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ON THE BRYANT-SCHNEIDER GROUP OF RIGHT CHEBAN LOOP

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Abstract. In this study, we investigate the Bryant-Schneider group of right Cheban loops. The study started with establishing some results on the algebraic properties of right Cheban loop. Consequently, it is established that every right pseudo-automorphism of a right Cheban loop is an element of the Bryant-Schneider group. It is shown that the middle inner mapping T_a is an automorphism of a right Cheban loop if and only if $a \in N_\mu$ of the right Cheban loop. R_a and L_a are subsequently shown to be elements of the Bryant-Schneider group. The Crypto-automorphism group of right Cheban loop is investigated. A two-middle pseudo-automorphism group, the crypto-automorphism group and the Bryant-Scheider group of a loop are found to coincide.

1. Introduction

Cheban loops are loops of the generalized Bol-Moufang type. Both left Cheban loops and Cheban loops were introduced by Cheban [19]. The author also showed that Cheban loops are generalized Moufang loops. The structural properties of left Cheban loops and Cheban loops were studied in [12, 19]. Construction of a right Cheban loop of small orders was presented in [10] and the holomorph of the right Cheban loops was studied in [11]. Some recent results in the theory of loops, especially Bol-Moufang and non-Bol-Moufang types, can be found in [4]. Robinson [21] introduced the idea of Bryant-Schneider group of a loop and the study was motivated by the work of Bryant and Schneider [5]. Since the advent of the Bryant-Schneider group, some studies by

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Adeniran [2,3], and Chiboka [9] have been done on it relative to CC-loops, C-loops, and extra loops after Robinson [21] studied the Bryant-Schneider group of a Bol loop. The study of subgroup of Bryant-Schneider group of a Basarab loop was recently announced by Osoba et al. [17]. As mentioned in [21], the Bryant-Schneider group of a loop is extremely useful in investigating isotopy condition(s) in loops. Adeniran, in [1,2], established a condition under which an element of the Bryant Schneider group of a C-loop will form an automorphism. Chiboka in [8] got a similar result for extra loops but the condition was necessary and sufficient which is not the case for C-loops in [2]. Adeniran [2] went further to show that elements of the Bryant-Schneider group of a C-loop can be expressed as a product of pseudo-automorphism and right translations of elements of the nucleus of the loop but Chiboka [8] corresponding result on extra loops was a general form of this and was also possible for left translations. The two authors confirmed that the Bryant-Schneider group of the two loops is a kind of generalized holomorph of the loops. In fact, this result is true for Bol loops [21]. The crypto-automorphism groups of some quasigroups were studied in [16] and the set of all crypto-automorphism groups of a quasigroup with left and right identity elements was shown to form a group. Consequently, different characterizations of subgroups of the crypto-automorphism groups of a middle Bol loop were presented and the crypto-automorphism groups and the Bryant-Schneider groups of a loop were found to coincide. In this study, we investigate the Bryant-Schneider groups of right Cheban loops. It is established that every right pseudo-automorphism of a right Cheban loop is an element of the Bryant-Schneider group. It is shown that the middle inner mapping T_a is an automorphism of right Cheban loop if and only if $a \in N_\mu$ of the right Cheban loop. R_a and L_a are subsequently shown to be elements of the Bryant-Schneider group. Crypto-automorphism group of right Cheban loop is investigated. A two-middle pseudo-automorphism groups, the crypto-automorphism groups and the Bryant-Schneider groups of a loop are found to coincide.

2. Preliminaries

In this section, we give definitions of terminologies and present some previous results used in this work. Let Q be a non-empty set. Define a binary operation $\cdot$ on Q. If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid. If the system of equations:

