On certain mappings onto Ricci symmetric manifolds

Volodymyr E. Berezovskii[1], Irena Hinterleitner[2], Josef Mikeš[3], and Lenka Rýparová[3]

[1]Department of Mathematics and Physics, Uman National University of Horticulture, berez.volod@rambler.ru
[2]Department of Mathematics, Brno University of Technology, hinterleitner.i@fce.vutbr.cz
[3]Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, josef.mikes@upol.cz, lenka.ryparova01@upol.cz

1 On mappings onto Ricci symmetric manifolds

Let us recall that manifolds with affine connection and Riemannian manifolds are called *Ricci symmetric* if in them the Ricci tensor is absolutely parallel.

Let $f\colon M_n \to \bar{M}_n$ be a diffeomorphism (possibly a bijection of "sufficiently high" differentiability class) between n-dimensional manifolds M_n and $\bar{M}_n$. The above considerations allow us to suppose that the manifolds in fact coincide, $\bar{M} \equiv M$.

Hence $A_n = (M, \nabla)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla}) \equiv (\bar{M}, \nabla)$ be manifolds M and $\bar{M}$ with affine connections ∇ and $\bar{\nabla}$, respectively. Then the type (1,2) tensor field

$$P = \bar{\nabla} - \nabla$$

is called the *deformation tensor* of the connections ∇ and $\bar{\nabla}$ with respect to f on M.

We proved that A_n admits mapping f onto Ricci symmetric manifolds $\bar{A}_n$ if and only if it satisfies the following equation

$$\nabla_m \bar{R}_{ij} = P^{\alpha}_{mi} \bar{R}_{\alpha j} + P^{\alpha}_{mj} \bar{R}_{i\alpha},$$

where $\bar{R}_{ij}$ are components of the Ricci tensor on $\bar{A}_n$.

Here we considered certain special mappings onto Ricci symmetric manifolds.

2 Conformal mappings onto Ricci symmetric manifolds

The following Theorem 2.1 ([1]) is devoted to conformal mappings onto Ricci symmetric manifolds. These results are practically generalized results which we have obtained for conformal mappings of Riemannian manifolds onto Einstein spaces [3, 5] and geodesic mappings of manifolds with affine connection onto symmetric Riemannian manifolds [4, 6, 7].

Theorem 2.1. *n-dimensinal (pseudo-) Riemannian manifold V_n admits conformal mapping onto Ricci symmetric (pseudo-) Riemannian manifold $\bar{V}_n$ if and only if on V_n exists a solution of the following closed Cauchy type equations system in covariant derivative respective unknown functions $\psi(x)$, $\psi_i(x)$, $\mu(x)$, and $\bar{R}_{ij}(x)$ $\big(= \bar{R}_{ji}(x)\big)$:*

$$\begin{gathered}
\psi_{,i} = \psi_i \\
\psi_{i,j} = \frac{\mu}{n-2}\, g_{ij} + \psi_i\psi_j - \frac{1}{n-2}\left(\bar{R}_{ij} - R_{ij}\right), \\
\bar{R}_{ij,k} = 2\psi_k \bar{R}_{ij} + \psi_i \bar{R}_{jk} + \psi_j \bar{R}_{ik} - \psi^\alpha \bar{R}_{i\alpha} g_{jk} - \psi^\alpha \bar{R}_{j\alpha} g_{ik}, \\
(n-1)\mu_{,k} = g^{\alpha\beta}\big((n-2)\psi_\gamma R^\gamma_{\beta k\alpha} - (n-1)\psi_\beta \bar{R}_{\alpha k} - \psi_\beta R_{\alpha k}\big) + \left(R + (n-1)\mu\right)\psi_k - \frac{R_{,k}}{2},
\end{gathered}$$

where comma denotes covariant derivative on V_n.

Here functions $\bar{R}_{ij}(x)$ are components of the Ricci tensor on manifold $\bar{V}_n$.

3 Geodesic mappings onto Ricci symmetric manifolds

Analogical results were obtained for geodesic mappings [2].

Theorem 3.1. *n-dimensinal manifold A_n with affine connection admits geodesic mapping onto Ricci symmetric manifold $\bar{A}_n$ with affine connection if and only if on A_n exists a solution of the following closed Cauchy type equations system in covariant derivative respective unknown functions $\psi_i(x)$ and $\bar{R}_{ij}(x)$*

$$\begin{gathered}
\bar{R}_{ij,m} = 2\psi_m \bar{R}_{ij} + \psi_i \bar{R}_{mj} + \psi_j \bar{R}_{im}, \\
\psi_{i,j} = \frac{1}{n^2-1}\big(n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\big) + \psi_i\psi_j,
\end{gathered}$$

where comma denotes covariant derivative on A_n.

Here functions $\bar{R}_{ij}(x)$ are components of the Ricci tensor on manifold $\bar{A}_n$.

We set a number of principian parameters on which depend the general solutions of the Cauchy type equation systems in Theorems 2.1 and 3.1.

It is interesting to note that if Ricci symmetric manifold $\bar{A}_n$ is equiaffine (i.e., the Ricci tensor is symmetric), then integrability condition of equations in Theorem 3.1 is linear respective unknown functions $\psi_i(x)$ and $\bar{R}_{ij}(x)$.

References

[1] V. E. Berezovskii, I. Hinterleitner, N. I. Guseva and J. Mikeš, Conformal mappings of Riemannian spaces onto Ricci symmetric spaces, Math. Notes **103** (2018), 304–307.

[2] V. Berezovskii, I. Hinterleitner and J. Mikeš, Geodesic mappings of manifolds with affine connection onto the Ricci symmetric manifolds, Filomat **32**(2) (2018), 379–385.

[3] H. W. Brinkmann, Einstein spaces which mapped conformally on each other, Math. Ann. **94** (1925), 119–145.

[4] J. Mikeš and V. Berezovski, Geodesic mappings of affine-connected spaces onto Riemannian spaces, Colloquia mathematica Societatis János Bolyai 56 (1992), 491–494.

[5] J. Mikeš, M. L. Gavril'chenko and E. I. Gladysheva, Conformal mappings onto Einstein spaces, Moscow Univ. Math. Bull. **49** (1994), 10–14.

[6] J. Mikeš et al. Differential Geometry of Special Mappings, Palacky University Press, Olomouc, 2015.

[7] N. S. Sinyukov, Geodesic Mappings of Riemannian Spaces, Nauka, Moscow, 1979.