# Computer classification of fundamental domains of plane discontinuous groups

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H. Poincaré (1882) attempted to describe a plane crystallographic group in the Bolyai-Lobachevsky hyperbolic plane $\mathbf{H}^2$ by appropriate fundamental polygon. This initiative he extended also to space. B. N. Delone (Delaunay) in 1960's refreshed this very hard topic for Euclidean space groups by the so-called stereohedron problem: to give all fundamental domains for a given space group, with few partial results.

A. M. Macbeath (1967) completed the initiative of H. Poincaré in classifying the 2-orbifolds by giving each with a signature. That is by a base surface with orientable or non-orientable genus, by some singular points on it, as rotational centres with given periods, by some boundary components, in each with given dihedral corners. All these are characterized up to an equivariant isomorphism, also indicated in this talk. There is a nice curvature formula that describes whether the above (good) orbifold, i.e., co-compact plane group (with compact fundamental domain) is realizable either in the sphere $\mathbf{S}^2$, or in the Euclidean plane $\mathbf{E}^2$, or in the hyperbolic plane $\mathbf{H}^2$, respectively.

Our initiative in 1990's was to combine the two above descriptions. Namely, how to give all the combinatorially different fundamental domains for any above plane group. Z. Lučić and E. Molnár completed this by a graph theoretical tree enumeration algorithm. That time N. Vasiljević implemented this algorithm to computer (program COMCLASS), of super-exponential complexity, by certain new ideas as well.

In the time of the Yugoslav war we lost our manuscript, then the new one has been surprisingly rejected (?!). Now we have refreshed our manuscript to submit again and that is to appear as [1]. Here we intend to present a report on it, also with some new problems.

## References

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