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KEMENY'S CONSTANT OF A CYLINDER OCTAGONAL CHAIN

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Abstract. If $A(G)$ is the adjacency matrix of a graph G with n vertices and $D^{-1/2}(G)$ is the diagonal matrix of reciprocals of square roots of vertex degrees, then the Kemeny's constant of G is $K(G) = \sum_{i=2}^{n} \frac{1}{1-\lambda_i}$, where $\lambda_2, \lambda_3, \dots, \lambda_n$ are all but the largest eigenvalue of $D^{-1/2}(G)A(G)D^{-1/2}(G)$. We use an approach based on determinants of particular tridiagonal matrices admitting certain periodicity to provide a closed formula for the Kemeny's constant of a cylinder octagonal chain graph, where a graph in question is obtained from a linear octagonal chain graph by identifying the lateral edges. In this way we present the correct result of [S. Zaman, A. Ullah, Kemeny's constant and global mean first passage time of random walks on octagonal cell network, Math. Meth. Appl. Sci., 46 (2023), 9177–9186] that for the graphs in question calculated the multiple of Kirchhoff index instead.

1. Introduction

Let $G = (V, E)$ be an unoriented graph without loops or multiple edges. We write n for its order (i.e., the number of vertices), $A(G)$ for its adjacency matrix, and

$$D(G) = \operatorname{diag}(1/\sqrt{d_1}, 1/\sqrt{d_2}, \dots, 1/\sqrt{d_n}),$$

for the diagonal matrix of reciprocals of square roots of vertex degrees.

Kemeny's constant is a graph invariant that provides an interplay between Markov chains, random walks, and spectral invariants. It measures the expected number of time steps required for a Markov chain to transition from a starting state to a random destination state sampled from the Markov chain's distribution. Equivalently, it measures an average of the mean first passage times in a random walk on the vertices of a graph. Consequently, it provides an information on graph shape and

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connectivity [4]. Actually, there are two definitions of this constant. One of them says that if π_i is the stationary probability for the vertex i and m_{ji} is the expected number of steps before the vertex i is visited in a random walk starting from the vertex j, then $\widetilde{K}(G) = \sum_{i=1}^{n} \pi_i m_{ji}$ is a constant not depending on the starting vertex j. The other definition says that $K(G) = \widetilde{K}(G) - 1$. Obviously, there is no essential difference between these definitions, and to avoid confusion in this study we take that the *Kemeny's constant* is $K(G)$. This approach agrees with the classical monograph of Kemeny and Snell [11]. It occurs that

$$K(G) = \sum_{i=2}^{n} \frac{1}{1-\lambda_i}, \tag{1.1}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are non-increasingly indexed eigenvalues of the symmetric matrix $D(G)^{-1/2}A(G)D(G)^{-1/2}$ [10, 12]. We know from [10] that the eigenvalues of the previous matrix lie in the segment $[-1, 1]$, along with $1 = \lambda_1 > \lambda_2$, which means that (1.1) is defined correctly.

We denote by L_n the linear octagonal chain of n octagons. The *cylinder octagonal chain* M'_n is obtained from L_n by identifying the lateral edges (see Figure 1). The main result of this paper reads as follows.

Theorem 1.1. *Let M'_n be a cylinder octagonal chain. The Kemeny's constant of M'_n is given by*

$$K(M'_n) = \frac{147n^2 - 19}{84} + \frac{37nU_{n-1}(8)}{8U_{n-1}(8) - 2U_{n-2}(8) - 2},$$

where $U_n(8) = \frac{(4+\sqrt{15})^{n+1} - (4-\sqrt{15})^{n+1}}{2\sqrt{15}}$.

We will see in the next sections that $U_n(\cdot)$, $n \geq 0$, is the Chebyshev polynomial of the second kind; in our result computed in the point 8.

In [9], the Kirchhoff index $n\sum_{i=2}^{n} \frac{1}{\mu_i}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix of a cylinder octagonal chain, was computed. Afterwards, in [16], the authors computed the Kemeny's constant of the same graph using a wrong expression stating that the Kemeny's constant of a connected graph is the Kirchhoff index multiplied by n. In addition, their computations go through identical lines as those in [9]. The same constant is dealt correctly in [17].