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b.$$

have unique solutions in Q for x and y respectively, then $(Q, \cdot)$ is called a quasigroup. If there exists a unique element $e \in Q$, called the identity element, such that for all $x \in Q$

$$x \cdot e = e \cdot x = x.$$

$(Q, \cdot)$ is called a loop. Let x be a fixed element in a groupoid $(Q, \cdot)$. The left and right translation maps of Q, L_x and R_x are, respectively, defined by

$$yL_x = x \cdot y \quad \text{and} \quad yR_x = y \cdot x.$$

Also, a quasigroup is a groupoid in which the left and right translations defined on the groupoids are bijections. Since left and right translation mappings of a loop are bijectives, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x\backslash y = yL_x^{-1} \quad \text{and} \quad y/x = yR_x^{-1},$$

and note that, $x\backslash y = z$ if and only if $x \cdot z = y$ and $x/y = z$ if and only if $z \cdot y = x$. A loop identity is of Bol-Moufang type if two variables of its three variables occur once on each side, the third variable occurs twice on each side and the order in which the variables appear on both sides is the same, for instance, $(xy \cdot z)y = x(yz \cdot y)$. Phillips and Vojtechovsky in [20] showed that there are fourteen such varieties, they also extended and completed Fenyves work in [13]. The identity of Bol-Moufang type can be generalized by assigning different variable orders on either side of the identity, for instance, $(y \cdot xx)z = (y \cdot zx)x$. The varieties of loops axiomatized by a single identity of this type are said to be of generalized Bol-Moufang type [12]. Right, left and Cheban loops are loops of the generalized Bol-Moufang type. Both Left Cheban and Cheban loops were introduced by Cheban in [6]. He showed that Cheban loops are generalized Moufang loops. Phillips and Shcherbacov in [19] also studied on the structural properties of left Cheban and Cheban loops. They established that left Cheban loops are left conjugacy closed (LCC). Furthermore, they proved that Cheban loops have weak inverse property, are power associative and are conjugacy closed loops.

Definition 2.1. A loop satisfying the identity

$$(z \cdot yx)x = zx \cdot xy \tag{2.1}$$

is called a *Right Cheban loop* (RChL). *Left Cheban loops* on the other hand are loops satisfying the mirror identity

$$x(xy \cdot z) = yx \cdot xz. \tag{2.2}$$

Loops that are both right and left Cheban are called *Cheban loops*. Cheban loops can also be characterized as those loops that satisfy the identity

$$x(xy \cdot z) = (y \cdot zx)x. \tag{2.3}$$

Definition 2.2. A loop $(Q, \cdot)$ is called a left inverse property loop if it satisfies the left inverse property (LIP) given by: $x^\lambda(xy) = y$.

Definition 2.3. A loop $(Q, \cdot)$ is called a right inverse property loop if it satisfies the right inverse property (RIP) given by: $(yx)x^\rho = y$.

A loop is called an IP loop if it is both LIP-loop and RIP-loop.

Definition 2.4. Let $(Q, \cdot)$ be a loop and $BS(Q, \cdot)$ be a set of all permutations θ of Q such that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta)$ is an autotopism of $(Q, \cdot)$ for some $f, g \in Q$. Then, $BS(Q, \cdot)$ is called the *Bryant-Schneider group* of the loop $(Q, \cdot)$.

Definition 2.5. Let $BS_\lambda(Q,\cdot) = \{\theta \in SYM(Q,\cdot) : \exists g \in Q \ni (\theta R_g^{-1},\theta,\theta) \in AUT(Q,\cdot)\}$, that is the set of all special maps in a loop, then $BS_\lambda \leq BS(Q,\cdot)$ is called the left Bryant-Schneider group of the $(Q,\cdot)$ and

$$BS_\rho(Q,\cdot) = \{\theta \in SYM(Q,\cdot) : \exists\, g \in Q \ni (\theta,\theta L_f^{-1},\theta) \in AUT(Q,\cdot)\},$$

that is the set of all special maps in a loop, then $BS_\rho \leq BS(Q,\cdot)$ is called the right Bryant-Schneider group of the $(Q,\cdot)$.

Definition 2.6. A loop $(Q,\cdot)$ is called an automorphic inverse property loop if it satisfies the automorphic inverse identity given by $(xy)^{-1} = x^{-1}y^{-1}$.

Definition 2.7. A triple (U,V,W) of bijections from a set G onto a set H is called an *isotopism* of a groupoid $(G,\cdot)$ onto groupoid $(H,\circ)$ provided $xU \circ yV = (xy)W$ for all $x,y \in G$. $(H,\circ)$ is then called an isotope of $(G,\cdot)$, and groupoids $(G,\cdot)$ and $(H,\circ)$ are said to be *isotopic* to each other. An isotopism of $(G,\cdot)$ onto $(G,\cdot)$ is called an *autotopism* of $(G,\cdot)$.

