$ and we have*

$$\lim_{(i', l') \rightarrow (+\infty, +\infty)} \sup_{y \in B'} \left\| F_{i'}(\cdot + b_{k_{l'}}; y + x_{k_{l'}}) - F(\cdot + b_{k_{l'}}; y + x_{k_{l'}}) \right\|_P = 0,$$

then $F(\cdot; \cdot)$ is $(\Sigma, R_X, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic of type 1, if:

- (i) $\Sigma(f - g) = \Sigma f - \Sigma g$, $f, g \in P$;
- (ii) there exists $d > 0$ such that $\|\Sigma f\|_P \leq d\|f\|_P$, $f \in P$.

For example, this condition holds if $I = \mathbb{R}^n$, $P = C_b(\mathbb{R}^n : Y)$ and $\Sigma(\cdot)$ is given as in the formulation of Proposition 2.1. Suppose now P is a vector space, the metric d has the property of translation invariance and, for every $d \in \mathbb{C}$, one has $df \in P$, $f \in P$ and the existence of a real number $\phi(d) > 0$ such that $\|df\|_P \leq \phi(d)\|f\|_P$ for all $f \in P$. If we assume that for each sequence in R_X any its subsequence also belongs to R_X as well as that the mapping Σ is linear, then the space consisting of all (strongly) $(\Sigma, R, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic [(strongly) $(\Sigma, R_X, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic] functions is a vector space; see also [17, Remark(ii), pp. 234–235].

The reader may consult [17, Definition 3.1] for the notions of a Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$ -almost periodic function and a $(\mathcal{B}, I', \rho, \mathcal{P})$ -uniformly recurrent function. The following notion could be also introduced (we can similarly examine some other classes of (metrically) Σ -almost periodic type functions considered in [16, Chapter 4–Chapter 7] in the case that Σ is the identity mapping).

Definition 2.3. Let $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $\Sigma : Y^{I \times X} \rightarrow Y^{I \times X}$, $F : I \times X \rightarrow Y$, ρ is a binary relation on Y , $R(\Sigma(F)) \subseteq D(\rho)$ and $I + I' \subseteq I$. Then, it is said that:

- (i) $F(\cdot; \cdot)$ is Bohr $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$ -almost periodic if $\Sigma(F)$ is Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$ -almost periodic;
- (ii) $F(\cdot; \cdot)$ is $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$ -uniformly recurrent if $\Sigma(F)$ is $(\mathcal{B}, I', \rho, \mathcal{P})$ -uniformly recurrent.

The structural results established for Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$ -almost periodic functions and $(\mathcal{B}, I', \rho, \mathcal{P})$ -uniformly recurrent functions can be straightforwardly generalized for Bohr $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$ -almost periodic functions and $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$ -uniformly recurrent functions, provided that the mapping $\Sigma(\cdot)$ has suitable properties; we can slightly extend the assertions of [17, Proposition 3.7, Corollary 3.8, Proposition 3.10] in this manner, for example. We will not examine such results here.

2.2. Vectorially Stepanov almost periodic type functions. We denote here the region I by Λ and the region I' by Λ' . We assume the following:

(SM1-1v): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is Lebesgue measurable, $m(\Omega) > 0$, $\Lambda' + \Lambda \subseteq \Lambda$ and $\Lambda + \Omega \subseteq \Lambda$.

Now we introduce the following notion.

Definition 2.4. Suppose that (SM1-1v) holds, ρ is a binary relation on Y , $F : \Lambda \times X \rightarrow Y$, for each $x \in X$ the mapping $F(\cdot; x)$ is locally integrable,

$$(2.4) \quad [\Sigma(F)](\mathbf{s}; x) := \int_{\Omega} F(\mathbf{s} + \mathbf{w}; x) d\mathbf{w}, \quad \mathbf{s} \in \Lambda, \quad x \in X,$$

and $R(\Sigma(F)) \subseteq D(\rho)$. Then, $F(\cdot; \cdot)$ is said to be vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$ -almost periodic, resp. vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$ -uniformly recurrent, if $F(\cdot; \cdot)$ is Bohr $(\Sigma, \mathcal{B}, \Lambda', \rho, \mathcal{P})$ -almost periodic, resp. $(\Sigma, \mathcal{B}, \Lambda', \rho, \mathcal{P})$ -uniformly recurrent, with $\Sigma(\cdot)$ being defined through (2.4).

If (2.3) holds, then it is said that $F(\cdot; \cdot)$ is vectorially Stepanov $(R, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic, resp. strongly vectorially Stepanov $(R, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic in the case that $\Lambda = \mathbb{R}^n$, if $\Sigma(F)$ is $(R, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic, resp. strongly $(R, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic, with $\Sigma(\cdot)$ being defined through (2.4).

If $F : \Lambda \times X \rightarrow Y$, $G : \Lambda \times X \rightarrow Y$ and for each $x \in X$ the mappings $F(\cdot; x)$ and $G(\cdot; x)$ are locally integrable, then we have $\Sigma(\alpha F + \beta G) = \alpha \Sigma(F) + \beta \Sigma(G)$ for all scalars $\alpha, \beta \in \mathbb{C}$, so that the collection of all vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$ -almost periodic functions is a linear vector space if the corresponding space of all $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$ -almost periodic functions is a linear vector space (we cannot expect the linearity of space of all vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$ -uniformly recurrent functions in any sense; cf. [15]). Furthermore, if $\Lambda = \Lambda' = \mathbb{R}^n$, $\rho = I$, R contains all sequences in $\mathbb{R}^n$ and $\mathcal{B}$ contains all compact sets in X , then the metrical Bochner criterion established in [19, Theorem 1] immediately clarifies the coincidence of the class of all vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$ -almost periodic functions and the class consisting of all vectorially Stepanov $(R, \mathcal{B}, \mathcal{P})$ -multi-almost periodic functions, provided that the pseudometric space P has the properties stated in this result.

Proposition 2.3. Assume (SM1-1v) holds, $\rho = T \in L(Y)$, $F_k : \Lambda \times X \rightarrow Y$ and the function $F_k(\cdot; \cdot)$ is vectorially Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$ -almost periodic, resp. vectorially Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$ -uniformly recurrent ($k \in \mathbb{N}$), where $P = C_b(\Lambda : Y)$. If $F : \Lambda \times X \rightarrow Y$, for each $y \in X$ the mapping $F(\cdot; y)$ is locally integrable, and for each $B \in \mathcal{B}$,

$$\lim_{l \rightarrow +\infty} \sup_{\mathbf{s} \in \Lambda, y \in B} \left\| \int_{\Omega} [F_l(\mathbf{s} + \mathbf{w}; x) - F(\mathbf{s} + \mathbf{w}; x)] d\mathbf{w} \right\| = 0,$$

then $F(\cdot; \cdot)$ is vectorially Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$ -almost periodic, resp. vectorially Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$ -uniformly recurrent.

Proof. We examine the class of vectorially Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$ -almost periodic functions, only. Let us fix $B \in \mathcal{B}$ and $\epsilon > 0$. Then there exist $k_0 \in \mathbb{N}$ and $W > 0$ such

that for each $\mathbf{w}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, W) \cap \Lambda'$ such that

$$\sup_{\mathbf{s} \in \Lambda, y \in B} \left\| \int_{\mathbf{s}+\Omega} [F_{k_0}(\mathbf{w} + \tau; y) - TF_{k_0}(\mathbf{w}; y)] d\mathbf{w} \right\| \leq \epsilon/3.$$

Then, for each $\mathbf{s} \in \Lambda$ and $y \in B$, we have the following:

$$\begin{aligned} & \left\| \int_{\mathbf{s}+\Omega} [F(\mathbf{u} + \tau; y) - TF(\mathbf{u}; y)] d\mathbf{u} \right\| \\ & \leq \left\| \int_{\mathbf{s}+\Omega} [F(\mathbf{u} + \tau; y) - F_{k_0}(\mathbf{u} + \tau; y)] d\mathbf{u} \right\| + \left\| \int_{\mathbf{s}+\Omega} [F_{k_0}(\mathbf{u} + \tau; y) - TF_{k_0}(\mathbf{u}; y)] d\mathbf{u} \right\| \\ & \quad + \left\| \int_{\mathbf{s}+\Omega} [TF_{k_0}(\mathbf{u}; y) - TF(\mathbf{u}; y)] d\mathbf{u} \right\| \\ & \leq \left\| \int_{\mathbf{s}+\Omega} [F(\mathbf{u} + \tau; y) - F_{k_0}(\mathbf{u} + \tau; y)] d\mathbf{u} \right\| + \left\| \int_{\mathbf{s}+\Omega} [F_{k_0}(\mathbf{u} + \tau; y) - TF_{k_0}(\mathbf{u}; y)] d\mathbf{u} \right\| \\ & \quad + \|T\| \cdot \left\| \int_{\mathbf{s}+\Omega} [F_{k_0}(\mathbf{u}; y) - F(\mathbf{u}; y)] d\mathbf{u} \right\|. \end{aligned}$$

The vectorial Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$ -almost periodicity of $F(\cdot; \cdot)$ follows thereof. $\square$

Suppose now that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is Lebesgue measurable, $m(\Omega) > 0$ and $\Lambda + \Omega \subseteq \Lambda$. Denote by $S_v^1(\Lambda \times X : Y)$ the collection of all functions $F : \Lambda \times X \rightarrow Y$ such that for each $x \in X$ the function $F(\cdot; x)$ is locally integrable and

$$(2.5) \quad \|F\|_{S_{v,B,\Omega}^1} := \sup_{\mathbf{t} \in \Lambda, x \in B} \frac{1}{m(\Omega)} \left\| \int_{\mathbf{t}+\Omega} F(\mathbf{s}; x) d\mathbf{s} \right\| < +\infty, \quad B \in \mathcal{B}.$$

For functions $F : \Lambda \rightarrow Y$, we exclude the term “ B ”; moreover, we exclude the term “ Ω ” if $\Omega = [0, 1]^n$.

Let us observe the following facts.

- (i) The assumption $\|F\|_{S_v^1} = 0$ does not imply $F(\mathbf{t}) = 0$ for a.e. $\mathbf{t} \in \Lambda$; for example, examine the function $F(t) = \cos(2\pi t)$, $t \in \mathbb{R}$.
- (ii) If $\lambda \in \mathbb{C}$, $B \in \mathcal{B}$ and $\|F\|_{S_{v,B,\Omega}^1} < +\infty$, then $\|\lambda \cdot F\|_{S_{v,B,\Omega}^1} = |\lambda| \cdot \|F\|_{S_{v,B,\Omega}^1}$.
- (iii) If $B \in \mathcal{B}$, $\|F\|_{S_{v,B,\Omega}^1} < +\infty$, $G : \Lambda \times X \rightarrow Y$ has the property that for each $y \in X$ the function $G(\cdot; y)$ is locally integrable and $\|G\|_{S_{v,B,\Omega}^1} < +\infty$, then

$$\|F + G\|_{S_{v,B,\Omega}^1} \leq \|F\|_{S_{v,B,\Omega}^1} + \|G\|_{S_{v,B,\Omega}^1}.$$

For each $B \in \mathcal{B}$, by $S_{v,B}^1(\Lambda \times B : Y)$ we denote the collection of all functions $F : \Lambda \times B \rightarrow Y$ such that for each $x \in B$ the function $F(\cdot; x)$ is locally integrable and (2.5) holds. Then $S_{v,B}^1(\Lambda \times B : Y)$ is a vector space and (i)-(iii) simply yield that $\|\cdot\|_{S_{v,B,\Omega}^1}$ is a seminorm on $S_{v,B}^1(\Lambda \times B : Y)$.

The next result can be compared to [20, Theorem 5.2.1, pp. 199–200].

Proposition 2.4. *Suppose that $B \in \mathcal{B}$, $(F_k(\cdot; \cdot))_{k \in \mathbb{N}}$ is a sequence of functions in $S_{v,B}^1(\Lambda \times B : Y)$, and the following holds.*

- (i) For every $\epsilon > 0$, there exists $s_0 \in \mathbb{N}$ such that, for every $s_1, s_2 \in \mathbb{N}$ with $\min(s_1, s_2) \geq s_0$, we have $\|F_{s_1} - F_{s_2}\|_{S_{v,B,\Omega}^1} \leq \epsilon$.
- (ii) There exists $F : \Lambda \times B \rightarrow Y$ such that for each $x \in B$ the function $F(\cdot; x)$ is locally integrable and $\lim_{k \rightarrow +\infty} \|F_k(\mathbf{t}; x) - F(\mathbf{t}; x)\| = 0$ for a.e. $\mathbf{t} \in \Lambda$.
- (iii) For each compact set $K \subseteq \Lambda$ and for each $x \in B$, there exists $g_x \in L^1(K; Y)$ such that $\|F_k(\mathbf{t}; x)\| \leq g_x(\mathbf{t})$ a.e. on K for all $k \in \mathbb{N}$.

Then, $\lim_{k \rightarrow +\infty} \|F_k - F\|_{S_{v,B,\Omega}^1} = 0$.

Proof. If $\epsilon > 0$, $x \in B$ and $\mathbf{t} \in \Lambda$, then [4, Theorem 1.1.8] and (ii)-(iii) together imply that

$$(2.6) \quad \lim_{m \rightarrow +\infty} \int_{\mathbf{t}+\Omega} F_m(\mathbf{s}; x) d\mathbf{s} = \int_{\mathbf{t}+\Omega} F(\mathbf{s}; x) d\mathbf{s}.$$

On the other hand, (i) implies that there exists $s_0 \in \mathbb{N}$ such that, for every $s_1, s_2 \in \mathbb{N}$ with $\min(s_1, s_2) \geq s_0$, one has:

$$(2.7) \quad \left\| \int_{\mathbf{r}+\Omega} [F_{s_1}(\mathbf{s}; y) - F_{s_2}(\mathbf{s}; y)] d\mathbf{s} \right\| \leq \epsilon, \quad \mathbf{r} \in \Lambda, y \in B.$$

The required conclusion follows by letting $m \rightarrow +\infty$ in (2.7) and using (2.6) after that. $\square$

3. VECTORIALLY WEYL ALMOST PERIODIC TYPE FUNCTIONS

If $F : \mathbb{R}^n \rightarrow X$ is Lebesgue measurable and for each bounded and closed set $K \subseteq \mathbb{R}^n$, we have $\int_K \|F(\mathbf{t})\|^p d\mathbf{t} < +\infty$, where $p > 0$, then $F(\cdot)$ is:

- (i) equi-Weyl- p -almost periodic if, for every $\epsilon > 0$, there exist two finite real numbers $l' > 0$ and $L' > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \mathbb{R}^n$ with

$$\sup_{\mathbf{t} \in \mathbb{R}^n} \left[(l')^{-n} \int_{\mathbf{t}+l'[0,1]^n} \|F(\tau + \mathbf{u}) - F(\mathbf{u})\|^p d\mathbf{u} \right] < \epsilon.$$

- (ii) Weyl- p -almost periodic if, for every $\epsilon > 0$, there exists a finite real number $L' > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \mathbb{R}^n$ with

$$\limsup_{l' \rightarrow +\infty} \sup_{\mathbf{t} \in \mathbb{R}^n} \left[(l')^{-n} \int_{\mathbf{t}+l'[0,1]^n} \|F(\tau + \mathbf{u}) - F(\mathbf{u})\|^p d\mathbf{u} \right] < \epsilon.$$

Now we would like to propose the following notion (we will not consider here the metrical generalizations of this definition; see [16, Section 4.3]).

Definition 3.1. Suppose that $p > 0$.

- (i) A locally Lebesgue integrable function $F : \mathbb{R}^n \rightarrow X$ is said to be vectorially equi-Weyl-almost periodic if, for every $\epsilon > 0$, there exist two finite real numbers

$l' > 0$ and $L' > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, L')$ with

$$\sup_{\mathbf{t} \in \mathbb{R}^n} \left\| \int_{\mathbf{t} + l'[0,1]^n} [F(\tau + \mathbf{u}) - F(\mathbf{u})] d\mathbf{u} \right\| < \epsilon(l')^n.$$

- (ii) A locally Lebesgue integrable function $F : \mathbb{R}^n \rightarrow X$ is said to be vectorially Weyl-almost periodic if, for every $\epsilon > 0$, there exists a finite real number $L' > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, L')$ with

$$\limsup_{l' \rightarrow +\infty} \sup_{\mathbf{t} \in \mathbb{R}^n} \left[(l')^{-n} \left\| \int_{\mathbf{t} + l'[0,1]^n} [F(\tau + \mathbf{u}) - F(\mathbf{u})] d\mathbf{u} \right\| \right] < \epsilon.$$

Any (equi-)Weyl-1-almost periodic function is clearly vectorially (equi-)Weyl-almost periodic and any vectorially Stepanov almost periodic function $F : \mathbb{R}^n \rightarrow X$ is clearly vectorially equi-Weyl-almost periodic.

Example 3.1. (see also [4, Example 4.6.5]) Suppose that $X := c_0$. Define $f(t) := ((1/k)\cos(t/k))_k$, $t \in \mathbb{R}$. Then $f : \mathbb{R} \rightarrow X$ is almost periodic but its first integral $F(t) := (\sin(t/k))_k$, $t \in \mathbb{R}$ is bounded, uniformly continuous but not Stepanov- p -almost automorphic for any exponent $p > 0$; furthermore, we know that $F(\cdot)$ is vectorially Weyl almost automorphic as well as that $F(\cdot)$ is not vectorially Weyl almost automorphic of type 1 nor jointly vectorially Weyl almost automorphic (see [1]).

Let us prove that $F(\cdot)$ is vectorially Weyl-almost periodic, i.e., the vectorially Weyl-1-almost periodic. To show this, notice that ($r, s \in \mathbb{R}$, $\tau \in \mathbb{R}$):

$$F(r + s + \tau) - F(r + s) = 2 \left(\sin \frac{\tau}{2k} \cdot \cos \frac{2r + 2s + 2\tau}{2k} \right)_k,$$

so that ($t \in \mathbb{R}$, $\tau \in \mathbb{R}$; $l > 0$)

$$\begin{aligned} & \frac{1}{l} \int_t^{t+l} [F(v + \tau) - F(v)] dv \\ &= \frac{2}{l} \left(k \sin \frac{\tau}{2k} \cdot \left[\sin \frac{2t + 2l + \tau}{2k} - \sin \frac{2t + \tau}{2k} \right] \right)_k \\ (3.1) \quad &= \frac{4}{l} \left(k \sin \frac{\tau}{2k} \cdot \cos \frac{2t + l + \tau}{2k} \cdot \sin \frac{l}{2k} \right)_k. \end{aligned}$$

This immediately implies the required statement since $|k \sin(\tau/2k)| \leq |\tau|/2$, $k \in \mathbb{N}$, and

$$\left| \cos \frac{4t + 2l + 2\tau}{4k} \cdot \sin \frac{l}{2k} \right| \leq 1, \quad k \in \mathbb{N}.$$

Let us prove now that $F(\cdot)$ is not Weyl- p -almost periodic for any $p > 0$. Suppose that $0 < \epsilon < 32^{-1} \cdot (\cos(7\pi/16) \cdot \sqrt{2}/2)^p$. Then, there exists $L > 0$ such that, for every $t_0 \in \mathbb{R}$, we can find $\tau \in [t_0 - L, t_0 + L]$ such that there exists $l_0(\tau) > 0$ with

$$(3.2) \quad \int_0^l \left\| \left(\cos \frac{2x + \tau}{2k} \cdot \sin \frac{\tau}{2k} \right)_k \right\|^p dx \leq \epsilon \cdot l, \quad l \geq l_0(\tau).$$

Suppose that $k_0 \in \mathbb{N}$ is sufficiently large and $2k_0\pi/4 \leq \tau < (2k_0 + 2)\pi/4$. Then $\pi/4 \leq \tau/2k_0 < 3\pi/8$ and therefore $\sin(\tau/2k_0) \geq \sqrt{2}/2$. Suppose, further, that $x_0/k_0 = \pi/16$; then $\cos((2x + \tau)/2k_0) \geq \cos(7\pi/16)$, $x \in [2m\pi k_0, 2m\pi k_0 + x_0]$ ($m \in \mathbb{N}_0$) and taking into account (3.2) with $l = 2m\pi k_0 + x_0$, where m is sufficiently large, we get

$$\begin{aligned} & \epsilon(2m\pi k_0 + x_0) \\ & \geq \int_0^{2m\pi k_0 + x_0} \left\| \left(\cos \frac{2x + \tau}{2k} \cdot \sin \frac{\tau}{2k} \right)_k \right\|^p dx \\ & \geq \int_0^{2m\pi k_0 + x_0} \left| \cos \frac{2x + \tau}{2k_0} \cdot \sin \frac{\tau}{2k_0} \right|^p dx \geq (m + 1)x_0 \left(\frac{\sqrt{2}}{2} \cdot \cos \frac{7\pi}{16} \right)^p, \quad l \geq l_0(\tau). \end{aligned}$$

Dividing by m and letting $m \rightarrow +\infty$, we get

$$2\pi\epsilon(k_0/x_0) = 32\epsilon \geq \left(\frac{\sqrt{2}}{2} \cdot \cos \frac{7\pi}{16} \right)^p,$$

which is a contradiction.