The aim of this paper is twofold. First, we present a correct closed formula for the Kemeny's constant of a cylinder octagonal chain. Secondly (better say, simultaneously), we show that some computations of [9, 16, 17] can be simplified by employing known but rarely used results of [15] concerning determinants of particular tridiagonal matrices admitting certain periodicity.

The remaining content is organized in the following way. In Section 2, we revisit two relevant results from [15]. In Section 3, we prove some particular results needed for the proof of Theorem 1.1. The proof of this theorem is finalized in Section 4.

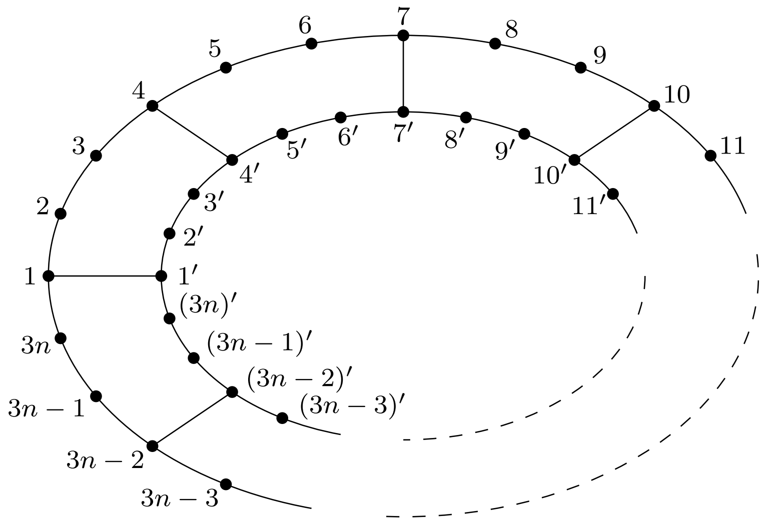

FIGURE 1. The cylinder octagonal chain M_n'.

2. Emphasizing some Results from Matrix Theory

In this section we recall some powerful, often neglected, results of [15] that give an explicit form for the computation of determinants of special kind of tridiagonal matrices, called periodic continuants. These results are also obtained in [8] using a different approach.

Accordingly,

$$D_n(a_ib_i) = \begin{pmatrix} a_1 & -b_1 & & & & \\ -b_1 & a_2 & -b_2 & & & \\ & -b_2 & & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & -b_{n-1} \\ & & & & -b_{n-1} & a_n \end{pmatrix}$$

is called a *symmetric continuant matrix.* Its determinant is called a *continuant.* By $D_{i_1,i_2,\dots,i_p}$ we denote the minor of $D_n(a_ib_i)$ obtained from its $i_1, i_2, \dots, i_p$ rows and columns.

A symmetric continuant matrix of order $nm + r$, $0 \le r \le m - 1$, in which the elements a_k, b_k are periodical modulo m, i.e., $a_{k+m} = a_k$ and $b_{k+m} = b_k$ hold, is called a *periodic continuant matrix.* These matrices are also known as m-Toeplitz tridiagonal matrices (see [7]) with applications in different contexts (see [1–3, 5, 14]). The determinant of the corresponding matrix is called a *periodic continuant* of class r and period m and it is denoted by $P_n^{(m,r)}(a_kb_k)$, $k = 1, 2, \dots, m$. According to [15], $P_n^{(m,r)}(a_kb_k)$ can be expressed using Chebyshev polynomials of the second kind $U_n(x)$.

We recall that

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}$$
$$= \frac{(x/2+\sqrt{(x/2)^2-1})^{n+1} - (x/2-\sqrt{(x/2)^2-1})^{n+1}}{2\sqrt{(x/2)^2-1}}, \quad x \neq 2,$$

as well as $U_1(x) = x$ and $U_0 = 1$. We also have $U_n(2) = n+1$.

The formula for $P_n^{(m,r)}(a_k b_k)$ reads as follows:

$$P_n^{(m,r)}(a_k b_k) = (b_1 b_2 \cdots b_m)^n \times \Big(D_{1,2,\ldots,r} U_n(x) + \frac{b_m b_1 b_2 \cdots b_r}{b_{r+1} b_{r+2} \cdots b_{m-1}} D_{r+2,r+3,\ldots,m-1} U_{n-1}(x) \Big), \tag{2.1}$$

where

$$x = \frac{D_{1,2,\ldots,m} - b_m^2 D_{2,\ldots,m-1}}{b_1 b_2 \cdots b_m}.$$

We set $D_{1,2,\ldots,r} = 1$, for $r = 0$, $D_{r+2,r+3,\ldots,m-1} = 1$, for $r = m-2$ and $D_{r+2,r+3,\ldots,m-1} = 0$, for $r = m-1$.