Definition 2.8. A bijection U on Q is called a *right pseudo-automorphism* of a quasigroup $(Q,\cdot)$ if there exists at least one element $c \in Q$ such that

$$xU \cdot (yU \cdot c) = (xy)U \cdot c, \quad \text{for all } x,y \in Q.$$

The element c is then called a *companion* of U.

Definition 2.9. The inner mapping group I is generated by the permutations $R_{(x,y)} = R_xR_yR_{(xy)^{-1}}$, $L_{(x,y)} = L_xL_yL_{(yx)^{-1}}$ and $T_x = R_xL_{x^{-1}}$. These permutations are called inner mappings.

Definition 2.10. The left nucleus of $(Q,\cdot)$ denoted by

$$N_\lambda(Q,\cdot) = \{a \in Q : a \cdot xy = ax \cdot y \text{ for all } x,y \in Q\}.$$

The right nucleus of $(Q,\cdot)$ denoted by

$$N_\rho(Q,\cdot) = \{a \in Q : xy \cdot a = x \cdot ya \text{ for all } x,y \in Q\}.$$

The middle nucleus of $(Q,\cdot)$ denoted by

$$N_\mu(Q,\cdot) = \{a \in Q : xa \cdot y = x \cdot ay \text{ for all } x,y \in Q\}.$$

The nucleus of $(Q,\cdot)$ denoted by

$$N(Q,\cdot) = N_\lambda(Q,\cdot) \bigcap N_\rho(Q,\cdot) \bigcap N_\mu(Q,\cdot).$$

The centrum of $(Q,\cdot)$ denoted by

$$C(Q,\cdot) = \{a \in Q : ax = xa \text{ for all } x \in Q\}.$$

The center of $(Q,\cdot)$ denoted by

$$Z(Q,\cdot) = N(Q,\cdot) \bigcap C(Q,\cdot).$$

Definition 2.11. Let $(Q,\cdot)$ be a quasigroup. A mapping $\phi \in SYM(Q)$ will be called a two-middle pseudo-automorphism if there exist $a,b \in Q$ such that $(\phi R_a^{-1}, \phi L_{b^\lambda}^{-1}, \phi) \in AUT(Q,\cdot)$. The set of two-middle pseudo-automorphisms will be represented by $PS_{\mu_2}(Q,\cdot)$. Note that $PS_\mu(Q,\cdot) \subseteq PS_{\mu_2}(Q,\cdot)$.

Lemma 2.1. *Let Q be a loop and a its element. Then,*

(i) $a \in N_\lambda$ *if and only if* $(L_a, I, L_a) \in AUT(Q)$*;*
(ii) $a \in N_\rho$ *if and only if* $(I, R_a, R_a) \in AUT(Q)$*;*
(iii) $a \in N_\mu$ *if and only if* $(R^{-1}, L_a, I) \in AUT(Q)$*.*

Theorem 2.1 ([18])**.** *Let (U, V, W) be an autotopism of a loop $(Q, \cdot)$.*

(i) *If $(Q, \cdot)$ is an RIPL, then (W, JVJ, U) is also an autotopism of $(Q, \cdot)$.*
(ii) *If $(Q, \cdot)$ is an LIPL, then (JUJ, W, V) is also an autotopism of $(Q, \cdot)$.*

Theorem 2.2. *A loop Q is a Right Conjugacy Closed if and only if (R_x, T_x, R_x) is an autotopism for all $x \in Q$.*

Theorem 2.3 ([7])**.** *A loop Q is left Cheban if and only if it is Left Conjugacy Closed (LCC) and $R_x^2 = L_x^2$, for all $x \in Q$.*

Theorem 2.4 ([16])**.** *The set of crypto-automorphisms $CAUM(Q, \cdot)$ of a quasigroup $(Q, \cdot)$ with right and left identity elements forms a group.*

Theorem 2.5 ([16])**.** *Let $(Q, \cdot)$ be a loop. Then, $BS(Q, \cdot) = CAUM(Q, \cdot)$.*

3. Algebraic Properties of Right Cheban Loop

Theorem 3.1. *Let $(Q, \cdot)$ be a RChL. Then, the left and right inverses coincide and thus, $x^\lambda = x^\rho = x^{-1}$.*

