Groups of automorphisms and integrability in finite terms

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Elementary functions are easy to differentiate but hard to integrate: an indefinite integral of an elementary function is usually not an elementary function [1].

Theorem 1 (Liouville's theorem). *An integral y of $f \in K$ where K is a differential field belongs to an elementary extension of K if and only if y is representable in the form*

$$y(x) = \int_{x_0}^{x} f(t)\, dt = A_0(x) + \sum_{i=1}^{n} \lambda_i \ln A_i(x),$$

where A_i are functions in the field K for $i = 0, \dots, n$.

For large classes of functions algorithms based on this theorem make it possible to either evaluate an integral or to prove that the integral cannot be "evaluated in finite terms".

In the talk I will discuss a proof of the Liouville's theorem. I will show that it can be proved by the Galois theory arguments. Liouvill's theorem is based on two statements. The first one suggested by Abel deals with a finite group of automorphisms. The second statement deals with an n-dimensional commutative Lie group of automorphisms.

References

[1] J. F. Ritt, Integration in Finite Terms, Columbia Univ. Press, New York, 1948.