Remark 2.1. We point out that present definition of $U_n(x)$ gives monic polynomials and does not coincide with the one used in Wolfram Mathematica. There $U_n(x)$ is computed using `ChebyshevU [n,\frac{x}{2}]`.

3. Initial Results

In this and the next section, we write $A[i]$ and $A[i,j]$ to denote the matrix obtained from a matrix A by deleting the ith row and column, and the ith and the jth rows and columns, respectively. The following proposition is needed.

Proposition 3.1. ([6]) *Given an $n \times n$ symmetric matrix $A = (a_{ij})$ whose graph G is a cycle, say $(1, \ldots, n, 1)$, the characteristic polynomial of A is*

$$\phi_A(x) = (x - a_{ii})\phi_{A[i]}(x) - |a_{i-1,i}|^2 \phi_{A[i-1,i]}(x) - |a_{i,i+1}|^2 \phi_{A[i,i+1]}(x) - 2(a_{12} \cdots a_{n-1,n} a_{n,1}).$$

For the purpose of the current paper we will consider the periodic continuant matrices with period $m = 3$. Our starting point is the matrix of the following form:

$$A = \begin{pmatrix} a & -b & & & & & -b \\ -b & c & -d & & & & \\ & -d & c & -b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -d & a & -b & \\ & & & & -b & c & -d \\ -b & & & & & -d & c \end{pmatrix}_{3n}. \tag{3.1}$$

In the sequel we compute $\det(A[i])$, and $\det(A[i,j])$; particular principal minors of A.

Lemma 3.1. *Let A be the matrix given in* (3.1)*. Then,*

$$\det(A[i]) = (b^2d)^{n-1}\begin{cases}(ac-b^2)U_{n-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big), & \text{if } i\equiv 0 \bmod 3 \text{ or } i\equiv 2 \bmod 3;\\ (c^2-d^2)U_{n-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big), & \text{if } i\equiv 1 \bmod 3.\end{cases}$$

Proof. The underlying graph of A is the cycle C_{3n}. The removal of the ith row and ith column in A results in a matrix whose graph is the path P_{3n-1}. We distinguish three cases depending on i.

(i) If $i \equiv 0 \bmod 3$, then $A[i]$ is permutation similar to

$$\begin{pmatrix} a & -b & & & & & \\ -b & c & -d & & & & \\ & -d & c & -b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & c & -b & \\ & & & & -b & a & -b \\ & & & & & -b & c \end{pmatrix}_{3n-1}. \tag{3.2}$$

In this case we also have $3n-1 = 3(n-1)+2$, and therefore $m = 3$ and $r = 2$. Therefore, by employing (2.1) (for these m and r), we arrive at

$$\det(A[i]) = (b^2d)^{n-1}(ac-b^2)U_{n-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big).$$

(ii) If $i \equiv 1 \bmod 3$, then $A[i]$ is permutation similar to

$$\begin{pmatrix} c & -d & & & & & \\ -d & c & -b & & & & \\ & -b & a & -b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -b & a & -b & \\ & & & & -b & c & -d \\ & & & & & -d & c \end{pmatrix}_{3n-1}. \tag{3.3}$$

By (2.1),

$$\det(A[i]) = (b^2d)^{n-1}(c^2-d^2)U_{n-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big).$$

(iii) For $i \equiv 2 \bmod 3$, $A[i]$ is similar to

$$(3.4)\qquad \begin{pmatrix} c & -b & & & & & \\ -b & a & -b & & & & \\ & -b & c & -d & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & c & -d & \\ & & & & -d & c & -b \\ & & & & & -b & a \end{pmatrix}_{3n-1},$$

which leads to

$$\det(A[i]) = (b^2d)^{n-1}(ac-b^2)U_{n-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big).$$