Proof. If $(Q, \cdot)$ is a RChL, then we have $(z \cdot yx)x = zx \cdot xy$. Setting $y = x^\rho$ in the RChL identity, we obtain

$$(z \cdot x^\rho x)x = zx \cdot xx^\rho, \quad (z \cdot x^\rho x)x = zx, \quad (z \cdot x^\rho x) = z, \quad x^\rho x = e, \quad x^\rho x = e.$$

This implies $x^\rho = x^\lambda$. Therefore, $x^\lambda = x^\rho = x^{-1}$. □

Remark 3.1. This result implies that every element in a right Cheban loop has a unique inverse.

Theorem 3.2. *Commutative RChL are Abelian groups.*

Proof. Let $(Q, \cdot)$ be a commutative RChL. Then,

$$(z \cdot yx)x = zx \cdot xy = zx \cdot yx \quad \text{(by commutativity)}. \tag{3.1}$$

Setting $yx = a$ in (3.1), we have $az \cdot x = za \cdot x = zx \cdot a$. This implies $az \cdot x = a \cdot zx$ for all $a, x, z \in Q$. This shows that associativity holds in $(Q, \cdot)$. Hence, $(Q, \cdot)$ is an Abelian group. □

Theorem 3.3. *A loop $(Q, \cdot)$ is a RChL if and only if $(R_x, R_x^{-1}L_x, R_x) \in AUT(Q, \cdot)$.*

Proof. Suppose $(Q, \cdot)$ is a RChL. Then,

$$(z \cdot yx)x = zx \cdot xy, \quad (z \cdot yR_x)R_x = zR_x \cdot yL_x.$$

Setting $y = yR_x^{-1}$, we have

$$(zy)R_x = zR_x \cdot yR_x^{-1}L_x, \quad (R_x, R_x^{-1}L_x, R_x) \in AUT(Q, \cdot).$$

Conversely, suppose $(R_x, R_x^{-1}L_x, R_x) \in AUT(Q, \cdot)$. Then, letting this autotopism take on the product yz as argument, we have $yR_x \cdot zR_x^{-1}L_x = (yz)R_x$. Setting $z = zR_x$ to give

$$yR_x \cdot zL_x = (y \cdot zR_x)R_x, \quad yx \cdot xz = (y \cdot zx)x.$$

□

Theorem 3.4. *Let $(Q, \cdot)$ is a RChL. Then, $[R_{x^2}, R_x] = I$ for all $x \in Q$.*

Proof. Let $(Q, \cdot)$ be a $RChL$. Then, $(z \cdot yx)x = zx \cdot xy$. Setting $y = x$ in the above equation, we have $(z \cdot x^2)x = zx \cdot x^2$. This implies $zR_{x^2}R_x = zR_xR_{x^2}$. Thus, $[R_{x^2}, R_x] = I$ (i.e., R_{x^2} and R_x commute). □

Corollary 3.1. *Right Cheban loops are 3-power associative.*

Proof. By Theorem 3.4, $R_{x^2}R_x = R_xR_{x^2}$. Then, $eR_{x^2}R_x = eR_xR_{x^2}$. Thus, $x^2 \cdot x = x \cdot x^2$. It then follows that $x^3 = xxx$. □

Remark 3.2. Theorem 3.4 and Corollary 3.1 demonstrate a weaker form of Power associativity.

4. Bryant-Schneider Group of Right Cheban Loop

In the foregoing, we give different characterisations of right Cheban loops in terms of its automorphisms and pseudo-automorphisms with associated companion(s). These characterisations are later used to study Bryant-Schneider group of right Cheban loop.