Now we will prove that $F(\cdot)$ is not vectorially equi-Weyl-almost periodic. Suppose that $0 < \epsilon < \sqrt{2}/2$. Then there exist $l > 0$ and $L > 0$ such that, for every $t_0 \in \mathbb{R}$, there exists $\tau \in [t_0 - L, t_0 + L]$ such that

$$(3.3) \quad \left\| \frac{4}{l} \left(k \sin \frac{\tau}{2k} \cdot \cos \frac{2t + l + \tau}{2k} \cdot \sin \frac{l}{2k} \right)_k \right\| \leq \epsilon, \quad t \in \mathbb{R};$$

see (3.1). Furthermore, there exists $k_0(\epsilon, l) \in \mathbb{N}$ such that

$$(3.4) \quad 4k \sin \frac{l}{2k} \geq l, \quad k \geq k_0(\epsilon, l).$$

After that, take any $t_0 > L + \pi k_0(\epsilon, l)/2$. Then there exists $k_0 \geq k_0(\epsilon, l)$ such that $2k_0\pi/4 \leq \tau < (2k_0 + 2)\pi/4$. Then, as above, $\pi/4 \leq \tau/2k_0 < 3\pi/8$ and $\sin(\tau/2k_0) \geq \sqrt{2}/2$. Since (3.3) holds for all $t \in \mathbb{R}$, we can take $t = -(l + \tau)/2$ in order to see that $\cos((2t + l + \tau)/2k_0) = 1$ so that (3.4) and the above estimates enable one to see that the right hand side of (3.3) is greater than or equal to $\sqrt{2}/2$, which is a contradiction.

Let us finally observe that, for every $l > 0$, we have

$$(3.5) \quad \sup_{r \in \mathbb{R}} \frac{1}{l} \left\| \int_r^{r+l} F(s) ds \right\| = 1,$$

so that the vectorial Weyl seminorm of $F(\cdot)$, defined through

$$\|F\|_{W,v} := \lim_{l \rightarrow +\infty} \sup_{r \in \mathbb{R}} \frac{1}{l} \left\| \int_r^{r+l} F(s) ds \right\|,$$

exists and equals 1. In order to see that (3.5) holds, notice that a simple computation shows that

$$\frac{1}{l} \left\| \int_r^{r+l} F(s) ds \right\| = \sup_{k \in \mathbb{N}} \left[\frac{2k}{l} \cdot \left| \sin \frac{2r + l}{2k} \cdot \sin \frac{l}{2k} \right| \right], \quad r \in \mathbb{R}.$$

Therefore, for every $l > 0$,

$$\begin{aligned} \frac{1}{l} \left\| \int_t^{t+l} F(s) ds \right\| &\geq \sup_{t \in (\pi\mathbb{Z}-l)/2} \sup_{k \in \mathbb{N}} \left[\frac{2k}{l} \cdot \left| \sin \frac{2t+l}{2k} \cdot \sin \frac{l}{2k} \right| \right] \\ &\geq \sup_{k \in \mathbb{N}} \left[\frac{2k}{l} \cdot \left| \sin \frac{l}{2k} \right| \right] = 1, \end{aligned}$$

since $\lim_{k \rightarrow +\infty} (2k/l) \cdot |\sin(l/2k)| = 1$.

If $F(\cdot)$ is locally integrable, then the existence of vectorial Weyl seminorm of $F(\cdot)$ in $[0, +\infty]$ cannot be proved with the help of the argumentation given on [15, pp. 375–376]. Moreover, this seminorm does not necessarily exists.

Example 3.2. Define the function $F : \mathbb{R} \rightarrow \mathbb{R}$ by $F(t) := -2k$ if $t \in [2k, 2k+1)$ for some $k \in \mathbb{Z}$ and $F(t) := 2k+2$ if $t \in [2k+1, 2k+2)$ for some $k \in \mathbb{Z}$. Then,

$$\limsup_{l \rightarrow +\infty} \sup_{t \in \mathbb{R}} \frac{1}{l} \left| \int_t^{t+l} F(s) ds \right| = +\infty \quad \text{and} \quad \liminf_{l \rightarrow +\infty} \sup_{t \in \mathbb{R}} \frac{1}{l} \left| \int_t^{t+l} F(s) ds \right| = 1,$$

and therefore, $\|F\|_{W,v}$ does not exist in $[0, +\infty]$. In order to see this, it suffices to show that

$$(3.6) \quad \sup_{t \in \mathbb{R}} \left| \int_t^{t+2k} F(s) ds \right| = 2k \quad \text{and} \quad \sup_{t \in \mathbb{R}} \left| \int_t^{t+2k+1} F(s) ds \right| = +\infty, \quad k \in \mathbb{N}.$$

By our construction, we have $\int_{2m+1}^{2m+1+2k} F(s) ds = 0$ for all $m, k \in \mathbb{Z}$. Hence,

$$\sup_{t \in \mathbb{R}} \left| \int_t^{t+2k+1} F(s) ds \right| \geq \sup_{t \in 2\mathbb{Z}+1} \left| \int_t^{t+2k+1} F(s) ds \right|.$$

If $t = 2m+1$ for some $m \in \mathbb{Z}$, then we have

$$\int_t^{t+2k+1} F(s) ds = \int_{2m+2k+1}^{2m+2k+2} F(s) ds = 2m+2k+2,$$

which simply implies the second equality in (3.6). On the other hand, we have

$$(3.7) \quad \sup_{t \in \mathbb{R}} \left| \int_t^{t+2k} F(s) ds \right| \geq \int_0^{2k} F(s) ds = 2k, \quad k \in \mathbb{N}.$$

If $t \in [2m, 2m+1)$ for $m \in \mathbb{Z}$, then one has

$$\begin{aligned} \int_t^{t+2k} F(s) ds &= \int_t^{2m+1} F(s) ds - \int_{t+2k}^{2m+2k+1} F(s) ds \\ &= (-2)(2m+1-t)m + (2m+1-t)(2m+2k) = 2(2m+1-t)k, \end{aligned}$$

so that

$$(3.8) \quad \left| \int_t^{t+2k} F(s) ds \right| \leq 2k.$$

Similarly, if $t \in [2m+1, 2m+2)$ for an integer $m \in \mathbb{Z}$, then one has

$$\begin{aligned} \int_t^{t+2k} F(s) ds &= \int_t^{2m+1} F(s) ds + \int_{2m+2k+1}^{t+2k} F(s) ds \\ &= (2m+1-t)(2m+2) - (2m+1-t)(2m+2k+2) = 2(2m+1-t)k, \end{aligned}$$

so that (3.8) again holds. By (3.7)-(3.8), we deduce the first equality in (3.6).

The uniform convergence of a sequence of locally integrable functions implies the convergence of this sequence in the vectorial Weyl seminorm; furthermore, the standard evidence shows that, if the sequence of vectorially (equi)-Weyl-almost periodic functions converges in the vectorial Weyl seminorm, then the limit function is likewise vectorially (equi) Weyl-almost periodic.

It is not so simple to deduce a proper analogue of Proposition 2.1 for vectorially Weyl almost periodic type functions. After clarifying this fact, we will generalize the notion introduced in Definition 3.1 as follows (cf. also [16, Definition 3.1.1–Definition 3.1.6]).

Definition 3.2. Suppose that $F : \Lambda \times X \rightarrow Y$, ρ is a binary relation on Y and the following condition holds.

(WM1-1v): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is Lebesgue measurable, $m(\Omega) > 0$, $\Lambda' + \Lambda \subseteq \Lambda$ and $\Lambda + l\Omega \subseteq \Lambda$ for $l > 0$.

(i) It is said that $F(\cdot; \cdot)$ is vectorially equi-Weyl- $(\mathcal{B}, \Lambda', \rho, \Omega)$ -almost periodic if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l' > 0$ and $L' > 0$ such that for each $\mathbf{w}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ there exists an integrable mapping $y_{\mathbf{t};x} : \mathbf{t} + l'\Omega \rightarrow Y$ such that $y_{\mathbf{t};x}(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$ for a.e. $\mathbf{u} \in \mathbf{t} + l'\Omega$ and

$$(3.9) \quad \sup_{\mathbf{t} \in \Lambda; x \in B} \left\| \int_{\mathbf{t} + l'\Omega} [F(\tau + \mathbf{u}; x) - y_{\mathbf{t};x}(\mathbf{u})] d\mathbf{u} \right\| < \epsilon(l')^n.$$

(ii) It is said that $F(\cdot; \cdot)$ is vectorially Weyl- $(\mathcal{B}, \Lambda', \rho, \Omega)$ -almost periodic if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L' > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ there exists an integrable mapping $y_{\mathbf{t};x} : \mathbf{t} + l'\Omega \rightarrow Y$ such that $y_{\mathbf{t};x}(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$ for a.e. $\mathbf{u} \in \mathbf{t} + l'\Omega$ and

$$\limsup_{l' \rightarrow +\infty} \sup_{\mathbf{t} \in \Lambda; x \in B} \left[(l')^{-n} \left\| \int_{\mathbf{t} + l'\Omega} [F(\tau + \mathbf{u}; x) - y_{\mathbf{t};x}(\mathbf{u})] d\mathbf{u} \right\| \right] < \epsilon.$$

We have the following result.

Proposition 3.1. (cf. also [16, pp. 23–24]) Suppose that $m \in \mathbb{N}$, $F : \Lambda \times X \rightarrow Y$, $\rho = T \in L(Y)$ and (WM1-1v) holds. If $F(\cdot; \cdot)$ is vectorially (equi)-Weyl- $(\mathcal{B}, \Lambda', T, \Omega)$ -almost periodic, then $F(\cdot; \cdot)$ is vectorially (equi)-Weyl- $(\mathcal{B}, m\Lambda', T^m, \Omega)$ -almost periodic, as well.

Proof. We examine the class of vectorially equi-Weyl- $(\mathcal{B}, \Lambda', T, \Omega)$ -almost periodic functions, only. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed. Then there exist two finite real

numbers $l' > 0$ and $L' > 0$ such that for each $\mathbf{w}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ we have that (3.9) holds with $y_{\mathbf{t};x} = TF(\mathbf{u}; x)$ for a.e. $\mathbf{u} \in \mathbf{t} + l'\Omega$. Let $\mathbf{t} \in \Lambda$ and $x \in B$ be fixed, and let $\mathbf{w}_0 \in \Lambda'$ and $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ be as above. Then, it is clear that $s\Lambda' + \Lambda \subseteq \Lambda$, $s \in \mathbb{N}$ and

$$F(\mathbf{u} + m\tau; x) - T^m F(\mathbf{u}; x) = \sum_{j=0}^{m-1} T^j \left[F(\mathbf{u} + (m-j)\tau; x) - TF(\mathbf{u} + (m-j-1)\tau; x) \right],$$

for any $\mathbf{u} \in \mathbf{t} + l'\Omega$. This implies

$$\begin{aligned} & \left\| \int_{\mathbf{t}+l'\Omega} [F(\mathbf{u} + m\tau; x) - T^m F(\mathbf{u}; x)] d\mathbf{u} \right\| \\ & \times \sum_{j=0}^{m-1} \|T\|^j \cdot \left\| \int_{\mathbf{t}+l'\Omega} [F(\mathbf{u} + (m-j)\tau; x) - T^m F(\mathbf{u} + (m-j-1)\tau; x)] d\mathbf{u} \right\| \\ & = \sum_{j=0}^{m-1} \|T\|^j \cdot \left\| \int_{\mathbf{t}+(m-j-1)\tau+l'\Omega} [F(\mathbf{u} + \tau; x) - T^m F(\mathbf{u}; x)] d\mathbf{u} \right\| \leq \sum_{j=0}^{m-1} \|T\|^j \epsilon. \end{aligned}$$

The required conclusion follows from this estimate. $\square$

We close this section by observing that Proposition 3.1 can be also formulated for the corresponding classes of vectorially Stepanov almost periodic functions.

4. INVARIANCE OF VECTORIAL STEPANOV ALMOST PERIODICITY AND VECTORIAL WEYL ALMOST PERIODICITY UNDER THE ACTIONS OF CONVOLUTION PRODUCTS

We open this section by clarifying the following result.

Proposition 4.1. *Assume $\nu : \mathbb{R} \rightarrow (0, +\infty)$, $1/\nu(\cdot)$ is locally bounded and there exists a Lebesgue measurable function $\psi : \mathbb{R} \rightarrow (0, +\infty)$ such that $\nu(x+y) \leq \psi(x)\nu(y)$ for all $x, y \in \mathbb{R}$. Suppose further that $f : \mathbb{R} \rightarrow X$ is a bounded, vectorially $(\Lambda', T, \mathcal{P})$ -almost periodic function, resp. a bounded, vectorially $(\Lambda', T, \mathcal{P})$ -uniformly recurrent function, where $\rho = T \in L(X)$ and $P = C_{b,\nu}(\mathbb{R} : X)$. If $(R(t))_{t>0}$ is any strongly continuous operator family in $L(X)$ such that $R(t)T = TR(t)$ for all $t > 0$ and $\int_0^{+\infty} \|R(r)\| \cdot (1 + \psi(r)) dr < +\infty$, then the function $F : \mathbb{R} \rightarrow X$, given by*

$$(4.1) \quad F(t) := \int_{-\infty}^t R(t-s)f(s) ds, \quad t \in \mathbb{R},$$

is bounded, continuous and vectorially $(\Lambda', T, \mathcal{P})$ -almost periodic, resp. bounded, continuous and vectorially $(\Lambda', T, \mathcal{P})$ -uniformly recurrent.

Proof. Suppose that $\epsilon > 0$. Then there exists $l' > 0$ such that, for every $w_0 \in \Lambda'$ there exists $\tau \in [w_0 - l', w_0 + l'] \cap \Lambda'$ such that

$$(4.2) \quad \left\| \left[\int_0^1 f(t + \tau + s) ds - T \int_0^1 f(t + s) ds \right] \cdot \nu(t) \right\| \leq \epsilon, \quad t \in \mathbb{R}.$$

Clearly, $F(\cdot)$ is well-defined, bounded and continuous and $F(t) = \int_0^{+\infty} R(r)f(t-r) dr$ for all $t \in \mathbb{R}$. Furthermore, we can use (4.2), the Fubini theorem and the prescribed assumptions to show that:

$$\begin{aligned}
& \left\| \left[\int_0^1 F(t+\tau+s) ds - T \int_0^1 F(t+s) ds \right] \cdot \nu(t) \right\| \\
&= \left\| \left[\int_0^1 \int_0^{+\infty} R(r)f(t+\tau+s-r) dr ds - T \int_0^1 \int_0^{+\infty} R(r)f(t+s-r) dr ds \right] \cdot \nu(t) \right\| \\
&= \left\| \left[\int_0^1 \int_0^{+\infty} R(r)[f(t+\tau+s-r) dr ds - Tf(t+s-r)] dr ds \right] \cdot \nu(t) \right\| \\
&= \left\| \left[\int_0^{+\infty} R(r) \left[\int_0^1 [f(t+\tau+s-r) dr ds - Tf(t+s-r)] ds \right] dr \right] \cdot \nu(t) \right\| \\
&\leq \int_0^{+\infty} \|R(r)\| \cdot \left\| \int_0^1 [f(t+\tau+s-r) dr ds - Tf(t+s-r)] ds \right\| dr \cdot \nu(t) \\
&\leq \int_0^{+\infty} \|R(r)\| \cdot \frac{\epsilon}{\nu(t-r)} \cdot \nu(t) dr \leq \epsilon \int_0^{+\infty} \|R(r)\| \psi(r) dr.
\end{aligned}$$

This simply implies the desired conclusion. $\square$

The statement of [14, Proposition 2.6.11] cannot be reconsidered for the vectorial Stepanov almost periodicity. Concerning the vectorial Weyl almost periodicity, we will clarify the following result; cf. also the proof of [14, Proposition 2.11.1 (i)].

Proposition 4.2. *Suppose that $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that $\int_0^{+\infty} \|R(s)\| ds < +\infty$, $T \in L(X)$ and $R(t)T = TR(t)$ for all $t > 0$. If $f : \mathbb{R} \rightarrow X$ is bounded and vectorially (equi-) Weyl- $(\Lambda', T, \mathcal{P})$ -almost periodic, where $P = C_b(\mathbb{R} : X)$, then the function $F(\cdot)$, given by (4.1), is bounded, continuous and vectorially (equi-) Weyl- $(\Lambda', T, \mathcal{P})$ -almost periodic.*

The statement of [14, Theorem 2.11.4] cannot be reconsidered for the vectorial Weyl almost periodicity, unfortunately. Let us finally note that we can employ different pivot spaces X and Y in the formulations of Proposition 4.1 and Proposition 4.2, provided that $T = cI$, where $c \in \mathbb{C} \setminus \{0\}$.

5. CONCLUSIONS AND FINAL REMARKS

In this research article, we have introduced and analyzed several new classes of vectorial Stepanov- p -almost periodic type functions and vectorial (equi-)Weyl- p -almost periodic type functions, where $p > 0$. In contrast with the usual research studies of generalized almost periodic functions, we have used the vector-valued integration in our approach. We have also analyzed Σ -almost periodic type functions and the invariance of vectorial Stepanov almost periodicity and vectorial Weyl almost periodicity under the actions of convolution products.

The notion introduced in Definition 2.4 (ii) can be extended in the following way (the metrical generalizations of this definition can be also introduced; see [16, Section 4.2] for more details).

Definition 5.1. Suppose that (SM1-1v) holds, $p > 0$, ρ is a binary relation on Y , $F : \Lambda \times X \rightarrow Y$ and $R(F) \subseteq D(\rho)$. The function $F(\cdot; \cdot)$ is said to be vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, p)$ -almost periodic if for each $x \in X$ the mapping $F(\cdot; x)$ is p -locally integrable as well as, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $l' > 0$ such that for each $\mathbf{w}_0 \in \Lambda$ there exists a point $\tau \in B(\mathbf{w}_0, l') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ there exists a p -locally integrable function $y_{\mathbf{t};x}(\cdot)$ on Ω such that $y_{\mathbf{t};x}(\mathbf{u}) \in \rho(F(\mathbf{t} + \mathbf{u}; x))$ for a.e. $\mathbf{u} \in \Omega$ and

$$\left| \int_{\Omega} [F(\mathbf{t} + \tau + \mathbf{u}; x) - y_{\mathbf{t};x}(\mathbf{u})]^p d\mathbf{u} \right| \leq \epsilon^p, \quad \mathbf{t} \in \Lambda, x \in B.$$

The notion of a vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, p)$ -uniformly recurrent function can be similarly introduced in the scalar-valued setting. Let us finally note that we can also consider the class of scalar-valued vectorially (equi-)Weyl- p -almost periodic functions and the class of scalar-valued vectorially (equi-)Weyl- $(\mathcal{B}, \Lambda', \rho, p)$ -almost periodic functions following our approaches from Definition 3.1 and Definition 3.2 as well as many other classes of generalized vectorially almost periodic functions. More details will appear somewhere else.

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BI-UNIVALENT FUNCTIONS CLASSES DEFINED BY POISSON DISTRIBUTION SERIES

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ABSTRACT. By using Poisson distribution series we introduce and derive different subclasses of regular and bi-univalent functions in the open unit disk. We then present special estimates for the Taylor coefficient inequalities $|a_2|$ and $|a_3|$ of functions belonging to this new subclass. In addition, several consequences of our results are pointed out which are new and have not yet been discussed in association with bounded boundary rotation.