The proof is completed. □

For the sake of conciseness, we introduce the following functions:

$$F_1^\ell(x) = (b^2d)^{n-2}(ac-b^2)^2U_{\ell-1}(x)U_{n-\ell-1}(x),$$
$$F_2^\ell(x) = (b^2d)^{n-2}(c^2-d^2)^2U_{\ell-1}(x)U_{n-\ell-1}(x),$$
$$F_3^\ell(x) = (b^2d)^{n-1}\Big(cU_\ell(x)+dU_{\ell-1}(x)\Big)\Big(U_{n-\ell-1}(x)+\frac{c}{d}U_{n-\ell-2}(x)\Big),$$
$$F_4^\ell(x) = (b^2d)^{n-1}\Big(aU_\ell(x)+\frac{b^2}{d}U_{\ell-1}(x)\Big)\Big(U_{n-\ell-1}(x)+\frac{ad}{b^2}U_{n-\ell-2}(x)\Big).$$

Lemma 3.2. *Let A be the matrix given in* (3.1), $x = \frac{ac^2-ad^2-2b^2c}{b^2d}$ *and* $\ell = \lfloor \frac{j-i}{3} \rfloor$, *for* $i<j$. *Then* $\det(A[i,j])$ *is equal to*

$$\begin{cases} F_1^\ell(x), & \text{if } i,j \equiv 0 \bmod 3 \text{ or } i,j \equiv 2 \bmod 3;\\ F_2^\ell(x), & \text{if } i,j \equiv 1 \bmod 3;\\ F_3^\ell(x), & \text{if } i \equiv 1 \bmod 3, j \equiv 0 \bmod 3 \text{ or } i \equiv 2 \bmod 3, j \equiv 1 \bmod 3;\\ F_3^{n-\ell-1}(x), & \text{if } i \equiv 0 \bmod 3, j \equiv 1 \bmod 3 \text{ or } i \equiv 1 \bmod 3, j \equiv 2 \bmod 3;\\ F_4^\ell(x), & \text{if } i \equiv 0 \bmod 3, j \equiv 2 \bmod 3;\\ F_4^{n-\ell-1}(x), & \text{if } i \equiv 2 \bmod 3, j \equiv 0 \bmod 3. \end{cases}$$

Proof. The underlying graph of $A[i,j]$ is either a path or two disjoint paths. We distinguish the following cases.

(i) $i,j \equiv 0 \bmod 3$, or $i,j \equiv 2 \bmod 3$.

Let $i=k$ and $j=k+3\ell$, for some $\ell \in \{1,2,\dots,n-1\}$. In this case, the graph of $A[i,j]$ is the disjoint union of two paths, whose matrices are of the from (3.2) with size $3(\ell-1)+2$ and (3.3) with size $3(n-\ell-1)+2$, respectively. Then,

$$\det(A[i,j]) = (b^2d)^{n-2}(ac-b^2)^2U_{\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)U_{n-\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big).$$

(ii) $i,j \equiv 1 \bmod 3$.

The graph of $A[i,j]$ is the union of two paths with matrices of the form given in (3.4). Their sizes are $3\ell+2$ and $3(n-\ell-2)+2$, respectively. By fixing i and j to be as in the previous item, we obtain

$$\det(A[i,j]) = (b^2d)^{n-2}(c^2-d^2)^2 U_{\ell-1}\Big(\frac{ac^2-2b^2c-ad^2}{b^2d}\Big)U_{n-\ell-1}\Big(\frac{ac^2-2b^2c-ad^2}{b^2d}\Big).$$

(iii) $i \equiv 1 \bmod 3$, $j \equiv 0 \bmod 3$.

Let $i=k$ and $j=k+3\ell+2$, for some $\ell \in \{0,1,\dots,n-1\}$. The underlying graph of $A[i,j]$ is either the union of an isolated vertex and a path or the union of two paths whose matrices are of the form

$$\begin{pmatrix} c & -d & & & & & \\ -d & c & -b & & & & \\ & -b & a & -b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -d & c & -b & \\ & & & & -b & a & -b \\ & & & & & -b & c \end{pmatrix}_{3\ell+1}, \quad \begin{pmatrix} a & -b & & & & & \\ -b & c & -d & & & & \\ & -d & c & -b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & a & -b & \\ & & & & -b & c & -d \\ & & & & & -d & c \end{pmatrix}_{3n-3\ell-3}.$$

In both cases, the determinant is given by

$$\begin{aligned}\det(A[i,j]) =&(b^2d)^{n-1}\Big(cU_\ell\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+dU_{\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big)\\ &\times\Big(U_{n-\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{c}{d}U_{n-\ell-2}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big).\end{aligned}$$

(iv) $i \equiv 2 \bmod 3$, $j \equiv 1 \bmod 3$.