Lemma 4.1. *Let $(Q, \cdot)$ be a RChL. Then, the following are equivalent:*

(1) *flexibility;*
(2) *left inverse property.*

Proof. By setting $y = y/x$ in RChL identity (2.1), one obtain

$$(z \cdot y)x = zx \cdot x(y/x). \tag{4.1}$$

(1) if and only if (2). Set $z = e$ in a RChL (4.1),

$$yR_x = yR_x^{-1}L_xL_x, \quad R_xL_x^{-1} = R_x^{-1}L_x, \quad L_x^{-1}R_x = L_xR_x^{-1}. \tag{4.2}$$

Now, setting $z = x^\lambda$ in (4.1), to get

$$(x^\lambda y)x = x(y/x), \quad R_x^{-1}L_x = L_{x^\lambda}R_x. \tag{4.3}$$

Then, in a RChL, (4.2) = (4.3) if and only if $L_xR_x^{-1} = R_x^{-1}L_x$ if and only if $yL_xR_x^{-1} = yR_x^{-1}L_x$ if and only if $(xy)/x = x \cdot (y/x)$ if and only if $xy \cdot x = x \cdot yx$. So, a RChL is flexible if and only if $L_x^{-1}R_x = L_{x^\lambda}R_x$, if and only if $L_x^{-1} = L_{x^\lambda}$ if and only if $yL_x^{-1} = yL_{x^\lambda}$ if and only if $x^\lambda y = x\backslash y$ if and only if $x^\lambda \cdot xy = y$. □

Theorem 4.1. *Let $(Q, \cdot)$ be a RChL with flexibility property and let $x \in N_\mu$ for all $x \in Q$. if $\alpha \in BS(Q, \cdot)$. Then, $\alpha \in AUM(Q, \cdot)$.*

Proof. Let $(Q,\cdot)$ be a RChL and $x \in N_\mu$ implies that $H = (R_x, L_x^{-1}, I) \in AUT(Q,\cdot)$ for all $x \in Q$. $\alpha \in BS(Q,\cdot)$ if and only if $J = (\alpha R_t^{-1}, \alpha L_k^{-1}, \alpha) \in AUT(Q,\cdot)$ for some $k,t \in Q$. Then, the product $HJ = (\alpha R_t^{-1} R_x, \alpha L_k^{-1} L_x^{-1}, \alpha) \in AUT(Q,\cdot)$ for some $k,t \in Q$ and for all $x \in Q$.

Setting $t = x$, we have $(\alpha, \alpha L_k^{-1} L_x^{-1}, \alpha) \in AUT(Q,\cdot)$ for some $k,t \in Q$ and for all $x \in Q$. Since $(Q,\cdot)$ is flexible if and only if it satisfies LIP from Lemma 4.1, then $(\alpha, \alpha L_k^{-1} L_x^{-1}, \alpha) \in AUT(Q,\cdot)$ for all $x \in Q$ and for some $k \in Q$. Setting $k = x^{-1}$ to get $(\alpha, \alpha L_{x^{-1}}^{-1} L_x^{-1}, \alpha) \in AUT(Q,\cdot)$. Hence, $\alpha \in AUM(Q,\cdot)$. □

Theorem 4.2. *Let $(Q,\cdot)$ be a RChL with flexibility property. If $\alpha \in BS(Q,\cdot)$, then α is a pseudo-automorphism with companion x^{-1} only if $t = k = x$.*

Proof. Suppose $(Q,\cdot)$ is a RChL. Then, $(Q,\cdot)$ is an RCC and this implies that, $H = (R_x, R_x L_x^{-1}, R_x)$ is an autotopism for all $x \in Q$. Then, $H^{-1} = (R_x^{-1}, L_x R_x^{-1}, R_x^{-1})$ is also an autotopism for all $x \in Q$. But for a right Cheban loop $(Q,\cdot)$, $N = (R_x, R_x^{-1} L_x, R_x) \in AUT(Q,\cdot)$. Hence, $NH^{-1} = (I, R_x^{-1} L_x L_x R_x^{-1}, I)$ is an autotopism for all $x \in Q$.

Let $\alpha \in BS(Q,\cdot)$. Then, $F = (\alpha R_t^{-1}, \alpha L_k^{-1}, \alpha)$ is an autotopism of $(Q,\cdot)$ for some $t,k \in Q$. Thus,

$$FNH^{-1} = (\alpha R_t^{-1}, \alpha L_k^{-1} R_x^{-1} L_x L_x R_x^{-1}, \alpha).$$

Now using $t = k = x$ and by using the dual of Theorem 2.3, we have,

$$FNH^{-1} = (\alpha R_x^{-1}, \alpha L_x^{-1} R_x^{-1} L_x^2 R_x^{-1}, \alpha) = (\alpha R_x^{-1}, \alpha L_x^{-1} R_x^{-1} R_x^2 R_x^{-1}, \alpha).$$