1. INTRODUCTION

By ρ we denote the class of functions $\text{Im}(\varsigma)$ in the open unit disk $\mathcal{O} = \{\varsigma : |\varsigma| < 1\}$ and normalized by the conditions $\text{Im}(0) = 0$ and $\text{Im}'(0) = 1$. Each $\text{Im} \in \rho$ has the form

$$(1.1) \quad \text{Im}(\varsigma) = \varsigma + \sum_{r=2}^{+\infty} x_r \varsigma^r, \quad \varsigma \in \mathcal{O}.$$

Indicate by $\mathcal{S}$ the class of all univalent functions in $\mathcal{O}$.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{J}$ include the class $\mathcal{J}^*(\tau)$ of starlike functions of order τ in $\mathcal{O}$ and the class $k(\tau)$ of convex functions of order τ , $0 \leq \tau < 1$, in $\mathcal{O}$.

It is well known that every function $\text{Im} \in \mathcal{J}$ has an inverse Im^{-1} defined by

$$\text{Im}^{-1}(\text{Im}(\varsigma)) = \varsigma, \quad \varsigma \in \mathcal{O},$$

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and

$$\operatorname{Im}^{-1}(\operatorname{Im}(\varrho)) = \varrho, \quad |\varrho| < n_0(\operatorname{Im}), \quad n_0(\operatorname{Im}) \geq 1/4,$$

where

$$(1.2) \quad \operatorname{Im}^{-1}(\varrho) = \mathcal{G}(\varrho) = \varrho - a_2\varrho^2 + (2x_2^2 - x_3)\varrho^3 - (5x_2^3 - 5x_2x_3 + x_4)\varrho^4 + \cdots.$$

A function $\operatorname{Im}(\varsigma) \in \rho$ is said to be bi-univalent in $\mathcal{O}$ if both $\operatorname{Im}(\varsigma)$ and $\operatorname{Im}^{-1}(\varsigma)$ are univalent in $\mathcal{O}$. Indicated by $\mathcal{H}$, the class of bi-univalent functions in $\mathcal{O}$ is given by (1.1).

Brannan and Taha [1] present bi-starlike functions of order τ , $0 \leq \tau < 1$, indicated by $\mathcal{J}_{\mathcal{H}}^*(\tau)$ and bi-convex functions of order τ indicated by $\mathcal{K}_{\mathcal{H}}(\tau)$.

Srivastava et al. [14], Xu et al. [16, 17], Frasin and Aouf [3] and Hayami and Owa [5] defined more interesting examples of bi-univalent function in the class $\mathcal{H}$ (also [2, 3, 15] and [17]).

A regular function $\operatorname{Im}(\varsigma)$ is subordinate to an analytic function $\mathcal{G}(\varsigma)$, written $\operatorname{Im}(\varsigma) \prec \mathcal{G}(\varsigma)$, provided there is an analytic function ϱ defined on $\mathcal{O}$ with $\varrho(0) = 0$ and $|\varrho(\varsigma)| < 1$ satisfying $\operatorname{Im}(\varsigma) = \mathcal{G}(\varrho(\varsigma))$. Ma and Minda [6] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{\varsigma \operatorname{Im}'(\varsigma)}{\operatorname{Im}(\varsigma)}$ or $1 + \frac{\varsigma \operatorname{Im}''(\varsigma)}{\operatorname{Im}'(\varsigma)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\mathcal{O}$, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps $\mathcal{O}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike-functions consists of functions $\operatorname{Im}(\varsigma) \in \rho$ satisfying the subordination $\frac{\varsigma \operatorname{Im}'(\varsigma)}{\operatorname{Im}(\varsigma)} \prec \phi(\varsigma)$. Similarly, the class of Ma-Minda convex functions of functions $\operatorname{Im}(\varsigma) \in \rho$ satisfying the subordination $1 + \frac{\varsigma \operatorname{Im}''(\varsigma)}{\operatorname{Im}'(\varsigma)} \prec \phi(\varsigma)$.

A function $\operatorname{Im}(\varsigma)$ is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both $\operatorname{Im}(\varsigma)$ and $\operatorname{Im}^{-1}(\varsigma)$ are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{J}_{\mathcal{H}}^*(\phi)$ and $\mathcal{K}_{\mathcal{H}}(\phi)$. In the sequel, it is assumed that is an analytic function with positive real part in the unit disk $\mathcal{O}$, satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(\mathcal{O})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$(1.3) \quad \phi(\varsigma) = 1 + V_1\varsigma + V_2\varsigma^2 + V_3\varsigma^3 + \cdots, \quad V_1 > 0.$$

A parameter σ is called Poisson distributed taken the values $0, 1, 2, 3, \dots$, with $e^{-\mu}$, $\mu \frac{e^{-\mu}}{1!}$, $\mu^2 \frac{e^{-\mu}}{2!}$, $\mu^3 \frac{e^{-\mu}}{3!}$, $\dots$, respectively, probabilities, where μ is called the parameter. Thus,

$$(1.4) \quad \mathcal{P}(\sigma = n) = \mu^n \frac{e^{-\mu}}{n!}, \quad n = 0, 1, 2, 3, \dots$$

Lately, in [10] ([7, 8]) Porwal presented

$$(1.5) \quad \mathcal{K}(\mu, \varsigma) = \varsigma + \sum_{r=2}^{+\infty} \frac{\mu^{r-1}}{(r-1)!} e^{-\mu} \varsigma^r, \quad \varsigma \in \mathcal{O},$$

the coefficients of this power series are probabilities of Poisson distribution, where $\mu > 0$.

In [10], Porwal also defined series

$$(1.6) \quad \mathfrak{g}(\mu, \varsigma) = 2\varsigma - \mathcal{K}(\mu, \varsigma) = \varsigma - \sum_{r=2}^{+\infty} \frac{\mu^{r-1}}{(r-1)!} e^{-\mu} \varsigma^r, \quad \varsigma \in \mathcal{O}.$$

Using the convolution, in [11] Porwal and Kumar introduced a linear operator $\varpi_\mu : \chi \rightarrow \chi$ defined by

$$(1.7) \quad \varpi_\mu \text{Im}(\varsigma) = \mathcal{K}(\mu, \varsigma) * \text{Im}(\varsigma) = \varsigma + \sum_{r=2}^{+\infty} \frac{\mu^{r-1}}{(r-1)!} e^{-\mu} x_r \varsigma^r, \quad \varsigma \in \mathcal{O}.$$

In [4] Goodman defined uniformly convex $\mathcal{UCV}$ functions so that its regular characterization is $\phi \in \mathcal{UCV}$ if and only if

$$\text{Re} \left\{ 1 + \frac{\varsigma \phi''(\varsigma)}{\phi'(\varsigma)} \right\} \geq \text{Re} \frac{\zeta \phi''(\zeta)}{\phi'(\zeta)}, \quad (\varsigma, \zeta) \in \mathcal{O} \times \mathcal{O}.$$

Ronning [12] defined the class $\mathcal{S}_P$ consisting of functions $\psi(\varsigma) = \varsigma \phi'(\varsigma)$, so $\phi \in \mathcal{UCV}$. By choosing $\zeta = e^{i\gamma} \varsigma$ in suitable way, Rosy et al. [13] wrote the regular description of $\mathcal{S}_P$ as $\psi \in \mathcal{UCV}$ if and only if

$$\text{Re} \left\{ \frac{\varsigma \psi'(\varsigma)}{\psi(\varsigma)} + e^{i\gamma} \left(\frac{\varsigma \psi'(\varsigma)}{\psi(\varsigma)} - 1 \right) \right\} \geq 0,$$

or equivalently $\mathcal{S}_P$ is the subclass of ρ consisting of functions described in (1.1) satisfying

$$\text{Re} \left\{ (1 + e^{i\gamma}) \frac{\varsigma \text{Im}'(\varsigma)}{\text{Im}(\varsigma)} - e^{i\gamma} \right\} \geq 0.$$

2. PREPARATION

Definition 2.1. A function $\text{Im}(\varsigma) \in \mathcal{H}$ described in (1.1) is said to belong to class $\mathcal{S}_\lambda^\mu(\phi, \varsigma)$ if it satisfies

$$(2.1) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varsigma (\varpi_\mu \text{Im}(\varsigma))'}{\varpi_\mu \text{Im}(\varsigma)} + \lambda \left(1 + \frac{\varsigma (\varpi_\mu \text{Im}(\varsigma))''}{(\varpi_\mu \text{Im}(\varsigma))'} \right) \right\} - \beta e^{i\gamma} \prec \phi(\varsigma),$$

where $\beta \geq 0$, $0 \leq \lambda \leq 1$, $-\pi \leq \gamma < \pi$, $\mu > 0$ and $\varsigma \in \mathcal{O}$, and

$$(2.2) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varrho (\varpi_\mu \text{Im}(\varrho))'}{\varpi_\mu \text{Im}(\varrho)} + \lambda \left(1 + \frac{\varrho (\varpi_\mu \text{Im}(\varrho))''}{(\varpi_\mu \text{Im}(\varrho))'} \right) \right\} - \beta e^{i\gamma} \prec \phi(\varrho),$$

where $\beta \geq 0$, $0 \leq \lambda \leq 1$, $-\pi \leq \gamma < \pi$, $\mu > 0$ and $\varrho \in \mathcal{O}$.

Definition 2.2. A function $\text{Im}(\varsigma) \in \mathcal{H}$, as described in (1.1), is said to belong to the class $\mathcal{K}_\lambda^\mu(\phi, \varsigma)$ if it satisfies

$$(2.3) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varpi_\mu \text{Im}(\varsigma)}{\varsigma} + \lambda (\varpi_\mu \text{Im}(\varsigma))' \right\} - \beta e^{i\gamma} \prec \phi(\varsigma),$$

where $\beta \geq 0$, $0 \leq \lambda \leq 1$, $-\pi \leq \gamma < \pi$, $\mu > 0$ and $\varsigma \in \mathcal{O}$, and

$$(2.4) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varpi_\mu \text{Im}(\varrho)}{\varrho} + \lambda (\varpi_\mu \text{Im}(\varrho))' \right\} - \beta e^{i\gamma} \prec \phi(\varrho),$$

where $\beta \geq 0$, $0 \leq \lambda \leq 1$, $-\pi \leq \gamma < \pi$, $\mu > 0$ and $\varrho \in \mathcal{O}$.

By specializing the parameter λ in Definitions 2.1 and 2.2, we obtain the following subfamilies.

Remark 2.1. A function $\text{Im}(\varsigma) \in \mathcal{H}$ described in (1.1) and for $\lambda = 0$, we note that $\mathcal{S}_\lambda^\mu(\phi, \varsigma) \equiv \mathcal{S}^\mu(\phi, \varsigma)$ in Definition 2.1, satisfies the following conditions

$$(1 + \beta e^{i\gamma}) \left\{ \frac{\varsigma (\varpi_\mu \text{Im}(\varsigma))'}{\varpi_\mu \text{Im}(\varsigma)} \right\} - \beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1 + \beta e^{i\gamma}) \left\{ \frac{\varrho (\varpi_\mu \text{Im}(\varrho))'}{\varpi_\mu \text{Im}(\varrho)} \right\} - \beta e^{i\gamma} \prec \phi(\varrho).$$

Remark 2.2. A function $\text{Im}(\varsigma) \in \mathcal{H}$ given by (1.1) and for $\lambda = 1$, we note that $\mathcal{S}_\lambda^\mu(\phi, \varsigma) \equiv \mathcal{S}_1^\mu(\phi, \varsigma)$ in Definition 2.1, satisfies the following conditions

$$(1 + \beta e^{i\gamma}) \left[1 + \frac{\varsigma (\varpi_\mu \text{Im}(\varsigma))''}{(\varpi_\mu \text{Im}(\varsigma))'} \right] - \beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1 + \beta e^{i\gamma}) \left[1 + \frac{\varrho (\varpi_\mu \text{Im}(\varrho))''}{(\varpi_\mu \text{Im}(\varrho))'} \right] - \beta e^{i\gamma} \prec \phi(\varrho).$$

Remark 2.3. A function $\text{Im}(\varsigma) \in \mathcal{H}$ given by (1.1) and for $\lambda = 0$, we note that $\mathcal{K}_\lambda^\mu(\phi, \varsigma) \equiv \mathcal{K}^\mu(\phi, \varsigma)$ in Definition 2.2, satisfies the following conditions

$$(1 + \beta e^{i\gamma}) \left\{ \frac{\varpi_\mu \text{Im}(\varsigma)}{\varsigma} \right\} - \beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1 + \beta e^{i\gamma}) \left\{ \frac{\varpi_\mu \text{Im}(\varrho)}{\varrho} \right\} - \beta e^{i\gamma} \prec \phi(\varrho).$$

Remark 2.4. A function $\text{Im}(\varsigma) \in \mathcal{H}$ given by (1.1) and for $\lambda = 1$, we note that $\mathcal{K}_\lambda^\mu(\phi, \varsigma) \equiv \mathcal{K}_1^\mu(\phi, \varsigma)$ in Definition 2.2, satisfies the following conditions

$$(1 + \beta e^{i\gamma}) (\varpi_\mu \text{Im}(\varsigma))' - \beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1 + \beta e^{i\gamma}) (\varpi_\mu \text{Im}(\varrho))' - \beta e^{i\gamma} \prec \phi(\varrho).$$

3. COEFFICIENT ESTIMATES FOR $\mathcal{S}_\lambda^\mu(\phi, \varsigma)$ AND $\mathcal{K}_\lambda^\mu(\phi, \varsigma)$

Before starting and proving our main results, we need the following lemma for deriving our main results.

Lemma 3.1 ([9]). *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where $\mathcal{P}$ is the family of all functions h analytic in $\mathcal{O}$ for which $\Re(h(\varsigma)) > 0$ and*

$$h(z) = 1 + c_1\varsigma + c_2\varsigma^2 + \cdots, \quad \text{for } \varsigma \in \mathcal{O}.$$

Define the functions $\mathfrak{s}(\varsigma)$ and $\mathfrak{t}(\varsigma)$ by

$$\mathfrak{s}(\varsigma) = \frac{1 + u(\varsigma)}{1 - u(\varsigma)} = 1 + \mathfrak{s}_1\varsigma + \mathfrak{s}_2\varsigma^2 + \cdots$$

and

$$\mathfrak{t}(\varsigma) = \frac{1 + v(\varsigma)}{1 - v(\varsigma)} = 1 + \mathfrak{t}_1\varsigma + \mathfrak{t}_2\varsigma^2 + \cdots.$$

It follows that

$$u(\varsigma) = \frac{\mathfrak{s}(\varsigma) - 1}{\mathfrak{s}(\varsigma) + 1} = \frac{1}{2} \left[\mathfrak{s}_1\varsigma + \left(\mathfrak{s}_2 - \frac{\mathfrak{s}_1^2}{2} \right) \varsigma^2 + \cdots \right]$$

and

$$v(\varsigma) = \frac{\mathfrak{t}(\varsigma) - 1}{\mathfrak{t}(\varsigma) + 1} = \frac{1}{2} \left[\mathfrak{t}_1\varsigma + \left(\mathfrak{t}_2 - \frac{\mathfrak{t}_1^2}{2} \right) \varsigma^2 + \cdots \right].$$

Then, $\mathfrak{s}(\varsigma)$ and $\mathfrak{t}(\varsigma)$ are analytic in $\mathcal{O}$ with $\mathfrak{s}(0) = \mathfrak{t}(0) = 1$.

Since $u, v : \mathcal{O} \rightarrow \mathcal{O}$, the functions $\mathfrak{s}(\varsigma)$ and $\mathfrak{t}(\varsigma)$ have a positive real part in $\mathcal{O}$, and $|\mathfrak{s}_i| \leq 2$ and $|\mathfrak{t}_i| \leq 2$ for each i .

Now, we will obtain the coefficient estimates for the class $\mathcal{S}_\lambda^\mu(\phi, \varsigma)$ in Theorem 3.1 and the class $\mathcal{K}_\lambda^\mu(\phi, \varsigma)$ in Theorem 3.2.

Theorem 3.1. *If $\text{Im}(\varsigma)$ belongs to $\mathcal{S}_\lambda^\mu(\phi, \varsigma)$ and has the series representation described in (1.1), then*

$$(3.1) \quad |x_2| \leq \frac{V_1 \sqrt{V_1}}{\mu \sqrt{(1 + \beta e^{i\gamma}) e^{-\mu} \{ [(1 + 2\lambda) - (1 + 3\lambda)e^{-\mu}] V_1^2 + (V_1 - V_2)(1 + \beta e^{i\gamma})(1 + \lambda)^2 e^{-\mu} \}}}$$

and

$$(3.2) \quad |x_3| \leq \frac{V_1}{(1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu}} + \frac{V_1^2}{(1 + \beta e^{i\gamma})^2(1 + \lambda)^2 \mu^2 e^{-2\mu}}.$$

Proof. From (2.1) and (2.2) we can get

$$(3.3) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varsigma(\varpi_\mu \text{Im}(\varsigma))'}{\varpi_\mu \text{Im}(\varsigma)} + \lambda \left[1 + \frac{\varsigma(\varpi_\mu \text{Im}(\varsigma))''}{(\varpi_\mu \text{Im}(\varsigma))'} \right] \right\} - \beta e^{i\gamma} = \phi(u(\varsigma))$$

and

$$(3.4) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varrho(\varpi_\mu \operatorname{Im}(\varrho))'}{\varpi_\mu \operatorname{Im}(\varrho)} + \lambda \left[1 + \frac{\varrho(\varpi_\mu \operatorname{Im}(\varrho))''}{(\varpi_\mu \operatorname{Im}(\varrho))'} \right] \right\} - \beta e^{i\gamma} = \phi(v(\varrho)),$$

where $\mathfrak{s}(\varsigma), \mathfrak{t}(\varsigma) \in \mathcal{P}$ and

$$\phi(u(\varsigma)) = \phi \left(\frac{1}{2} \left[\mathfrak{s}_1 \varsigma + \left(\mathfrak{s}_2 - \frac{\mathfrak{s}_1^2}{2} \right) \varsigma^2 + \cdots \right] \right)$$

and

$$\phi(v(\varrho)) = \phi \left(\frac{1}{2} \left[\mathfrak{t}_1 \varrho + \left(\mathfrak{t}_2 - \frac{\mathfrak{t}_1^2}{2} \right) \varrho^2 + \cdots \right] \right).$$

Now, from (3.3) and (3.4), we get

$$(3.5) \quad (1 + \beta e^{i\gamma})(1 + \lambda)\mu e^{-\mu} x_2 = \frac{1}{2} V_1 \mathfrak{s}_1,$$

$$(3.6) \quad \begin{aligned} & (1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu} x_3 - (1 + \beta e^{i\gamma})(1 + 3\lambda)\mu^2 e^{-2\mu} x_2^2 \\ &= \frac{1}{2} V_1 \left(\mathfrak{s}_2 - \frac{\mathfrak{s}_1^2}{2} \right) + \frac{1}{4} V_2^2 \mathfrak{s}_1, \end{aligned}$$

$$(3.7) \quad - (1 + \beta e^{i\gamma})(1 + \lambda)\mu e^{-\mu} x_2 = \frac{1}{2} V_1 \mathfrak{t}_1$$

and

$$(3.8) \quad \begin{aligned} & (1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu} (2x_2^2 - x_3) - (1 + \beta e^{i\gamma})(1 + 3\lambda)\mu^2 e^{-2\mu} x_2^2 \\ &= \frac{1}{2} V_1 \left(\mathfrak{t}_2 - \frac{\mathfrak{t}_1^2}{2} \right) + \frac{1}{4} V_2^2 \mathfrak{t}_1. \end{aligned}$$

From (3.5) and (3.7), we get

$$(3.9) \quad \mathfrak{s}_1 = -\mathfrak{t}_1$$

and

$$(3.10) \quad 8(1 + \beta e^{i\gamma})^2 (1 + \lambda)^2 \mu^2 e^{-2\mu} x_2^2 = V_1^2 (\mathfrak{s}_1^2 + \mathfrak{t}_1^2).$$