Supposing that $i=k$ and $j=k+3\ell+2$, for some $\ell \in \{0,1,\dots,n-2\}$, similarly to the previous case we obtain

$$\begin{aligned}\det(A[i,j]) =&(b^2d)^{n-1}\Big(cU_\ell\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+dU_{\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big)\\ &\times\Big(U_{n-\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{c}{d}U_{n-\ell-2}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big).\end{aligned}$$

(v) $i \equiv 0 \bmod 3$, $j \equiv 2 \bmod 3$ or $i \equiv 2 \bmod 3$, $j \equiv 0 \bmod 3$.

Assume that $i=k$ and $j=k+3\ell+2$, for some $\ell \in \{0,1,\dots,n-2\}$ or $i=k$ and $j=k+3\ell+1$, for some $\ell \in \{0,1,\dots,n-1\}$. The underlying graph of $A[i,j]$ is either a path or the union of two paths whose matrices are

$$\begin{pmatrix} a & -b & & & & & \\ -b & c & -d & & & & \\ & -d & c & -b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & c & -d & \\ & & & & -d & c & -b \\ & & & & & -b & a \end{pmatrix}_{3\ell+1}, \quad \begin{pmatrix} c & -b & & & & & \\ -b & a & -b & & & & \\ & -b & c & -d & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -d & c & -b & \\ & & & & -b & a & -b \\ & & & & & -b & c \end{pmatrix}_{3n-3\ell-3}.$$

In the former case, we have

$$\det(A[i,j]) = (b^2d)^{n-1}\Big(aU_\ell\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{b^2}{d}U_{\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big)$$
$$\times\Big(U_{n-\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{ad}{b^2}U_{n-\ell-2}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big),$$

whereas in the latter one, it holds

$$\det(A[i,j]) =(b^2d)^{n-1}\Big(aU_{n-\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{b^2}{d}U_{n-\ell-2}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big)$$
$$\times\Big(U_\ell\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{ad}{b^2}U_{\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big).$$

(vi) $i \equiv 1 \bmod 3$, $j \equiv 2 \bmod 3$.

Let $i = k$ and $j = k + 3\ell + 1$, for some $\ell \in \{0, 1, \dots, n-1\}$. The matrices of the corresponding paths are

$$\begin{pmatrix} c & -d & & & & & \\ -d & c & -b & & & & \\ & -b & a & -b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -b & c & -d & \\ & & & & -d & c & -b \\ & & & & & -b & a \end{pmatrix}_{3\ell}, \quad \begin{pmatrix} c & -b & & & & & \\ -b & a & -b & & & & \\ & -b & c & -d & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & a & -b & \\ & & & & -b & c & -d \\ & & & & & -d & ce \end{pmatrix}_{3n-3\ell-2},$$

and, similarly to the previous cases, we obtain

$$\det(A[i,j]) =(b^2d)^{n-1}\Big(U_\ell\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{c}{d}U_{\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big)$$
$$\times\Big(cU_{n-\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+dU_{n-\ell-2}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big).$$

(vii) $i \equiv 0 \bmod 3$, $j \equiv 1 \bmod 3$.

Let $i = k$ and $j = k + 3\ell + 1$, for some $\ell \in \{0, 1, \dots, n-2\}$. Then

$$\det(A[i,j]) =(b^2d)^{n-1}\Big(cU_{n-\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+dU_{n-\ell-2}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big)$$
$$\times\Big(U_\ell\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)+\frac{c}{d}U_{\ell-1}\Big(\frac{ac^2-ad^2-2b^2c}{b^2d}\Big)\Big).$$