Hence, we have

$$(\alpha R_x^{-1}, \alpha L_x^{-1} R_x^{-1} R_x R_x R_x^{-1}, \alpha) = (\alpha R_x^{-1}, \alpha L_{x^{-1}}, \alpha),$$

and by Theorem 2.1, we have

$$(J\alpha R_x^{-1} J, \alpha, \alpha L_{x^{-1}}) \in AUT(Q,\cdot), \quad \text{for all } x \in Q. \tag{4.4}$$

Therefore, for all $y,z \in Q$ we get, $yJ\alpha R_x^{-1} J \cdot z\alpha = (yz)\alpha L_{x^{-1}}$, and this becomes $y\alpha R_x^{-1} J = \alpha L_{x^{-1}}$ when $z = e$. This implies that $(\alpha L_{x^{-1}}, \alpha, \alpha L_{x^{-1}}) \in AUT(Q,\cdot)$ for all $x \in Q$. Thus, α is an automorphism of $(Q,\cdot)$. □

Remark 4.1. Robinson [21] considered the Bryant-Schneider group of a Bol loop and found out that they can be expressed as a product of pseudo-automorphisms and right translations. Theorems 4.1 and 4.2 show that the Bryant-Schneider group of right Cheban loop can also be expressed in the same way.

Theorem 4.3. *Let $(Q,\cdot)$ be a RChL and let θ be a bijection on Q such that θR_x^{-1} is an automorphism of Q, then θ is an element of the Bryant-Schneider group of $(Q,\cdot)$, $BS(Q,\cdot)$.*

Proof. Suppose $(Q,\cdot)$ is a RChL and θ is a bijection on Q such that θR_x^{-1} is an automorphism of Q, then $(\theta R_x^{-1}, \theta R_x^{-1}, \theta R_x^{-1}) \in AUT(Q,\cdot)$. By Theorem 2.2, it follows that the product

$$(\theta R_x^{-1}, \theta R_x^{-1}, \theta R_x^{-1})(R_x, R_x L_x^{-1}, R_x) = (\theta, \theta L_x^{-1}, \theta) \in AUT(Q,\cdot).$$

Hence, $\theta \in BS_\rho(Q,\cdot)$, but $BS_\rho(Q\cdot) \leq BS(Q,\cdot)$ and thus, $\theta \in BS(Q,\cdot)$. □

Theorem 4.4. *Let $(Q,\cdot)$ be a flexible RChL. Then, $a \in N_\mu(Q,\cdot)$ if and only if the middle inner mapping of Q, $T_a = R_a L_a^{-1}$, is an automorphism of $(Q,\cdot)$.*

Proof. Suppose $(Q,\cdot)$ is a flexible RChL and let $a \in N_\mu(Q,\cdot)$. Then it is a left inverse property loop and thus, $(JR_aJ, R_a, R_a^{-1}L_a) = (L_{a^{-1}}, R_a, R_a^{-1}L_a) = (L_a^{-1}, R_a, R_a^{-1}L_a) \in AUT(Q,\cdot)$ and it follows that $(R_a, L_a^{-1}, I) \in AUT(Q,\cdot)$ for every $a \in Q$. By Theorem 2.2, it follows also

$$(L_a^{-1}, R_a, R_a^{-1}L_a)(R_a, L_a^{-1}, I) = (L_a^{-1}R_a, R_aL_a^{-1}, R_a^{-1}L_a) \in AUT(Q,\cdot).$$

Thus,

$$(R_aL_a^{-1}, R_aL_a^{-1}, R_a^{-1}L_a) = (T_a, T_a, T_a) \in AUT(Q,\cdot)$$

Hence, it follows that $T_a \in AUM(Q,\cdot)$. The converse is achieved by reversing the previous steps. □

Corollary 4.1. *Let $(Q,\cdot)$ be a RChL. Then, for every $a \in N_\mu(Q,\cdot)$, L_a is an element of $BS(Q,\cdot)$.*