Now, from (3.6), (3.8) and (3.10), we obtain

$$(3.11) \quad \begin{aligned} & 4(1 + \beta e^{i\gamma})\mu^2 e^{-\mu} \left\{ \left[(1 + 2\lambda) - (1 + 3\lambda)e^{-\mu} \right] V_1^2 \right. \\ & \left. + (V_1 - V_2)(1 + \beta e^{i\gamma})(1 + \lambda)^2 e^{-\mu} \right\} x_2^2 = V_1^3 (\mathfrak{s}_2 + \mathfrak{t}_2). \end{aligned}$$

By using Lemma 3.1 to the coefficients $\mathfrak{s}_2$ and $\mathfrak{t}_2$, we have

$$|x_2| \leq \frac{V_1 \sqrt{V_1}}{\mu \sqrt{(1 + \beta e^{i\gamma})e^{-\mu} \{ [(1 + 2\lambda) - (1 + 3\lambda)e^{-\mu}] V_1^2 + (V_1 - V_2)(1 + \beta e^{i\gamma})(1 + \lambda)^2 e^{-\mu} \}}}.$$

To find $|x_3|$, we subtract (3.6) from (3.8) and using (3.9), we have

$$2(1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu} (x_3 - x_2^2) = \frac{1}{2} V_1 (\mathfrak{s}_2 - \mathfrak{t}_2).$$

Upon substituting the value of x_2^2 from (3.10), we get

$$x_3 = \frac{V_1(\mathfrak{s}_2 - \mathfrak{t}_2)}{4(1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu}} + \frac{V_1^2(\mathfrak{s}_1^2 + \mathfrak{t}_1^2)}{8(1 + \beta e^{i\gamma})^2(1 + \lambda)^2\mu^2 e^{-2\mu}}.$$

By using Lemma 3.1 once again to the coefficients $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{t}_1$ and $\mathfrak{t}_2$, we get

$$|x_3| \leq \frac{V_1}{(1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu}} + \frac{V_1^2}{(1 + \beta e^{i\gamma})^2(1 + \lambda)^2\mu^2 e^{-2\mu}},$$

we obtain the estimate given by (3.2). $\square$

Theorem 3.2. *If $\text{Im}(\varsigma)$ belongs to $\mathcal{K}_\lambda^\mu(\phi, \varsigma)$ and has the series representation described in (1.1), then*

$$(3.12) \quad |x_2| \leq \frac{V_1 \sqrt{V_1}}{\sqrt{2\mu(1 + \beta e^{i\gamma})e^{-\mu} \{ (2\lambda + 1)V_1^2 + 2\mu(V_1 - V_2)(1 + \beta e^{i\gamma})(\lambda + 1)^2 e^{-\mu} \}}}$$

and

$$(3.13) \quad |x_3| \leq \frac{2V_1}{\mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu}} + \frac{V_1^2}{\mu^2(1 + \beta e^{i\gamma})^2(\lambda + 1)^2 e^{-2\mu}}.$$

Proof. From (2.3) and (2.4), we have

$$(3.14) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varpi_\mu \text{Im}(\varsigma)}{\varsigma} + \lambda \varpi_\mu \text{Im}(\varsigma)' \right\} - \beta e^{i\gamma} = \phi(u(\varsigma))$$

and

$$(3.15) \quad (1 + \beta e^{i\gamma}) \left\{ (1 - \lambda) \frac{\varpi_\mu \text{Im}(\varrho)}{\varrho} + \lambda (\varpi_\mu \text{Im}(\varrho))' \right\} - \beta e^{i\gamma} = \phi(v(\varrho)),$$

where $\mathfrak{s}(\varsigma), \mathfrak{t}(\varsigma) \in \mathcal{P}$ and

$$\phi(u(\varsigma)) = \phi \left(\frac{1}{2} \left[\mathfrak{s}_1 \varsigma + \left(\mathfrak{s}_2 - \frac{\mathfrak{s}_1^2}{2} \right) \varsigma^2 + \cdots \right] \right)$$

and

$$\phi(v(\varrho)) = \phi \left(\frac{1}{2} \left[\mathfrak{t}_1 \varrho + \left(\mathfrak{t}_2 - \frac{\mathfrak{t}_1^2}{2} \right) \varrho^2 + \cdots \right] \right).$$

From (3.14) and (3.15), we get

$$(3.16) \quad (1 + \beta e^{i\gamma})(\lambda + 1)\mu e^{-\mu} x_2 = \frac{1}{2} V_1 \mathfrak{s}_1,$$

$$(3.17) \quad \mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu} x_3 = V_1 \mathfrak{s}_2 + \left(\frac{V_2 - V_1}{2} \right) \mathfrak{s}_1^2,$$

$$(3.18) \quad - (1 + \beta e^{i\gamma})(\lambda + 1)\mu e^{-\mu} x_2 = \frac{1}{2} V_1 \mathfrak{t}_1$$

and

$$(3.19) \quad \mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu}(2x_2^2 - x_3) = V_1 \mathfrak{t}_2 + \left(\frac{V_2 - V_1}{2} \right) \mathfrak{t}_1^2.$$

From (3.16) and (3.18), we get

$$(3.20) \quad \mathfrak{s}_1 = -\mathfrak{t}_1$$

and

$$(3.21) \quad 8(1 + \beta e^{i\gamma})^2(\lambda + 1)^2 \mu^2 e^{-2\mu} x_2^2 = V_1^2(\mathfrak{s}_1^2 + \mathfrak{t}_1^2).$$

Now, from (3.17), (3.19) and (3.11), we obtain

$$(3.22) \quad \begin{aligned} & 2(1 + \beta e^{i\gamma})\mu e^{-\mu} \left\{ (2\lambda + 1)V_1^2 + 2\mu(V_1 - V_2)(1 + \beta e^{i\gamma})(\lambda + 1)^2 e^{-\mu} \right\} x_2^2 \\ & = V_1^3(\mathfrak{s}_2 + \mathfrak{t}_2). \end{aligned}$$

By using Lemma 3.1 to the coefficients $\mathfrak{s}_2$ and $\mathfrak{t}_2$, we have

$$|x_2| \leq \frac{V_1 \sqrt{V_1}}{\sqrt{2\mu(1 + \beta e^{i\gamma})e^{-\mu} \left\{ (2\lambda + 1)V_1^2 + 2\mu(V_1 - V_2)(1 + \beta e^{i\gamma})(\lambda + 1)^2 e^{-\mu} \right\}}}.$$

To find $|x_3|$, we subtract (3.17) from (3.19) and using (3.20), we have

$$2\mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu}(x_3 - x_2^2) = \frac{1}{2}V_1(\mathfrak{s}_2 - \mathfrak{t}_2).$$

Upon replacing the value of x_2^2 from (3.21), we get

$$x_3 = \frac{V_1(\mathfrak{s}_2 - \mathfrak{t}_2)}{2\mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu}} + \frac{V_1^2(\mathfrak{s}_1^2 + \mathfrak{t}_1^2)}{8\mu^2(1 + \beta e^{i\gamma})^2(\lambda + 1)^2 e^{-2\mu}}.$$

By using Lemma 3.1 once again to the coefficients $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{t}_1$ and $\mathfrak{t}_2$, we have

$$|x_3| \leq \frac{2V_1}{\mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu}} + \frac{V_1^2}{\mu^2(1 + \beta e^{i\gamma})^2(\lambda + 1)^2 e^{-2\mu}},$$

we obtain the estimate given by (3.13). $\square$

By specializing the parameter $\lambda = 0$ in Theorems 3.1 and 3.2, we get the coefficients for the functions in $\mathcal{S}^\mu(\phi, \varsigma)$ and $\mathcal{K}^\mu(\phi, \varsigma)$ defined in Remark 2.1 and Remark 2.3.

Corollary 3.1. *If $\text{Im}(\varsigma)$ belongs to the class $\mathcal{S}^\mu(\phi, \varsigma)$ and has the series representation described in (1.1), then*

$$|x_2| \leq \frac{V_1 \sqrt{V_1}}{\mu \sqrt{(1 + \beta e^{i\gamma})e^{-\mu} \left\{ (1 - e^{-\mu})V_1^2 + (V_1 - V_2)(1 + \beta e^{i\gamma})e^{-\mu} \right\}}}$$

and

$$|x_3| \leq \frac{V_1}{(1 + \beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{V_1^2}{(1 + \beta e^{i\gamma})^2 \mu^2 e^{-2\mu}}.$$

Corollary 3.2. *If $\text{Im}(\varsigma)$ belongs to the class $\mathcal{K}^\mu(\phi, \varsigma)$ and has the series representation described in (1.1), then*

$$|x_2| \leq \frac{V_1 \sqrt{V_1}}{\sqrt{2\mu(1 + \beta e^{i\gamma})e^{-\mu} \{V_1^2 + 2\mu(V_1 - V_2)(1 + \beta e^{i\gamma})e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{2V_1}{\mu(1 + \beta e^{i\gamma})e^{-\mu}} + \frac{V_1^2}{\mu^2(1 + \beta e^{i\gamma})^2 e^{-2\mu}}.$$

Putting $\lambda = 1$ in Theorems 3.1 and 3.2, the coefficient estimates for the functions in $\mathcal{S}_1^\mu(\phi, \varsigma)$ and $\mathcal{K}_1^\mu(\phi, \varsigma)$ was defined in Remark 2.2 and 2.4.

Corollary 3.3. *If $\text{Im}(\varsigma)$ belongs to the class $\mathcal{S}_1^\mu(\phi, \varsigma)$ and has the series representation described in (1.1), then*

$$|x_2| \leq \frac{V_1 \sqrt{V_1}}{\mu \sqrt{(1 + \beta e^{i\gamma})e^{-\mu} \{(3 - 4e^{-\mu})V_1^2 + 4(V_1 - V_2)(1 + \beta e^{i\gamma})e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{V_1}{3(1 + \beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{V_1^2}{4(1 + \beta e^{i\gamma})^2 \mu^2 e^{-2\mu}}.$$

Corollary 3.4. *Let $\text{Im}(\varsigma)$ given by (1.1) be in the class $\mathcal{K}_1^\mu(\phi, \varsigma)$. Then,*

$$|x_2| \leq \frac{V_1 \sqrt{V_1}}{\sqrt{2\mu(1 + \beta e^{i\gamma})e^{-\mu} \{3V_1^2 + 8\mu(V_1 - V_2)(1 + \beta e^{i\gamma})e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{2V_1}{3\mu(1 + \beta e^{i\gamma})e^{-\mu}} + \frac{V_1^2}{4\mu^2(1 + \beta e^{i\gamma})^2 e^{-2\mu}}.$$

4. COROLLARIES AND CONSEQUENCES

For the function ϕ , given by

$$(4.1) \quad \phi(\varsigma) = \left(\frac{1 + \varsigma}{1 - \varsigma} \right)^\alpha = 1 + 2\alpha\varsigma + 2\alpha^2\varsigma^2 + \cdots, \quad 0 < \alpha \leq 1,$$

we have $V_1 = 2\alpha$ and $V_2 = 2\alpha^2$.

On the other hand if we take

$$(4.2) \quad \phi(\varsigma) = \frac{1 + (1 - 2\zeta)\varsigma}{1 - \varsigma} = 1 + 2(1 - \zeta)\varsigma + 2(1 - \zeta)\varsigma^2 + \cdots, \quad 0 \leq \zeta < 1,$$

then $V_1 = V_2 = 2(1 - \zeta)$.

Corollary 4.1. *By using $\phi(\varsigma)$ of the form (4.1) for functions $\text{Im}(\varsigma) \in \mathcal{S}_\lambda^\mu(\phi, \varsigma)$ as given by Theorem 3.1, we state the following results:*

$$|x_2| \leq \frac{2\alpha}{\mu \sqrt{(1 + \beta e^{i\gamma})e^{-\mu} \{2\alpha [(1 + 2\lambda) - (1 + 3\lambda)e^{-\mu}] + (1 - \alpha)(1 + \beta e^{i\gamma})(1 + \lambda)^2 e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{2\alpha}{(1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu}} + \frac{4\alpha^2}{(1 + \beta e^{i\gamma})^2(1 + \lambda)^2 \mu^2 e^{-2\mu}}.$$

Corollary 4.2. *By choosing $\phi(\varsigma)$ of the form (4.1) for functions $\text{Im}(\varsigma) \in \mathcal{K}_\lambda^\mu(\phi, \varsigma)$ as given by Theorem 3.2, we state the following results:*

$$|x_2| \leq \frac{2\alpha}{\sqrt{2\mu(1 + \beta e^{i\gamma})e^{-\mu} \{2\alpha(2\lambda + 1) + 2(1 - \alpha)\mu(1 + \beta e^{i\gamma})(\lambda + 1)^2 e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{4\alpha}{\mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu}} + \frac{4\alpha^2}{\mu^2(1 + \beta e^{i\gamma})^2(\lambda + 1)^2 e^{-2\mu}}.$$

Remark 4.1. From Corollary 4.1, taken $\lambda = 0$, we can get

$$|x_2| \leq \frac{2\alpha}{\mu \sqrt{(1 + \beta e^{i\gamma})e^{-\mu} \{2\alpha(1 - e^{-\mu}) + (1 - \alpha)(1 + \beta e^{i\gamma})e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{2\alpha}{(1 + \beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{4\alpha^2}{(1 + \beta e^{i\gamma})^2 \mu^2 e^{-2\mu}}.$$

Remark 4.2. From Corollary 4.2, taken $\lambda = 0$, we obtain the following results:

$$|x_2| \leq \frac{2\alpha}{\sqrt{2\mu(1 + \beta e^{i\gamma})e^{-\mu} \{2\alpha + 2(1 - \alpha)\mu(1 + \beta e^{i\gamma})e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{4\alpha}{\mu(1 + \beta e^{i\gamma})e^{-\mu}} + \frac{4\alpha^2}{\mu^2(1 + \beta e^{i\gamma})^2 e^{-2\mu}}.$$

Remark 4.3. From Corollary 4.1, taken $\lambda = 1$, we obtain the following results:

$$|x_2| \leq \frac{2\alpha}{\mu \sqrt{(1 + \beta e^{i\gamma})e^{-\mu} \{2\alpha(3 - 4e^{-\mu}) + 4(1 - \alpha)(1 + \beta e^{i\gamma})e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{2\alpha}{3(1 + \beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{\alpha^2}{(1 + \beta e^{i\gamma})^2 \mu^2 e^{-2\mu}}.$$

Remark 4.4. From Corollary 4.2, taken $\lambda = 1$, we obtain the following results:

$$|x_2| \leq \frac{\alpha}{\sqrt{\mu(1 + \beta e^{i\gamma})e^{-\mu} \{3\alpha + 4\mu(1 - \alpha)(1 + \beta e^{i\gamma})e^{-\mu}\}}}$$

and

$$|x_3| \leq \frac{4\alpha}{3\mu(1 + \beta e^{i\gamma})e^{-\mu}} + \frac{\alpha^2}{\mu^2(1 + \beta e^{i\gamma})^2 e^{-2\mu}}.$$

Corollary 4.3. By using $\phi(\varsigma)$ of the form (4.2) for functions $\text{Im}(\varsigma) \in \mathcal{S}_\lambda^\mu(\phi, \varsigma)$ as given by Theorem 3.1, we acquire the results:

$$|x_2| \leq \frac{1}{\mu} \sqrt{\frac{2(1 - \zeta)}{(1 + \beta e^{i\gamma})e^{-\mu} [(1 + 2\lambda) - (1 + 3\lambda)e^{-\mu}]}}$$

and

$$|x_3| \leq \frac{2(1 - \zeta)}{(1 + \beta e^{i\gamma})(1 + 2\lambda)\mu^2 e^{-\mu}} + \frac{4(1 - \zeta)^2}{(1 + \beta e^{i\gamma})^2 (1 + \lambda)^2 \mu^2 e^{-2\mu}}.$$

Corollary 4.4. By using $\phi(\varsigma)$ of the form (4.2) for functions $\text{Im}(\varsigma) \in \mathcal{K}_\lambda^\mu(\phi, \varsigma)$ as given by Theorem 3.2, we acquire the following results:

$$|x_2| \leq \sqrt{\frac{2(1 - \zeta)}{2\mu(1 + \beta e^{i\gamma})e^{-\mu}(2\lambda + 1)}}$$

and

$$|x_3| \leq \frac{4(1 - \zeta)}{\mu(1 + \beta e^{i\gamma})(2\lambda + 1)e^{-\mu}} + \frac{4(1 - \zeta)^2}{\mu^2(1 + \beta e^{i\gamma})^2 (\lambda + 1)^2 e^{-2\mu}}.$$

Remark 4.5. From Corollary 4.3, taken $\lambda = 0$, we obtain the following results:

$$|x_2| \leq \frac{1}{\mu} \sqrt{\frac{2(1 - \zeta)}{(1 + \beta e^{i\gamma})e^{-\mu} (1 - e^{-\mu})}}$$

and

$$|x_3| \leq \frac{2(1 - \zeta)}{(1 + \beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{4(1 - \zeta)^2}{(1 + \beta e^{i\gamma})^2 \mu^2 e^{-2\mu}}.$$

Remark 4.6. From Corollary 4.4, taken $\lambda = 0$, we obtain the results:

$$|x_2| \leq \sqrt{\frac{2(1 - \zeta)}{2\mu^2(1 + \beta e^{i\gamma})e^{-\mu^2}}}$$

and

$$|x_3| \leq \frac{4(1 - \zeta)}{\mu(1 + \beta e^{i\gamma})e^{-\mu}} + \frac{4(1 - \zeta)^2}{\mu^2(1 + \beta e^{i\gamma})^2 e^{-2\mu}}.$$

Remark 4.7. From Corollary 4.3, taken $\lambda = 1$, we acquire the following results:

$$|x_2| \leq \frac{1}{\mu} \sqrt{\frac{2(1-\zeta)}{(1+\beta e^{i\gamma})e^{-\mu}(3-4e^{-\mu})}}$$

and

$$|x_3| \leq \frac{2(1-\zeta)}{3(1+\beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{(1-\zeta)^2}{(1+\beta e^{i\gamma})^2 \mu^2 e^{-2\mu}}.$$

Remark 4.8. From Corollary (4.4), taken $\lambda = 1$, we obtain the following results:

$$|x_2| \leq \sqrt{\frac{2(1-\zeta)}{6\mu(1+\beta e^{i\gamma})e^{-\mu}}}$$

and

$$|x_3| \leq \frac{4(1-\zeta)}{3\mu(1+\beta e^{i\gamma})e^{-\mu}} + \frac{(1-\zeta)^2}{\mu^2(1+\beta e^{i\gamma})^2 e^{-2\mu}}.$$

NOTATION

There are many applications of Poisson distribution such as a space-time spectral order since collocation method for the fourth-order nonlocal heat model arising in viscoelasticity, a high-order and efficient numerical technique for reactors, a robust error analysis of the OSC method for multi-term fourth-order sub-diffusion equation, an efficient ADI difference scheme for the nonlocal evolution problem in three-dimensional space.

CONCLUSION

We successfully used the Poisson distribution series to introduce and derive different subclasses of regular and bi-univalent functions in the open unit disk in part two. In part three we present special estimates for the Taylor coefficients' inequalities $|a_2|$ and $|a_3|$ of the functions belonging to the new two classes $\mathcal{S}_\lambda^\mu(\phi, \varsigma)$ and $\mathcal{K}_\lambda^\mu(\phi, \varsigma)$. Also, in part four we investigated certain corollaries and consequences of the results by choosing $V_{2081} = 23b1$, $V_{2082} = 23b1$, $V_{2081} = V_{2082} = 2(1-\zeta)$. The results presented in this paper have been beneficial supplement for the research of geometric function theory of complex analysis.

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A SURVEY ON STRONGLY REGULAR GRAPHS WITH $m_2 = qm_3$ AND $m_3 = qm_2$

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ABSTRACT. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j , and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j , where S_k denotes the neighborhood of the vertex k . Let $\lambda_1 = r$, λ_2 and λ_3 be the distinct eigenvalues of a connected strongly regular graph. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r , λ_2 and λ_3 , respectively. We here survey results related to the parameters n , r , τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, \dots, 12$.