The proof is completed. □

4. Proof of Theorem 1.1

In this section, $\sigma(M)$ denotes the multiset of eigenvalues of a symmetric matrix M. We recall from the introductory section that M'_n denotes the cylinder octagonal chain with $6n$ vertices. If its vertices are labelled as in Figure 1, then its adjacency matrix is

$$A(M_n') = \left(\begin{array}{ccccccc|ccccccc}
0 & 1 & & & & & 1 & 1 & & & & & & \\
1 & 0 & 1 & & & & & & 0 & & & & & \\
 & 1 & 0 & 1 & & & & & & 0 & & & & \\
 & & \ddots & \ddots & \ddots & & & & & & \ddots & & & \\
 & & & 1 & 0 & 1 & & & & & & 1 & & \\
 & & & & 1 & 0 & 1 & & & & & & 0 & \\
1 & & & & & 1 & 0 & & & & & & & 0 \\
\hline
1 & & & & & & & 0 & 1 & & & & & 1 \\
 & 0 & & & & & & 1 & 0 & 1 & & & & \\
 & & 0 & & & & & & 1 & 0 & 1 & & & \\
 & & & \ddots & & & & & & \ddots & \ddots & \ddots & & \\
 & & & & 1 & & & & & & 1 & 0 & 1 & \\
 & & & & & 0 & & & & & & 1 & 0 & 1 \\
 & & & & & & 0 & 1 & & & & & 1 & 0
\end{array}\right),$$

while the diagonal matrix of vertex degrees is

$$D(M_n') = \operatorname{diag}(\underbrace{\underbrace{3,2,2}, \underbrace{3,2,2}, \ldots, \underbrace{3,2,2}}_{2n}).$$

It can be easily verified that $I - D(M_n')^{-1/2} A(M_n') D(M_n')^{-1/2}$ is equal to $S(M_n')$ that is of the following form:

$$\left(\begin{array}{ccccccc|ccccccc}
1 & -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{\sqrt{6}} & -\frac{1}{3} & & & & & & \\
-\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & & & 0 & & & & & \\
 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & & & & 0 & & & & \\
 & & \ddots & \ddots & \ddots & & & & & & \ddots & & & \\
 & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & & & & -\frac{1}{3} & & \\
 & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & & & 0 & \\
-\frac{1}{\sqrt{6}} & & & & & -\frac{1}{2} & 1 & & & & & & & 0 \\
\hline
-\frac{1}{3} & & & & & & & 1 & -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{\sqrt{6}} \\
 & 0 & & & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & \\
 & & 0 & & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\
 & & & \ddots & & & & & & \ddots & \ddots & \ddots & & \\
 & & & & -\frac{1}{3} & & & & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & \\
 & & & & & 0 & & & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\
 & & & & & & 0 & -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{2} & 1
\end{array}\right).$$

In a condensed form,

$$S(M_n') = \begin{pmatrix} P'(M_n') & R'(M_n') \\ R'(M_n') & P'(M_n') \end{pmatrix},$$

according to the suggested block partition. Taking into account that both $P'(M_n')$ and $R'(M_n')$ are symmetric, it follows from [13, Lemma 4.1] that

$$\sigma(S(G)) = \sigma(P'(M_n') + R'(M_n')) \sqcup \sigma(P'(M_n') - R'(M_n')),$$

where $\sqcup$ denotes the sum of multisets, i.e., the multiset in which the multiplicity of an element is the sum of its multiplicities in the summands. We compute

$$P(M_n') = P'(M_n') + R'(M_n') = \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{\sqrt{6}} & \\ & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\ -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{2} & 1 \end{pmatrix}$$

and

$$R(M_n') = P'(M_n') - R'(M_n') = \begin{pmatrix} \frac{4}{3} & -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{2} & \frac{4}{3} & -\frac{1}{\sqrt{6}} & \\ & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\ -\frac{1}{\sqrt{6}} & & & & & -\frac{1}{2} & 1 \end{pmatrix}.$$

For both $P(M_n')$ and $R(M_n')$, the underlying graph is the cycle C_{3n}. In addition, $P(M_n')$ is singular (confirmed later in this section). Let $\sigma(P(M_n')) = \{\mu_1, \mu_2, \dots, \mu_{3n-1}, 0\}$ and $\sigma(R(M_n')) = \{\nu_1, \nu_2, \dots, \nu_{3n}\}$. Taking into account (1.1), we obtain

$$K(M_n') = \sum_{i=2}^{3n} \frac{1}{\mu_i} + \sum_{i=1}^{3n} \frac{1}{\nu_i}.$$

Let

$$\phi_{P(M_n')}(t) = \det(tI - P(M_n')) = t^{3n} + a_{3n-1}t^{3n-1} + \cdots + a_1 t$$

and

$$\phi_{R(M_n')}(t) = \det(tI - R(M_n')) = t^{3n} + b_{3n-1}t^{3n-1} + \cdots + b_1 t + b_0.$$