Proof. By Theorem 4.4, $R_aL_a^{-1} \in AUM(Q,\cdot)$ implies that $(R_aL_a^{-1})^{-1} = L_aR_a^{-1} \in AUM(Q,\cdot)$. Since $AUM(Q,\cdot)$ is a group then it follows, from Theorem 4.3, that $L_a \in BS(Q,\cdot)$. □

Corollary 4.2. *Let $(Q,\cdot)$ be a RChL. Then, for every $a \in N_\mu(Q,\cdot)$, $R_aL_a^{-1} \in BS(Q,\cdot)$.*

Proof. By Theorem 4.4, $R_aL_a^{-1} \in AUM(Q,\cdot)$. Then, the result follows from the fact that every automorphism of $(Q,\cdot)$ is an element of BS$(Q,\cdot)$. □

Corollary 4.3. *Let $(Q,\cdot)$ be a RChL. Then, for every $a \in N_\mu(Q,\cdot)$, R_a is an element of $BS(Q,\cdot)$.*

Proof. Since $BS(Q,\cdot)$ is a group, the result follows from Corollaries 4.1 and 4.2 that $R_aL_a^{-1} \cdot L_a = R_a \in BS(Q,\cdot)$. □

5. Crypto-Automorphism Group of Right Cheban Loops

Lemma 5.1. *Let $(Q,\cdot)$ be a quasigroup. A map $\phi \in SYM(Q)$ is a crypto-automorphism of $(Q,\cdot)$ with companion (a, b^λ) if and only if $(R_a\phi, L_{b^\lambda}\phi, \phi) \in AUT(Q,\cdot)$.*

Proof. It follows from Definition 2.11. □

Theorem 5.1. *Let $(Q,\cdot)$ be a quasigroup. Then, $PS_{\mu_2}(Q,\cdot) = CAUM(Q,\cdot)$.*

Proof. If ϕ is a two-middle pseudo-automorphism of $(Q,\cdot)$, then $(\phi R_a^{-1}, \phi L_{b^\lambda}^{-1}, \phi) \in AUT(Q,\cdot)$ with companion (a,b). For all $u, v \in Q$, we have

$$u\phi R_a^{-1} \cdot v\phi L_{b^\lambda}^{-1} = (uv)\phi. \tag{5.1}$$

Let $u\phi R_a^{-1} = f$ if and only if $fa = u\phi$ if and only if $u = (fa)\phi^{-1}$. Analogously, set $v\phi L_{b^\lambda}^{-1} = g$ if and only if $v\phi = b^\lambda g$ if and only if $v = (b^\lambda g)\phi^{-1}$. Substitute u and v in (5.1), to get $(fa)R_a^{-1} \cdot (b^\lambda g)L_{b^\lambda}^{-1} = \big((fa)\phi^{-1} \cdot (b^\lambda g)\phi^{-1}\big)\phi$ if and only if $(f \cdot g)\phi^{-1} = (f \cdot a)\phi^{-1} \cdot (b^\lambda \cdot g)\phi^{-1}$ if and only if $(R_a\phi^{-1}, L_{b^\lambda}\phi^{-1}, \phi^{-1}) \in AUT(Q, \cdot)$. Hence, $\phi^{-1} \in CAUM(Q, \cdot)$. So, by Theorem 2.4, $\phi \in CAUM(Q, \cdot)$. Conversely, we reverse the steps to get $\phi \in PS_{\mu_2}(Q, \cdot)$. □

Theorem 5.2. *Let U be a crypto-automorphism of RChL $(Q, \cdot)$ with companion (a, b). Then, $R_xU = \varphi \in BS(Q, \cdot)$.*

Proof. $(Q, \cdot)$ is a RChL if and only if $P = (R_x, R_x^{-1}L_x, R_x)$ is an autotopism of $(Q, \cdot)$. By Theorem 3.3, $P = (R_x, R_xL_x^{-1}, R_x) \in AUT(Q, \cdot)$ for all $x \in Q$. Let U be a crypto-automorphism of $(Q, \cdot)$. Then, $T = (R_aU, L_bU, U)$ is an autotopism of $(Q, \cdot)$. The product

(5.2)

$$PT = (R_x, R_xL_x^{-1}, R_x)(R_aU, L_bU, U) = (R_xR_aU, R_xL_x^{-1}L_bU, R_xU) \in AUT(Q, \cdot).$$