1. INTRODUCTION

Let G be a simple graph of order n with vertex set $V(G) = \{1, 2, \dots, n\}$. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of its $(0,1)$ adjacency matrix A and is denoted by $\sigma(G)$. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph K_n) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j , and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j , where $S_k \subseteq V(G)$ denotes the neighborhood of the vertex k . We know that a regular connected graph G is strongly regular if and only if it has exactly three distinct eigenvalues [1] (see also [3]). Let $\lambda_1 = r$, λ_2 and λ_3 denote the distinct eigenvalues of a connected strongly regular graph G . Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r , λ_2 and λ_3 . Further, let $\bar{r} = (n - 1) - r$, $\bar{\lambda}_2 = -\lambda_3 - 1$ and $\bar{\lambda}_3 = -\lambda_2 - 1$ denote the distinct

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eigenvalues of the strongly regular graph $\overline{G}$, where $\overline{G}$ denotes the complement of G . Then $\bar{\tau} = n - 2r - 2 + \theta$ and $\bar{\theta} = n - 2r + \tau$, where $\bar{\tau} = \tau(\overline{G})$ and $\bar{\theta} = \theta(\overline{G})$.

Remark 1.1. (i) If G is a disconnected strongly regular graph of degree r , then $G = mK_{r+1}$, where mH denotes the m -fold union of the graph H .

(ii) G is a disconnected strongly regular graph if and only if $\theta = 0$.

Remark 1.2. (i) A strongly regular graph G of order $n = 4k + 1$ and degree $r = 2k$ with $\tau = k - 1$ and $\theta = k$ is called a conference graph.

(ii) A strongly regular graph is a conference graph if and only if $m_2 = m_3$.

(iii) If $m_2 \neq m_3$, then G is an integral graph.

Note. We say that a connected or disconnected graph G is integral if its spectrum $\sigma(G)$ consists only of integral values.

We have recently started to investigate strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, where q is a positive integer [4]. In the same work we have described the parameters n , r , τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, 4$. In particular, we have described in [5], [6] and [7] the parameters n , r , τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 5, 6, \dots, 12$. We here survey results related to the parameters of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, \dots, 12$, as follows.

Proposition 1.1 ([2]). *Let G be a connected or disconnected strongly regular graph of order n and degree r . Then,*

$$(1.1) \quad r^2 - (\tau - \theta + 1)r - (n - 1)\theta = 0.$$

Proposition 1.2 ([2]). *Let G be a connected strongly regular graph of order n and degree r . Then,*

$$(1.2) \quad 2r + (\tau - \theta)(m_2 + m_3) + \delta(m_2 - m_3) = 0,$$

where $\delta = \lambda_2 - \lambda_3$.

Remark 1.3 ([4]). Using the same procedure applied in [4] we can establish the parameters n , r , τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for any fixed value $q \in \mathbb{N}$, as follows. First, let $m_3 = p$, $m_2 = qp$ and $n = (q + 1)p + 1$, where $q \in \mathbb{N}$. Using (1.2) we obtain $r = p(|\lambda_3| - q\lambda_2)$. Let $|\lambda_3| - q\lambda_2 = t$ where $t = 1, 2, \dots, q$. Let $\lambda_2 = k$ where k is a positive integer. Then, (i) $\lambda_3 = -(qk + t)$; (ii) $\tau - \theta = -((q - 1)k + t)$; (iii) $\delta = (q + 1)k + t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 - kt$. Using (ii), (iv) and (v) we can easily see that (1.1) is reduced to

$$(1.3) \quad (p + 1)t^2 - ((q + 1)p + 1)t + q(q + 1)k^2 + 2qkt = 0.$$

Second, let $m_2 = p$, $m_3 = qp$ and $n = (q + 1)p + 1$, where $q \in \mathbb{N}$. Using (1.2) we obtain $r = p(q|\lambda_3| - \lambda_2)$. Let $q|\lambda_3| - \lambda_2 = t$, where $t = 1, 2, \dots, q$. Let $\lambda_3 = -k$, where k is a positive integer. Then, (i) $\lambda_2 = qk - t$; (ii) $\tau - \theta = (q - 1)k - t$; (iii) $\delta = (q + 1)k - t$;

(iv) $r = pt$ and (v) $\theta = pt - qk^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1.1) is reduced to

$$(1.4) \quad (p+1)t^2 - ((q+1)p+1)t + q(q+1)k^2 - 2qkt = 0.$$

Using (1.3) and (1.4) we can obtain for $t = 1, 2, \dots, q$ the corresponding classes of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, respectively.

2. MAIN RESULTS

Remark 2.1. Since $m_2(\overline{G}) = m_3(G)$ and $m_3(\overline{G}) = m_2(G)$ we note that if $m_2(G) = qm_3(G)$, then $m_3(\overline{G}) = qm_2(\overline{G})$.

Remark 2.2. In Theorems 2.1, 2.2, ..., 2.11 the complements of strongly regular graphs appear in pairs in (k^0) and $(\overline{k}^0)$ classes, where k denotes the corresponding number of a class.

Remark 2.3. $\overline{\alpha K_\beta}$ is a strongly regular graph of order $n = \alpha\beta$ and degree $r = (\alpha-1)\beta$ with $\tau = (\alpha-2)\beta$ and $\theta = (\alpha-1)\beta$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -\beta$ with $m_2 = \alpha(\beta-1)$ and $m_3 = \alpha-1$.

In order to demonstrate a method which is applied to describe the parameters n , r , τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, we shall establish the parameters of strongly regular graphs with $m_2 = 2m_3$ and $m_3 = 2m_2$, as follows.

Proposition 2.1. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 2m_3$. Then G belongs to the class $(\overline{2}^0)$ or (3^0) represented in Theorem 2.1.*

Proof. Let $m_3 = p$, $m_2 = 2p$ and $n = 3p + 1$ where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(|\lambda_3| - 2\lambda_2)$. Let $|\lambda_3| - 2\lambda_2 = t$ where $t = 1, 2$. Let $\lambda_2 = k$ where k is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_3 = -(2k+t)$; (ii) $\tau - \theta = -(k+t)$; (iii) $\delta = 3k+t$; (iv) $r = pt$ and (v) $\theta = pt - 2k^2 - kt$. In this case we can easily see that (1.3) is reduced to

$$(2.1) \quad (p+1)t^2 - (3p+1)t + 6k^2 + 4kt = 0.$$

CASE 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(2k+1)$, $\tau - \theta = -(k+1)$, $\delta = 3k+1$, $r = p$ and $\theta = p - 2k^2 - k$. Using (2.1) we find that $p = k(3k+2)$. So, we obtain that G is a strongly regular graph of order $n = (3k+1)^2$ and degree $r = k(3k+2)$ with $\tau = k^2 - 1$ and $\theta = k(k+1)$.

CASE 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(2k+2)$, $\tau - \theta = -(k+2)$, $\delta = 3k+2$, $r = 2p$ and $\theta = 2p - 2k^2 - 2k$. Using (2.1) we find that $p = (k+1)(3k+1)$. Replacing k with $k-1$ we arrive at $p = k(3k-2)$. So, we obtain that G is a strongly regular graph of order $n = (3k-1)^2$ and degree $r = 2k(3k-2)$ with $\tau = (k-1)(4k+1)$ and $\theta = 2k(2k-1)$. $\square$

Proposition 2.2. *Let G be a connected strongly regular graph of order n and degree r with $m_3 = 2m_2$. Then G belongs to the class (2^0) or $(\bar{3}^0)$ represented in Theorem 2.1.*

Proof. Let $m_2 = p$, $m_3 = 2p$ and $n = 3p + 1$ where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(2|\lambda_3| - \lambda_2)$. Let $2|\lambda_3| - \lambda_2 = t$ where $t = 1, 2$. Let $\lambda_3 = -k$ where k is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_2 = 2k - t$; (ii) $\tau - \theta = k - t$; (iii) $\delta = 3k - t$; (iv) $r = pt$ and (v) $\theta = pt - 2k^2 + kt$. In this case we can easily see that (1.4) is reduced to

$$(2.2) \quad (p+1)t^2 - (3p+1)t + 6k^2 - 4kt = 0.$$

CASE 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k - 1$ and $\lambda_3 = -k$, $\tau - \theta = k - 1$, $\delta = 3k - 1$, $r = p$ and $\theta = p - 2k^2 + k$. Using (2.2) we find that $p = k(3k - 2)$. So we obtain that G is a strongly regular graph of order $n = (3k - 1)^2$ and degree $r = k(3k - 2)$ with $\tau = k^2 - 1$ and $\theta = k(k - 1)$.

CASE 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k - 2$ and $\lambda_3 = -k$, $\tau - \theta = k - 2$, $\delta = 3k - 2$, $r = 2p$ and $\theta = 2p - 2k^2 + 2k$. Using (2.2) we find that $p = (k - 1)(3k - 1)$. Replacing k with $k + 1$ we arrive at $p = k(3k + 2)$. So we obtain that G is a strongly regular graph of order $n = (3k + 1)^2$ and degree $r = 2k(3k + 2)$ with $\tau = (k + 1)(4k - 1)$ and $\theta = 2k(2k + 1)$. $\square$

Theorem 2.1. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 2m_3$ or $m_3 = 2m_2$. Then, G is one of the following strongly regular graphs.*

- (1⁰) G is the complete bipartite graph of order $n = 4$ and degree $r = 2$ with $\tau = 0$ and $\theta = 2$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -2$ with $m_2 = 2$ and $m_3 = 1$.
- (2⁰) G is a strongly regular graph of order $n = (3k - 1)^2$ and degree $r = k(3k - 2)$ with $\tau = k^2 - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 2k - 1$ and $\lambda_3 = -k$ with $m_2 = k(3k - 2)$ and $m_3 = 2k(3k - 2)$.
- ($\bar{2}^0$) G is a strongly regular graph of order $n = (3k - 1)^2$ and degree $r = 2k(3k - 2)$ with $\tau = (k - 1)(4k + 1)$ and $\theta = 2k(2k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -2k$ with $m_2 = 2k(3k - 2)$ and $m_3 = k(3k - 2)$.
- (3⁰) G is a strongly regular graph of order $n = (3k + 1)^2$ and degree $r = k(3k + 2)$ with $\tau = k^2 - 1$ and $\theta = k(k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(2k + 1)$ with $m_2 = 2k(3k + 2)$ and $m_3 = k(3k + 2)$.
- ($\bar{3}^0$) G is a strongly regular graph of order $n = (3k + 1)^2$ and degree $r = 2k(3k + 2)$ with $\tau = (k + 1)(4k - 1)$ and $\theta = 2k(2k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 2k$ and $\lambda_3 = -(k + 1)$ with $m_2 = k(3k + 2)$ and $m_3 = 2k(3k + 2)$.

Proof. First, according to Remark 2.3 we have $\alpha(\beta - 1) = 2(\alpha - 1)$, from which we find that $\alpha = 2$, $\beta = 2$. In view of this we obtain the strongly regular graph represented in Theorem 2.1 (1⁰). Next, according to Proposition 2.1 it turns out that G belongs to the class ($\bar{2}^0$) or (3⁰) if $m_2 = 2m_3$. According to Proposition 2.2 it turns out that G belongs to the class (2⁰) or ($\bar{3}^0$) if $m_3 = 2m_2$. $\square$

Theorem 2.2 ([4]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 3m_3$ or $m_3 = 3m_2$. Then, G is one of the following strongly regular graphs.*

- (1⁰) G is the strongly regular graph $\overline{3K_3}$ of order $n = 9$ and degree $r = 6$ with $\tau = 3$ and $\theta = 6$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 6$ and $m_3 = 2$.
- (2⁰) G is a strongly regular graph of order $n = (4k - 1)^2$ and degree $r = 2k(2k - 1)$ with $\tau = k^2 + k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 3k - 1$ and $\lambda_3 = -k$ with $m_2 = 2k(2k - 1)$ and $m_3 = 6k(2k - 1)$.
- (2⁰) G is a strongly regular graph of order $n = (4k - 1)^2$ and degree $r = 6k(2k - 1)$ with $\tau = 9k^2 - 5k - 1$ and $\theta = 3k(3k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -3k$ with $m_2 = 6k(2k - 1)$ and $m_3 = 2k(2k - 1)$.
- (3⁰) G is a strongly regular graph of order $n = (4k + 1)^2$ and degree $r = 2k(2k + 1)$ with $\tau = k^2 - k - 1$ and $\theta = k(k + 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(3k + 1)$ with $m_2 = 6k(2k + 1)$ and $m_3 = 2k(2k + 1)$.
- (3⁰) G is a strongly regular graph of order $n = (4k + 1)^2$ and degree $r = 6k(2k + 1)$ with $\tau = 9k^2 + 5k - 1$ and $\theta = 3k(3k + 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(k + 1)$ with $m_2 = 2k(2k + 1)$ and $m_3 = 6k(2k + 1)$.

Theorem 2.3 ([4]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 4m_3$ or $m_3 = 4m_2$. Then, G is one of the following strongly regular graphs.*

- (1⁰) G is the complete bipartite graph $K_{3,3}$ of order $n = 6$ and degree $r = 3$ with $\tau = 0$ and $\theta = 3$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 4$ and $m_3 = 1$.
- (2⁰) G is the strongly regular graph $\overline{4K_4}$ of order $n = 16$ and degree $r = 12$ with $\tau = 8$ and $\theta = 12$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -4$ with $m_2 = 12$ and $m_3 = 3$.
- (3⁰) G is a strongly regular graph of order $n = (5k - 1)^2$ and degree $r = k(5k - 2)$ with $\tau = k^2 + 2k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 4k - 1$ and $\lambda_3 = -k$ with $m_2 = k(5k - 2)$ and $m_3 = 4k(5k - 2)$.
- (3⁰) G is a strongly regular graph of order $n = (5k - 1)^2$ and degree $r = 4k(5k - 2)$ with $\tau = 16k^2 - 7k - 1$ and $\theta = 4k(4k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -4k$ with $m_2 = 4k(5k - 2)$ and $m_3 = k(5k - 2)$.
- (4⁰) G is a strongly regular graph of order $n = (5k + 1)^2$ and degree $r = k(5k + 2)$ with $\tau = k^2 - 2k - 1$ and $\theta = k(k + 1)$, where $k \geq 3$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(4k + 1)$ with $m_2 = 4k(5k + 2)$ and $m_3 = k(5k + 2)$.
- (4⁰) G is a strongly regular graph of order $n = (5k + 1)^2$ and degree $r = 4k(5k + 2)$ with $\tau = 16k^2 + 7k - 1$ and $\theta = 4k(4k + 1)$, where $k \geq 3$. Its eigenvalues are $\lambda_2 = 4k$ and $\lambda_3 = -(k + 1)$ with $m_2 = k(5k + 2)$ and $m_3 = 4k(5k + 2)$.
- (5⁰) G is a strongly regular graph of order $n = 6(5k - 1)^2$ and degree $r = 2(30k^2 - 12k + 1)$ with $\tau = 24k^2 - 15k + 1$ and $\theta = 6k(4k - 1)$, where $k \in \mathbb{N}$. Its

eigenvalues are $\lambda_2 = 3k - 1$ and $\lambda_3 = -(12k - 2)$ with $m_2 = 4(30k^2 - 12k + 1)$ and $m_3 = 30k^2 - 12k + 1$.

- ($\bar{5}^0$) G is a strongly regular graph of order $n = 6(5k - 1)^2$ and degree $r = 3(30k^2 - 12k + 1)$ with $\tau = 18k(3k - 1)$ and $\theta = 3(3k - 1)(6k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 12k - 3$ and $\lambda_3 = -3k$ with $m_2 = 30k^2 - 12k + 1$ and $m_3 = 4(30k^2 - 12k + 1)$.
- (6^0) G is a strongly regular graph of order $n = 6(5k + 1)^2$ and degree $r = 2(30k^2 + 12k + 1)$ with $\tau = 24k^2 + 15k + 1$ and $\theta = 6k(4k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 12k + 2$ and $\lambda_3 = -(3k + 1)$ with $m_2 = 30k^2 + 12k + 1$ and $m_3 = 4(30k^2 + 12k + 1)$.
- ($\bar{6}^0$) G is a strongly regular graph of order $n = 6(5k + 1)^2$ and degree $r = 3(30k^2 + 12k + 1)$ with $\tau = 18k(3k + 1)$ and $\theta = 3(3k + 1)(6k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(12k + 3)$ with $m_2 = 4(30k^2 + 12k + 1)$ and $m_3 = 30k^2 + 12k + 1$.

Theorem 2.4 ([5]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 5m_3$ or $m_3 = 5m_2$. Then, G is one of the following strongly regular graphs.*

- (1^0) G is the strongly regular graph $\overline{5K_5}$ of order $n = 25$ and degree $r = 20$ with $\tau = 15$ and $\theta = 20$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -5$ with $m_2 = 20$ and $m_3 = 4$.
- (2^0) G is a strongly regular graph of order $n = (6k - 1)^2$ and degree $r = 2k(3k - 1)$ with $\tau = k^2 + 3k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 5k - 1$ and $\lambda_3 = -k$ with $m_2 = 2k(3k - 1)$ and $m_3 = 10k(3k - 1)$.
- ($\bar{2}^0$) G is a strongly regular graph of order $n = (6k - 1)^2$ and degree $r = 10k(3k - 1)$ with $\tau = 25k^2 - 9k - 1$ and $\theta = 5k(5k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -5k$ with $m_2 = 10k(3k - 1)$ and $m_3 = 2k(3k - 1)$.
- (3^0) G is a strongly regular graph of order $n = (6k + 1)^2$ and degree $r = 2k(3k + 1)$ with $\tau = k^2 - 3k - 1$ and $\theta = k(k + 1)$, where $k \geq 4$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(5k + 1)$ with $m_2 = 10k(3k + 1)$ and $m_3 = 2k(3k + 1)$.
- ($\bar{3}^0$) G is a strongly regular graph of order $n = (6k + 1)^2$ and degree $r = 10k(3k + 1)$ with $\tau = 25k^2 + 9k - 1$ and $\theta = 5k(5k + 1)$, where $k \geq 4$. Its eigenvalues are $\lambda_2 = 5k$ and $\lambda_3 = -(k + 1)$ with $m_2 = 2k(3k + 1)$ and $m_3 = 10k(3k + 1)$.

Theorem 2.5 ([5]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 6m_3$ or $m_3 = 6m_2$. Then, G is one of the following strongly regular graphs.*

- (1^0) G is the complete bipartite graph $K_{4,4}$ of order $n = 8$ and degree $r = 4$ with $\tau = 0$ and $\theta = 4$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -4$ with $m_2 = 6$ and $m_3 = 1$.
- (2^0) G is the strongly regular graph $\overline{3K_5}$ of order $n = 15$ and degree $r = 10$ with $\tau = 5$ and $\theta = 10$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -5$ with $m_2 = 12$ and $m_3 = 2$.