Then

$$\sum_{i=2}^{3n} \frac{1}{\mu_i} = -\frac{a_2}{a_1} \quad \text{and} \quad \sum_{i=1}^{3n} \frac{1}{\nu_i} = -\frac{b_1}{b_0}.$$

In addition,

$$a_1 = (-1)^{3n-1} \textstyle\sum_{i=1}^{3n} \det(P(M_n')[i]) \quad \text{and} \quad a_2 = (-1)^{3n-2} \sum_{1\leq i<j\leq 3n} \det(P(M_n')[i,j]),$$

along with

$$b_1 = (-1)^{3n-1} \textstyle\sum_{i=1}^{3n} \det(R(M_n')[i]) \quad \text{and} \quad b_0 = (-1)^{3n} \det(R(M_n')).$$

Computations of these coefficients are separated in the following itemization. We apply Lemmas 3.1 and 3.2 for $a = \frac{2}{3}$ in $P(M_n')$, and $a = \frac{4}{3}$ in $R(M_n')$, and $b = \frac{1}{\sqrt{6}}$, $c = 1$, $d = \frac{1}{2}$ in both.

- By Lemma 3.1, for $\frac{ac^2-ad^2-2b^2c}{b^2d}=2$, we obtain

$$\begin{aligned}
a_1 &= (-1)^{3n-1}\sum_{i=1}^{3n}\det(P(M'_n)[i]) \\
&= (-1)^{3n-1}\Big(\frac{n}{12^{n-1}}U_{n-1}(2)+\frac{3n}{4\cdot 12^{n-1}}U_{n-1}(2)\Big) \\
&= (-1)^{3n-1}\frac{7n}{4\cdot 12^{n-1}}U_{n-1}(2) = (-1)^{3n-1}\frac{7n^2}{4\cdot 12^{n-1}} = (-1)^{3n-1}\frac{21n^2}{12^n}.
\end{aligned}$$

- Similarly,

$$\begin{aligned}
b_1 &= (-1)^{3n-1}\sum_{i=1}^{3n}\det(R(M'_n)[i]) \\
&= (-1)^{3n-1}\frac{1}{12^{n-1}}\Big(2n\cdot\frac{7}{6}U_{n-1}(8)+\frac{3}{4}nU_{n-1}(8)\Big) \\
&= (-1)^{3n-1}\frac{37n}{12^n}U_{n-1}(8).
\end{aligned}$$

- By Lemma 3.2,

$$\begin{aligned}
(-1)^{3n-2}a_2 &= \sum_{\ell=1}^{n-1}(n-\ell)\Big(2F_1^{\ell}(2)+F_2^{\ell}(2)\Big) \\
&\quad+\sum_{\ell=0}^{n-1}(n-\ell)\Big(F_3^{\ell}(2)+F_3^{n-\ell-1}(2)+F_4^{n-\ell-1}(2)\Big) \\
&\quad+\sum_{\ell=0}^{n-2}(n-\ell-1)\Big(F_3^{\ell}(2)+F_3^{n-\ell-1}(2)+F_4^{\ell}(2)\Big) \\
&= \sum_{\ell=1}^{n-1}\frac{17\ell(n-\ell)^2}{16\cdot 12^{n-2}} \\
&\quad+\sum_{\ell=0}^{n-1}\frac{(n-\ell)}{12^{n-1}}\Big(\frac{(3\ell+2)(3(n-\ell)-2)}{2}+\frac{5(3\ell+1)(3(n-\ell)-1)}{6}\Big) \\
&\quad+\sum_{\ell=0}^{n-2}\frac{(n-\ell-1)}{12^{n-1}}\Big(\frac{5(3\ell+2)(3(n-\ell)-2)}{6}+\frac{(3\ell+1)(3(n-\ell)-1)}{2}\Big) \\
&= \frac{17}{16\cdot 12^{n-2}}\sum_{\ell=1}^{n-1}\ell(n-\ell)^2 \\
&\quad+\frac{1}{12^{n-1}}\sum_{\ell=0}^{n-2}\Big(\frac{(3\ell+2)(3(n-\ell)-2)(8(n-\ell)-5)}{6} \\
&\quad+\frac{(3\ell+1)(3(n-\ell)-1)(8(n-\ell)-3)}{6}\Big)+\frac{39n-23}{6\cdot 12^{n-1}} \\
&= \frac{n^2(147n^2-19)}{48\cdot 12^{n-1}}.
\end{aligned}$$

