Distributed order fractional wave equation

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In this work, the classical wave equation is generalized for the case of viscoelastic materials by the use of distributed order fractional model, and describe wave propagation in infinite viscoelastic media. We consider, analyze and solve the distributed order wave equation given as system:

$$\begin{aligned} \frac{\partial}{\partial x}\sigma(x,t) &= \rho\,\frac{\partial^2}{\partial t^2}u(x,t), \\ \int_0^1 \phi_\sigma(\alpha)\,{}_0D_t^\alpha\sigma(x,t)\,d\alpha, &= E\int_0^1 \phi_\epsilon(\alpha)\,{}_0D_t^\alpha\epsilon(x,t)\,d\alpha \\ \epsilon(x,t) &= \frac{\partial}{\partial x}u(x,t), \end{aligned}$$

where u, σ and ϵ are displacement, stress and strain, x real number and $t > 0$, $\rho = \text{const.}$ is the density of the media, $E = \text{const.}$ is the generalized Young modulus of elasticity, and ϕ_σ and ϕ_ϵ are constitutive functions or distributions, describing material properties. The left, resp. the right hand side in the second equation is a distributed order fractional derivative of σ, resp. ϵ, with ${}_0D_t^\alpha$ being Riemann-Liouville fractional derivative of order α .

The first equation is the equation of motion and it is a consequence of the Second Newton Law. The second equation is the constitutive equation of distributed order fractional type, and the third equation is the strain measure for small local deformations. In fact, the system is derived from the basic equations of elasticity, where the equation of motion and the strain measure are preserved, since they hold true for any type of deformable body, and only the constitutive equation, which is the Hooke law for an elastic body, is changed by distributed order fractional model, and thus adapted for viscoelastic type media.

We study existence and uniqueness of fundamental solutions for the generalized Cauchy problem corresponding to distributed order wave equation. As consequence, we establish existence, uniqueness, and obtain explicit form of the solution to a class of wave equations, corresponding to the linear fractional order constitutive models, and we also study a genuine distributed order wave equation. The wave speed is found to be connected with the material properties at initial time instant, more precisely with the glass modulus.

References

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