- (3⁰) G is the strongly regular graph $\overline{6K_6}$ of order $n = 36$ and degree $r = 30$ with $\tau = 24$ and $\theta = 30$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -6$ with $m_2 = 30$ and $m_3 = 5$.
- (4⁰) G is a strongly regular graph of order $n = (7k - 1)^2$ and degree $r = k(7k - 2)$ with $\tau = k^2 + 4k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 6k - 1$ and $\lambda_3 = -k$ with $m_2 = k(7k - 2)$ and $m_3 = 6k(7k - 2)$.
- (4⁰) G is a strongly regular graph of order $n = (7k - 1)^2$ and degree $r = 6k(7k - 2)$ with $\tau = 36k^2 - 11k - 1$ and $\theta = 6k(6k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -6k$ with $m_2 = 6k(7k - 2)$ and $m_3 = k(7k - 2)$.
- (5⁰) G is a strongly regular graph of order $n = (7k + 1)^2$ and degree $r = k(7k + 2)$ with $\tau = k^2 - 4k - 1$ and $\theta = k(k + 1)$, where $k \geq 5$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(6k + 1)$ with $m_2 = 6k(7k + 2)$ and $m_3 = k(7k + 2)$.
- (5⁰) G is a strongly regular graph of order $n = (7k + 1)^2$ and degree $r = 6k(7k + 2)$ with $\tau = 36k^2 + 11k - 1$ and $\theta = 6k(6k + 1)$, where $k \geq 5$. Its eigenvalues are $\lambda_2 = 6k$ and $\lambda_3 = -(k + 1)$ with $m_2 = k(7k + 2)$ and $m_3 = 6k(7k + 2)$.
- (6⁰) G is a strongly regular graph of order $n = 2(7k - 2)^2$ and degree $r = 3(14k^2 - 8k + 1)$ with $\tau = 18k^2 - 16k + 2$ and $\theta = 6k(3k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 2k - 1$ and $\lambda_3 = -(12k - 3)$ with $m_2 = 6(14k^2 - 8k + 1)$ and $m_3 = 14k^2 - 8k + 1$.
- (6⁰) G is a strongly regular graph of order $n = 2(7k - 2)^2$ and degree $r = 4(14k^2 - 8k + 1)$ with $\tau = 2k(16k - 7)$ and $\theta = 4(2k - 1)(4k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 12k - 4$ and $\lambda_3 = -2k$ with $m_2 = 14k^2 - 8k + 1$ and $m_3 = 6(14k^2 - 8k + 1)$.
- (7⁰) G is a strongly regular graph of order $n = 2(7k + 2)^2$ and degree $r = 3(14k^2 + 8k + 1)$ with $\tau = 18k^2 + 16k + 2$ and $\theta = 6k(3k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 12k + 3$ and $\lambda_3 = -(2k + 1)$ with $m_2 = 14k^2 + 8k + 1$ and $m_3 = 6(14k^2 + 8k + 1)$.
- (7⁰) G is a strongly regular graph of order $n = 2(7k + 2)^2$ and degree $r = 4(14k^2 + 8k + 1)$ with $\tau = 2k(16k + 7)$ and $\theta = 4(2k + 1)(4k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 2k$ and $\lambda_3 = -(12k + 4)$ with $m_2 = 6(14k^2 + 8k + 1)$ and $m_3 = 14k^2 + 8k + 1$.
- (8⁰) G is a strongly regular graph of order $n = 15(7k - 1)^2$ and degree $r = 2(105k^2 - 30k + 2)$ with $\tau = 60k^2 - 35k + 3$ and $\theta = 10k(6k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k - 1$ and $\lambda_3 = -(30k - 4)$ with $m_2 = 6(105k^2 - 30k + 2)$ and $m_3 = 105k^2 - 30k + 2$.
- (8⁰) G is a strongly regular graph of order $n = 15(7k - 1)^2$ and degree $r = 5(105k^2 - 30k + 2)$ with $\tau = 5(5k - 1)(15k - 1)$ and $\theta = 5(5k - 1)(15k - 2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 30k - 5$ and $\lambda_3 = -5k$ with $m_2 = 105k^2 - 30k + 2$ and $m_3 = 6(105k^2 - 30k + 2)$.
- (9⁰) G is a strongly regular graph of order $n = 15(7k + 1)^2$ and degree $r = 2(105k^2 + 30k + 2)$ with $\tau = 60k^2 + 35k + 3$ and $\theta = 10k(6k + 1)$, where $k \in \mathbb{N}$. Its

eigenvalues are $\lambda_2 = 30k + 4$ and $\lambda_3 = -(5k + 1)$ with $m_2 = 105k^2 + 30k + 2$ and $m_3 = 6(105k^2 + 30k + 2)$.

- ($\bar{9}^0$) G is a strongly regular graph of order $n = 15(7k + 1)^2$ and degree $r = 5(105k^2 + 30k + 2)$ with $\tau = 5(5k + 1)(15k + 1)$ and $\theta = 5(5k + 1)(15 + 2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k$ and $\lambda_3 = -(30k + 5)$ with $m_2 = 6(105k^2 + 30k + 2)$ and $m_3 = 105k^2 + 30k + 2$.

Theorem 2.6 ([5]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 7m_3$ or $m_3 = 7m_2$. Then, G is one of the following strongly regular graphs.*

- (1^0) G is the strongly regular graph $\overline{7K_7}$ of order $n = 49$ and degree $r = 42$ with $\tau = 35$ and $\theta = 42$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -7$ with $m_2 = 42$ and $m_3 = 6$.
- (2^0) G is a strongly regular graph of order $n = (8k - 1)^2$ and degree $r = 2k(4k - 1)$ with $\tau = k^2 + 5k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 7k - 1$ and $\lambda_3 = -k$ with $m_2 = 2k(4k - 1)$ and $m_3 = 14k(4k - 1)$.
- ($\bar{2}^0$) G is a strongly regular graph of order $n = (8k - 1)^2$ and degree $r = 14k(4k - 1)$ with $\tau = 49k^2 - 13k - 1$ and $\theta = 7k(7k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -7k$ with $m_2 = 14k(4k - 1)$ and $m_3 = 2k(4k - 1)$.
- (3^0) G is a strongly regular graph of order $n = (8k + 1)^2$ and degree $r = 2k(4k + 1)$ with $\tau = k^2 - 5k - 1$ and $\theta = k(k + 1)$, where $k \geq 6$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(7k + 1)$ with $m_2 = 14k(4k + 1)$ and $m_3 = 2k(4k + 1)$.
- ($\bar{3}^0$) G is a strongly regular graph of order $n = (8k + 1)^2$ and degree $r = 14k(4k + 1)$ with $\tau = 49k^2 + 13k - 1$ and $\theta = 7k(7k + 1)$, where $k \geq 6$. Its eigenvalues are $\lambda_2 = 7k$ and $\lambda_3 = -(k + 1)$ with $m_2 = 2k(4k + 1)$ and $m_3 = 14k(4k + 1)$.
- (4^0) G is a strongly regular graph of order $n = 105(8k - 3)^2$ and degree $r = 3(840k^2 - 630k + 118)$ with $\tau = 945k^2 - 765k + 153$ and $\theta = 15(3k - 1)(21k - 8)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 15k - 6$ and $\lambda_3 = -(105k - 39)$ with $m_2 = 7(840k^2 - 630k + 118)$ and $m_3 = 840k^2 - 630k + 118$.
- ($\bar{4}^0$) G is a strongly regular graph of order $n = 105(8k - 3)^2$ and degree $r = 5(840k^2 - 630k + 118)$ with $\tau = 5(525k^2 - 387k + 71)$ and $\theta = 15(5k - 2)(35k - 13)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 105k - 40$ and $\lambda_3 = -(15k - 5)$ with $m_2 = 840k^2 - 630k + 118$ and $m_3 = 7(840k^2 - 630k + 118)$.
- (5^0) G is a strongly regular graph of order $n = 105(8k + 3)^2$ and degree $r = 3(840k^2 + 630k + 118)$ with $\tau = 945k^2 + 765k + 153$ and $\theta = 15(3k + 1)(21k + 8)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 105k + 39$ and $\lambda_3 = -(15k + 6)$ with $m_2 = 840k^2 + 630k + 118$ and $m_3 = 7(840k^2 + 630k + 118)$.
- ($\bar{5}^0$) G is a strongly regular graph of order $n = 105(8k + 3)^2$ and degree $r = 5(840k^2 + 630k + 118)$ with $\tau = 5(525k^2 + 387k + 71)$ and $\theta = 15(5k + 2)(35k + 13)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 15k + 5$ and $\lambda_3 = -(105k + 40)$ with $m_2 = 7(840k^2 + 630k + 118)$ and $m_3 = 840k^2 + 630k + 118$.

Theorem 2.7 ([5]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 8m_3$ or $m_3 = 8m_2$. Then, G is one of the following strongly regular graphs.*

- (1⁰) G is the complete bipartite graph $K_{5,5}$ of order $n = 10$ and degree $r = 5$ with $\tau = 0$ and $\theta = 5$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -5$ with $m_2 = 8$ and $m_3 = 1$.
- (2⁰) G is the strongly regular graph $\overline{4K_7}$ of order $n = 28$ and degree $r = 21$ with $\tau = 14$ and $\theta = 21$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -7$ with $m_2 = 24$ and $m_3 = 3$.
- (3⁰) G is the strongly regular graph $\overline{8K_8}$ of order $n = 64$ and degree $r = 56$ with $\tau = 48$ and $\theta = 56$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -8$ with $m_2 = 56$ and $m_3 = 7$.
- (4⁰) G is a strongly regular graph of order $n = (9k - 1)^2$ and degree $r = k(9k - 2)$ with $\tau = k^2 + 6k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 8k - 1$ and $\lambda_3 = -k$ with $m_2 = k(9k - 2)$ and $m_3 = 8k(9k - 2)$.
- (4⁰) G is a strongly regular graph of order $n = (9k - 1)^2$ and degree $r = 8k(9k - 2)$ with $\tau = 64k^2 - 15k - 1$ and $\theta = 8k(8k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -8k$ with $m_2 = 8k(9k - 2)$ and $m_3 = k(9k - 2)$.
- (5⁰) G is a strongly regular graph of order $n = (9k + 1)^2$ and degree $r = k(9k + 2)$ with $\tau = k^2 - 6k - 1$ and $\theta = k(k + 1)$, where $k \geq 7$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(8k + 1)$ with $m_2 = 8k(9k + 2)$ and $m_3 = k(9k + 2)$.
- (5⁰) G is a strongly regular graph of order $n = (9k + 1)^2$ and degree $r = 8k(9k + 2)$ with $\tau = 64k^2 + 15k - 1$ and $\theta = 8k(8k + 1)$, where $k \geq 7$. Its eigenvalues are $\lambda_2 = 8k$ and $\lambda_3 = -(k + 1)$ with $m_2 = k(9k + 2)$ and $m_3 = 8k(9k + 2)$.
- (6⁰) G is a strongly regular graph of order $n = 4(9k - 4)^2$ and degree $r = 3(2k - 1)(18k - 7)$ with $\tau = 18k(2k - 1)$ and $\theta = (3k - 2)(12k - 5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 24k - 11$ and $\lambda_3 = -(3k - 1)$ with $m_2 = (2k - 1)(18k - 7)$ and $m_3 = 8(2k - 1)(18k - 7)$.
- (6⁰) G is a strongly regular graph of order $n = 4(9k - 4)^2$ and degree $r = 6(2k - 1)(18k - 7)$ with $\tau = 3(48k^2 - 45k + 10)$ and $\theta = 2(3k - 1)(24k - 11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k - 2$ and $\lambda_3 = -(24k - 10)$ with $m_2 = 8(2k - 1)(18k - 7)$ and $m_3 = (2k - 1)(18k - 7)$.
- (7⁰) G is a strongly regular graph of order $n = 4(9k + 4)^2$ and degree $r = 3(2k + 1)(18k + 7)$ with $\tau = 18k(2k + 1)$ and $\theta = (3k + 2)(12k + 5)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 3k + 1$ and $\lambda_3 = -(24k + 11)$ with $m_2 = 8(2k + 1)(18k + 7)$ and $m_3 = (2k + 1)(18k + 7)$.
- (7⁰) G is a strongly regular graph of order $n = 4(9k + 4)^2$ and degree $r = 6(2k + 1)(18k + 7)$ with $\tau = 3(48k^2 + 45k + 10)$ and $\theta = 2(3k + 1)(24k + 11)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 24k + 10$ and $\lambda_3 = -(3k + 2)$ with $m_2 = (2k + 1)(18k + 7)$ and $m_3 = 8(2k + 1)(18k + 7)$.

- (8⁰) G is a strongly regular graph of order $n = 10(9k - 1)^2$ and degree $r = 4(90k^2 - 20k + 1)$ with $\tau = 160k^2 - 55k + 3$ and $\theta = 20k(8k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k - 1$ and $\lambda_3 = -(40k - 4)$ with $m_2 = 8(90k^2 - 20k + 1)$ and $m_3 = 90k^2 - 20k + 1$.
- (8⁰) G is a strongly regular graph of order $n = 10(9k - 1)^2$ and degree $r = 5(90k^2 - 20k + 1)$ with $\tau = 10k(25k - 4)$ and $\theta = 5(5k - 1)(10k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 40k - 5$ and $\lambda_3 = -5k$ with $m_2 = 90k^2 - 20k + 1$ and $m_3 = 8(90k^2 - 20k + 1)$.
- (9⁰) G is a strongly regular graph of order $n = 10(9k + 1)^2$ and degree $r = 4(90k^2 + 20k + 1)$ with $\tau = 160k^2 + 55k + 3$ and $\theta = 20k(8k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 40k + 4$ and $\lambda_3 = -(5k + 1)$ with $m_2 = 90k^2 + 20k + 1$ and $m_3 = 8(90k^2 + 20k + 1)$.
- (9⁰) G is a strongly regular graph of order $n = 10(9k + 1)^2$ and degree $r = 5(90k^2 + 20k + 1)$ with $\tau = 10k(25k + 4)$ and $\theta = 5(5k + 1)(10k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k$ and $\lambda_3 = -(40k + 5)$ with $m_2 = 8(90k^2 + 20k + 1)$ and $m_3 = 90k^2 + 20k + 1$.
- (10⁰) G is a strongly regular graph of order $n = 28(9k - 1)^2$ and degree $r = 2(252k^2 - 56k + 3)$ with $\tau = 112k^2 - 63k + 5$ and $\theta = 14k(8k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k - 1$ and $\lambda_3 = -(56k - 6)$ with $m_2 = 8(252k^2 - 56k + 3)$ and $m_3 = 252k^2 - 56k + 3$.
- (10⁰) G is a strongly regular graph of order $n = 28(9k - 1)^2$ and degree $r = 7(252k^2 - 56k + 3)$ with $\tau = 14(7k - 1)(14k - 1)$ and $\theta = 7(7k - 1)(28k - 3)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 56k - 7$ and $\lambda_3 = -7k$ with $m_2 = 252k^2 - 56k + 3$ and $m_3 = 8(252k^2 - 56k + 3)$.
- (11⁰) G is a strongly regular graph of order $n = 28(9k + 1)^2$ and degree $r = 2(252k^2 + 56k + 3)$ with $\tau = 112k^2 + 63k + 5$ and $\theta = 14k(8k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 56k + 6$ and $\lambda_3 = -(7k + 1)$ with $m_2 = 252k^2 + 56k + 3$ and $m_3 = 8(252k^2 + 56k + 3)$.
- (11⁰) G is a strongly regular graph of order $n = 28(9k + 1)^2$ and degree $r = 7(252k^2 + 56k + 3)$ with $\tau = 14(7k + 1)(14k + 1)$ and $\theta = 7(7k + 1)(28k + 3)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k$ and $\lambda_3 = -(56k + 7)$ with $m_2 = 8(252k^2 + 56k + 3)$ and $m_3 = 252k^2 + 56k + 3$.

Theorem 2.8 ([6]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 9m_3$ or $m_3 = 9m_2$. Then, G is one of the following strongly regular graphs.*

- (1⁰) G is the strongly regular graph $\overline{3K_7}$ of order $n = 21$ and degree $r = 14$ with $\tau = 7$ and $\theta = 14$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -7$ with $m_2 = 18$ and $m_3 = 2$.
- (2⁰) G is the strongly regular graph $\overline{9K_9}$ of order $n = 81$ and degree $r = 72$ with $\tau = 63$ and $\theta = 72$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -9$ with $m_2 = 72$ and $m_3 = 8$.

- (3⁰) G is a strongly regular graph of order $n = (10k - 1)^2$ and degree $r = 2k(5k - 1)$ with $\tau = k^2 + 7k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 9k - 1$ and $\lambda_3 = -k$ with $m_2 = 2k(5k - 1)$ and $m_3 = 18k(5k - 1)$.
- (3⁰) G is a strongly regular graph of order $n = (10k - 1)^2$ and degree $r = 18k(5k - 1)$ with $\tau = 81k^2 - 17k - 1$ and $\theta = 9k(9k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -9k$ with $m_2 = 18k(5k - 1)$ and $m_3 = 2k(5k - 1)$.
- (4⁰) G is a strongly regular graph of order $n = (10k + 1)^2$ and degree $r = 2k(5k + 1)$ with $\tau = k^2 - 7k - 1$ and $\theta = k(k + 1)$, where $k \geq 8$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(9k + 1)$ with $m_2 = 18k(5k + 1)$ and $m_3 = 2k(5k + 1)$.
- (4⁰) G is a strongly regular graph of order $n = (10k + 1)^2$ and degree $r = 18k(5k + 1)$ with $\tau = 81k^2 + 17k - 1$ and $\theta = 9k(9k + 1)$, where $k \geq 8$. Its eigenvalues are $\lambda_2 = 9k$ and $\lambda_3 = -(k + 1)$ with $m_2 = 2k(5k + 1)$ and $m_3 = 18k(5k + 1)$.
- (5⁰) G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 2(90k^2 - 54k + 8)$ with $\tau = 36k^2 + 4k - 5$ and $\theta = (2k - 1)(18k - 5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k - 11$ and $\lambda_3 = -(4k - 1)$ with $m_2 = 90k^2 - 54k + 8$ and $m_3 = 9(90k^2 - 54k + 8)$.
- (5⁰) G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 8(90k^2 - 54k + 8)$ with $\tau = 4(4k - 1)(36k - 13)$ and $\theta = 4(4k - 1)(36k - 11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 4k - 2$ and $\lambda_3 = -(36k - 10)$ with $m_2 = 9(90k^2 - 54k + 8)$ and $m_3 = 90k^2 - 54k + 8$.
- (6⁰) G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 5(90k^2 - 54k + 8)$ with $\tau = 225k^2 - 155k + 25$ and $\theta = (5k - 1)(45k - 14)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k - 2$ and $\lambda_3 = -(45k - 13)$ with $m_2 = 9(90k^2 - 54k + 8)$ and $m_3 = 90k^2 - 54k + 8$.
- (6⁰) G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 5(90k^2 - 54k + 8)$ with $\tau = 225k^2 - 115k + 13$ and $\theta = (5k - 2)(45k - 13)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 45k - 14$ and $\lambda_3 = -(5k - 1)$ with $m_2 = 90k^2 - 54k + 8$ and $m_3 = 9(90k^2 - 54k + 8)$.
- (7⁰) G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 2(90k^2 + 54k + 8)$ with $\tau = 36k^2 - 4k - 5$ and $\theta = (2k + 1)(18k + 5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 4k + 1$ and $\lambda_3 = -(36k + 11)$ with $m_2 = 9(90k^2 + 54k + 8)$ and $m_3 = 90k^2 + 54k + 8$.
- (7⁰) G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 8(90k^2 + 54k + 8)$ with $\tau = 4(4k + 1)(36k + 13)$ and $\theta = 4(4k + 1)(36k + 11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k + 10$ and $\lambda_3 = -(4k + 2)$ with $m_2 = 90k^2 + 54k + 8$ and $m_3 = 9(90k^2 + 54k + 8)$.
- (8⁰) G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 5(90k^2 + 54k + 8)$ with $\tau = 225k^2 + 115k + 13$ and $\theta = (5k + 2)(45k + 13)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 5k + 1$ and $\lambda_3 = -(45k + 14)$ with $m_2 = 9(90k^2 + 54k + 8)$ and $m_3 = 90k^2 + 54k + 8$.
- (8⁰) G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 5(90k^2 + 54k + 8)$ with $\tau = 225k^2 + 155k + 25$ and $\theta = (5k + 1)(45k + 14)$, where $k \geq 0$.