- By Proposition 3.1 applied for $i=1$, we have

$$\begin{aligned}
b_0 &= (-1)^{3n}\Big(-\frac{4}{3}(-1)^{3n-1}\det(R(M_n')[1]) - \frac{1}{6}(-1)^{3n-2}\det(R(M_n')[1,3n]) \\
&\quad -\frac{1}{6}(-1)^{3n-2}\det(R(M_n')[1,2]) - \frac{2(-1)^n}{12^n}\Big) \\
&= (-1)^{3n}\Big(-\frac{4}{3}(-1)^{3n-1}\frac{3}{4\cdot 12^{n-1}}U_{n-1}(8) - \frac{(-1)^{3n-2}}{6\cdot 12^{n-1}}\Big(U_{n-1}(8)+\frac{1}{2}U_{n-2}(8)\Big) \\
&\quad -\frac{(-1)^{3n-2}}{6\cdot 12^{n-1}}\Big(U_{n-1}(8)+\frac{1}{2}U_{n-2}(8)\Big) - \frac{2(-1)^n}{12^n}\Big) \\
&= \frac{1}{12^{n-1}}U_{n-1}(8) - \frac{1}{6\cdot 12^{n-1}}(2U_{n-1}(8)+U_{n-2}(8)) - \frac{2}{12^n} \\
&= \frac{1}{12^n}\Big(8U_{n-1}(8) - 2U_{n-2}(8) - 2\Big).
\end{aligned}$$

We also verify that $a_0=0$, i.e., that $P(M_n')$ is singular:

$$\begin{aligned}
a_0 &= (-1)^{3n}\Big(-\frac{2}{3}(-1)^{3n-1}\det(R(M_n')[1]) - \frac{1}{6}(-1)^{3n-2}\det(R(M_n')[1,3n]) \\
&\quad -\frac{1}{6}(-1)^{3n-2}\det(R(M_n')[1,2]) - \frac{2(-1)^n}{12^n}\Big) \\
&= (-1)^{3n}\Big(-\frac{2}{3}(-1)^{3n-1}\frac{3}{4\cdot 12^{n-1}}U_{n-1}(2) - \frac{(-1)^{3n-2}}{6\cdot 12^{n-1}}\Big(U_{n-1}(2)+\frac{1}{2}U_{n-2}(2)\Big) \\
&\quad -\frac{(-1)^{3n-2}}{6\cdot 12^{n-1}}\Big(U_{n-1}(2)+\frac{1}{2}U_{n-2}(2)\Big) - \frac{2}{12^n}\Big) \\
&= (-1)^{3n}\Big(\frac{n(-1)^{3n-1}}{2\cdot 12^{n-1}} - \frac{(-1)^{3n-2}}{6\cdot 12^{n-1}}(2n+n-1) - \frac{2}{12^n}\Big) \\
&= \frac{1}{6\cdot 12^{n-1}} - \frac{2}{12^n} = 0.
\end{aligned}$$

Gathering the previous results, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Taking into account the previous computation, we deduce

$$K(M_n') = -\frac{a_2}{a_1} - \frac{b_1}{b_0} = \frac{147n^2-19}{84} + \frac{37(-1)^{3n}nU_{n-1}(8)}{8U_{n-1}(8)-2U_{n-2}(8)-2},$$

which completes the proof. □

We conclude this section with an example.

Example 4.1. For $n=4$, we obtain $K(M_4') = \frac{9847}{210} \approx 46.8905$, whereas the value obtained in [16] from

$$\frac{9n^2-1}{12} + \frac{17\sqrt{15}n}{30}\cdot\frac{(4+\sqrt{15})^n-(4-\sqrt{15})^n}{(4+\sqrt{15})^n+(4-\sqrt{15})^n+2}$$

gives approximately 20.6909. These values are different, due to the incorrect expression that was employed in [16]. It is worth mentioning that in [17], the correct expression was used, and there one can find the periodic continuants, as well. We believe that our approach can also simplify calculations of that paper.

We emphasise that a similar approach could be applied to compute the Kemeny's constant of cylinder k-gonal chains, for k even, constructed analogously. These graphs would be of order $(k-2)n$ for some n. Clearly, all computations will be more complex due to increased number of cases modulo $k-1$ (see Lemma 3.2) that should be considered.

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