$R_xU = \varphi$ in the last autotopism implies that

$$(R_xR_aU, R_xL_x^{-1}L_bU, \varphi) \in AUT(Q, \cdot). \tag{5.3}$$

Writing this in identical relation, for all $y, z \in Q$, we have

$$yR_xR_aU \cdot zR_xL_x^{-1}L_bU = (y \cdot z)\varphi. \tag{5.4}$$

Put $y = e$ in (5.4) to obtain $eR_xR_aU \cdot zR_xL_x^{-1}L_bU = e \cdot z\varphi$, this implies

$$(ex)R_aU \cdot zR_xL_x^{-1}L_bU = z\varphi$$

and

$$zR_xL_x^{-1}L_bUL_{xaU} = z\varphi, \quad zR_xL_x^{-1}L_bU = z\varphi L_{xaU}^{-1}, \quad R_xL_x^{-1}L_bU = \varphi L_{xaU}^{-1}. \tag{5.5a}$$

Also, put $z = e$ in (5.4) to have $yR_xR_aU \cdot eR_xL_x^{-1}L_bU = y\varphi$. Then,

$$yR_xR_aU \cdot (x\backslash(ex))L_bU = y\varphi$$

and

$$yR_xR_aUR_{bU} = y\varphi, \quad yR_xR_aU = y\varphi R_{bU}^{-1}, \quad R_xR_aU = \varphi R_{bU}^{-1}. \tag{5.5}$$

Using (5.5a) and (5.5) in (5.3), we have $(\varphi R_{bU}^{-1}, \varphi L_{xaU}^{-1}, \varphi) \in AUT(Q, \cdot)$. Thus, $\varphi \in BS(Q, \cdot)$ with $(a, b) = (bU, xaU)$. □

Corollary 5.1. *Let $(Q, \cdot)$ be a RChL, and let $R_xU = \varphi$ be a middle pseudo-automorphism with companion (a, b). Then, φ^{-1} is a crypto-automorphism with companion (bU, xaU).*

Proof. Consequence of Theorem 5.1 and Theorem 5.2. □

Theorem 5.3. *Let U be a crypto-automorphism of RChL $(Q, \cdot)$ with companion (a, b). Then, $R_xR_aU = \varphi$ is a right pseudo-automorphism with companion xU, where $R_x^{-1}\varphi = R_aU$.*

Proof. Let $R_xR_aU = \varphi$ in (5.2) of Theorem 5.2 which implies that

$$(\varphi, R_xL_x^{-1}L_bU, R_xU) \tag{5.6}$$

is an autotopism of $(Q,\cdot)$. For all $y, z \in Q$, we have

$$y\varphi \cdot zR_xL_x^{-1}L_bU = (y \cdot z)R_xU. \tag{5.7}$$

Put $y = e$ in (5.6) the identity element of Q to get

$$e\varphi \odot zR_xL_x^{-1}L_bU = zR_xU, \quad zR_xL_x^{-1}L_bU = zR_xU, \quad R_xL_x^{-1}L_bU = R_xU,$$

for all $z \in Q$. Substituting the last equality in (5.6) to get

$$(\varphi, R_xU, R_xU) \in AUT(Q,\cdot). \tag{5.8}$$

Note that $e = eR_xR_aU = eR_xR_x^{-1}\varphi = e\varphi$. Thus, $e\varphi = e$. For all $y, z \in Q$, we have $y\varphi \cdot zR_xU = (y \cdot z)R_xU$. Put $z = e$ to get

$$y\varphi \cdot eR_xU = yR_xU, \quad y\varphi \cdot (exU) = yR_xU, \quad y\varphi R_{(xU)} = yR_xU, \quad \varphi R_{(xU)} = R_xU.$$

Substituting the last equality in (5.8), we get that $(\varphi, \varphi R_{xU}, \varphi R_{xU})$ is an autotopism of $(Q,\cdot)$. Thus, φ is a right pseudo-automorphism with companion xU. □

Theorem 5.4. *Let $(Q,\cdot)$ be a loop. Then, $PS_{\mu_2}(Q,\cdot) = CAUM(Q,\cdot) = BS(Q,\cdot)$.*

Proof. This follows from Theorem 5.1 and Theorem 2.5. □

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