- Its eigenvalues are $\lambda_2 = 45k + 13$ and $\lambda_3 = -(5k + 2)$ with $m_2 = 90k^2 + 54k + 8$ and $m_3 = 9(90k^2 + 54k + 8)$.*
- (9⁰) *G is a strongly regular graph of order $n = 21(10k - 1)^2$ and degree $r = 3(210k^2 - 42k + 2)$ with $\tau = 189k^2 - 77k + 5$ and $\theta = 21k(9k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k - 1$ and $\lambda_3 = -(63k - 6)$ with $m_2 = 9(210k^2 - 42k + 2)$ and $m_3 = 210k^2 - 42k + 2$.*
- ($\bar{9}$ ⁰) *G is a strongly regular graph of order $n = 21(10k - 1)^2$ and degree $r = 7(210k^2 - 42k + 2)$ with $\tau = 7(147k^2 - 27k + 1)$ and $\theta = 7(7k - 1)(21k - 2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 63k - 7$ and $\lambda_3 = -7k$ with $m_2 = 210k^2 - 42k + 2$ and $m_3 = 9(210k^2 - 42k + 2)$.*
- (10⁰) *G is a strongly regular graph of order $n = 21(10k + 1)^2$ and degree $r = 3(210k^2 + 42k + 2)$ with $\tau = 189k^2 + 77k + 5$ and $\theta = 21k(9k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 63k + 6$ and $\lambda_3 = -(7k + 1)$ with $m_2 = 210k^2 + 42k + 2$ and $m_3 = 9(210k^2 + 42k + 2)$.*
- ($\bar{10}$ ⁰) *G is a strongly regular graph of order $n = 21(10k + 1)^2$ and degree $r = 7(210k^2 + 42k + 2)$ with $\tau = 7(147k^2 + 27k + 1)$ and $\theta = 7(7k + 1)(21k + 2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k$ and $\lambda_3 = -(63k + 7)$ with $m_2 = 9(210k^2 + 42k + 2)$ and $m_3 = 210k^2 + 42k + 2$.*

Theorem 2.9 ([6]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 10m_3$ or $m_3 = 10m_2$. Then, G is one of the following strongly regular graphs.*

- (1⁰) *G is the complete bipartite graph $K_{6,6}$ of order $n = 12$ and degree $r = 6$ with $\tau = 0$ and $\theta = 6$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -6$ with $m_2 = 10$ and $m_3 = 1$.*
- (2⁰) *G is the strongly regular graph $\overline{5K_9}$ of order $n = 45$ and degree $r = 36$ with $\tau = 27$ and $\theta = 36$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -9$ with $m_2 = 40$ and $m_3 = 4$.*
- (3⁰) *G is the strongly regular graph $\overline{10K_{10}}$ of order $n = 100$ and degree $r = 90$ with $\tau = 80$ and $\theta = 90$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -10$ with $m_2 = 90$ and $m_3 = 9$.*
- (4⁰) *G is a strongly regular graph of order $n = (11k - 1)^2$ and degree $r = k(11k - 2)$ with $\tau = k^2 + 8k - 1$ and $\theta = k(k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 10k - 1$ and $\lambda_3 = -k$ with $m_2 = k(11k - 2)$ and $m_3 = 10k(11k - 2)$.*
- ($\bar{4}$ ⁰) *G is a strongly regular graph of order $n = (11k - 1)^2$ and degree $r = 10k(11k - 2)$ with $\tau = 100k^2 - 19k - 1$ and $\theta = 10k(10k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -10k$ with $m_2 = 10k(11k - 2)$ and $m_3 = k(11k - 2)$.*
- (5⁰) *G is a strongly regular graph of order $n = (11k + 1)^2$ and degree $r = k(11k + 2)$ with $\tau = k^2 - 8k - 1$ and $\theta = k(k + 1)$, where $k \geq 9$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(10k + 1)$ with $m_2 = 10k(11k + 2)$ and $m_3 = k(11k + 2)$.*

- ($\bar{5}^0$) G is a strongly regular graph of order $n = (11k+1)^2$ and degree $r = 10k(11k+2)$ with $\tau = 100k^2 + 19k - 1$ and $\theta = 10k(10k+1)$, where $k \geq 9$. Its eigenvalues are $\lambda_2 = 10k$ and $\lambda_3 = -(k+1)$ with $m_2 = k(11k+2)$ and $m_3 = 10k(11k+2)$.
- (6^0) G is a strongly regular graph of order $n = 3(11k-2)^2$ and degree $r = 5(33k^2 - 12k + 1)$ with $\tau = 75k^2 - 42k + 4$ and $\theta = 15k(5k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k-1$ and $\lambda_3 = -(30k-5)$ with $m_2 = 10(33k^2 - 12k + 1)$ and $m_3 = 33k^2 - 12k + 1$.
- ($\bar{6}^0$) G is a strongly regular graph of order $n = 3(11k-2)^2$ and degree $r = 6(33k^2 - 12k + 1)$ with $\tau = 27k(4k-1)$ and $\theta = 6(3k-1)(6k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 30k-6$ and $\lambda_3 = -3k$ with $m_2 = 33k^2 - 12k + 1$ and $m_3 = 10(33k^2 - 12k + 1)$.
- (7^0) G is a strongly regular graph of order $n = 3(11k+2)^2$ and degree $r = 5(33k^2 + 12k + 1)$ with $\tau = 75k^2 + 42k + 4$ and $\theta = 15k(5k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 30k+5$ and $\lambda_3 = -(3k+1)$ with $m_2 = 33k^2 + 12k + 1$ and $m_3 = 10(33k^2 + 12k + 1)$.
- ($\bar{7}^0$) G is a strongly regular graph of order $n = 3(11k+2)^2$ and degree $r = 6(33k^2 + 12k + 1)$ with $\tau = 27k(4k+1)$ and $\theta = 6(3k+1)(6k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(30k+6)$ with $m_2 = 10(33k^2 + 12k + 1)$ and $m_3 = 33k^2 + 12k + 1$.
- (8^0) G is a strongly regular graph of order $n = 5(11k-3)^2$ and degree $r = 2(55k^2 - 30k + 4)$ with $\tau = 20k^2 - 33k + 7$ and $\theta = 2k(10k-3)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 3k-1$ and $\lambda_3 = -(30k-8)$ with $m_2 = 10(55k^2 - 30k + 4)$ and $m_3 = 55k^2 - 30k + 4$.
- ($\bar{8}^0$) G is a strongly regular graph of order $n = 5(11k-3)^2$ and degree $r = 9(55k^2 - 30k + 4)$ with $\tau = 27(3k-1)(5k-1)$ and $\theta = 9(3k-1)(15k-4)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 30k-9$ and $\lambda_3 = -3k$ with $m_2 = 55k^2 - 30k + 4$ and $m_3 = 10(55k^2 - 30k + 4)$.
- (9^0) G is a strongly regular graph of order $n = 5(11k+3)^2$ and degree $r = 2(55k^2 + 30k + 4)$ with $\tau = 20k^2 + 33k + 7$ and $\theta = 2k(10k+3)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 30k+8$ and $\lambda_3 = -(3k+1)$ with $m_2 = 55k^2 + 30k + 4$ and $m_3 = 10(55k^2 + 30k + 4)$.
- ($\bar{9}^0$) G is a strongly regular graph of order $n = 5(11k+3)^2$ and degree $r = 9(55k^2 + 30k + 4)$ with $\tau = 27(3k+1)(5k+1)$ and $\theta = 9(3k+1)(15k+4)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(30k+9)$ with $m_2 = 10(55k^2 + 30k + 4)$ and $m_3 = 55k^2 + 30k + 4$.
- (10^0) G is a strongly regular graph of order $n = 15(11k-5)^2$ and degree $r = 3(165k^2 - 150k + 34)$ with $\tau = 3(45k^2 - 54k + 15)$ and $\theta = 3(3k-1)(15k-7)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 6k-3$ and $\lambda_3 = -(60k-27)$ with $m_2 = 10(165k^2 - 150k + 34)$ and $m_3 = 165k^2 - 150k + 34$.
- ($\bar{10}^0$) G is a strongly regular graph of order $n = 15(11k-5)^2$ and degree $r = 8(165k^2 - 150k + 34)$ with $\tau = 2(480k^2 - 429k + 95)$ and $\theta = 24(2k-1)(20k-9)$,

- where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 60k - 28$ and $\lambda_3 = -(6k - 2)$ with $m_2 = 165k^2 - 150k + 34$ and $m_3 = 10(165k^2 - 150k + 34)$.
- (11⁰) G is a strongly regular graph of order $n = 15(11k + 5)^2$ and degree $r = 3(165k^2 + 150k + 34)$ with $\tau = 3(45k^2 + 54k + 15)$ and $\theta = 3(3k + 1)(15k + 7)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 60k + 27$ and $\lambda_3 = -(6k + 3)$ with $m_2 = 165k^2 + 150k + 34$ and $m_3 = 10(165k^2 + 150k + 34)$.
- (11⁰) G is a strongly regular graph of order $n = 15(11k + 5)^2$ and degree $r = 8(165k^2 + 150k + 34)$ with $\tau = 2(480k^2 + 429k + 95)$ and $\theta = 24(2k + 1)(20k + 9)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 6k + 2$ and $\lambda_3 = -(60k + 28)$ with $m_2 = 10(165k^2 + 150k + 34)$ and $m_3 = 165k^2 + 150k + 34$.
- (12⁰) G is a strongly regular graph of order $n = 70(11k - 5)^2$ and degree $r = 4(770k^2 - 700k + 159)$ with $\tau = 2(560k^2 - 469k + 97)$ and $\theta = 28(2k - 1)(20k - 9)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 140k - 64$ and $\lambda_3 = -(14k - 6)$ with $m_2 = 770k^2 - 700k + 159$ and $m_3 = 10(770k^2 - 700k + 159)$.
- (12⁰) G is a strongly regular graph of order $n = 70(11k - 5)^2$ and degree $r = 7(770k^2 - 700k + 159)$ with $\tau = 14(245k^2 - 226k + 52)$ and $\theta = 14(7k - 3)(35k - 16)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 14k - 7$ and $\lambda_3 = -(140k - 63)$ with $m_2 = 10(770k^2 - 700k + 159)$ and $m_3 = 770k^2 - 700k + 159$.
- (13⁰) G is a strongly regular graph of order $n = 70(11k + 5)^2$ and degree $r = 4(770k^2 + 700k + 159)$ with $\tau = 2(560k^2 + 469k + 97)$ and $\theta = 28(2k + 1)(20k + 9)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 14k + 6$ and $\lambda_3 = -(140k + 64)$ with $m_2 = 10(770k^2 + 700k + 159)$ and $m_3 = 770k^2 + 700k + 159$.
- (13⁰) G is a strongly regular graph of order $n = 70(11k + 5)^2$ and degree $r = 7(770k^2 + 700k + 159)$ with $\tau = 14(245k^2 + 226k + 52)$ and $\theta = 14(7k + 3)(35k + 16)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 140k + 63$ and $\lambda_3 = -(14k + 7)$ with $m_2 = 770k^2 + 700k + 159$ and $m_3 = 10(770k^2 + 700k + 159)$.

Theorem 2.10 ([7]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 11m_3$ or $m_3 = 11m_2$. Then, G is one of the following strongly regular graphs.*

- (1⁰) G is the strongly regular graph $\overline{11K_{11}}$ of order $n = 121$ and degree $r = 110$ with $\tau = 99$ and $\theta = 110$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -11$ with $m_2 = 110$ and $m_3 = 10$.
- (2⁰) G is a strongly regular graph of order $n = (12k - 1)^2$ and degree $r = 2k(6k - 1)$ with $\tau = k^2 + 9k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 11k - 1$ and $\lambda_3 = -k$ with $m_2 = 2k(6k - 1)$ and $m_3 = 22k(6k - 1)$.
- (2⁰) G is a strongly regular graph of order $n = (12k - 1)^2$ and degree $r = 22k(6k - 1)$ with $\tau = 121k^2 - 21k - 1$ and $\theta = 11k(11k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -11k$ with $m_2 = 22k(6k - 1)$ and $m_3 = 2k(6k - 1)$.
- (3⁰) G is a strongly regular graph of order $n = (12k + 1)^2$ and degree $r = 2k(6k + 1)$ with $\tau = k^2 - 9k - 1$ and $\theta = k(k + 1)$, where $k \geq 10$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(11k + 1)$ with $m_2 = 22k(6k + 1)$ and $m_3 = 2k(6k + 1)$.

- ($\bar{3}^0$) G is a strongly regular graph of order $n = (12k+1)^2$ and degree $r = 22k(6k+1)$ with $\tau = 121k^2 + 21k - 1$ and $\theta = 11k(11k+1)$, where $k \geq 10$. Its eigenvalues are $\lambda_2 = 11k$ and $\lambda_3 = -(k+1)$ with $m_2 = 2k(6k+1)$ and $m_3 = 22k(6k+1)$.
- (4^0) G is a strongly regular graph of order $n = 385(12k-5)^2$ and degree $r = 5(4620k^2 - 3850k + 802)$ with $\tau = 5(1925k^2 - 1645k + 351)$ and $\theta = 35(5k-2)(55k-23)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 35k-15$ and $\lambda_3 = -(385k-160)$ with $m_2 = 11(4620k^2 - 3850k + 802)$ and $m_3 = 4620k^2 - 3850k + 802$.
- ($\bar{4}^0$) G is a strongly regular graph of order $n = 385(12k-5)^2$ and degree $r = 7(4620k^2 - 3850k + 802)$ with $\tau = 7(2695k^2 - 2225k + 459)$ and $\theta = 35(7k-3)(77k-32)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 385k - 161$ and $\lambda_3 = -(35k-14)$ with $m_2 = 4620k^2 - 3850k + 802$ and $m_3 = 11(4620k^2 - 3850k + 802)$.
- (5^0) G is a strongly regular graph of order $n = 385(12k+5)^2$ and degree $r = 5(4620k^2 + 3850k + 802)$ with $\tau = 5(1925k^2 + 1645k + 351)$ and $\theta = 35(5k+2)(55k+23)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 385k + 160$ and $\lambda_3 = -(35k+15)$ with $m_2 = 4620k^2 + 3850k + 802$ and $m_3 = 11(4620k^2 + 3850k + 802)$.
- ($\bar{5}^0$) G is a strongly regular graph of order $n = 385(12k+5)^2$ and degree $r = 7(4620k^2 + 3850k + 802)$ with $\tau = 7(2695k^2 + 2225k + 459)$ and $\theta = 35(7k+3)(77k+32)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 35k+14$ and $\lambda_3 = -(385k+161)$ with $m_2 = 11(4620k^2 + 3850k + 802)$ and $m_3 = 4620k^2 + 3850k + 802$.

Theorem 2.11 ([7]). *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 12m_3$ or $m_3 = 12m_2$. Then, G is one of the following strongly regular graphs.*

- (1^0) G is the complete bipartite graph $K_{7,7}$ of order $n = 14$ and degree $r = 7$ with $\tau = 0$ and $\theta = 7$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -7$ with $m_2 = 12$ and $m_3 = 1$.
- (2^0) G is the strongly regular graph $\overline{3K_9}$ of order $n = 27$ and degree $r = 18$ with $\tau = 9$ and $\theta = 18$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -9$ with $m_2 = 24$ and $m_3 = 2$.
- (3^0) G is the strongly regular graph $\overline{4K_{10}}$ of order $n = 40$ and degree $r = 30$ with $\tau = 20$ and $\theta = 30$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -10$ with $m_2 = 36$ and $m_3 = 3$.
- (4^0) G is the strongly regular graph $\overline{6K_{11}}$ of order $n = 66$ and degree $r = 55$ with $\tau = 44$ and $\theta = 55$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -11$ with $m_2 = 60$ and $m_3 = 5$.
- (5^0) G is the strongly regular graph $\overline{12K_{12}}$ of order $n = 144$ and degree $r = 132$ with $\tau = 120$ and $\theta = 132$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -12$ with $m_2 = 132$ and $m_3 = 11$.
- (6^0) G is a strongly regular graph of order $n = (13k-1)^2$ and degree $r = k(13k-2)$ with $\tau = k^2 + 10k - 1$ and $\theta = k(k-1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 12k-1$ and $\lambda_3 = -k$ with $m_2 = k(13k-2)$ and $m_3 = 12k(13k-2)$.

- ($\bar{6}^0$) G is a strongly regular graph of order $n = (13k-1)^2$ and degree $r = 12k(13k-2)$ with $\tau = 144k^2 - 23k - 1$ and $\theta = 12k(12k-1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k-1$ and $\lambda_3 = -12k$ with $m_2 = 12k(13k-2)$ and $m_3 = k(13k-2)$.
- (7^0) G is a strongly regular graph of order $n = (13k+1)^2$ and degree $r = k(13k+2)$ with $\tau = k^2 - 10k - 1$ and $\theta = k(k+1)$, where $k \geq 11$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(12k+1)$ with $m_2 = 12k(13k+2)$ and $m_3 = k(13k+2)$.
- ($\bar{7}^0$) G is a strongly regular graph of order $n = (13k+1)^2$ and degree $r = 12k(13k+2)$ with $\tau = 144k^2 + 23k - 1$ and $\theta = 12k(12k+1)$, where $k \geq 11$. Its eigenvalues are $\lambda_2 = 12k$ and $\lambda_3 = -(k+1)$ with $m_2 = k(13k+2)$ and $m_3 = 12k(13k+2)$.
- (8^0) G is a strongly regular graph of order $n = 3(13k-3)^2$ and degree $r = 4(39k^2 - 18k + 2)$ with $\tau = 48k^2 - 45k + 7$ and $\theta = 12k(4k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k-1$ and $\lambda_3 = -(36k-8)$ with $m_2 = 12(39k^2 - 18k + 2)$ and $m_3 = 39k^2 - 18k + 2$.
- ($\bar{8}^0$) G is a strongly regular graph of order $n = 3(13k-3)^2$ and degree $r = 9(39k^2 - 18k + 2)$ with $\tau = 3(81k^2 - 34k + 3)$ and $\theta = 9(3k-1)(9k-2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k-9$ and $\lambda_3 = -3k$ with $m_2 = 39k^2 - 18k + 2$ and $m_3 = 12(39k^2 - 18k + 2)$.
- (9^0) G is a strongly regular graph of order $n = 3(13k+3)^2$ and degree $r = 4(39k^2 + 18k + 2)$ with $\tau = 48k^2 + 45k + 7$ and $\theta = 12k(4k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k+8$ and $\lambda_3 = -(3k+1)$ with $m_2 = 39k^2 + 18k + 2$ and $m_3 = 12(39k^2 + 18k + 2)$.
- ($\bar{9}^0$) G is a strongly regular graph of order $n = 3(13k+3)^2$ and degree $r = 9(39k^2 + 18k + 2)$ with $\tau = 3(81k^2 + 34k + 3)$ and $\theta = 9(3k+1)(9k+2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(36k+9)$ with $m_2 = 12(39k^2 + 18k + 2)$ and $m_3 = 39k^2 + 18k + 2$.
- (10^0) G is a strongly regular graph of order $n = 10(13k-2)^2$ and degree $r = 3(130k^2 - 40k + 3)$ with $\tau = 90k^2 - 70k + 8$ and $\theta = 15k(6k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k-1$ and $\lambda_3 = -(60k-9)$ with $m_2 = 12(130k^2 - 40k + 3)$ and $m_3 = 130k^2 - 40k + 3$.
- ($\bar{10}^0$) G is a strongly regular graph of order $n = 10(13k-2)^2$ and degree $r = 10(130k^2 - 40k + 3)$ with $\tau = 5(200k^2 - 59k + 4)$ and $\theta = 10(5k-1)(20k-3)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 60k-10$ and $\lambda_3 = -5k$ with $m_2 = 130k^2 - 40k + 3$ and $m_3 = 12(130k^2 - 40k + 3)$.
- (11^0) G is a strongly regular graph of order $n = 10(13k+2)^2$ and degree $r = 3(130k^2 + 40k + 3)$ with $\tau = 90k^2 + 70k + 8$ and $\theta = 15k(6k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 60k+9$ and $\lambda_3 = -(5k+1)$ with $m_2 = 130k^2 + 40k + 3$ and $m_3 = 12(130k^2 + 40k + 3)$.
- ($\bar{11}^0$) G is a strongly regular graph of order $n = 10(13k+2)^2$ and degree $r = 10(130k^2 + 40k + 3)$ with $\tau = 5(200k^2 + 59k + 4)$ and $\theta = 10(5k+1)(20k+3)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k$ and $\lambda_3 = -(60k+10)$ with $m_2 = 12(130k^2 + 40k + 3)$ and $m_3 = 130k^2 + 40k + 3$.

- (12⁰) G is a strongly regular graph of order $n = 14(13k-1)^2$ and degree $r = 6(182k^2 - 28k + 1)$ with $\tau = 504k^2 - 119k + 5$ and $\theta = 42k(12k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k - 1$ and $\lambda_3 = -(84k - 6)$ with $m_2 = 12(182k^2 - 28k + 1)$ and $m_3 = 182k^2 - 28k + 1$.
- (12⁰) G is a strongly regular graph of order $n = 14(13k - 1)^2$ and degree $r = 7(182k^2 - 28k + 1)$ with $\tau = 14k(49k - 5)$ and $\theta = 7(7k - 1)(14k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 84k - 7$ and $\lambda_3 = -7k$ with $m_2 = 182k^2 - 28k + 1$ and $m_3 = 12(182k^2 - 28k + 1)$.
- (13⁰) G is a strongly regular graph of order $n = 14(13k + 1)^2$ and degree $r = 6(182k^2 + 28k + 1)$ with $\tau = 504k^2 + 119k + 5$ and $\theta = 42k(12k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 84k + 6$ and $\lambda_3 = -(7k + 1)$ with $m_2 = 182k^2 + 28k + 1$ and $m_3 = 12(182k^2 + 28k + 1)$.
- (13⁰) G is a strongly regular graph of order $n = 14(13k + 1)^2$ and degree $r = 7(182k^2 + 28k + 1)$ with $\tau = 14k(49k + 5)$ and $\theta = 7(7k + 1)(14k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k$ and $\lambda_3 = -(84k + 7)$ with $m_2 = 12(182k^2 + 28k + 1)$ and $m_3 = 182k^2 + 28k + 1$.
- (14⁰) G is a strongly regular graph of order $n = 30(13k - 6)^2$ and degree $r = 5(390k^2 - 360k + 83)$ with $\tau = 10(75k^2 - 76k + 19)$ and $\theta = 10(5k - 2)(15k - 7)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 10k - 5$ and $\lambda_3 = -(120k - 55)$ with $m_2 = 12(390k^2 - 360k + 83)$ and $m_3 = 390k^2 - 360k + 83$.
- (14⁰) G is a strongly regular graph of order $n = 30(13k - 6)^2$ and degree $r = 8(390k^2 - 360k + 83)$ with $\tau = 2(960k^2 - 865k + 194)$ and $\theta = 40(2k - 1)(24k - 11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 120k - 56$ and $\lambda_3 = -(10k - 4)$ with $m_2 = 390k^2 - 360k + 83$ and $m_3 = 12(390k^2 - 360k + 83)$.
- (15⁰) G is a strongly regular graph of order $n = 30(13k + 6)^2$ and degree $r = 5(390k^2 + 360k + 83)$ with $\tau = 10(75k^2 + 76k + 19)$ and $\theta = 10(5k + 2)(15k + 7)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 120k + 55$ and $\lambda_3 = -(10k + 5)$ with $m_2 = 390k^2 + 360k + 83$ and $m_3 = 12(390k^2 + 360k + 83)$.
- (15⁰) G is a strongly regular graph of order $n = 30(13k + 6)^2$ and degree $r = 8(390k^2 + 360k + 83)$ with $\tau = 2(960k^2 + 865k + 194)$ and $\theta = 40(2k + 1)(24k + 11)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 10k + 4$ and $\lambda_3 = -(120k + 56)$ with $m_2 = 12(390k^2 + 360k + 83)$ and $m_3 = 390k^2 + 360k + 83$.
- (16⁰) G is a strongly regular graph of order $n = 66(13k - 1)^2$ and degree $r = 2(858k^2 - 132k + 5)$ with $\tau = 264k^2 - 143k + 9$ and $\theta = 22k(12k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 11k - 1$ and $\lambda_3 = -(132k - 10)$ with $m_2 = 12(858k^2 - 132k + 5)$ and $m_3 = 858k^2 - 132k + 5$.
- (16⁰) G is a strongly regular graph of order $n = 66(13k - 1)^2$ and degree $r = 11(858k^2 - 132k + 5)$ with $\tau = 22(11k - 1)(33k - 2)$ and $\theta = 11(11k - 1)(66k - 5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 132k - 11$ and $\lambda_3 = -11k$ with $m_2 = 858k^2 - 132k + 5$ and $m_3 = 12(858k^2 - 132k + 5)$.

- (17⁰) G is a strongly regular graph of order $n = 66(13k + 1)^2$ and degree $r = 2(858k^2 + 132k + 5)$ with $\tau = 264k^2 + 143k + 9$ and $\theta = 22k(12k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 132k + 10$ and $\lambda_3 = -(11k + 1)$ with $m_2 = 858k^2 + 132k + 5$ and $m_3 = 12(858k^2 + 132k + 5)$.
- (17⁰) G is a strongly regular graph of order $n = 66(13k + 1)^2$ and degree $r = 11(858k^2 + 132k + 5)$ with $\tau = 22(11k + 1)(33k + 2)$ and $\theta = 11(11k + 1)(66k + 5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 11k$ and $\lambda_3 = -(132k + 11)$ with $m_2 = 12(858k^2 + 132k + 5)$ and $m_3 = 858k^2 + 132k + 5$.

3. CONCLUDING REMARKS

Using equations (1.3) and (1.4), and applying the same procedure as in articles [4–7], we can establish the parameters n , r , τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for any fixed value $q \in \mathbb{N}$, by using only one parameter k . All results presented in this work has been verified using a computer program `srgpar.exe`, which was written by the author in the programming language `Borland C++ Builder 5.5`. using only one parameter k .

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ON GENERATING RELATIONS ASSOCIATED WITH THE EXTENDED GAUSS AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this research note, we establish a new class of generating relations associated with the extended Gauss and confluent hypergeometric functions using the concept of Hadamard product. Some deductions of our main results are also indicated.

1. INTRODUCTION AND PRELIMINARIES

The generating relations play a very important role in the study of the functions. In particular, in the field of special functions several important and useful properties have been investigated by making use of the generating relations (see, for example, [3, 4, 7, 9, 13] and [14]). Due to great importance of such type of relations, in this paper, we establish some new interesting generating relations of the extended Gauss and confluent hypergeometric functions using the Hadamard product of two functions.

The Hadamard product of two functions is defined as follows.

Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ be two power series whose radii of convergence are given by R_f and R_g , respectively. Then, the Hadamard product is defined by (see [16])

$$(1.1) \quad (f * g)(z) = \sum_{n=0}^{+\infty} a_n b_n z^n,$$

whose radius of convergence R satisfies $R_f R_g \leq R$.

Key words and phrases. Gauss hypergeometric function, extended Gauss hypergeometric function, confluent hypergeometric function, extended confluent hypergeometric function, Hadamard product.

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In recent years, numerous extensions of some well known special functions have been introduced and investigated by a number of authors (see, for example, [1, 2, 5–8, 10–13, 17, 18] and [19]). In particular, in 2014, Srivastava et al. [3] proposed a very interesting generalization of the Gauss hypergeometric function as follows:

$$(1.2) \quad F_p^{(\alpha, \beta, u, v)}(a, b; c; z) = \sum_{n=0}^{+\infty} (a)_n \frac{B_p^{(\alpha, \beta, u, v)}(b+n, c-b)}{B(b, c-b) n!} z^n$$

$$(\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(u), \operatorname{Re}(v)\} > 0; \operatorname{Re}(c) > \operatorname{Re}(b) > 0; \operatorname{Re}(p) \geq 0; |z| < 1),$$

where $B_p^{(\alpha, \beta, u, v)}$ is the extended beta function defined by

$$(1.3) \quad B_p^{(\alpha, \beta, u, v)}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \Phi\left(\alpha; \beta; -\frac{p}{t^u(1-t)^v}\right) dt$$

$$(\min\{\operatorname{Re}(x), \operatorname{Re}(y), \operatorname{Re}(\alpha), \operatorname{Re}(\beta)\} > 0; \min\{\operatorname{Re}(u), \operatorname{Re}(v)\} > 0; \operatorname{Re}(p) \geq 0).$$

With the help of (1.3), we can write the integral representation of (1.2) as follows:

$$(1.4) \quad F_p^{(\alpha, \beta, u, v)}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \\ \times \Phi\left(\alpha; \beta; \frac{-p}{t^u(1-t)^v}\right) dt,$$

where $(\operatorname{Re}(p) \geq 0; |\arg(1-z)| < \pi; \operatorname{Re}(c) > \operatorname{Re}(b) > 0; \min\{\operatorname{Re}(u), \operatorname{Re}(v)\} > 0)$.

Moreover, Khan et al. [12] proposed a new generalized form of the confluent hypergeometric function as follows:

$$(1.5) \quad \Phi_p^{(\alpha, \beta; u, v)}(b; c; z) = \sum_{n=0}^{+\infty} \frac{B_p^{(\alpha, \beta; u, v)}(b+n, c-b)}{B(b, c-b)} \cdot \frac{z^n}{n!}.$$

They also defined its integral representation as

$$(1.6) \quad \Phi_p^{(\alpha, \beta; u, v)}(b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \exp(z t) \Phi\left(\alpha; \beta; \frac{-p}{t^u(1-t)^v}\right) dt$$

$$(\operatorname{Re}(c) > \operatorname{Re}(b) > 0; \operatorname{Re}(u) > 0, \operatorname{Re}(v) > 0; \operatorname{Re}(p) \geq 0).$$

The case $u = v$ in (1.4) and (1.6), respectively, yields the extended Gauss and confluent hypergeometric functions defined by Parmar [15], which further gives the known generalization of Gauss and confluent hypergeometric functions given by Özergin et al. [2] by taking $v = 1$. Also, it is noticed that, if we set $\alpha = \beta$ and $u = v$ in (1.4) and (1.6), respectively, we get the extended Gauss and confluent hypergeometric functions defined by Lee et al. [1] and if we set $\alpha = \beta$ and $u = v = 1$ in (1.4) and (1.6), respectively, then we get the extended Gauss and confluent hypergeometric functions defined by Chaudhry et al. [8]. Clearly, for $p = 0$, (1.4) and (1.6) immediately reduces to the classical Gauss and confluent hypergeometric functions (see [4]).

Also, in 2014, Choi et al. [5] introduced the following extension of Gauss and confluent hypergeometric functions by introducing another parameter:

$$(1.7) \quad F_{p,q}(a, b; c; z) = \sum_{n=0}^{+\infty} (a)_n \frac{B_{p,q}(b+n; c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$$(\operatorname{Re}(c) > \operatorname{Re}(b) > 0; \operatorname{Re}(p) \geq 0, \operatorname{Re}(q) \geq 0; |z| < 1).$$

and

$$(1.8) \quad \Phi_{p,q}(b; c; z) = \sum_{n=0}^{+\infty} \frac{B_{p,q}(b+n; c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$$(\operatorname{Re}(c) > \operatorname{Re}(b) > 0; \operatorname{Re}(p) \geq 0, \operatorname{Re}(q) \geq 0; |z| < 1),$$

where

$$(1.9) \quad B_{p,q}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt$$

$$\operatorname{Re}(p), \operatorname{Re}(q), \operatorname{Re}(x), \operatorname{Re}(y) > 0.$$

They also give their integral representation by

$$(1.10) \quad F_{p,q}(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt$$

$$(\operatorname{Re}(p) > \operatorname{Re}(q) > 0; \operatorname{Re}(c) > \operatorname{Re}(b) > 0)$$

and

$$(1.11) \quad \Phi_{p,q}(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \exp\left(zt - \frac{p}{t} - \frac{q}{1-t}\right) dt$$

$$(\operatorname{Re}(p) > \operatorname{Re}(q) > 0; \operatorname{Re}(c) > \operatorname{Re}(b) > 0).$$

For $q = p$, (1.7) and (1.8) reduces to the known extension of Gauss and confluent hypergeometric functions defined by Chaudhary et al. [8], which again for $p = 0$, yields the classical Gauss and confluent hypergeometric functions (see [4]).

2. MAIN RESULTS

In this section, we derive a new class of generating relations of extended Gauss and confluent hypergeometric functions by using the Hadamard product of two functions.

Theorem 2.1. *For $\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(u), \operatorname{Re}(v)\} > 0$; $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, the following result holds true:*

$$(2.1) \quad (1+t)^{-\lambda} F_p^{(\alpha, \beta, u, v)}\left(a, b; c; \frac{z}{1+t}\right)$$

$$= \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r F_p^{(\alpha, \beta, u, v)}(a, b; c; z) * {}_2F_1(\lambda+r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where ${}_2F_1(a, b; c; z)$ represents the classical Gauss hypergeometric function (see [4, p. 29, Eq. (4)]).

Proof. In order to derive (2.1), we denote the left-hand side of (2.1) by $f(s)$ and $(1+t)$ by s , then we get

$$(2.2) \quad f(s) = s^{-\lambda} F_p^{(\alpha, \beta, u, v)} \left(a, b; c; \frac{z}{s} \right).$$

On expanding $F_p^{(\alpha, \beta, u, v)}$ with the help of (1.2), we arrive at

$$(2.3) \quad f(s) = \sum_{n=0}^{+\infty} \frac{(a)_n B_p^{(\alpha, \beta, u, v)}(b+n, c-b)}{B(b, c-b)} s^{-\lambda-n} \frac{z^n}{n!}.$$

Now differentiating (2.3) r times with respect to s and after some simplification, we get

$$(2.4) \quad f^{(r)}(s) = (-1)^r s^{-\lambda-r} (\lambda)_r \sum_{n=0}^{+\infty} \frac{(a)_n B_p^{(\alpha, \beta, u, v)}(b+n, c-b)}{B(b, c-b)n!} \cdot \frac{(\lambda+r)_n (1)_n}{(\lambda)_n n!} \left(\frac{z}{s} \right)^n.$$

By applying (1.1) in (2.4), we have

$$(2.5) \quad f^{(r)}(s) = (-1)^r s^{-\lambda-r} (\lambda)_r F_p^{(\alpha, \beta, u, v)} \left(a, b; c; \frac{z}{s} \right) * {}_2F_1 \left(\lambda+r, 1; \lambda; \frac{z}{s} \right).$$

Now, on replacing s by $s+t$ in (2.2), and then expanding $f(s+t)$ by Taylor's series, we get

$$(2.6) \quad (s+t)^{-\lambda} F_p^{(\alpha, \beta, u, v)} \left(a, b; c; \frac{z}{s+t} \right) = \sum_{r=0}^{+\infty} \frac{t^r}{r!} f^{(r)}(s).$$

Using (2.5) in (2.6), we obtain

$$(2.7) \quad \begin{aligned} (s+t)^{-\lambda} F_p^{(\alpha, \beta, u, v)} \left(a, b; c; \frac{z}{s+t} \right) &= \sum_{r=0}^{+\infty} (-1)^r s^{-\lambda-r} (\lambda)_r \\ &\times F_p^{(\alpha, \beta, u, v)} \left(a, b; c; \frac{z}{s} \right) * {}_2F_1 \left(\lambda+r, 1; \lambda; \frac{z}{s} \right) \frac{t^r}{r!}. \end{aligned}$$

Finally, on setting $s=1$ in (2.7), we get our claimed result (2.1). $\square$

Theorem 2.2. For $\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(u), \operatorname{Re}(v)\} > 0$; $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, the following result holds true:

$$(2.8) \quad \begin{aligned} (1+t)^{-\lambda} \Phi_p^{(\alpha, \beta, u, v)} \left(b; c; \frac{z}{1+t} \right) &= \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r \Phi_p^{(\alpha, \beta, u, v)}(b; c; z) * {}_2F_1(\lambda+r, 1; \lambda; z) \frac{t^r}{r!} \\ (z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1). \end{aligned}$$

Theorem 2.3. For $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, $\operatorname{Re}(p) \geq 0$ and $\operatorname{Re}(q) \geq 0$, the following result holds true:

$$(2.9) \quad (1+t)^{-\lambda} F_{p,q} \left(a, b; c; \frac{z}{1+t} \right) = \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r F_{p,q}(a, b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1).$$

Theorem 2.4. For $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, $\operatorname{Re}(p) \geq 0$ and $\operatorname{Re}(q) \geq 0$, the following result holds true:

$$(2.10) \quad (1+t)^{-\lambda} \Phi_{p,q} \left(b; c; \frac{z}{1+t} \right) = \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r \Phi_{p,q}(b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1).$$

Proof. The proofs of Theorem 2.2, Theorem 2.3 and Theorem 2.4 are similar to Theorem 2.1. $\square$

3. SPECIAL CASES

Corollary 3.1. For $\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(v)\} > 0$, $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.1) \quad (1+t)^{-\lambda} F_p^{(\alpha, \beta, v)} \left(a, b; c; \frac{z}{1+t} \right)$$

$$= \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r F_p^{(\alpha, \beta, v)}(a, b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $F_p^{(\alpha, \beta, v)}(\cdot)$ is the extended Gauss hypergeometric function defined by Parmar [15].

This corollary can be established with the help of (2.1) by putting $u = v$.

Corollary 3.2. For $\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta)\} > 0$, $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.2) \quad (1+t)^{-\lambda} F_p^{(\alpha, \beta)} \left(a, b; c; \frac{z}{1+t} \right)$$

$$= \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r F_p^{(\alpha, \beta)}(a, b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $F_p^{(\alpha, \beta)}(\cdot)$ represents the extended Gauss hypergeometric function given by Özergin et al. [2].

This corollary can be derived with the help of (2.1) by putting $u = v = 1$.

Corollary 3.3. For $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, $\operatorname{Re}(v) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.3) \quad (1+t)^{-\lambda} F_p^v \left(a, b; c; \frac{z}{1+t} \right) = \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r F_p^v(a, b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $F_p^v(\cdot)$ denotes the extended Gauss hypergeometric function introduced by Lee et al. [1].

The above corollary can be obtained with the help of (2.1) by setting $\alpha = \beta$ and $u = v$.

Corollary 3.4. For $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.4) \quad (1+t)^{-\lambda} F_p \left(a, b; c; \frac{z}{1+t} \right) = \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r F_p(a, b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $F_p(\cdot)$ is the extended Gauss hypergeometric function defined by Chaudhry et al. [8].

This corollary can be proved with the help of (2.1) by setting $\alpha = \beta$ and $u = v = 1$.

Corollary 3.5. For $\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(v)\} > 0$; $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.5) \quad (1+t)^{-\lambda} \Phi_p^{(\alpha, \beta, v)} \left(b; c; \frac{z}{1+t} \right) = \sum_{r=0}^{\infty} (-1)^r (\lambda)_r \Phi_p^{(\alpha, \beta, v)}(b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $\Phi_p^{(\alpha, \beta, v)}(\cdot)$ is the extended confluent hypergeometric function defined by Parmar [15].

This corollary can be established with the help of (2.8) by putting $u = v$.

Corollary 3.6. For $\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta)\} > 0$, $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.6) \quad (1+t)^{-\lambda} \Phi_p^{(\alpha, \beta)} \left(b; c; \frac{z}{1+t} \right) = \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r \Phi_p^{(\alpha, \beta)}(b; c; z) * {}_2F_1(\lambda + r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $\Phi_p^{(\alpha, \beta)}(\cdot)$ represents the extended confluent hypergeometric function given by Özergin et al. [2].

This corollary can be derived with the help of (2.8) by putting $u = v = 1$.

Corollary 3.7. For $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, $\operatorname{Re}(v) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.7) \quad (1+t)^{-\lambda} \Phi_p^v\left(b; c; \frac{z}{1+t}\right) = \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r \Phi_p^v(b; c; z) * {}_2F_1(\lambda+r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $\Phi_p^v(\cdot)$ denotes the extended confluent hypergeometric function introduced by Lee et al. [1].

This corollary can be established with the help of (2.8) by setting $\alpha = \beta$ and $u = v$.

Corollary 3.8. For $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $\operatorname{Re}(p) \geq 0$, we have

$$(3.8) \quad (1+t)^{-\lambda} \Phi_p\left(b; c; \frac{z}{1+t}\right) = \sum_{r=0}^{+\infty} (-1)^r (\lambda)_r \Phi_p(b; c; z) * {}_2F_1(\lambda+r, 1; \lambda; z) \frac{t^r}{r!}$$

$$(z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus Z_0^-, |t| < 1),$$

where $\Phi_p(\cdot)$ is the extended confluent hypergeometric function defined by Chaudhry et al. [8].

This corollary can be obtained with the help of (2.8) by setting $\alpha = \beta$ and $u = v = 1$.

Remark 3.1. If we set $q = p$ in (2.9) and (2.10), then we can easily get our known results given in (3.4) and (3.8), respectively.

CONCLUSION

In this paper, we have derived a set of generating relations associated with the extended Gauss and confluent hypergeometric functions, which are defined by some well known authors namely Srivastava et al. [3], Khan et al. [12] and Choi et al. [5]. Also, we have investigated the special cases of our main findings. Very recently, numerous authors have introduced and studied several extensions of Gauss and confluent hypergeometric functions (see, for example, [1, 2] and [3]). Therefore, it is noticed that by using the method outlined in this paper we can establish some more interesting generating relations for the functions given in [1, 2] and [3]. Thus, the technique adopted in this article provides a very flexible and powerful tool of yielding the new results in the theory of special functions.

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